Valuing American and European options using Binomial model.

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1. Introduction

The market value of a stock option is decided by active trading in exchanges. The price, or value, of stock options is influenced by the strike price, the initial value of the stock, the stocks volatility, dividends, the risk-free interest rate and the time to maturity. Based, on theoretical stochastic models for the movement of the underlying stock, a theoretical value for the stock option can be determined. In this report, the focus is on pricing American and European stock options using the Binomial model for the stock movement.

In this model, the stock price is modeled to be a constant S(k) in the interval [(k-1) Td, k Td), and across intervals S(k+1) = u S(k) or S(k+1) = d S(k), where u > 1 and d < 1. The values of u and d are related to the volatility of the (11.6) and (11.7) in the book ``Options, Futures and Other Derivatives'' by J. Hull (henceforth referred to as `the book'). The Binomial pricing model is obtained from this model for the movement of the stock using no arbitrage arguments. We refer the reader to chapter 11 of the book for the exact details.

The price determined by the Binomial model mainly depends on the number of time-intervals(n) the Time period T has been divided into. We will later see in part (2) that the price found using the binomial model converge to the price obtained by the renowned Black-Scholes-Merton model for large n.

2. Algorithm and Implementation

We start by discussing the algorithm. The program takes the following inputs

1. Spot price of stock
2. Volatility of stock
3. Continuous dividend yield of the stock
4. Risk free interest rate
5. Strike price of the option
6. Time to maturity

7. No. of steps of the Binomial model

The program outputs the price of the option. The program starts with an array of size n+1, in which each element corresponds to a leaf of the Binomial tree. The algorithm consists of n iterations and in iteration i an array of size n+1 - i is generated, in which each element corresponds to a node that in the tree that is at height i-1 from the leaves. In the next iteration, neighboring cells are combined to generate another array of size i-1 according to the principles outlined in the book. We next discuss the algorithm.

Algorithm:

Step 1: Find n possible values for the stock price

$$S_i = S_o u^i d^{n-i}, \qquad i \in \{1,..n\}$$

Step 2: Evaluate the payoff for each possible stock price using the payoff function f defined in the Chapter 11 of the book.

$$f_i(j) = f(S_i), \qquad i,j \in \{1,..n\}$$

Note that the payoff function depends on the nature of the option, i.e., whether the option is Call or Put, and American or European.

Step 3: Use the one-step binomial model and determine the price at the previous time instant. Use f_i(j) and $f_{i+1}(j)$ and find the new $f_i(j-1)$. Reduce the size of array by 1(by placing zeros)

$$f_i - e^{-r\left(\frac{T}{n}\right)}(f_i p^* + f_{i+1}(1-p^*), \qquad i \in \{1,..n-1\}$$

Where p* is risk-neutral probability defined as

$$p^* = \frac{e^{(r-q)T/n} - d}{u-d}$$

Step 4: If the option is American find $max(f_i, S_i)$ overwrite to f_i.

Step 5: Repeat Step 3 and Step 4 till the size of the array is 1.(ignoring 0's)

Code:

The algorithm is implemented in 2 different programming languages, Matlab and C++. Both the programs are submitted for the project.

Memory usage:

The memory used is an array of size n+1. Though n(n+1)/2 variables have to be saved in total during the execution of the program, only a few of them have to be saved at any point because of the recursive algorithm. The i-th recursive step depends only on the data obtained from the (i-1)th step and doesn't depend on the old data. Hence we can overwrite the existing variables and avoid using new memory allocations.

3. Convergence of the Binomial model to the Black-Scholes-Merton Model [Question 2]

Goal: To verify that as n increases the price using the Binomial model converges to the price obtained from the BSM model.

Using Black-Scholes-Merton Model: The price of a European call option with continuous dividend yield using Black-Scholes-Merton Model is given by

$$c = e^{-rT}(F\Phi(d_1) - K\Phi(d_2))$$

where $F = Se^{(r-q)T}$,

$$d_1 = \frac{(\ln(\frac{S}{K}) + (r - q + \frac{\sigma^2}{2})T)}{\sigma\sqrt{T}}, \quad d_2 = \frac{(\ln(\frac{S}{K}) + (r - q - \frac{\sigma^2}{2})T)}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

The price of a European call option with continuous dividend yield is computed to be **c=$8.1026**.

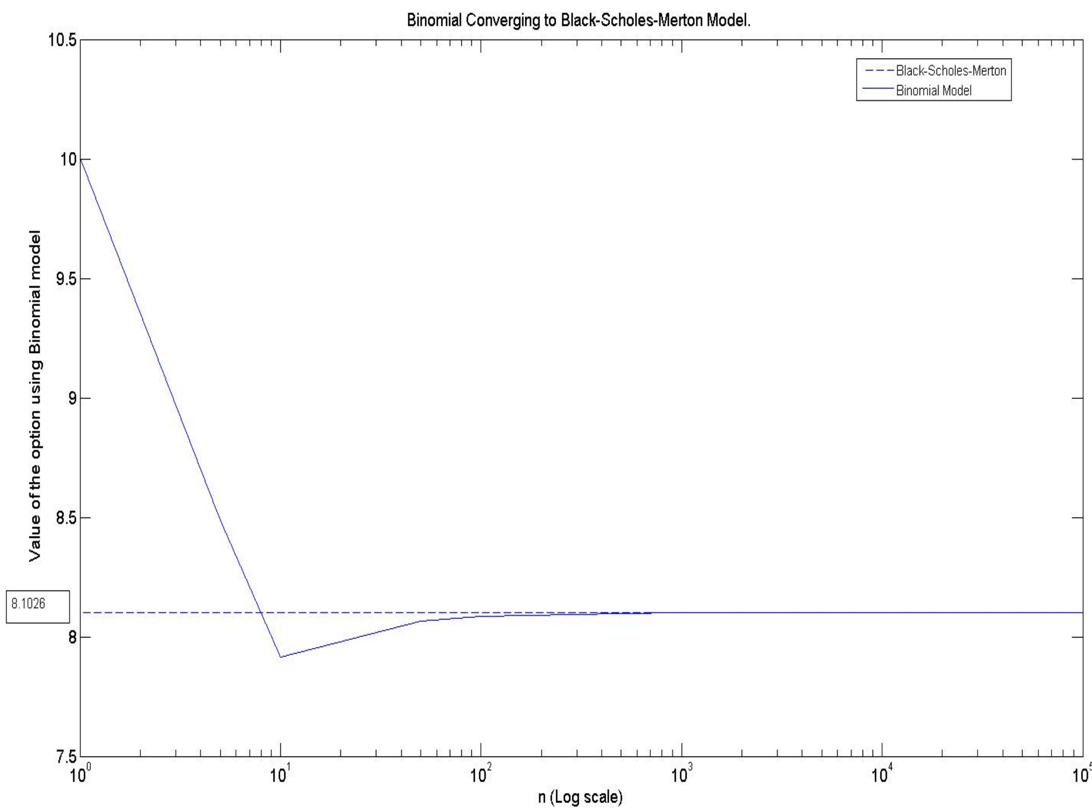
Using Binomial Model:

The price of the call option using the Binomial model implemented in part (a) for varying values of n is shown below:

N	Call price	Error(w.r.t B-S-M model)
1	10.0064	1.9038

5	8.4864	0. 3838
10	7.9147	-0.1879
50	8.0646	-0.0380
100	8.0836	-0.0190
1000	8.1007	-0.0019
10000	8.1025	-0.0001
100000	8.1026	0

The Plot showing the convergence of Binomial model to Black-Scholes-Merton model is shown below.



Conclusion: Error decreases as the number of steps used (n) in pricing the call with the Binomial model increases. For n>1000, there is almost no error in determining the price.

4. Relation between American put option value and the initial stock price [Ques 3A]

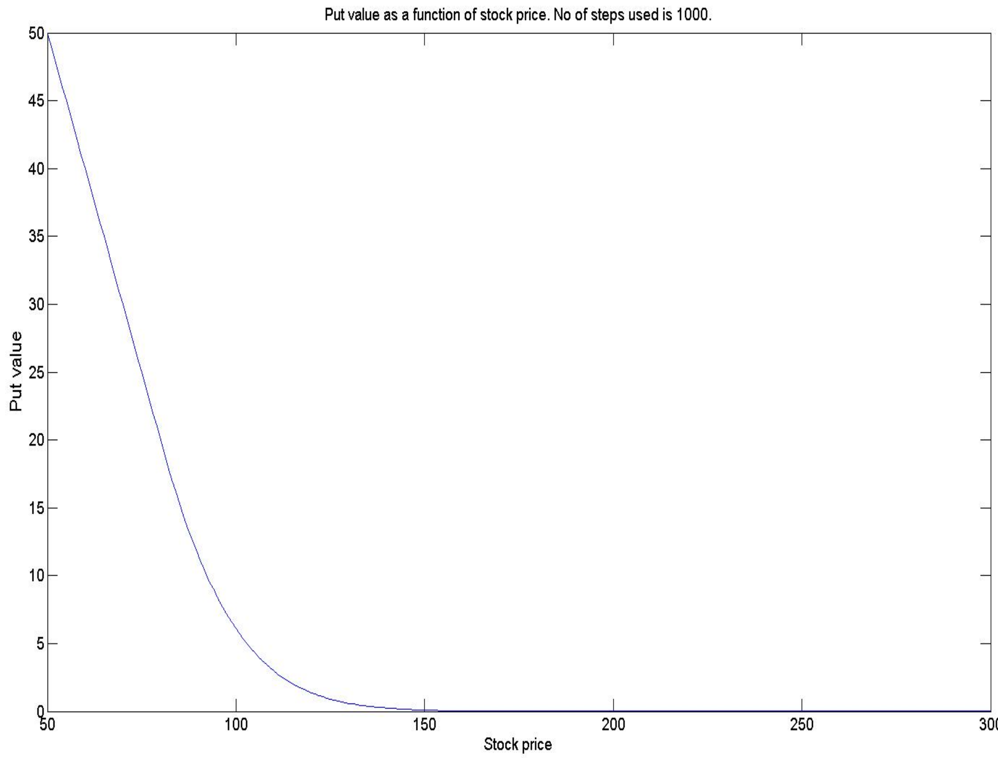
As the initial stock price increases much beyond the strike price K, the final stock price will be greater than K with a high probability reiterating that the price of the put option should be very small for high values of S. This can be seen in the following figure which shows the variation of put price p as a function of the S, the initial stock price.

A. Result:

The following table summarizes the results.

Stock Value	Put price	Stock Value	Put price
50	50	125	0.87148
55	45	130	0.578051
60	40	135	0.397247
65	35	140	0.219787
70	30	145	0.13167
75	25	150	0.0921047
80	20	155	0.0530279
85	15.3348	160	0.0212524
90	11.4351	165	0.0156553
95	8.54121	170	0.0103056
100	6.01855	175	0.00495591
105	4.40164	180	0.00135926
110	3.03539	185	0.00101833
115	2.02434	190	0.000677402

For S= $190, K=$100, the option price is almost zero.



5. Effect of the dividend on the critical early-exercise initial stock price for an American put option [Ques 3b]

A. Algorithm:

Issues: To find the exact value of S*(12) (upto 2 digits), we need to check for every S possible, if there is an early exercise possible. This takes a lot of time computer time. Instead we do the search in 2 steps.

Coarse tuning: For values of S with wider intervals, [70-80,80-90,90-95 etc], check if an early exercise is possible and find the interval where the S* lies.

This can be found by seeing where the "indicator function" of early exercise goes off as S varies.

Finer tuning: We have a comparatively smaller interval to check and we split the interval into widths of 0.1 and find the value where the "indicator function" goes off.

Finest tuning: Repeat the same for the finer interval by dividing it into subintervals of width of 0.01

Implementation:

Indicator function: The indicator function tells when an early exercise occurs. This can be implemented by comparing the option value(f) at a node to the option intrinsic value (the value if the option is exercised at that point.). This has been included in the Binomialearly.m function of matlab code given. And the code for fine tuning is given in earlyput.m. This is also implemented in C++. Both the programs are attached.

B. Results:

Q3b(i) Intitial stock price S*(12) for which it becomes optimal to early exercise is 179.5500

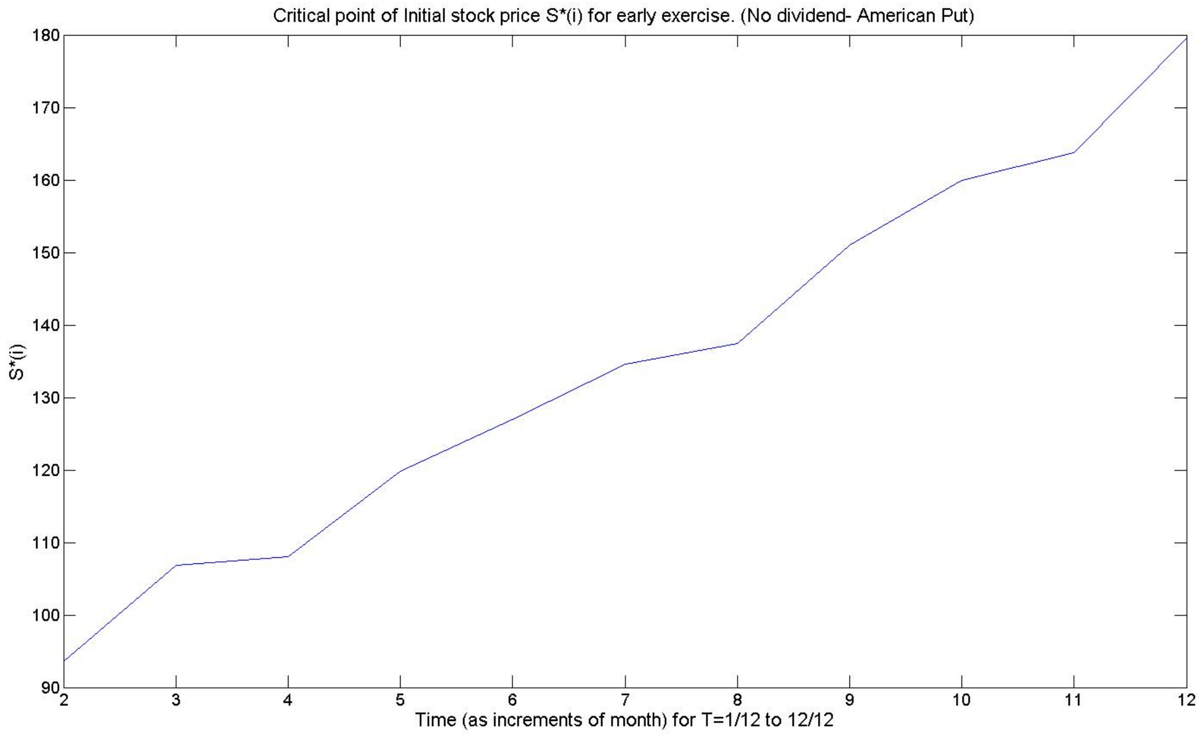
Q3b(ii) Initial Stock price S*(i) as a function of Time T=i/12.

In this case, we fix the time interval dT=1 month and look at the variation of S*(i) as a function of T =i months till T=12 months.

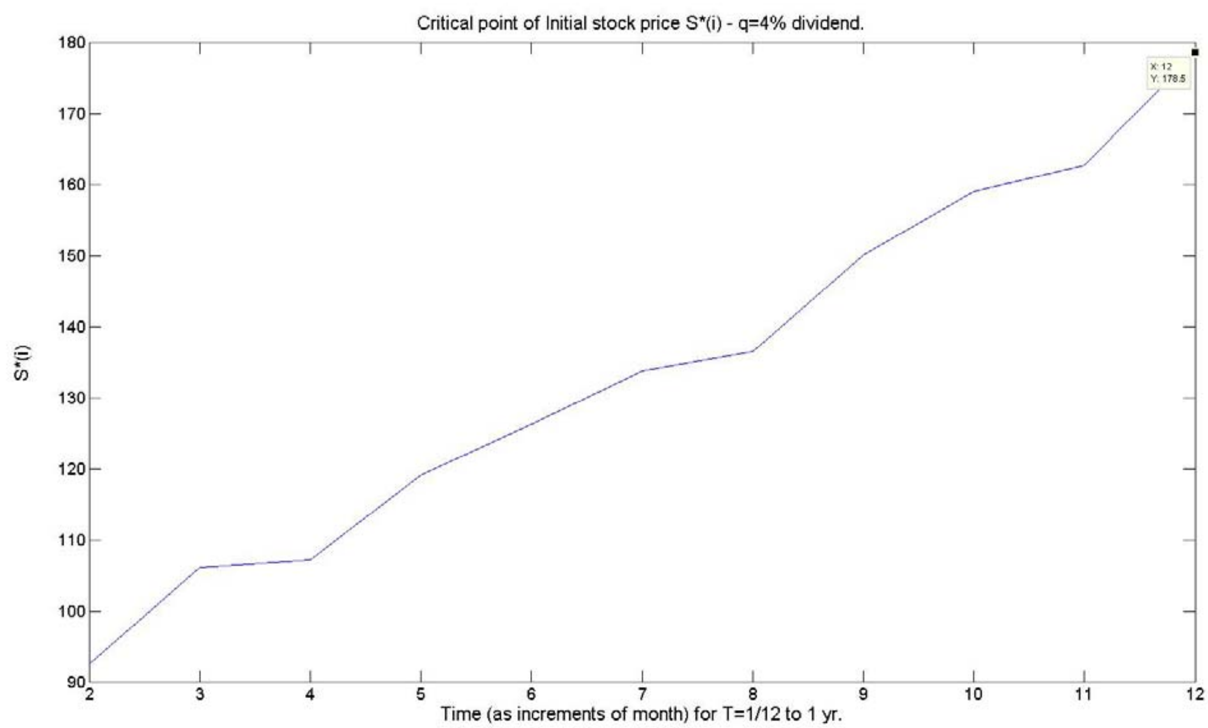
Stock price S*(i) as a function of i for no-dividend American put option and a continuous dividend yield American put option are given below.

Time in months	Stock price-No dividend	Stock price q=0.04
2	93.61	92.56
3	106.79	106.16
4	108.02	107.19
5	119.86	119.15
6	126.98	126.23
7	134.53	133.74
8	137.39	136.47
9	151	150.11
10	159.97	159.03
11	163.75	162.69
12	179.55	178.49

Plot of the stock price as a function of i for no dividend option.



Plot of the stock price as a function of i with a continuous dividend yield q=0.04.



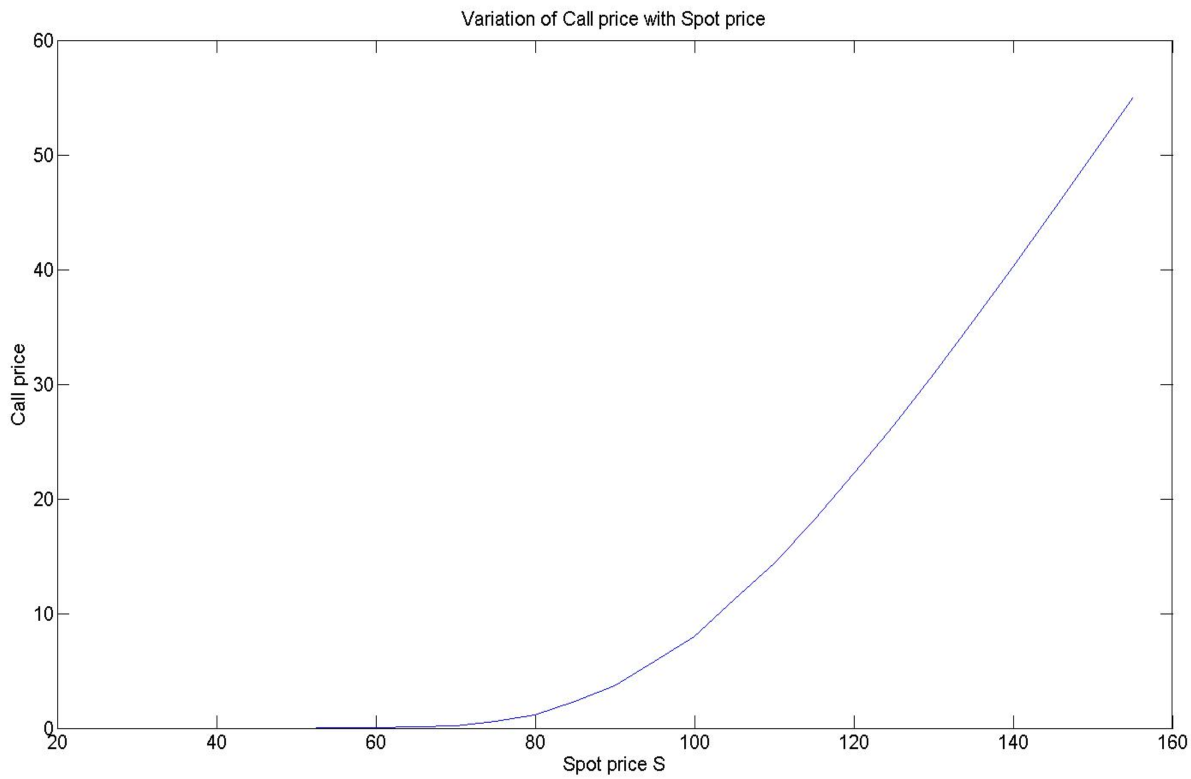
C. **Conclusions**: We observe that the dividend has the effect of reducing the optimal spot price vale. An American option is exercised when its current value is higher than its risk-neutral expected value. The current value of the put option increases with the dividend. The risk-neutral expected value increases with a decrease in the spot price. Thus when the dividend is higher we can allow for a lower spot price, and still exercise early.

6. Call value as a function of the initial stock price for an American Call option [Ques 4].

This problem is very similar to the previous problem. The only difference is that this concerns a call option, rather than a put option. The implementation details are identical, with suitable changes made to account for the fact that the option is now a call. The results are summarized below.

Results: Call price as a function of Spot price

Spot Price	Call Price	Spot Price	Call Price
30	0	95	5.80576
35	0	100	7.96044
40	0	105	11.1911
45	0	110	14.4298
50	0	115	18.1473
55	0.001944	120	22.25
60	0.020467	125	26.3766
65	0.086958	130	30.9058
70	0.218609	135	35.5528
75	0.633357	140	40.236
80	1.18088	145	45.0898
85	2.32676	150	50
90	3.65499	155	55



Intitial stock price S*(12) for which it becomes optimal to early exercise is $66.2 with a 0.04 continuous dividend yield

Initial Stock price S*(i) as a function of Time T=i/12.

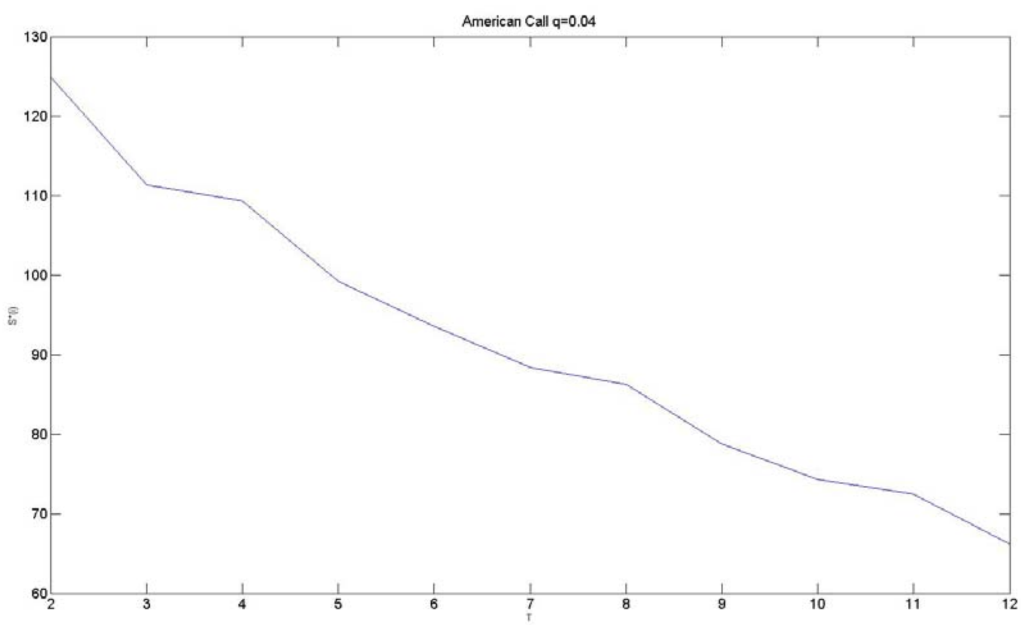
In this case, we fix the time interval dT=1 month and look at the variation of S*(i) as a function of T =i months till T=12 months.

Stock price S*(i) as a function of i for q=0.04 dividend American call option and a q=0.08 continuous dividend yield American call option are given below.

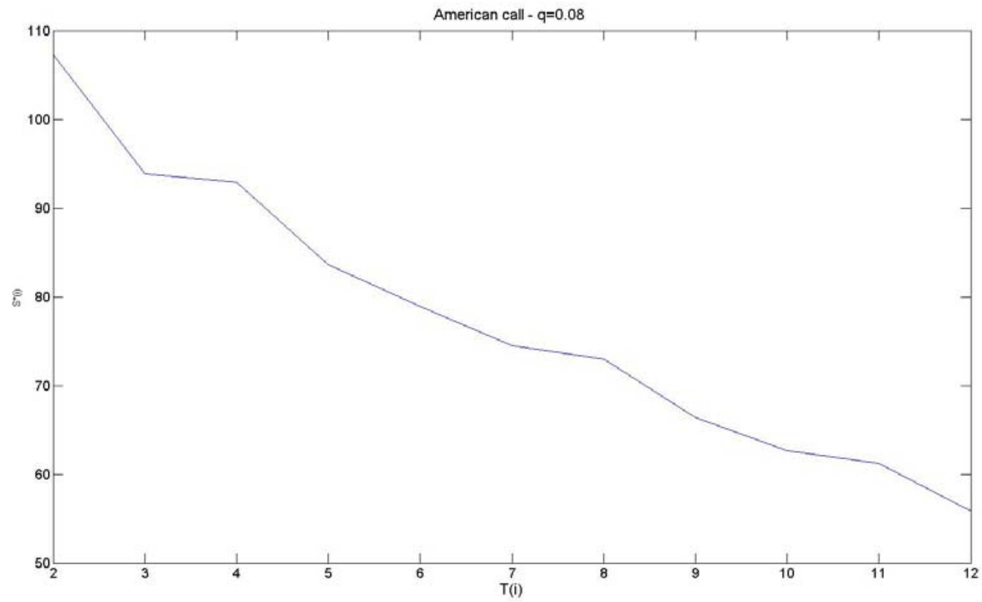
Time in months	Stock price-q=0.04	Stock price q=0.08
2	124.89	107.33
3	111.32	93.88
4	109.33	92.88
5	99.18	83.64
6	93.61	78.95
7	88.36	74.52
8	86.27	72.99
9	78.72	66.39
10	74.31	62.67
11	72.44	61.24
12	66.2	55.83

Critial values of Stock price for early exercise as a function of T

a) With q = 0.04



b) With q=0.08



Conclusion: We observe that the dividend has the effect of reducing the optimal spot price vale. An American option is exercised when its current value is higher than its risk-neutral expected value. The current value of the call option decreases with the dividend. The risk-neutral expected value decreases with a decrease in the spot price. Thus when the dividend is higher we can allow for a lower spot price, and still exercise the call option early.