Importance Sampling Examples

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I. Exponential RV

Problem: For an exponential random variable $X \sim \exp(1)$, compute the probability of the event that $X \geq \frac{1}{\epsilon}$.

Measure Transformation: Define $A_\epsilon = \{x : x \geq \frac{1}{\epsilon}\}$. Let $\mathbb{P}$ and $\mathbb{Q}$ be two measures such that "X under measure $\mathbb{P}$" is $\exp(1)$ and "X under measure $\mathbb{Q}$" is $\exp(\epsilon)$. Then,

$$\mathbb{E}^{\mathbb{P}}\left[A_\epsilon\right] = \mathbb{E}^{\mathbb{Q}}\left[A_\epsilon \frac{d\mathbb{P}}{d\mathbb{Q}}\right] = \mathbb{E}^{\mathbb{Q}}\left[A_\epsilon \frac{e^{-(1-\epsilon)X}}{\epsilon}\right]$$

For ease of notation, lets denote "X under measure $\mathbb{Q}$" by Y. Thus Y is an $\exp(\epsilon)$ r.v. Let $\{\xi_i\}_{i=1}^N$ be realizations of Y generated using a computer program. The following term gives the required probability,

$$\Xi_N = \frac{1}{N}\sum_{i=1}^{N} 1_{\{\xi_i > \frac{1}{\epsilon}\}} \frac{e^{-(1-\epsilon)\xi_i}}{\epsilon}$$

Results: —

Num of samples generated	Num of $X > \frac{1}{\epsilon}$	Num of $Y > \frac{1}{\epsilon}$	estimate of $\mathbb{P}(X > \frac{1}{\epsilon})$ using X	estimate of $\mathbb{P}(X > \frac{1}{\epsilon})$ using Y
10	0	2	0	0.0
10^2	0	37	0	1.517×10^{-44}
10^3	0	356	0	1.85×10^{-44}
10^4	0	3618	0	3.96×10^{-44}
10^5	0	36937	0	3.774×10^{-44}
10^6	0	368007	0	3.767×10^{-44}

TABLE I

Calculating $\mathbb{P}(X > \frac{1}{\epsilon})$ where $X \sim \exp(1)$ by simulating i) X directly ii) $Y \sim \exp(\epsilon)$ and taking a measure transformation. For $\epsilon = 0.01$, actual value of $\mathbb{P}(X > \frac{1}{\epsilon}) = 3.72 \times 10^{-44}$. Simulating Y gives a good estimate within 10^6 simulations as seen above.

II. Brownian Motion - Minimum energy path

Problem: For a stochastic process $X_t = \epsilon W_t$, find the probability that X_t hits 1 in $t \in [2,5]$ where W_t is a standard brownian motion.

Measure Transformation: Instead of simulating $X_t = \epsilon W_t$, simulate Y_t on the minimum energy path $Y_t = \frac{1}{5}t + \epsilon W_t$ and take measure transformation

$$\mathbb{P}(X_t \leq 1 \leq X_{t+\delta}) = \mathbb{E}^{\mathbb{Q}}\left[1_{\{Y_t \leq 1 \leq Y_{t+\delta}\}}\left(\frac{d\mathbb{P}}{d\mathbb{Q}}\right)_t\right] \quad (1)$$

$$\left(\frac{d\mathbb{P}}{d\mathbb{Q}}\right)_t = \exp\left(-\frac{1}{5\epsilon}W_t + \frac{1}{50\epsilon^2}t\right) \quad (2)$$

This measure transform is obtained using the following result: Let W_t is a standard Brownian Motion under measure $\mathbb{P}$, then $\tilde{W}_t = W_t - \theta t$ is a standard Brownian Motion under measure $\mathbb{Q}$ if

$$\left(\frac{d\mathbb{P}}{d\mathbb{Q}}\right)_t = \exp\left(\theta\tilde{W}_t + \frac{1}{2}\theta^2 t\right)$$

Results: —

Num of sample paths generated	Num of X_t paths hitting 1	Num of Y_t paths hitting 1	Prob that X_t hits 1 in $t \in [2,5]$ using X_t	Prob that X_t hits 1 in $t \in [2,5]$ using Y_t
10	0	4	0	4.0276×10^{-6}
10^2	0	55	0	2.6770×10^{-5}
10^3	0	510	0	1.3114×10^{-5}
10^4	0	5064	0	1.3634×10^{-5}
10^5	0	51319	0	1.4068×10^{-5}
10^6	0	513956	0	1.3986×10^{-5}

TABLE II

CALCULATING PROB THAT X_t HITS VALUE 1 DURING $t \in [2,5]$ WHERE $X_t = \epsilon W_t$ BY SIMULATING I) X_t DIRECTLY II) Y_t AND TAKING A MEASURE TRANSFORMATION. FOR $\epsilon = 0.05$, AS SEEN ABOVE SIMULATING Y REQUIRES FEWER SIMULATIONS THAN SIMULATING X DIRECTLY.

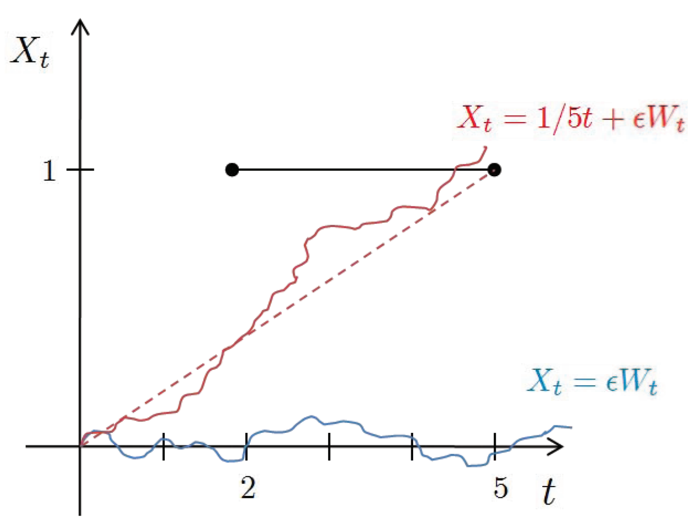

Fig. 1. Minimum Energy Path

III. Pricing Deep out-of-the money options

Problem: Suppose the value of option at $t = T$ is $(S_T - K)^+$. And the stock price S_t follows a log-normal distribution with parameters (μ, σ) under measure $\mathbb{P}$ and has an initial value S_0 at $t = 0$,

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

Then, the discounted price of the option at $t = 0$ is

$$C_0 = \mathbb{E}^{\mathbb{P}}\left[e^{-rT}(S_T - K)^+\right]$$

Measure transformation: The price can also be computed as

$$C_0 = \mathbb{E}^{\mathbb{P}}\left[e^{-rT}(S_T - K)^+\right] = \mathbb{E}^{\mathbb{Q}_\theta}\left[e^{-rT}(S_T - K)^+ \frac{d\mathbb{P}}{d\mathbb{Q}_\theta}\right]$$

where

$$\left(\frac{d\mathbb{P}}{d\mathbb{Q}_\theta}\right)_t = \exp\left(\theta \tilde{W}_t + \frac{1}{2}\theta^2 t\right)$$

which results in

$$\frac{dS_t}{S_t} = (\mu + \theta\sigma)dt + \sigma d\tilde{W}_t$$

Method: Implemented the paper “Optimal importance sampling in securities pricing”, Yi Su and M.C.Fu. [1]

- ***Optimization Step:*** Find the optimal measure transform θ^* for which variance in monte-carlo simulation is minimal.
- ***Pricing Stage:*** Simulate at $\theta = \theta^*$ and price the European/Asian call option.

Results: Asian Call Options: $S_0 = 100; K = 100; \sigma = \sqrt{0.2}; r = 0.05; T = 1.0Yr$ - Cost = \$7.52. Takes less than 50000 simulations.
Asian Call Options: $S_0 = 100; K = 110; \sigma = \sqrt{0.2}; r = 0.05; T = 1.0Yr$ - Cost = \$2.81. Takes around 10^6 simulations.

Comments: Not able to simulate for higher values of K. The algorithm needs improvements. The programming is done is java. A change of programming environment could lead to faster implementation.

The components of the code and the algorithm are given below:

- Stage I: Optimization stage - Find θ^* that minimizes the variance of MonteCarlo estimate.
  Let $V(\theta)$ denote the expected variance.
  Initialization: Set $\theta = \theta_0$
  Loop: For $n = 1$ to **numIter**
  - Estimate $\nabla V(\theta)$ using Monte Carlo simulation.
  - $\theta_{n+1} = \theta_n - a_n \nabla V(\theta_n)$
  - If $a_n \nabla V(\theta_n) < \epsilon$, exit loop.
- Stage II: Pricing Stage - Simulate at $\theta = \theta^*$. Loop: For $n = 1$ to **numSimul**
  - Generate sample path S_t
  - compute C_0 using payoff and $\frac{d\mathbb{P}}{d\mathbb{Q}_\theta}$

  Average of all **numSimul** values gives the final estimate of C_0

References

[1] Yi Su and Michael C. Fu. Optimal importance sampling in securities pricing. *Journal of Computational Finance*, 5:27–50, 2002.

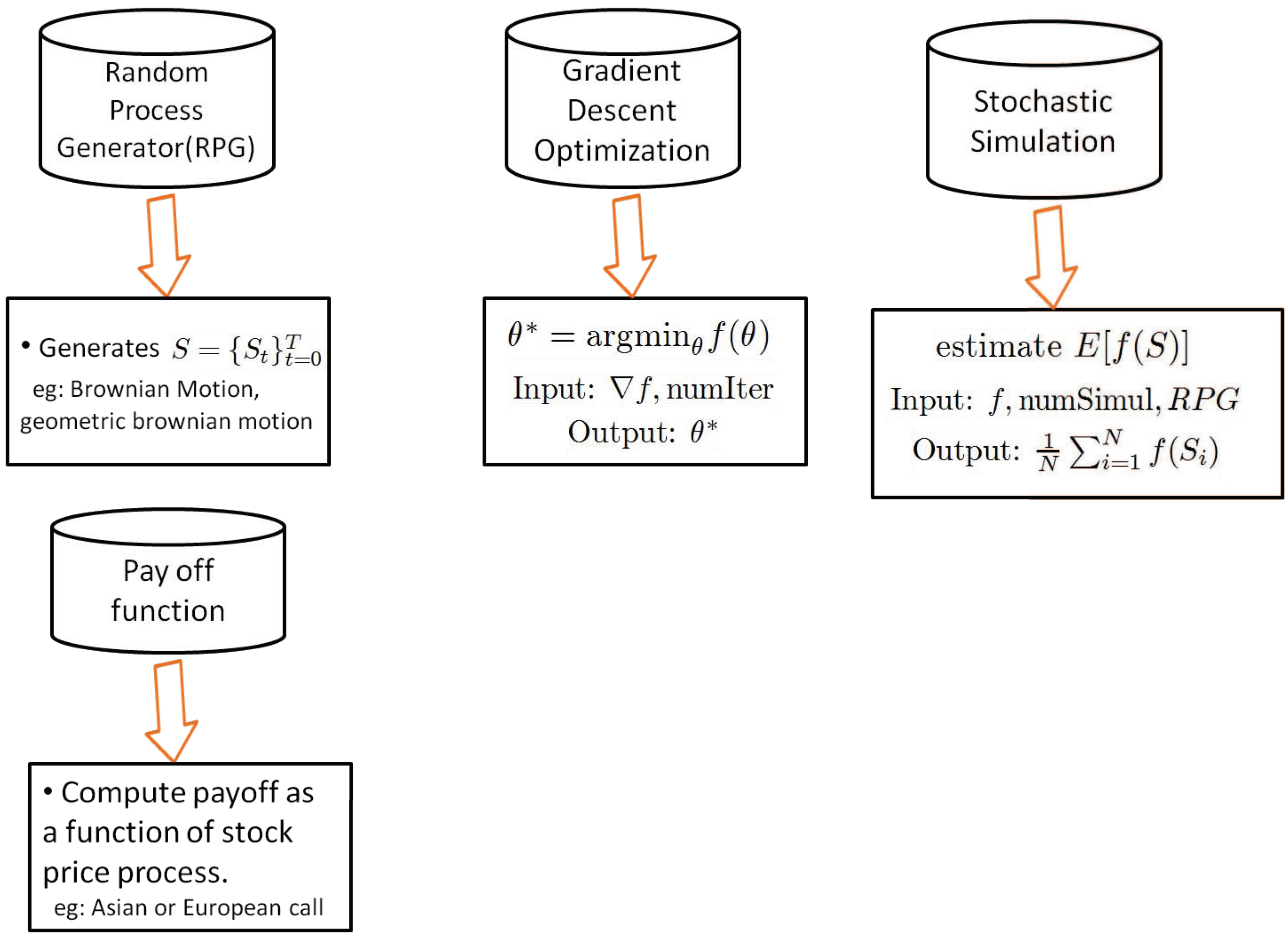

Fig. 2. Components of the code