Brownian Motion and Stochastic Flow Systems

J.M Harrison

Report written by Siva K. Gorantla

I. Introduction

Brownian motion is the seemingly random movement of particles suspended in a fluid or a mathematical model used to describe such random movements. This mathematical model has several real-world applications including the stock market fluctuations. The book written by J.M Harrison, aims at solving certain highly structured problems in the theory of buffered flow or stochastic flow systems, by the application of Brownian motion.

A significant development of mathematical models needed to analyze processes related to Brownian motion has been done in the first part of the book. Only the details relevant to the problem setup and results on stochastic flow systems would be presented in this report.

In general, a Brownian motion can take both positive and negative values. But some processes, like a queue process, cannot take negative values. A regulated Brownian motion, is fundamentally driven by an underlying Brownian motion, but like a queue process it cannot take negative values. A regulated BM can be expressed as a function of the underlying Brownian motion. After formally defining a one-dimensional regulated BM and Ito's formula for Brownian motion, the main results from the book are presented in three sections:

- Sec II presents a systematic analysis of 1D regulated Brownian motion. (Chapter 5 of the book). The markov property and the regenerative structure of the regulated BM are explored. These properties are then used to calculate i) the steady state distribution of the regulated BM and ii) the expected discounted cost of a stochastic flow system with underlying state as a regulated BM.
- Sec III considers a certain problem in the optimal control of Brownian motion. (Chapter 6). This problem, motivated by flow system applications, involves a discounted *linear* cost structure and a nonnegativity constraint on the state variable. The optimal policy is found to involve imposition of a regulated Brownian motion as a system model. Later, this result is extended to Cash management problem for a bank's cash fund.
- Sec IV considers flow system optimization problem. (Chapter 7). Using results from Chapter 5 and 6 (Sections II and III), the manufacturer's two-stage decision problem is recast as one of optimizing the

parameters of a regulated Brownian motion. Numerical solutions are worked out and the use of regulated BM as a flow system model is illustrated.

- A possible extension of the results is proposed at the end.

A. A simple flow system model

Consider the production of a commodity where produced goods flow into the inventory and the demand is an exogenous source of uncertainty. This can be represented as a two-stage flow system where the input process, the output process and the inventory(storage) process can be modeled as continuous stochastic processes. Let $\{A_t\}$ and $\{B_t\}$ be two increasing, continuous stochastic processes which represent the cumulative *potential* input and cumulative *potential* output till time t. Let X_t denote the netput process, then

$$X_t = X_0 + A_t - B_t$$

For a stationary, continuous, balanced high-volume flow system, the netput process can be approximated by a Brownian motion.

B. Regulated Brownian Motion

Let the buffer size of the inventory process described above be b. The following issues can occur when the buffer is full or empty.

- When the buffer is empty, demand cannot be met and the actual output of the system will be less than the demand. If L_t denotes the amount of potential output lost up to time t, then the actual cumulative output $= B_t - L_t$.
- When the buffer is full, some of the *potential* input may be lost. If U_t denotes the amount of potential input lost up to time t, then the actual cumulative input is $= A_t - U_t$.

Thus the inventory process is given by

$$Z_t = X_0 + (A_t - U_t) - (B_t - L_t) \tag{1}$$

$$= X_t + L_t - U_t \tag{2}$$

where

- L and Z are continuous and increasing processes with $L_0 = Z_0 = 0$.
- $Z_t \in [0, b]$ for all $t \geq 0$, and
- L and U increase only when $Z = 0$ and $Z = b$ respectively.

We will now see how the regulated BM (Z) an be expressed as a function of Brownian motion X.

Proposition 1: The stochastic processes (L,U,Z) can be obtained by applying a two-side regulator function (f,g,h) to the Brownian motion X_t.

Proof: $\{L_t\}$ and $\{U_t\}$ can be obtained from $\{X_t\}$ as

$$L = f(X) = \{\sup_{0\leq s\leq t}(X_s - U_s)^-, t \geq 0\} \quad U = g(X) = \{\sup_{0\leq s\leq t}(b - X_s - L_s)^-, t \geq 0\}$$

$\{Z_t\}$ can be obtained as a function of $\{X_t\}$ through imposition of *lower control barrier at zero and an upper control barrier at b*.

$$Z = h(X) = X + f(X) - g(X)$$

■

Z is called a **regulated Brownian motion** and (L, U, Z) a Brownian flow system. The processes are shown in the figure 1

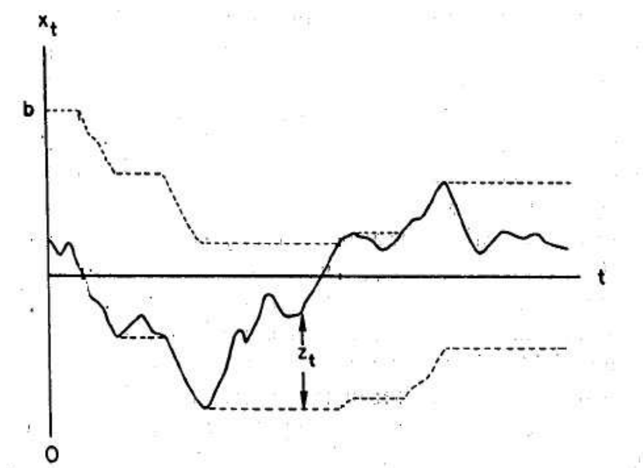

Fig. 1. Regulated Brownian Motion

C. Ito's formula

We use only the following two results about Ito's lemma throughout the report. For an Ito's process Z, and a twice continuously differentiable function f, $f(Z)$ is an Ito's process with

$$df(Z) = f'(Z)dZ + \frac{1}{2}f''(Z)(dZ)^2 \tag{3}$$

Integration by parts result:

$$e^{-\lambda t}f(Z_t) = f(Z_0) + \int_0^t e^{\lambda s}df(Z) - \lambda\int_0^t e^{-\lambda s}f(Z)ds \tag{4}$$

For a regulated Brownian motion $dZ = dX + dL - dU$ where $dX = \mu dt + \sigma dW$, substituting in (3)

$$df(Z) = \sigma f'(Z)dW + [\Gamma f(Z)dt + f'(0)dL - f'(b)dU] \tag{5}$$

where $\Gamma f = \frac{1}{2}\sigma^2 f'' + \mu f'$.

II. Regulated Brownian Motion

This section presents a systematic analysis of one-dimensional regulated Brownian motion. Using Ito's formula, both discounted performance measures and the steady-state distribution of Z are calculated. Let X be a (μ, σ) BM with starting state $x \in [0, b]$.

The regulated BM Z satisfies the strong Markov property and has regenerative structure which allows us to compute the steady state distribution of Z. The second problem is to calculate expected discounted cost

$$k(x) = E_x \left\{ \int_0^\infty e^{\lambda t}[u(Z)dZ - cdL + rdU] \right\}$$

for some continuous rate function $u(.)$, and some constants c and r.

A. Properties

1) Strong Markov Property: Given the value of regulated BM Z at time T, the expected discounted cost of the future process depends only on Z_T.

Proposition 2: Let $Z^*_t = Z_{T+t}$, $L^*_t = L_{T+t} - L_T$, and $U^*_t = U_{T+t} - U_T$. Define $k(x) = E_x[K(L, U, Z)]$. Then

$$E_x[K(L*, U*, Z*)|\mathcal{F}_T] = k(Z_T)$$

Proof: Define $X^*_t = Z_T + (X_{T+t} - X_T)$. Hence, (L^*, U^*, Z^*) is obtained by applying a two-side regulator(f,g,h) to X^*. Then by definition of $k(x)$, the proof follows. ■

2) Regenerative Structure of Regulated BM: Define

$$Z^*_{n+1}(t) = Z(T_n + t) \tag{6}$$

$$L^*_{n+1}(t) = L(T_n + t) - L(T_n) \tag{7}$$

$$U^*_{n+1}(t) = U(T_n + t) - U(T_n) \tag{8}$$

where T_{n+1} = smallest $t > T_n$ such that $Z(t) = 0$ and $Z(s) = b$ for some $s \in (T_n, t)$.

In words, T_{n+1} is the first time after T_n at which Z returns to level zero after first visiting level b. Let $\tau_n = T_n - T_{n-1}$. See figure 2. The regenerative structure of our Brownian flow system (L,U,Z) can be noted and the regeneration times $T_1, T_2 ...$ divide the evolution of (L, U, Z) into independent and identically distributed blocks or regenerative cycles of duration $\tau_1, \tau_2, ...$ Also,

- $E_x(K(L_n^*, U_n^*, Z_n^*)|\mathcal{F}(T_{n-1})) = E_0(K(L, U, Z))$.
- $\{L_1^*(\tau_1), L_2^*(\tau_2), ...\}$ and $\{U_1^*(\tau_1), U_2^*(\tau_2), ...\}$ are IID sequences and do not depend on starting value x.

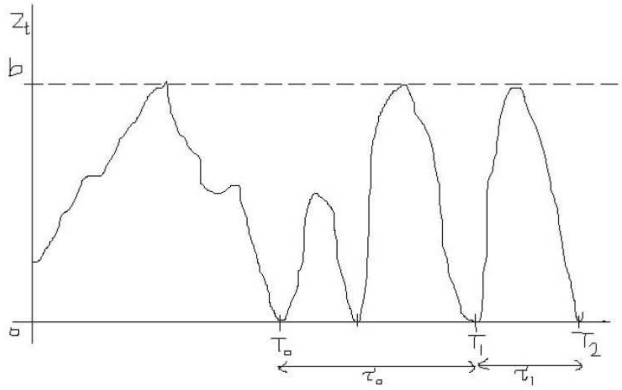

Fig. 2. Regenerative structure of Regulated Brownian Motion

B. Expected Discounted costs

Given a continuous cost rate function $u(.)$, and real constants c and r, we wish to calculate expected discounted cost

$$k(x) = E_x\left\{\int_0^\infty e^{\lambda t}[u(Z)dZ - cdL + rdU]\right\}$$

For any twice differentiable function f, the following proposition holds.

Proposition 3: Given $\lambda > 0$, set $u = \lambda f - \Gamma f$ on $[0, b]$. Also, let $c = f'(0)$ and $r = f'(b)$. Then

$$f(x) = E_x\left\{\int_0^\infty e^{\lambda t}[u(Z)dZ - cdL + rdU]\right\} \tag{9}$$

Proof: Apply the regulated BM version of Ito's lemma in (5) to the Integration by parts equation in (4). The result follows for infinite-horizon case $(t \to \infty)$. ■

Thus, to solve for $k(x)$ in the expected discounted cost problem, it is sufficient to solve the following differential equations.

$$\lambda k(x) - \Gamma k = u(x) \tag{10}$$

with boundary conditions $k'(0) = c$ and $k'(b) = r$.

C. Steady State Distribution

We want to find the steady state distribution of Z. We have already seen from the regenerative structure of (L,U,Z) that the evolution of (L, U, Z) can be divided into IID blocks of duration $\tau_1, \tau_2, ...$ Thus the evolution does not depend on the initial value $X_0 = x$ after the first time the state X hits a zero at T_0. Specifically, we are interested in finding the following terms:

$$\alpha = \frac{E_0[L(\tau)]}{E_0[\tau]} \qquad \beta = \frac{E_0[U(\tau)]}{E[\tau]}$$

$$\pi(A) = \frac{E_0[\int_0^\infty 1_A(Z_t)dt]}{E_0[\tau]} \tag{11}$$

α and β represent the expected rate of increase in L and U respectively. $\pi(A)$ is the expected amount of time Z spends in the set A. The π which is represented as the expected occupancy measure in a regenerative cycle is also the steady state distribution.

Proposition 4: If the drift $\mu = 0$, then $\alpha = \beta = \frac{\sigma^2}{2b}$ and π is the uniform distribution on $[0, b]$. Otherwise, setting $\theta = \frac{2\mu}{\sigma^2}$

$$\alpha = \frac{\mu}{e^{\theta b} - 1} \qquad \beta = \frac{\mu}{1 - e^{-\theta b}}$$

and π is the *truncated exponential* distribution

$$\pi(dz) = p(z)dz, \qquad \text{where} \qquad p(z) = \frac{\theta e^{\theta z}}{e^{\theta b} - 1}$$

The proof involves Ito's lemma for Brownian motion in (5)

III. Optimal Control of Brownian Motion

In this section, we consider a fundamental problem of linear stochastic control. This problem, motivated by flow system applications, involves a discounted linear cost structure and a nonnegativity constraint on the state variable. The optimal policy is found to involve imposition of a lower control barrier at zero and upper control barrier at b, where b is calculated explicitly as the unique solution of a certain equation. Thus optimization leads to regulated Brownian motion as a system model.

The problem is stated as follows: We want to design a *controller* who continuously monitors the content of a storage system. For example, an inventory or a bank account. In the absence of control, the content process $\{Z_t, t \geq 0\}$ follows a (μ, σ) Brownian motion. The controller can increase or decrease the contents by any amount at any time but should keep $Z_t \geq 0$. The following costs are involved: α is the cost of increasing the content by one unit. β is the cost of decreasing the content by one unit. h is the holding cost continuously incurred at rate hZ_t. Thus we have *linear holding costs* and *linear costs of control.*

A policy is defined as a pair of processes L and U such that i) L and U are adapted and right-continuous, increasing and positive. L_t represents the cumulative increase in system content effected by controller up to time t, and U_t the cumulative decrease. For every policy (L, U), there is a controlled process $Z = X + L - U$ with a cost function

$$k(x) = E_x \left\{ \int_0^\infty e^{-\lambda t} [hZ_t dt + \alpha dL + \beta dU] \right\} \tag{12}$$

An optimal policy (L, U) is one which minimizes the cost k(x). Note that a policy (L, U) and corresponding Z need not represent to the regulated brownian motion. L and U are not restricted to increase when $Z = 0$ and $Z = b$.

A. The Barrier policy

Given the linear structure of costs and rewards, it is natural to consider the following sort of *barrier policy*. For some parameter $b > 0$, make only such withdrawals as required to keep $Z_t \leq b$, and make deposits only to meet the constraint $Z_t \geq 0$. Thus the barrier policy (L,U) is obtained by applying to X the two-sided regulator, so that L and U are continuous, increasing process and increase only when $Z = 0$ and $Z = b$ respectively.

B. Optimal Policy

Among all policies (L, U), a barrier policy for a particular value of b can be shown to be optimal for the linear stochastic flow problem considered in (12). The proof follows a *policy improvement logic*. Assuming that a barrier policy is optimal, we find the value of b which minimizes k(x) over all possible barrier policies. Lets call this $k_b(x)$. Beginning with this candidate optimal policy having know the value function $k_b(x)$, we examine the effect of inserting some other policy over an interval [0,T] and reverting to use of the candidate policy thereafter. The question is whether the candidate policy can be improved by such a modification. If not, optimality of the candidate policy follows.

The solution to the problem is given by the barrier policy with $b > 0$ such that,

$$k(x) = \frac{hx}{\lambda} + \frac{h\mu}{\lambda^2} - \left(r\frac{g(x)}{g'(b)} + c\frac{g(x-b)}{g'(-b)} \right) \tag{13}$$

$$\frac{g(0)}{g(-b)} = \frac{r}{c} \tag{14}$$

where b is the unique solution given by equation (14), $c = \frac{h}{\lambda} + \alpha$, $r = \frac{h}{\lambda} + \beta$ and $g(x) = \alpha_*(\lambda)e^{\alpha^*(\lambda)x} + \alpha^*(\lambda)e^{-\alpha_*(\lambda)x}$. $\alpha_*(\lambda)$ and $\alpha^*(\lambda)$ are the roots of equation $\mu\beta + \frac{\sigma^2\beta^2}{2} = \lambda$.

In short, the solution to a stochastic flow system with linear costs is a barrier policy with b given by (14) and the value of the cost function is given by (13).

C. Cash Management

Consider the following stochastic cost management problem. Z_t is the content of the cash fund into which various types of income and operating disbursements are automatically channeled. In absence of managerial intervention, the content of the fund fluctuates as a (μ, σ) Brownian motion X. Wealth not held as cash can be invested in securities(bonds) to get an interest continuously at rate λ. A transaction of money to buy bonds has a cost of β dollars per each dollar invested. Thus the management gets $(1-\beta)$ dollars worth of bonds for a dollar invested. Similarly, bonds sold have an α transaction cost rate. That is for each dollar obtained, management has to give $(1+\alpha)$ dollars worth of bonds. Let S_t be dollar value of bonds, then

$$dS_t = \lambda S_t dt + (1-\beta)dU_t - (1+\alpha)dL_t$$

and we want to maximize the total value at some specified distant time T.

$$\text{Maximize} \quad e^{-\lambda T} E(S_T + Z_T)$$

This problem can be modified using integration by parts results and the definition of cash fund process $Z = X + L - U$ and can be equivalently posed as (for $T \to \infty$)

$$\text{Minimize} \quad E\left\{ \int_0^\infty e^{-\lambda t}[\lambda Z_t dt + \alpha dL + \beta dU] \right\}$$

This is in the form of the linear stochastic flow problem considered in (12). Thus the optimal solution to the stochastic cash management problem is to *convert bonds to dollars when the cash reserves are zero* $Z = 0$ *and invest in bonds when the cash fund reaches a threshold value* $Z = b$. The value of b is a function of interest rate λ, transaction costs α and β and computed using (14).

IV. Optimizing Flow System Performance

In this section, we consider the system optimization problem. Using results from Chapter 5 and 6 (Sections II and III), the manufacturer's two-stage decision problem is recast as one of optimizing the parameters of a regulated Brownian motion. Numerical solutions are worked out and the use of regulated BM as a flow system model is illustrated.

Consider the following single-product firm that must fix its work force size, or production capacity at time zero. Having fixed its capacity, the firm may choose an actual production rate at or below this level in each future period, but overtime production is assumed impossible. Demand that cannot be met from stock on hand is lost with no adverse effect on future demand. Suppose that the number of units demanded are i.i.d. random variables with a mean $a = 1000$ and $\sigma = 200$. Let $\pi = \$130$ be the selling price of the good, $w = \$20$ the labor cost, $m = \$50$ the materials cost per unit. The firm pays w dollars each week for each unit of potential production, regardless of whether that potential is fully exploited, so the variable cost of production after time zero is m. Assume an interest rate of $\lambda = 0.005$ per week.

If there is no uncertainty in the demand rate and $a = 1000$, then the firm would set its production capacity equal to the demand rate, and would realize a weekly profit of $a(\pi - w - m) = \$60000$ per week. Since the demand rate is random, the problem is to figure out how the production capacity should be chosen. Let B_t denote cumulative demand up to time t, and let μ be the excess capacity(possibly negative) decided on at time zero. Thus the cumulative potential production over $[0, t]$ is $A_t = (a+\mu)t$. The centered demand process $\{B_t - at, t \geq 0\}$ is approximated by a $(0, \sigma)$ Brownian motion. Thus, the netput process $X = A - B$, is a (μ, σ) Brownian motion. Maximizing the expected present value of total profit is equivalent to minimizing

$$\Delta = E\left\{\int_0^\infty e^{-\lambda t}[hZdt + w\mu dt + \delta dL]\right\} \tag{15}$$

where $Z_t = X_t + L_t - U_t$ is the inventory level and L_t is the cumulative potential sales lost over t and U_t is the cumulative amount of undertime employed (potential production foregone) up to time t.

The firm has to decide by choosing a value of μ at time zero and its dynamic operating policy is manifested in the choice of an undertime process U. These two aspects of management policy influence lost sales.

Solution: Given a choice of μ, the dynamic production control problem (15) can be formulated exactly as cost function in Section III (12). The optimal policy for this problem is a barrier policy- by enforcing lower control barrier at zero and an upper control barrier at b. In terms of the physical system, this means that potential production is foregone (capacity is underutilized) only as necessary to keep $Z \leq b$, and potential sales are lost when Z reaches zero.

Given that a barrier policy is optimal, our problem is in choosing the parameters of the brownian flow system by specifying the values of μ and b that minimize the cost Δ given in (15). At this point, we are interested in the following questions:

- Should μ be positive or negative, and of what order as percentage of average demand rate?
- Under the optimal production control policy, what fraction of demand will be lost?
- How big should the average inventory be as a multiple of average weekly demand?
- What is the cost of stochastic variability as a percentage of the weekly profit level achievable in the deterministic case?

For a given value of μ and b, using results (13) from Chapter 6, we have

$$\Delta = \frac{h\mu}{\lambda^2} + \frac{w\mu}{\lambda} - \frac{rg(0)}{g'(b)} - \frac{cg(-b)}{g'(-b)} \tag{16}$$

where $r = h/\lambda = \$70$, $c = h/\lambda + \delta = \$70 + \$80 = \$150$ and $g(x) = \alpha_*(\lambda)e^{\alpha^*(\lambda)x} + \alpha^*(\lambda)e^{-\alpha_*(\lambda)x}, x \in R$. The cost is computed for various values of b and μ and it can be verified that the optimal pair $\mu = 10, b = 2500$ has a performance degradation of $\$1200$ per week from the deterministic case. Thus the questions posed above can be answered as follows:

- An excess capacity of 10 units per week, about 1% of average demand rate(a=1000) is optimal maintaining a high degree of balance between production and demand.
- A performance degradation of $\$1200$ which is 0.2% of the net profit in deterministic case is seen.

In order to answer the other questions, we need to determine the steady state characteristics of the inventory process Z for the given values of μ and b. Using the steady state distribution computed in Section II, Proposition 4, where

$$\frac{E(L_t)}{t} \to \alpha \qquad \frac{E(U_t)}{t} \to \beta \qquad E(Z_t) \to \gamma \text{ as } t \to \infty$$

The values of α, β and γ are computed and

- $\alpha = 4.01$ representing the average rate at which demand is lost (in units per week).
- $\beta = 14.01$ represents the average amount of regular-time capacity that goes unused each week.
- $\gamma = 1503$ is the long-run average inventory level (which is 1.5 times the average demand rate).

Thus only 0.4% of demand is lost, about 1.4% of the labor paid for is not used, and that average inventory is about 1500.

This numerical examples shows how Brownian motion can be used to analyze stochastic flow systems. The key steps in solving this problem involved results from Chapters 5 and 6- The steady state distribution of regulated BM and optimum barrier policy for a linear stochastic flow system.

V. Extensions

One may think about the stochastic flow system as a queue, more precisely a Brownian queue in this case. In fact, further exploring the publications in this field, the same author has a result on quasi-reversibility of the Brownian queue. This makes it interesting from the communication perspective. If we want to communicate using the arrivals and departures of a queue, and compute the fundamental limits in terms of Capacity. Verdu has a result for the capacity of a M/M/1 queue “Bits Thru Queues”.The underlying principle which has driven this result is the Burke’s theorem and reversibility. Now that the Brownian queue has similar principles in terms of reversibility, it might be interesting to compute **the capacity of a brownian queue**.

The optimum control for linear costs has been shown as a regulated BM. Another idea is to find the optimum control when performing control over communication.