

A Completeness Proof for LJ

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1 Introduction

In this short lecture note we summarize how to show completeness for the multi-succedent intuitionistic propositional logic. The proof method we use is due to Mints. All the details can be found in his book “A Short Introduction to Intuitionistic Logic.”

2 The Canonical Model

In this section we define the canonical model and give some properties of this model. We call a sequent **underivable** if and only if there does not exist a derivation in the logic starting with the sequent.

To define the canonical model we first must define when a sequent is full, and when a sequent is saturated for invertible rules. We denote the set of subformulas of a set of formulas Γ as $Sub(\Gamma)$.

Definition 1 (Formula Full Sequent). *A sequent $\Gamma \vdash \Delta$ is **full for a formula** $\phi \in Sub(\Gamma, \Delta)$ if and only if it is underivable and either $\phi \in \Gamma \cup \Delta$ or $\Gamma \vdash \Delta, \phi$ and $\phi, \Gamma \vdash \Delta$ are derivable.*

Definition 2 (Full Sequent). *A sequent $\Gamma \vdash \Delta$ is **full** if and only if it is formula full for all formulas in $Sub(\Gamma, \Delta)$.*

Definition 3 (Saturated for Invertible Rules). *A sequent $\Gamma \vdash \Delta$ is **saturated for invertible rules** if it is underivable and the following conditions hold for any ϕ and ψ :*

Case. If $\phi \wedge \psi \in \Delta$ then $\phi \in \Delta$ or $\psi \in \Delta$.

Case. If $\phi \wedge \psi \in \Gamma$ then $\phi \in \Gamma$ and $\psi \in \Gamma$.

Case. If $\phi \vee \psi \in \Delta$ then $\phi \in \Delta$ and $\psi \in \Delta$.

Case. If $\phi \vee \psi \in \Gamma$ then $\phi \in \Gamma$ or $\psi \in \Gamma$.

Case. If $\phi \rightarrow \psi \in \Gamma$ then $\phi \in \Delta$ or $\psi \in \Gamma$.

Notice in the above definition that we do not have the following condition:

If $\phi \rightarrow \psi \in \Delta$ then $\phi \in \Gamma$ and $\psi \in \Delta$.

This is because the rule for right implication is not an invertible rule.

The following result holds for full sequents.

Lemma 4 (Saturation). *If $\Gamma \vdash \Delta$ is full, then it is saturated for invertible rules.*

It turns out that if a sequent is underivable then it may be extended into a new sequent which remains underivable, but is full. This is known as completion.

Definition 5 (The Completion Sequences). Suppose $\Gamma_0 \vdash \Delta_0$ is an underivable sequent, and that ϕ_0, \dots, ϕ_n is a fixed sequence of all the formulas in $\text{Sub}(\Gamma_0, \Delta_0)$. Then we define the **completion** sequences for $\Gamma_0 \vdash \Delta_0$ by constructing the sequences $\Gamma'_0 \subseteq \dots \subseteq \Gamma'_{n+1}$ and $\Delta'_0 \subseteq \dots \subseteq \Delta'_{n+1}$ of finite sets of formulas such that $\Gamma'_i \vdash \Delta'_i$ is underivable and full for formulas ϕ_j for all $j < i$. That is either $\phi_j \in \Gamma'_i \cup \Delta'_i$ or both $\Gamma'_i \vdash \Delta'_i, \phi_j$ and $\phi_j, \Gamma'_i \vdash \Delta'_i$ are derivable.

The construction is defined by mutual recursion as follows:

$$\begin{aligned}\Gamma'_0 &:= \Gamma_0 \\ \Gamma'_{i+1} &:= \phi_i, \Gamma'_i \quad \text{if } \phi_i, \Gamma'_i \vdash \Delta'_i \text{ is underivable} \\ \Gamma'_{i+1} &:= \Gamma'_i \quad \text{otherwise}\end{aligned}$$

and

$$\begin{aligned}\Delta'_0 &:= \Delta_0 \\ \Delta'_{i+1} &:= \phi_i, \Delta'_i \quad \text{if } \Gamma'_{i+1} \vdash \Delta'_i, \phi_i \text{ is underivable} \\ \Delta'_{i+1} &:= \Delta'_i \quad \text{otherwise.}\end{aligned}$$

Lemma 6. If $\Gamma_0 \subseteq \dots \subseteq \Gamma_n$ and $\Delta_0 \subseteq \dots \subseteq \Delta_n$ are completion sequences with respect to the sequence of subformulas $\phi_0, \dots, \phi_{n-1}$ then $\Gamma_i \vdash \Delta_i$ is full with respect to the formulas ϕ_j for all $j < i$.

Proof. This is a proof by induction on i .

Base Case. Then we must show that $\Gamma_0 \vdash \Delta_0$ is full with respect to the formulas ϕ_j for all $j < 0$. This is trivially the case.

Step Case. We must show that $\Gamma_{i+1} \vdash \Delta_{i+1}$ is full with respect to the formulas ϕ_j for all $j < i + 1$. Based on the definition of completion sequences we know one of two things about Γ .

Case. Suppose $\Gamma_{i+1} = \Gamma_i, \phi_i$. Then it must be the case that $\Gamma_{i+1} \vdash \Delta_i$ is underivable. Now we case split on Δ .

Case. Suppose $\Delta_{i+1} = \Delta_i, \phi_i$. Then it must be the case that $\Gamma_{i+1} \vdash \Delta_i, \phi_i$ is underivable. However, this is not the case. Thus we have arrived at a contradiction.

Case. Suppose $\Delta_{i+1} = \Delta_i$. Fullness of $\Gamma_{i+1} \vdash \Delta_{i+1}$ follows from the fact that we know by the IH that $\Gamma_i \vdash \Delta_i$ is full with respect to formulas ϕ_j for all $j < i$, $\phi_i \in \Gamma_i \cup \Delta_i$, and $\Gamma_{i+1} \vdash \Delta_i$ is underivable.

Case. Suppose $\Gamma_{i+1} = \Gamma_i$. We case split on the form of Δ .

Case. Suppose $\Delta_{i+1} = \Delta_i, \phi_i$. Then it must be the case that $\Gamma_{i+1} \vdash \Delta_{i+1}$ is underivable. In addition we know that $\phi_i \in \Gamma_{i+1} \cup \Delta_{i+1}$. By the IH we know that $\Gamma_i \vdash \Delta_i$ is full with respect to the formulas ϕ_j for all $j < i$. Thus, by definition we know $\Gamma_{i+1} \vdash \Delta_{i+1}$ is also full.

Case. Suppose $\Delta_{i+1} = \Delta_i$. Fullness of $\Gamma_{i+1} \vdash \Delta_{i+1}$ follows from the fact that we know $\Gamma_i, \phi_i \vdash \Delta_i$ and $\Gamma_i \vdash \Delta_i, \phi_i$ are derivable.

□

Lemma 7 (Completion). Any underivable sequent $\Gamma_0 \vdash \Delta_0$ may be extended to a full sequent $\Gamma \vdash \Delta$ for some Γ and Δ consisting of subformulas of Γ_0 and Δ_0 .

Proof. Using Definition 5 and Lemma 6 we can construct the completion sequence $\Gamma_0 \subseteq \dots \subseteq \Gamma_n$ and $\Delta_0 \subseteq \dots \subseteq \Delta_n$ with respect to some sequence of the formulas in $\text{Sub}(\Gamma_0, \Delta_0)$, $\phi_0, \dots, \phi_{n-1}$, such that $\Gamma_i \vdash \Delta_i$ is formula full with respect to the formulas ϕ_j for all $j < i$. Take $\Gamma = \Gamma_n$ and $\Delta = \Delta_n$. Clearly, $\Gamma \vdash \Delta$ is full. □

We now arrive at the definition of the canonical Kripke model. This will be the counter model we will use to show completeness.

Definition 8 (Canonical Model). The **canonical Kripke model** K is a tuple $\langle W, R_{\subseteq}, V_{\in} \rangle$ such that

- W is the set of all full sequents,

- $R_{\subseteq}(\Gamma \vdash \Delta, \Gamma' \vdash \Delta') = \Gamma \subseteq \Gamma'$, and
- $V_{\in}(p, \Gamma \vdash \Delta) = p \in \Gamma$.

We now show that the canonical model falsifies every underivable sequent.

Definition 9. A sequent $\Gamma \vdash \Delta$ is **falsified** in a world w of a Kripke model if $V(\wedge \Gamma, w) = 1$ and $V(\vee \Delta, w) = 1$. This implies that $V(\Gamma \vdash \Delta, w) = 0$.

Above we saw when a sequent is saturated for invertible rules. Next we define when a set of sequents is saturated for non-invertible rules. Following this definition is the definition of when a set is saturated.

Definition 10 (Saturated for Non-Invertible Rules). A set of sequents M is **saturated for non-invertible rules** if the following condition is satisfied for every $\Gamma \vdash \Delta$ in M :

if $\phi \rightarrow \psi \in \Delta$, then there is a sequent $\Gamma' \vdash \Delta' \in M$ such that $\phi, \Gamma \subseteq \Gamma'$ and $\psi \in \Delta'$.

Definition 11 (Saturated Set). A set of sequents M is **saturated** if every sequent in M is saturated for invertible rules and M is saturated for non-invertible rules.

The following result is important for completeness.

Lemma 12 (Canonical Model is Saturated). The set W of the canonical model is saturated.

Proof. By definition of W every sequent in W is full, hence by Lemma 4 they are saturated for invertible rules. It suffices to show that W is saturated for non-invertible rules. If $\Gamma \vdash \phi \rightarrow \psi, \Delta \in W$ then it must be the case that $\phi, \Gamma \vdash \psi, \Delta$ is underivable. So by Lemma 7 we may construct the full sequent $\Gamma' \vdash \Delta' \in W$ for some Γ' and Δ' , such that $\phi, \Gamma \subseteq \Gamma'$ and $\psi \in \Delta'$. \square

Theorem 13 (Falsification in K). Let $K = \langle W, R_{\subseteq}, V_{\in} \rangle$ be the canonical model. Then for $w \equiv \Gamma \vdash \Delta \in W$:

- If $\phi \in \Gamma$ then $V_{\in}(\phi, w) = 1$, and
- if $\phi \in \Delta$ then $V_{\in}(\phi, w) = 0$.
- This implies that $V_{\in}(\Gamma \vdash \Delta, w) = 0$, that is, w is falsified in K .

Proof. The conditions a and b imply the condition c. Thus it suffices to show conditions a and b. They can be proven by simultaneous induction on ϕ .

Case. Suppose $\phi \equiv p$.

- If $\phi \in \Gamma$ then by the definition of K we have $V_{\in}(\phi, w) = 1$.
- If $\phi \in \Delta$ then it must be the case that $\phi \notin \Gamma$ or $\Gamma \vdash \Delta$ would not be full (because it is derivable), but all worlds of K are full. Therefore, it must be the case that $V_{\in}(\phi, w) = 0$.

Case. Suppose $\phi \equiv \phi_1 \wedge \phi_2$.

- Suppose $\phi \in \Gamma$. Then we know $\phi_1 \in \Gamma$ and $\phi_2 \in \Gamma$, because $\Gamma \vdash \Delta$ is full, hence, saturated for invertible rules. Thus, $V_{\in}(\phi_1, w) = 1$ and $V_{\in}(\phi_2, w) = 1$ by the IH.
- Suppose $\phi \in \Delta$. Then it must be the case that either $\phi_1 \in \Delta$ or $\phi_2 \in \Delta$ because $\Gamma \vdash \Delta$ is saturated for invertible rules. Consider the former. Then by the IH $V_{\in}(\phi_1, w) = 0$. This implies that $V_{\in}(\phi, w) = 0$. The case where $\phi_2 \in \Delta$ is similar.

Case. Suppose $\phi \equiv \phi_1 \vee \phi_2$.

- If $\phi \in \Gamma$ then
- If $\phi \in \Delta$ then

Case. Suppose $\phi \equiv \phi_1 \rightarrow \phi_2$.

- a. If $\phi \in \Gamma$ then
- b. If $\phi \in \Delta$ then

□

This is all that is needed to prove completeness of LJ. We prove this in the next section.

3 Completeness

Theorem 14 (Completeness). *Each sequent underivable in LJ is falsified in the canonical model K . Hence every valid sequent is derivable in LJ.*

Proof. Suppose that $\Gamma \vdash \Delta$ is an underivable sequent in LJ, and W is the saturated set of all full sequents. That is W is the set of worlds of K . Then by completion (Lemma 7), there exists a Γ' and Δ' such that $\Gamma' \vdash \Delta'$ is the completed version of $\Gamma \vdash \Delta$. Since W is saturated we know $\Gamma' \vdash \Delta' \in W$, and we may apply Theorem 13 to obtain that $V_{\in}(\Gamma' \vdash \Delta', \Gamma' \vdash \Delta') = 0$. Hence, by monotonicity $V_{\in}(\Gamma \vdash \Delta, \Gamma' \vdash \Delta') = 0$. □

Corollary 15 (Admissibility of Cut). *If $\Gamma \vdash \Delta, \phi$ and $\phi, \Gamma \vdash \Delta$ are derivable, then $\Gamma \vdash \Delta$ is derivable.*

Proof. Suppose $\Gamma \vdash \Delta, \phi$ and $\phi, \Gamma \vdash \Delta$ are derivable for some Γ, Δ , and ϕ . Clearly, $\Gamma \vdash \phi$ is derivable by a simple application of the cut rule. Thus, by soundness + cut we know $\Gamma \vdash \phi$ is valid. Now by completeness (without cut) $\Gamma \vdash \Delta$ is derivable. □