A Completeness Proof for LJ

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April 2, 2013

1 Introduction

In this short lecture note we summarize how to show completeness for the multi-succedent intuitionistic propositional logic. The proof method we use is due to Mints. All the details can be found in his book "A Short Introduction to Intuitionistic Logic." We do not give any proofs. This is merely just an outline to facilitate learning how the completeness proof is done. For the details of the proofs see Mints p. 57.

2 The Canonical Model

In this section we define the canonical model and give some properties of this model. We call a sequent **underivable** if and only if there does not exist a derivation in the logic starting with the sequent.

To define the canonical model we first must define when a sequent is complete, and when a sequent is saturated for invertible rules.

Definition 1 (Complete Sequent)**.** *We denote the set of subformulas of a set of formulas* Γ *as* $Sub(\Gamma)$*. A sequent* $\Gamma \vdash \Delta$ *is* ***complete*** *if it is underivable and for any formula* $\phi \in Sub(\Gamma, \Delta)$ *either* $\phi \in \Gamma \cup \Delta$ *or* $\Gamma \vdash \Delta, \phi$ *and* $\phi, \Gamma \vdash \Delta$*.*

Definition 2 (Saturated for Invertible Rules)**.** *A sequent* $\Gamma \vdash \Delta$ *is* ***saturated for invertable rules*** *if it is underivable and the following conditions hold for any* ϕ *and* ψ*:*

Case. If $\phi \wedge \psi \in \Delta$ *then* $\phi \in \Delta$ *or* $\psi \in \Delta$*.*

Case. If $\phi \wedge \psi \in \Gamma$ *then* $\phi \in \Gamma$ *and* $\psi \in \Gamma$*.*

Case. If $\phi \vee \psi \in \Delta$ *then* $\phi \in \Delta$ *and* $\psi \in \Delta$*.*

Case. If $\phi \vee \psi \in \Gamma$ *then* $\phi \in \Gamma$ *or* $\psi \in \Gamma$*.*

Case. If $\phi \to \psi \in \Gamma$ *then* $\phi \in \Delta$ *or* $\psi \in \Gamma$*.*

Notice in the above definition that we do not have the following condition:

$$\text{If } \phi \to \psi \in \Delta \text{ then } \phi \in \Gamma \text{ and } \psi \in \Delta.$$

This is because the rule for right implication is not an invertible rule.
The following result holds for complete sequents.

Lemma 3 (Saturation)**.** *If* $\Gamma \vdash \Delta$ *is complete, then it is saturated for invertible rules.*

It turns out that if a sequent is underivable then it may be extended into a new sequent which remains underivable, but is complete. This is known as completion.

Definition 4 (The Completion Sequences)**.** *Suppose $\Gamma_0 \vdash \Delta_0$ is an underivable sequent, and that $\phi_0, \ldots, \phi_n$ is a fixed sequence of all the formulas in $Sub(\Gamma_0, \Delta_0)$. Then we define the* ***completion*** *sequences for $\Gamma_0 \vdash \Delta_0$ by constructing the sequences $\Gamma'_0 \subseteq \cdots \subseteq \Gamma'_{n+1}$ and $\Delta'_0 \subseteq \cdots \subseteq \Delta'_{n+1}$ of finite sets of formulas such that $\Gamma'_i \vdash \Delta'_i$ is underivable and complete for formulas ϕ_j for all $j < i$. That is either $\phi_j \in \Gamma'_i \cup \Delta'_i$ or both $\Gamma'_i \vdash \Delta'_i, \phi_j$ and $\phi_j, \Gamma'_i \vdash \Delta'_i$ are derivable.*

The construction is defined by mutual recursion as follows:

$$
\begin{array}{lcll}
\Gamma'_0 & := & \Gamma_0 & \\
\Gamma'_{i+1} & := & \phi_i, \Gamma'_i & \textit{if } \phi_i, \Gamma'_i \vdash \Delta'_i \textit{ is underivable} \\
\Gamma'_{i+1} & := & \Gamma'_i & \textit{otherwise}
\end{array}
$$

and

$$
\begin{array}{lcll}
\Delta'_0 & := & \Delta_0 & \\
\Delta'_{i+1} & := & \phi_i, \Delta'_i & \textit{if } \Gamma'_{i+1} \vdash \Delta'_i, \phi_i \textit{ is underivable} \\
\Delta'_{i+1} & := & \Delta'_i & \textit{otherwise.}
\end{array}
$$

Lemma 5. *If $\Gamma_0 \subseteq \cdots \subseteq \Gamma_n$ and $\Delta_0 \subseteq \cdots \subseteq \Delta_n$ are completion sequences with respect to the sequence of subformulas $\phi_0, \ldots, \phi_{n-1}$ then $\Gamma_i \vdash \Delta_i$ is complete with respect to the formulas ϕ_j for all $j < i$.*

Proof. This is a proof by induction on i.

Base Case. Then we must show that $\Gamma_0 \vdash \Delta_0$ is complete with respect to the formulas ϕ_j for all $j < 0$. This is trivially the case.

Step Case. We must show that $\Gamma_{i+1}\Delta_{i+1}$ is complete with respect to the formulas ϕ_j for all $j < i + 1$. Based on the definition of completion sequences we know one of two things about Γ.

- Case. Suppose $\Gamma_{i+1} = \Gamma_i, \phi_i$. Then it must be the case that $\Gamma_{i+1} \vdash \Delta_i$ is underivable. Now we case split on Δ.
  - Case. Suppose $\Delta_{i+1} = \Delta_i, \phi_i$. Then it must be the case that $\Gamma_{i+1} \vdash \Delta_i, \phi_i$ is underivable. However, this is not the case. Thus we have arrived at a contradiction.
  - Case. Suppose $\Delta_{i+1} = \Delta_i$. Completeness of $\Gamma_{i+1} \vdash \Delta_{i+1}$ follows from the fact that we know by the IH that $\Gamma_i \vdash \Delta_i$ is complete with respect to formulas ϕ_j for all $j < i$, $\phi_i \in \Gamma_i \cup \Delta_i$, and $\Gamma_{i+1} \vdash \Delta_i$ is underivable.
- Case. Suppose $\Gamma_{i+1} = \Gamma_i$. We case split on the form of Δ.
  - Case. Suppose $\Delta_{i+1} = \Delta_i, \phi_i$. Then it must be the case that $\Gamma_{i+1} \vdash \Delta_{i+1}$ is underivable. In addition we know that $\phi_i \in \Gamma_{i+1} \cup \Delta_{i+1}$. By the IH we know that $\Gamma_i \vdash \Delta_i$ is complete with respect to the formulas ϕ_j for all $j < i$. Thus, by definition we know $\Gamma_{i+1} \vdash \Delta_{i+1}$ is also complete.
  - Case. Suppose $\Delta_{i+1} = \Delta_i$. Completeness of $\Gamma_{i+1} \vdash \Delta_{i+1}$ follows from the fact that we know $\Gamma_i, \phi_i \vdash \Delta_i$ and $\Gamma_i \vdash \Delta_i, \phi_i$ are derivable.

□

Lemma 6 (Completion)**.** *Any underivable sequent $\Gamma_0 \vdash \Delta_0$ may be extended to a complete sequent $\Gamma \vdash \Delta$ for some Γ and Δ consisting of subformulas of Γ_0 and Δ_0.*

Proof. Using Definition 4 and Lemma 5 we can construct the completion sequence $\Gamma_0 \subseteq \cdots \Gamma_n$ and $\Delta_0 \subseteq \cdots \subseteq \Delta_n$ with respect to some sequence of the formulas in $Sub(\Gamma_0, \Delta_0)$, $\phi_0, \ldots, \phi_{n-1}$ such that $\Gamma_i \vdash \Delta_i$ is complete with respect to the formulas ϕ_j for all $j < i$. Take $\Gamma = \bigcup \Gamma_i$ and $\Delta = \bigcup \Delta_i$. Clearly, $\Gamma \vdash \Delta$ is complete. □

We know arrive at the definition of the canonical Kripke model. This will be the counter model we will use to show completeness.

Definition 7 (Canonical Model)**.** *The* ***canonical Kripke model*** *K is a tuple $\langle W, R_{\subseteq}, V_{\in} \rangle$ such that*

- *W is the set of all complete sequents,*
- *$R_{\subset}(\Gamma \vdash \Delta, \Gamma' \vdash \Delta') = \Gamma \subseteq \Gamma'$, and*
- *$V_{\in}(p, \Gamma \vdash \Delta) = p \in \Gamma$.*

Note that by completion the set of canonical worlds W must contain all completed underivable sequents. Hence, there exists a mapping between all underivable sequents and the underivable worlds of W. This mapping is defined by completion. This is a key point of which completeness depends.

We now show that the canonical model falsifies every underivable sequent.

Definition 8. *A sequent $\Gamma \vdash \Delta$ is **falsified** in a world w of a Kripke model if $V(\wedge\Gamma, w) = 1$ and $V(\vee\Delta, w) = 1$. This implies that $V(\Gamma \vdash \Delta, w) = 0$.*

Above we saw when a sequent is saturated for invertable rules. Next we define when a set of sequents is saturated for non-invertable rules. Following this definition is the definition of when a set is saturated.

Definition 9 (Saturated for Non-Invertable Rules)**.** *A set of sequents M is **saturated for non-invertable rules** if the following condition is satisfied for every $\Gamma \vdash \Delta$ in M:*

if $\phi \to \psi \in \Delta$, then there is a sequent $\Gamma' \vdash \Delta' \in M$ such that $\phi, \Gamma \subseteq \Gamma'$ and $\psi \in \Delta$.

Definition 10 (Saturated Set)**.** *A set of sequents M is **saturated** if every sequent in M is saturated for invertable rules and M is saturated for non-invertable rules.*

The following result is important for completeness.

Lemma 11 (Canonical Model is Saturated)**.** *The set W of the canonical model is saturated.*

Theorem 12 (Falsification in K)**.** *Let $K = \langle W, R_{\subseteq}, V_{\in}\rangle$ be the canonical model. Then for $w \equiv \Gamma \vdash \Delta \in W$:*

If $\phi \in \Gamma$ then $V_{\in}(\phi, w) = 1$, and

if $\phi \in \Delta$ then $V_{\in}(\phi, w) = 0$.

This implies that $V_{\in}(\Gamma \vdash \Delta, w) = 0$, that is, w is falsified in K.

This is all that is needed to prove completeness of LJ. We prove this in the next section.

3 Completeness

Theorem 13 (Completeness)**.** *Each sequent underivable in LJ is falsified in the canonical model K. Hence every valid sequent is derivable in LJ.*

Proof. Suppose that $\Gamma \vdash \Delta$ is an underivable sequent in LJ, and W is the saturated set of all complete sequents. That is W is the set of worlds of K. Then by completion (Lemma 6), there exists a Γ' and Δ' such that $\Gamma' \vdash \Delta'$ is the completed version of $\Gamma \vdash \Delta$. Sense W is saturated we know $\Gamma' \vdash \Delta' \in W$, and we may apply Theorem 12 to obtain that $V_{\in}(\Gamma' \vdash \Delta', \Gamma' \vdash \Delta') = 0$. Hence, by monotonicity $V_{\in}(\Gamma \vdash \Delta, \Gamma' \vdash \Delta') = 0$. □

Corollary 14 (Admissibility of Cut)**.** *If $\Gamma \vdash \Delta, \phi$ and $\phi, \Gamma \vdash \Delta$ are derivable, then $\Gamma \vdash \Delta$ is derivable.*

Proof. Suppose $\Gamma \vdash \Delta, \phi$ and $\phi, \Gamma \vdash \Delta$ are derivable for some Γ,Δ, and ϕ. Clearly, $\Gamma \vdash \phi$ is derivable by a simple application of the cut rule. Thus, by soundness + cut we know $\Gamma \vdash \phi$ is valid. Now by completeness (without cut) $\Gamma \vdash \Delta$ is derivable. □