# A Completeness Proof for LJ

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## 1 Introduction

In this short lecture note we summarize how to show completeness for the multi-succedent intuitionistic propositional logic. The proof method we use is due to Mints. All the details can be found in his book "A Short Introduction to Intuitionistic Logic."

## 2 The Canonical Model

In this section we define the canonical model and give some properties of this model. We call a sequent **underivable** if and only if there does not exist a derivation in the logic starting with the sequent.

To define the canonical model we first must define when a sequent is full, and when a sequent is saturated for invertible rules. We denote the set of subformulas of a set of formulas $\Gamma$ as $Sub(\Gamma)$.

**Definition 1** (Formula Full Sequent)**.** *A sequent $\Gamma \vdash \Delta$ is **full for a formula** $\phi \in Sub(\Gamma, \Delta)$ if and only if it is underivable and either $\phi \in \Gamma \cup \Delta$ or $\Gamma \vdash \Delta, \phi$ and $\phi, \Gamma \vdash \Delta$ are derivable.*

**Definition 2** (Full Sequent)**.** *A sequent $\Gamma \vdash \Delta$ is **full** if and only if it is formula full for all formulas in $Sub(\Gamma, \Delta)$.*

**Definition 3** (Saturated for Invertible Rules)**.** *A sequent $\Gamma \vdash \Delta$ is **saturated for invertable rules** if it is underivable and the following conditions hold for any $\phi$ and $\psi$:*

*Case.* *If $\phi \wedge \psi \in \Delta$ then $\phi \in \Delta$ or $\psi \in \Delta$.*

*Case.* *If $\phi \wedge \psi \in \Gamma$ then $\phi \in \Gamma$ and $\psi \in \Gamma$.*

*Case.* *If $\phi \vee \psi \in \Delta$ then $\phi \in \Delta$ and $\psi \in \Delta$.*

*Case.* *If $\phi \vee \psi \in \Gamma$ then $\phi \in \Gamma$ or $\psi \in \Gamma$.*

*Case.* *If $\phi \to \psi \in \Gamma$ then $\phi \in \Delta$ or $\psi \in \Gamma$.*

Notice in the above definition that we do not have the following condition:

$$\text{If } \phi \to \psi \in \Delta \text{ then } \phi \in \Gamma \text{ and } \psi \in \Delta.$$

This is because the rule for right implication is not an invertible rule.
The following result holds for full sequents.

**Lemma 4** (Saturation)**.** *If $\Gamma \vdash \Delta$ is full, then it is saturated for invertible rules.*

It turns out that if a sequent is underivable then it may be extended into a new sequent which remains underivable, but is full. This is known as completion.

**Definition 5** (The Completion Sequences)**.** *Suppose $\Gamma_0 \vdash \Delta_0$ is an underivable sequent, and that $\phi_0, \dots, \phi_n$ is a fixed sequence of all the formulas in $Sub(\Gamma_0, \Delta_0)$. Then we define the* ***completion*** *sequences for $\Gamma_0 \vdash \Delta_0$ by constructing the sequences $\Gamma'_0 \subseteq \cdots \subseteq \Gamma'_{n+1}$ and $\Delta'_0 \subseteq \cdots \subseteq \Delta'_{n+1}$ of finite sets of formulas such that $\Gamma'_i \vdash \Delta'_i$ is underivable and full for formulas $\phi_j$ for all $j < i$. That is either $\phi_j \in \Gamma'_i \cup \Delta'_i$ or both $\Gamma'_i \vdash \Delta'_i, \phi_j$ and $\phi_j, \Gamma'_i \vdash \Delta'_i$ are derivable.*

*The construction is defined by mutual recursion as follows:*

$$
\begin{array}{lcll}
\Gamma'_0 & := & \Gamma_0 & \\
\Gamma'_{i+1} & := & \phi_i, \Gamma'_i & \textit{if } \phi_i, \Gamma'_i \vdash \Delta'_i \textit{ is underivable} \\
\Gamma'_{i+1} & := & \Gamma'_i & \textit{otherwise}
\end{array}
$$

*and*

$$
\begin{array}{lcll}
\Delta'_0 & := & \Delta_0 & \\
\Delta'_{i+1} & := & \phi_i, \Delta'_i & \textit{if } \Gamma'_{i+1} \vdash \Delta'_i, \phi_i \textit{ is underivable} \\
\Delta'_{i+1} & := & \Delta'_i & \textit{otherwise.}
\end{array}
$$

**Lemma 6.** *If $\Gamma_0 \subseteq \cdots \subseteq \Gamma_n$ and $\Delta_0 \subseteq \cdots \subseteq \Delta_n$ are completion sequences with respect to the sequence of subformulas $\phi_0, \dots, \phi_{n-1}$ then $\Gamma_i \vdash \Delta_i$ is full with respect to the formulas $\phi_j$ for all $j < i$.*

*Proof.* This is a proof by induction on $i$.

Base Case. Then we must show that $\Gamma_0 \vdash \Delta_0$ is full with respect to the formulas $\phi_j$ for all $j < 0$. This is trivially the case.

Step Case. We must show that $\Gamma_{i+1}\Delta_{i+1}$ is full with respect to the formulas $\phi_j$ for all $j < i + 1$. Based on the definition of completion sequences we know one of two things about $\Gamma$.

- Case. Suppose $\Gamma_{i+1} = \Gamma_i, \phi_i$. Then it must be the case that $\Gamma_{i+1} \vdash \Delta_i$ is underivable. Now we case split on $\Delta$.
  - Case. Suppose $\Delta_{i+1} = \Delta_i, \phi_i$. Then it must be the case that $\Gamma_{i+1} \vdash \Delta_i, \phi_i$ is underivable. However, this is not the case. Thus we have arrived at a contradiction.
  - Case. Suppose $\Delta_{i+1} = \Delta_i$. Fullness of $\Gamma_{i+1} \vdash \Delta_{i+1}$ follows from the fact that we know by the IH that $\Gamma_i \vdash \Delta_i$ is full with respect to formulas $\phi_j$ for all $j < i$, $\phi_i \in \Gamma_i \cup \Delta_i$, and $\Gamma_{i+1} \vdash \Delta_i$ is underivable.
- Case. Suppose $\Gamma_{i+1} = \Gamma_i$. We case split on the form of $\Delta$.
  - Case. Suppose $\Delta_{i+1} = \Delta_i, \phi_i$. Then it must be the case that $\Gamma_{i+1} \vdash \Delta_{i+1}$ is underivable. In addition we know that $\phi_i \in \Gamma_{i+1} \cup \Delta_{i+1}$. By the IH we know that $\Gamma_i \vdash \Delta_i$ is full with respect to the formulas $\phi_j$ for all $j < i$. Thus, by definition we know $\Gamma_{i+1} \vdash \Delta_{i+1}$ is also full.
  - Case. Suppose $\Delta_{i+1} = \Delta_i$. Fullness of $\Gamma_{i+1} \vdash \Delta_{i+1}$ follows from the fact that we know $\Gamma_i, \phi_i \vdash \Delta_i$ and $\Gamma_i \vdash \Delta_i, \phi_i$ are derivable.

□

**Lemma 7** (Completion)**.** *Any underivable sequent $\Gamma_0 \vdash \Delta_0$ may be extended to a full sequent $\Gamma \vdash \Delta$ for some $\Gamma$ and $\Delta$ consisting of subformulas of $\Gamma_0$ and $\Delta_0$.*

*Proof.* Using Definition 5 and Lemma 6 we can construct the completion sequence $\Gamma_0 \subseteq \cdots \subseteq \Gamma_n$ and $\Delta_0 \subseteq \cdots \subseteq \Delta_n$ with respect to some sequence of the formulas in $Sub(\Gamma_0, \Delta_0)$, $\phi_0, \dots, \phi_{n-1}$, such that $\Gamma_i \vdash \Delta_i$ is formula full with respect to the formulas $\phi_j$ for all $j < i$. Take $\Gamma = \Gamma_n$ and $\Delta = \Delta_n$. Clearly, $\Gamma \vdash \Delta$ is full. □

We know arrive at the definition of the canonical Kripke model. This will be the counter model we will use to show completeness.

**Definition 8** (Canonical Model)**.** *The* ***canonical Kripke model*** *$K$ is a tuple $\langle W, R_{\subseteq}, V_{\in} \rangle$ such that*

- *$W$ is the set of all full sequents,*

- $R_{\subset}(\Gamma \vdash \Delta, \Gamma' \vdash \Delta') = \Gamma \subseteq \Gamma'$, *and*
- $V_{\in}(p, \Gamma \vdash \Delta) = p \in \Gamma$.

We now show that the canonical model falsifies every underivable sequent.

**Definition 9.** *A sequent $\Gamma \vdash \Delta$ is **falsified** in a world $w$ of a Kripke model if $V(\wedge\Gamma, w) = 1$ and $V(\vee\Delta, w) = 1$. This implies that $V(\Gamma \vdash \Delta, w) = 0$.*

Above we saw when a sequent is saturated for invertable rules. Next we define when a set of sequents is saturated for non-invertable rules. Following this definition is the definition of when a set is saturated.

**Definition 10** (Saturated for Non-Invertable Rules)**.** *A set of sequents $M$ is **saturated for non-invertable rules** if the following condition is satisfied for every $\Gamma \vdash \Delta$ in $M$:*

*if $\phi \to \psi \in \Delta$, then there is a sequent $\Gamma' \vdash \Delta' \in M$ such that $\phi, \Gamma \subseteq \Gamma'$ and $\psi \in \Delta$.*

**Definition 11** (Saturated Set)**.** *A set of sequents $M$ is **saturated** if every sequent in $M$ is saturated for invertable rules and $M$ is saturated for non-invertable rules.*

The following result is important for completeness.

**Lemma 12** (Canonical Model is Saturated)**.** *The set $W$ of the canonical model is saturated.*

**Theorem 13** (Falsification in $K$)**.** *Let $K = \langle W, R_{\subseteq}, V_{\in} \rangle$ be the canonical model. Then for $w \equiv \Gamma \vdash \Delta \in W$:*

*If $\phi \in \Gamma$ then $V_{\in}(\phi, w) = 1$, and*

*if $\phi \in \Delta$ then $V_{\in}(\phi, w) = 0$.*

*This implies that $V_{\in}(\Gamma \vdash \Delta, w) = 0$, that is, $w$ is falsified in $K$.*

This is all that is needed to prove completeness of LJ. We prove this in the next section.

# 3 Completeness

**Theorem 14** (Completeness)**.** *Each sequent underivable in LJ is falsified in the canonical model $K$. Hence every valid sequent is derivable in LJ.*

*Proof.* Suppose that $\Gamma \vdash \Delta$ is an underivable sequent in LJ, and $W$ is the saturated set of all full sequents. That is $W$ is the set of worlds of $K$. Then by completion (Lemma 7), there exists a $\Gamma'$ and $\Delta'$ such that $\Gamma' \vdash \Delta'$ is the completed version of $\Gamma \vdash \Delta$. Sense $W$ is saturated we know $\Gamma' \vdash \Delta' \in W$, and we may apply Theorem 13 to obtain that $V_{\in}(\Gamma' \vdash \Delta', \Gamma' \vdash \Delta') = 0$. Hence, by monotonicity $V_{\in}(\Gamma \vdash \Delta, \Gamma' \vdash \Delta') = 0$. ☐

**Corollary 15** (Admissibility of Cut)**.** *If $\Gamma \vdash \Delta, \phi$ and $\phi, \Gamma \vdash \Delta$ are derivable, then $\Gamma \vdash \Delta$ is derivable.*

*Proof.* Suppose $\Gamma \vdash \Delta, \phi$ and $\phi, \Gamma \vdash \Delta$ are derivable for some $\Gamma$,$\Delta$, and $\phi$. Clearly, $\Gamma \vdash \phi$ is derivable by a simple application of the cut rule. Thus, by soundness + cut we know $\Gamma \vdash \phi$ is valid. Now by completeness (without cut) $\Gamma \vdash \Delta$ is derivable. ☐