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Comparisons of control schemes for monitoring the means of processes subject to drifts

Changliang Zou · Yukun Liu · Zhaojun Wang

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Abstract Although statistical process control (SPC) techniques have been focused mostly on detecting step (constant) mean shift, drift which is a time-varying change frequently occurs in industrial applications. In this research, for monitoring drift change, the following five control schemes are compared: the exponentially weighted moving average (EWMA) chart and the cumulative sum (CUSUM) charts which are recommended detecting drift change in the literature; the generalized EWMA (GEWMA) chart proposed by Han and Tsung (2004); and two generalized likelihood ratio based schemes, GLR-S and GLR-L charts which are respectively under the assumption of step and linear trend shifts. Both the asymptotic estimation and the numerical simulation of the average run length (ARL) are presented. We show that when the in-control (IC) ARL is large (goes to infinity), the GLR-L chart has the best overall performance among the considered charts in detecting linear trend shift. From the viewpoint of practical IC ARL, based on the simulation results, we show that besides the GLR-L chart, the GEWMA chart offers a good balanced protection against drifts of different size. Some computational issues are also addressed.

Keywords Drift · Generalized likelihood ratio chart · Average run length · Exponentially weighted moving average control chart · Cumulative sum control chart · Generalized EWMA control chart

C. Zou · Y. Liu (✉) · Z. Wang
LPMC and Department of Statistics, School of Mathematical Sciences,
Nankai University, 300071 Tianjin, China
e-mail: nk_ykliu@yahoo.com.cn

1 Introduction

Statistical process control (SPC) has been widely used to monitor and improve the quality and productivity of manufacturing processes. Most of research in SPC focuses on the charting techniques. The practical applications of control charts now extend far beyond manufacturing into biology, genetics, medicine, finance and other areas (see Montgomery 2004). Although current methods focus mostly on monitoring and detection of constant (step) shift in the process mean, SPC methods for detecting nonconstant or time-varying shifts in the mean, such as drifts, also have drawn much attention.

With a drift, it is assumed that once a special cause initiates the drift of a process parameter away from its in-control (IC) value, the parameter continues to drift away at a constant rate until a control chart detects this drift. That is, after some change point, say τ, the process mean of the ith observation is $\mu_0 + (i - \tau)\theta$, where θ is the change in the process mean per observation, and μ_0 is the mean when the process is in-control. Drifts are usually due to causes such as gradual deterioration of equipment, catalyst aging, waste accumulation or human causes. Bissell (1984) studies the performance of the cumulative sum (CUSUM) chart and the Shewhart chart when there is a linear trend. Davis and Woodall (1988) evaluate the performance of the Shewhart chart when various trend rules are used and conclude that it is ineffective in detecting a drift in the process mean. Gan (1991) investigates the performance of the exponentially weighted moving average (EWMA) chart under a drift. Aerne et al. (1991) compare the Shewhart chart, the Shewhart chart supplemented with runs rules, the CUSUM chart and EWMA chart also for the case of a drift in the mean by the Markov chain and Monte Carlo simulation methods. They conclude that the results of average run length (ARL) under a drift in the mean are similar to those when the mean shifts abruptly. The CUSUM and EWMA charts are about equally effective in detecting a trend. For small to moderate linear trends, both are better than the Shewhart chart. Gan (1996) develops an integral equation method to accurately evaluate the ARL of CUSUM chart under a linear trend. Recently, Reynolds and Stoumbos (2001) consider the problem of simultaneously monitoring the mean μ and the standard deviation σ of the process subject to drifts. They show that the combinations of the EWMA charts (two charts for monitoring mean and variance respectively) can detect slow-rate and moderate-rate drifts much faster than the traditional combined Shewhart type mean (X) and moving-range (MR) charts. They also show that varying the sampling interval adaptively results in notable reduction in the detection delay of drifts in μ and/or σ. Davis and Krehbiel (2002) study the ARL performances of Shewhart charts with supplementary runs rules and zone control charts when the process mean changes linearly over time. Rainer et al. (2001) propose a Shewhart-type chart, Shewhart-UMP, which is derived from the UMP (uniformly most powerful) test for detecting the linear trend shifts. Based on the ARL analysis, they conclude that the difference in the ARLs of the classical $\bar{X}$ chart (with or without runs rules used) and of Shewhart-UMP chart is negligible and suggest the CUSUM and EWMA charts should be used for the detection of the drift. Fahmy and Elsayed (2006a) develop an approach based on the deviation between the target mean and the expected mean of the process. Koning and Does (2000) propose a CUSUM chart which is developed from the UMP test for the detection of linear trend in Phase I. Some other literatures also consider schemes of monitoring drifts, such as Domangue and Patch (1991) and Reynolds and Stoumbos (2004), etc. For the reason that the CUSUM and EWMA control charts have good performances for the detection of the drift which is shown in the foregoing literatures, in this paper we compare them with some other methods introduced later.

As we know, for the detection of step shift, both the EWMA and CUSUM control charts are based on some given reference values, say the pre-specified value of the shift, which is chosen to make the charts optimal. In fact, we rarely know the exact shift value. In order to offer a more balanced protection against shifts of different size, some magnitude-robust control schemes are developed in the literature. Among others, Capizzi and Masarotto (2003) propose an adaptive EWMA (AEWMA) chart which weights the past observations of the monitored process using a suitable function of the current "error". In Zhao et al. (2005) a quality control scheme, dual CUSUM (DCUSUM), is applied that combines two CUSUM charts to detect the range of shifts. Based on the generalized likelihood ratio approach for the on-line detection, Siegmund and Venkatraman (1995) propose a CUSUM-like control chart called Generalized Likelihood Ratio (GLR) scheme which doesn't depend on the reference value. Pignatiello and Simpson (2002) investigate a similar control chart and their simulation results show that this chart has a robust performance for various magnitude of shifts. By taking the maximum weighting parameter in the EWMA control chart, a Generalized EWMA (GEWMA) is proposed by Han and Tsung (2004). Through theoretical study, they prove that GLR control chart has the best overall performance in detecting mean shift among the four considered control charts (CUSUM, EWMA, GLR and GEWMA) when the IC ARL goes to infinity. Their numerical simulations show that the GEWMA chart has comparable performance with the GLR chart. For detecting drift which is time-varying change in the mean, it seems more reasonable to use the AEWMA, DCUSUM, GLR or GEWMA charts since the changes are nonconstant. In this paper, we choose the GLR and GEWMA charts for comparisons for convenience and simplicity. Note that Han and Tsung (2005, 2006) compare the Cuscore (cumulative score), GLR and CUSUM charts for detecting dynamic mean change by theoretical analysis of the ARLs. However, their theoretical and numerical analysis is based on the assumption that the mean change and reference pattern are bounded or finally approach some steady-stable values, which significantly differs from the case considered in this paper. Thus, the asymptotic estimation and comparison in Han and Tsung (2005, 2006) are not appropriate for drift change.

In the next section, we introduce the considered control charts for monitoring drifts. Theoretical comparisons for detecting the step shift and drift are presented in Sect. 3. Section 4 contains some numerical results. Some remarks are given in Sect. 5. Section 6 concludes this paper by summarizing its contributions. Proofs of the main results are given in the Appendix.

2 The control charts for monitoring drifts

Let X_i ($i = 1, 2, \ldots$) be the ith observation on an i.i.d. process. If after some change point τ, the distribution of X_i changes from $N(\mu_0, \sigma^2)$ to $N(\mu, \sigma^2)$ ($\mu \neq \mu_0$), the mean of X_i (for $i > \tau$) undergoes a persistent shift of size $\mu - \mu_0$ which is so-called step shift. If after the change point τ, the distribution of X_i changes from $N(\mu_0, \sigma^2)$ to $N(\mu_0 + (i - \tau)\theta, \sigma^2)$ ($\theta \neq 0$), then the mean of X_i ($i > \tau$) drifts away at a constant rate θ which is usually called a drift shift or linear trend shift. We assume that the change point τ, μ and θ are unknown, μ_0 and σ are known and without loss of generality, we take $\mu_0 = 0$ and $\sigma = 1$. Furthermore, for simplicity, in this paper we consider the one-sided upward control schemes, i.e., $\mu > 0$ or $\theta > 0$.

Using the classical likelihood formula for detecting the change-point under the alternative assumption of drift, we can obtain the following natural logarithm of the likelihood ratio

$$R(\tau, \theta | x) = 2 \ln \prod_{i=\tau+1}^{n} \frac{e^{-[X_i - (i-\tau)\theta]^2/2}}{e^{-X_i^2/2}}.$$

Since τ and θ are unknown, the test is conducted by maximizing $R(\tau, \theta | x)$ over all possible values of (τ, θ) given the observed sample. Since it is easy to verify that $\max_{0 \leq \tau < n, \theta} R(\tau, \theta | x) = \max_{0 \leq \tau < n} \max_{\theta} R(\tau, \theta | x)$, we have

$$\begin{aligned} R_n &= \max_{0 \leq \tau < n} \left[\sum_{i=\tau+1}^{n} (i - \tau) X_i \right]^2 \Big/ \sum_{i=\tau+1}^{n} (i - \tau)^2 \\ &= \max_{1 \leq k \leq n} \left[\sum_{i=n-k+1}^{n} (i - n + k) X_i \right]^2 \Big/ \sum_{i=n-k+1}^{n} (i - n + k)^2 = \max_{1 \leq k \leq n} [V_n(k)]^2, \end{aligned}$$

where $V_n(k) = \sum_{i=n-k+1}^{n} (i - n + k) X_i / a_k$ and $a_k = [\sum_{i=n-k+1}^{n} (i - n + k)^2]^{\frac{1}{2}} = \sqrt{\frac{(2k+1)(k+1)k}{6}}$. If R_n is sufficiently large, then an out-of-control signal is triggered. In fact, this control scheme is the GLR scheme under the assumption of linear trend shift (see Pollak and Siegmund 1975, Bassevile and Nikiforov 1993, Lai 1995 and Siegmund and Venkatraman 1995 for details). Fahmy and Elsayed (2006a) also mention this control scheme for detecting drift. In order to distinguish it from the GLR scheme for step shift change, we denote this control scheme as GLR-L chart in this paper. The upward stopping time outside a control limit c for the GLR-L chart can be expressed as

$$T_{\mathrm{GLR-L}}(c) = \inf\{n \geq 1 : \max_{1 \leq k \leq n} V_n(k) \geq c\}.$$

To compare the GLR-L control chart with other charts, the definitions of the optimal EWMA, CUSUM, GEWMA charts and GLR control scheme for step shift are given below. According to Wu (1994) the one-sided optimal EWMA control chart for step shift can be defined as

$$T_{\mathrm{E}}(c) = \inf\{n \geq 1 : W_n(r^*) \geq c\},$$
$$W_n(r^*) = \frac{\sqrt{2-r^*}}{\sqrt{r^*}} \sum_{i=0}^{n-1} r^*(1-r^*)^i X_{n-i}, \tag{1}$$

where $r^* = 2a^*\delta^2/c^2$, δ is the pre-specified value of the step shift and a^* is an optimal design constant for δ (See Srivastava and Wu 1993 for details). Srivastava and Wu (1997) recommend $a^* \approx 0.5117$ obtained by numerical search. The stopping time of the GEMWA chart proposed by Han and Tsung (2004) is defined as follows:

$$T_{\mathrm{GE}}(c) = \inf\{n \geq 1 : \max_{1 \leq k \leq n} \bar{W}_n\left(\frac{1}{k}\right) \geq c\},$$

where

$$\bar{W}_n(r) = \frac{\sqrt{2-r}}{\sqrt{r[1-(1-r)^{2n}]}} \sum_{i=0}^{n-1} r(1-r)^i X_{n-i}. \tag{2}$$

The classical one-sided stopping time of the CUSUM can be written as

$$T_{\mathrm{C}}(c) = \inf\{n \geq 1 : \max_{1 \leq k \leq n} \delta[\sum_{i=n-k+1}^{n} X_i - \delta k/2] \geq c\}, \quad \delta > 0.$$

It has been shown by Moustakides (1986) and Ritov (1990) that the performance in detecting the mean step shift of the one-sided CUSUM control chart with the reference value δ is optimal in the sense of Lorden (1971) if the real shift is δ.

Finally, according to Siegmund and Venkatraman (1995), the upward stopping time of the GLR for step shift (called GLR-S hereafter) is

$$T_{\mathrm{GLR-S}}(c) = \inf\{n \geq 1 : \max_{1 \leq k \leq n} U_n(k) \geq c\},$$

where $U_n(k) = \sum_{i=n-k+1}^{n} X_i/\sqrt{k}$. For detecting drift change, GLR-S chart can be regard as GLR control scheme with inaccuracy change pattern.

Note that for notation convenience, in the preceding definitions of various control schemes we use $c > 0$ generically to represent any control limit which may take a different value for each appearance.

Remark 1 Among these considered charts, the GLR-L chart is designed for detecting a drift change, but the other charts are constructed under the assumption of step shift change. So the comparison is somehow “unfair" when we concentrate on drift change. Nevertheless, these charts are commonly used for step shifts and also for drift changes, so that the theoretical and numerical comparisons in this paper are reasonable.

3 Comparisons of the EWMA, CUSUM, GEWMA, GLR-S and GLR-L schemes

When evaluating and comparing the ARL performances of various control charts, traditionally, we assume the change-point $\tau = 0$. The ARL performance is referred to as the zero-state ARL performance. In practice, it may be reasonable to assume that the process starts in control and then shifts at some random time τ in the future. For an arbitrarily $\tau > 0$, the ARL performance of a control chart is called steady-state ARL performance. In this paper, we only consider the zero-state ARL for convenience of discussion.

Let $P(\cdot)$ and $E(\cdot)$ denote the probability and expectation operators when there is no change and $P_\mu(\cdot)$ and $E_\mu(\cdot)$, the operators when the change point $\tau = 0$ and the true step shift value is μ; $P_\theta(\cdot)$ and $E_\theta(\cdot)$, the operators when the $\tau = 0$ and the rate of drift is θ. The most commonly used operating measures in SPC are the IC ARL and out-of-control (OC) ARL, i.e.

$$ARL_0(T) = E(T), \quad ARL_\mu(T) = E_\mu(T) \quad \text{or} \quad ARL_\theta(T) = E_\theta(T)$$

where T is a stopping time of a control scheme. The standard SPC terminology of the comparisons of different control charts is designing the charts to have a common ARL_0 and then comparing the OC ARLs for given a type and magnitude of shift. The chart with the smaller OC ARL is considered to have better performance.

Although it seems that the assessment on the performances of the foregoing stopping times is similar to those of the charts for step shift, a drift makes the OC process distribution change as time varies. Hence, theoretical derivation for the asymptotic behavior of OC ARL is different from that of a step shift which has been considered by Srivastava and Wu (1993), Siegmund and Venkatraman (1995) and Han and Tsung (2004). Also, because a drift doesn't approach a stable value, the method in Han and Tsung (2006) cannot be applied directly. Next we investigate the asymptotic estimations of IC and OC ARL for the considered charts.

The approximations of ARL_0 for the optimal EWMA, GEWMA, CUSUM and GLR-S control charts have been given by Wu (1994), Han and Tsung (2004), Taylor (1975) and Siegmund and Venkatraman (1995) respectively.

Proposition 1 *Let $c > 0$ be the control limit for EWMA, GEWMA, GLR-S and CUSUM control charts, then as $c \to \infty$,*

$$ARL_0(T_{\mathrm{E}}) \sim \rho_1 \frac{e^{c^2/2}}{c}, \quad ARL_0(T_{\mathrm{GE}}) \sim \rho_2 \frac{e^{c^2/2}}{c},$$
$$ARL_0(T_{\mathrm{GLR-S}}) \sim \rho_3 \frac{e^{c^2/2}}{c}, \quad ARL_0(T_{\mathrm{C}}) \sim \rho_4 e^c.$$

where $\rho_i > 0, i = 1, \ldots, 4$ are some constants and $\rho_1 > \rho_2$, $\rho_1 > \rho_3$.

This result can be obtained from Theorem 1 and Corollary 1 of Han and Tsung (2004).

In order to compare the GLR-L chart with the other charts, the first theorem is to give the approximation of ARL_0 for the GLR-L chart.

Theorem 1 *Let $c > 0$ be the control limit for the GLR-L chart. Then as $c \to \infty$, there exists a constant $\rho_5 > 0$ such that*

$$ARL_0(T_{\mathrm{GLR-L}}) \sim \rho_5 \frac{e^{c^2/2}}{c}, \tag{3}$$

where $\rho_5 < \rho_1$.

Han and Tsung (2004) give a thorough ARL_μ comparison of the optimal EWMA, GEWMA, CUSUM and GLR-S control charts under step shift. The next theorem and corollary show the comparison results between the GLR-L chart and the other charts for detecting the step shift. The approximation for $ARL_\mu(T_{\mathrm{GLR-L}})$ is given as follows.

Theorem 2 *If $ARL_0(T_{\mathrm{GLR-L}}) \to \infty$ or $c \to \infty$, then for $\mu > 0$ and $\theta = 0$,*

$$\begin{aligned} ARL_\mu(T_{\mathrm{GLR-L}}) &= \frac{4}{3} \cdot \frac{c^2}{\mu^2}\left(1 + o\left(\frac{\ln c}{c}\right)\right) \\ &= \frac{2\ln(ARL_0) + \ln(2\ln(ARL_0)) - 2\ln(\rho_5)}{\frac{3}{4}\mu^2} \\ &\quad \times \left(1 + o\left(\frac{\ln(2\ln(ARL_0))}{[\ln(ARL_0)]^{\frac{1}{2}}}\right)\right). \end{aligned} \tag{4}$$

Corollary 1 *If $ARL_0(T_{\mathrm{GE}}) = ARL_0(T_{\mathrm{GLR-S}}) = ARL_0(T_{\mathrm{GLR-L}}) = ARL_0(T_{\mathrm{C}}) = ARL_0(T_{\mathrm{E}}) \to \infty$, then for $\mu > 0$ and $\theta = 0$,*

(i) $ARL_\mu(T_{\mathrm{GLR-L}}) > ARL_\mu(T_{\mathrm{GE}}) > ARL_\mu(T_{\mathrm{GLR-S}})$.
(ii) $ARL_\mu(T_{\mathrm{C}}) > ARL_\mu(T_{\mathrm{GLR-L}})$ *if and only if* $0 < \mu < \frac{2}{3}\delta$ *or* $\mu > 2\delta$.
(iii) $ARL_\mu(T_{\mathrm{E}}) > ARL_\mu(T_{\mathrm{GLR-L}})$ *if and only if* $0 < \mu < 0.8298\delta$ *or* $\mu > 1.3283\delta$.

Remark 2 The GLR-L chart is designed to monitor linear trend shift, so, from Corollary 1 (i), we can see that when $ARL_0 \to \infty$, the GLR-L chart has the worst performance compared with the GEWMA and GLR-S charts. However, it follows from Corollary 1 (ii) and (iii) that the GLR-L chart is more efficient than the CUSUM and EWMA charts in detecting step shift except for μ being in some specified intervals when $ARL_0 \to \infty$.

Next, we consider the approximations for $ARL_\theta(T_{\mathrm{GE}})$, $ARL_\theta(T_{\mathrm{GLR-S}})$ and $ARL_\theta(T_{\mathrm{GLR-L}})$.

Theorem 3 *If $ARL_0(T_{\mathrm{GLR-L}}) \to \infty$ or $c \to \infty$, then for $\theta > 0$ and $\mu = 0$,*

$$ARL_\theta(T_{\mathrm{GLR-L}}) = \left(\frac{c}{\theta \upsilon_1}\right)^{\frac{2}{3}}\left(1 + o\left(\frac{1}{c^{\frac{2}{3}}}\right)\right), \tag{5}$$

where $\upsilon_1 = \frac{\sqrt{3}}{3}$.

Theorem 4 *If* $ARL_0(T_{\mathrm{GE}}) \to \infty$ *or* $c \to \infty$*, then for* $\theta > 0$ *and* $\mu = 0$*,*

$$ARL_\theta(T_{\mathrm{GE}}) = \left(\frac{c}{\theta \upsilon_2}\right)^{\frac{2}{3}} \left(1 + o\left(\frac{1}{c^{\frac{2}{3}}}\right)\right), \tag{6}$$

where $\upsilon_2 = \sqrt{\frac{2}{e^2-1}}$.

Theorem 5 *If* $ARL_0(T_{\mathrm{GLR-S}}) \to \infty$ *or* $c \to \infty$*, then for* $\theta > 0$ *and* $\mu = 0$*,*

$$ARL_\theta(T_{\mathrm{GLR-S}}) = \left(\frac{c}{\theta \upsilon_3}\right)^{\frac{2}{3}} \left(1 + o\left(\frac{1}{c^{\frac{2}{3}}}\right)\right), \tag{7}$$

where $\upsilon_3 = \sqrt{\frac{8}{27}}$.

By Theorems 3–5, we have the following corollary.

Corollary 2 *If* $ARL_0(T_{\mathrm{GE}}) = ARL_0(T_{\mathrm{GLR-S}}) = ARL_0(T_{\mathrm{GLR-L}}) \to \infty$*, then for* $\theta > 0$ *and* $\mu = 0$*,*

$$ARL_\theta(T_{\mathrm{GLR-L}}) < ARL_\theta(T_{\mathrm{GE}}) < ARL_\theta(T_{\mathrm{GLR-S}}). \tag{8}$$

The following theorems show the results of comparisons of the charts under drift.

Theorem 6 *If* $ARL_0(T_{\mathrm{C}}) = ARL_0(T_{\mathrm{GE}}) = ARL_0(T_{\mathrm{GLR-S}}) = ARL_0(T_{\mathrm{GLR-L}}) \to \infty$*, then for* $\theta > 0$ *and* $\mu = 0$*,*

$$ARL_\theta(T_{\mathrm{GLR-L}}) < ARL_\theta(T_{\mathrm{GE}}) < ARL_\theta(T_{\mathrm{GLR-S}}) < ARL_\theta(T_{\mathrm{C}}). \tag{9}$$

Theorem 7 *If* $ARL_0(T_{\mathrm{E}}) = ARL_0(T_{\mathrm{GE}}) = ARL_0(T_{\mathrm{GLR-S}}) = ARL_0(T_{\mathrm{GLR-L}}) \to \infty$*, then for* $\theta > 0$ *and* $\mu = 0$*,*

$$ARL_\theta(T_{\mathrm{GLR-L}}) < ARL_\theta(T_{\mathrm{GE}}) < ARL_\theta(T_{\mathrm{GLR-S}}) < ARL_\theta(T_{\mathrm{E}}). \tag{10}$$

Remark 3 Theorems 6 and 7 show that the GLR-L chart has the best performance in detecting a drift among the five control charts when $ARL_0 \to \infty$. Furthermore, the GEMWA and GLR-S charts are better than the optimal (for step shift detection) EWMA and CUSUM charts in detecting a drift uniformly when $ARL_0 \to \infty$.

4 Numerical results

In this section we present some simulation results of ARLs of the one-sided optimal EWMA, CUSUM, GEWMA, GLR-S and GLR-L charts. The numerical results of zero-state ARLs ($\tau = 0$) are tabulated in Table 1. These values are obtained based on 10,000 repetition Monte Carlo simulations. The IC ARL is chosen to be 1730. The pre-specified shift values δ for the EWMA and CUSUM charts are chosen to be 0.5, 1.0

Table 1 The ARL comparisons of the control charts with $ARL_0 \approx 1730$ under drift

Drift(θ)	EWMA			CUSUM			GEWMA	GLR-S	GLR-L
	$\delta = 0.5$	$\delta = 1.0$	$\delta = 1.5$	$\delta = 0.5$	$\delta = 1.0$	$\delta = 1.5$			
0.0005	317	377	440	345	412	470	375	381	368
0.001	215	253	297	231	275	317	252	257	249
0.005	83.6	92.6	106	86.6	98.6	112	96.2	97.8	95.4
0.01	55.6	58.8	66.1	56.9	61.8	69.3	62.1	63.3	62.0
0.05	22.6	21.1	22.0	22.6	21.6	22.7	22.4	22.7	22.5
0.1	15.5	13.9	13.8	15.4	14.7	14.2	14.4	14.6	14.5
0.5	6.65	5.56	5.09	6.60	5.54	5.17	5.10	5.23	5.18
1.0	4.67	3.83	3.43	4.63	3.80	3.45	3.26	3.38	3.31
2.0	3.21	2.74	2.32	3.17	2.67	2.32	2.09	2.16	2.12
3.0	2.86	2.06	1.98	2.79	2.04	1.96	1.69	1.75	1.72
4.0	2.14	2.00	1.83	2.10	1.98	1.74	1.31	1.37	1.34
RMI	0.254	0.169	0.182	0.263	0.200	0.210	0.067	0.092	0.070
c	2.711	3.033	3.161	9.660	5.620	3.904	3.500	3.670	3.580

and 1.5. So, the corresponding parameters r^* for the three EWMA charts are 0.03479, 0.11125 and 0.23052 (columns 1–3 respectively), and δ for the three CUSUM charts are 0.5, 1.0 and 1.5 (columns 4–6 respectively). The values of the control limit c for the considered charts are tabulated in the last row of Table 1. In order to assess the robustness of the charts to various magnitudes of drifts, here we use the relative mean index (RMI) which is given by Han and Tsung (2006). The RMI of a stopping time T is defined as

$$RMI(T) = \frac{1}{n} \sum_{i=1}^{n} \frac{ARL_{\theta_i}(T) - MARL_{\theta_i}}{MARL_{\theta_i}},$$

where $ARL_{\theta_i}(T)$ is the OC ARL of the stopping time T for drift size θ_i and $MARL_{\theta_i}$ is the smallest OC ARL among all the considered charts for drift size θ_i. In this numerical research, θ_i ranges from 0.0005 to 4 as given in Table 1. Obviously, small $RMI(T)$ means that the control chart T has a robust performance in detecting the drifts on the whole.

From Table 1, we can observe

– The GLR-L chart has uniformly better performance than the GLR-S chart.
– The GEWMA chart performs a slightly better than the GLR-L chart in detecting a drift which is not small, that is, $\theta \geq 0.05$. From the viewpoint of RMI, the GEWMA chart even outperforms the GLR-L chart. It's not surprising to us that the simulation results aren't consistent with the theoretical comparison results because the IC ARL is not large enough. Similar phenomenon can also be found in Han and Tsung (2004) for step shift.
– As is expected, the performances of the EWMA and CUSUM charts depend heavily on the pre-specified values. The EWMA and CUSUM charts with some given reference values can have smaller OC ARL than the GEWMA, GLR-S, GLR-L charts for some magnitude of drifts. Although the numerical results do not conform

Table 2 The averages, standard deviations and precisions of change-point estimates

θ	AVE	SD	$Pr(\lvert\widehat{\tau}-\tau\rvert \le 1)$	$Pr(\lvert\widehat{\tau}-\tau\rvert \le 3)$	$Pr(\lvert\widehat{\tau}-\tau\rvert \le 5)$
0.05	29.0	10.6	0.11	0.25	0.37
0.1	26.8	7.3	0.18	0.40	0.58
0.5	24.8	3.3	0.55	0.88	0.95
1.0	24.9	2.3	0.78	0.95	0.97

with the asymptotic results shown in Sect. 3, the CUSUM and EWMA charts are relatively inefficient compared with the other three charts based on the RMI.

Some other numerical results for $\tau > 0$ have been obtained (available from the authors) and similar conclusions are drawn.

It should be noted that the GLR-L control scheme possesses a unique advantage, namely after the chart signals, the user can obtain the maximum likelihood estimates (MLE) of θ and the change point τ without any more computational efforts. These estimates will help an engineer to identify and eliminate the root cause of a problem quickly and easily. Formally, assume that an out-of-control signal is triggered at the nth observation by the GLR-L chart, the estimates of τ and θ are given by

$$\widehat{\tau} = n - \arg\max_{1\le k\le n}\{V_n(k)\},$$
$$\widehat{\theta} = V_n(n-\widehat{\tau})/a_{n-\widehat{\tau}},$$

where $V_n(k)$ and a_k are defined in Sect. 2.

Here, we conduct simulations to evaluate the effectiveness of the change point estimator. The process change point is simulated at $\tau = 25$. Fifty thousand independent series are generated in the simulations. The IC ARL is again chosen to be 1730. Note that any series in which a signal occurs before the $(\tau+1)$th observation is discarded. In Table 2, we tabulate the averages (AVE) and standard deviations (SD) of the estimates $\widehat{\tau}$. Also, the observed frequencies with which the estimators are within a given number of samples around the actual τ, *i.e.*, the probabilities $Pr(|\widehat{\tau}-\tau| \le 1)$, $Pr(|\widehat{\tau}-\tau| \le 3)$ and $Pr(|\widehat{\tau}-\tau| \le 5)$ are presented. These probabilities may provide certain indications of the precision of the estimator. For simplicity, the drift sizes, $\theta = 0.05, 0.1, 0.5$ and 1.0 are considered.

Table 2 shows that the change point estimator performs well from the viewpoint of the average for any drift size. We can also see that $\widehat{\tau}$ has better precision as the magnitude of the drift increases. These findings about the MLE for drift monitoring are consistent with Pignatiello and Samuel's (2001) conclusion about step shift monitoring. Similar use of the MLE of drift time can also be found in Fahmy and Elsayed (2006b).

5 Some remarks

It should be noted that although the GEMWA and GLR-L charts have more robust performances in detecting drift change compared with the EWMA and CUSUM charts, they require more extensive computational efforts because both of them need search the maximum values from n statistics after the nth observation is observed. A remedy that comes into mind is to restrict the searching to a window of those statistics. Han and Tsung (2004) give a revised definition of GEWMA chart which not only lightens the computational burden but also keeps the effectiveness in detecting mean shift. For the control charts based on GLR algorithm, Willsky and Jones (1976) suggest replacing $\max_{1\leq k\leq n}$ in the GLR scheme by maximizing k over a moving window $n - M \leq k \leq n - \tilde{M}$. For simplicity, $\tilde{M}$ is usually fixed as zero. Lai (1995) gives a detailed theoretical discussion about such a window-limited scheme and presents some other modifications. We conduct some simulations to choose the parameter M for the GLR-L chart. Empirically, we find that $M = 100$ is large enough to provide similar performance of the GLR-L chart without window limited for monitoring the drift. Moreover, to ease the numerical calculation, appropriate programming can provide some recursive manner of storage and computation of the statistics $V_n(k)$ in the GLR-L scheme which may further alleviate the computational problem (The Fortran subroutine is available from authors upon request).

As we know, the ARL is not the only performance measure when we assess the effectiveness of a control chart. The standard deviation of the run length (SDRL) of a control scheme is also a popular indicator of the performance of a chart. When two charts have approximately equal IC and OC ARLs, the chart which has smaller OC SDRL can be regard as a better scheme. An ongoing effort of the authors is theoretically analyzing and comparing the SDRLs of various control schemes.

Moreover, it's interesting to consider the case for which the OC distribution after the change point τ is $N(\mu + (i - \tau)\theta, \sigma^2)$, where $\mu \neq \mu_0$ and $\theta \neq 0$. Then based on the GLR scheme, we would construct a control chart which may be able to effectively cope with drift and step shift simultaneously. We think that it warrants some future research on both theoretical and numerical properties.

6 Conclusions

Although statistical process control (SPC) techniques have been focused mostly on detecting step (constant) mean shift, the drift which is a time-varying change frequently occurs in industrial applications. In this research, we compare the EWMA, CUSUM, GEWMA and GLR-S and GLR-L charts in detecting drift change. Both the theoretical approximations and numerical simulations are presented. From theoretical study it is clear that when the IC ARL goes to infinity, the GEWMA, GLR-S, GLR-L charts outperform the EWMA and CUSUM charts for any magnitude of drifts. We also prove that the GLR-L chart has the best overall performance among the considered charts for monitoring drift changes. Since the IC ARL is not large enough in the numerical research, the simulation results show that the EWMA and CUSUM charts can have smaller OC ARL than the GEWMA, GLR-S, GLR-L charts for some magnitudes of drifts. However, the GEWMA, GLR-S, GLR-L charts have smaller RMI which indicates these three charts have better magnitude-robust performances. From the simulation results, we also find that when the control limit is not large enough, the

GEMWA chart performs a slightly better than the GLR-L chart in detecting a drift which is not small. From the viewpoint of RMI, the GEWMA chart even outperforms the GLR-L chart. Moreover, we use simulations to illustrate another advantage of the GLR-L scheme, say, MLE of the change point τ.

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Appendix

In order to prove Theorem 1, we need the following lemmas.

Lemma 1 *Let $c > 0$ be a control limit for the GLR-L control chart. Then*

$$ARL_0(T_{\text{GLR-L}}) < \rho_1 \frac{e^{c^2/2}}{c},$$

where ρ_1 is defined in Proposition 1.

Proof Since $V_m(1) = X_m$, we have

$$\begin{aligned}
P(T_{\text{E}} > n) &\geq P(X_k \leq c, 1 \leq k \leq n) \\
&= P(V_1(1) < c, V_2(1) < c, \ldots, V_n(1) < c) \\
&\geq P\left(\max_{1\leq k\leq m} V_m(k) < c, 1 \leq m \leq n\right) = P(T_{\text{GLR-L}} > n),
\end{aligned}$$

where the first inequality comes from the proof of Theorem 1 of Han and Tsung (2004). Hence $ET_{\text{GLR-L}} < ET_{\text{E}}$, say, $ARL_0(T_{\text{GLR-L}}(c)) < ARL_0(T_{\text{E}}(c))$. By Proposition 1, we can obtain Lemma 1 immediately. □

Lemma 2 *Let $\alpha_{ij}(m,n) = \text{Cov}(V_m(i), V_n(j))$ and $\beta_{ij}(m,n) = \text{Cov}(U_m(i), U_n(j))$. Then*

(i) $\alpha_{ij}(n,n) \geq \beta_{ij}(n,n)$;

(ii) *Let $b >> 0$, $m_0 = bc^2$, $m_1 = \sqrt{b}m_0$, $m \geq i \geq m_1$, $n \geq j \geq m_1$, and $0 < m - n < m_0$. Then, for large b, $\alpha_{ij}(m,n) > \beta_{ij}(m,n)$ as $c \to \infty$.*

Proof (i) Since $(U_n(1), U_n(2), \ldots, U_n(n))$ and $(V_n(1), V_n(2), \ldots, V_n(n))$ are two n-dimensional multivariate normal distributions with $EU_n(k) = EV_n(k) = 0$ and $\text{Var}(U_n(k)) = \text{Var}(V_n(k)) = 1$ for $1 \leq k \leq n$, it can be easily obtained

$$\beta_{ij}(n,n) = \sqrt{\frac{i}{j}} \quad i < j,$$

$$\alpha_{ij}(n,n) = \frac{(i+1)i(3j-i+1)}{\sqrt{(2i+1)(i+1)i(2j+1)(j+1)j}} \quad i < j.$$

Then

$$\left(\frac{\alpha_{ij}(n,n)}{\beta_{ij}(n,n)}\right)^2 = \frac{(i+1)(3j-i+1)^2}{(2i+1)(2j+1)(j+1)} > \frac{(i+1)(2j+1)}{(2i+1)(j+1)} > 1.$$

That is, $\alpha_{ij}(n,n) > \beta_{ij}(n,n)$ for $i < j$. By the same reasoning, we have $\alpha_{ij}(n,n) > \beta_{ij}(n,n)$ for $i > j$.

(ii) For $i \geq m-n+j$,

$$\beta_{ij}(m,n) = \sqrt{\frac{j}{i}},$$
$$\alpha_{ij}(m,n) = \frac{(j+1)j[3i-j+1-3(m-n)]}{\sqrt{(2i+1)(i+1)i(2j+1)(j+1)j}}.$$

Consequently,

$$\left(\frac{\alpha_{ij}(m,n)}{\beta_{ij}(m,n)}\right)^2 = \frac{(j+1)[3i-j+1-3(m-n)]^2}{(2i+1)(2j+1)(i+1)}.$$

Since $m-n < m_0$ and $i \geq m_1$, then $\frac{m-n}{i} \leq \frac{m_0}{m_1} = \frac{1}{\sqrt{b}} \to 0$ as $b \to \infty$. Note that

$$\begin{aligned}
&i^2\left[(j+1)\left(\frac{3i-j+1-3(m-n)}{i}\right)^2 - \left(2+\frac{1}{i}\right)(2j+1)\left(1+\frac{1}{i}\right)\right] \\
&= i^2 j\left[\left(1+\frac{1}{j}\right)\left(3-\frac{j}{i}+o(1)\right)^2 - \left(2+O\left(\frac{1}{i}\right)\right)\left(2+\frac{1}{j}\right)\right] \\
&> i^2 j\left[(3-1)^2-4\right] = 0.
\end{aligned}$$

For $i < m-n+j$, denote $k = i-(m-n)$, then

$$\begin{aligned}
&\beta_{ij}(m,n) = \frac{k}{\sqrt{ij}}, \\
&\alpha_{ij}(m,n) \\
&\quad = \frac{k\,[6(i+1)(j+1)-3(i+j-2k+1)(k+1)-3(m-n)(2j-k+1)]}{\sqrt{(2i+1)(i+1)i(2j+1)(j+1)j}}.
\end{aligned}$$

Then we have

$$\left(\frac{\alpha_{ij}(m,n)}{\beta_{ij}(m,n)}\right)^2 = \frac{(3j+1+3kj-k^2)^2}{(2i+1)(2j+1)(i+1)(j+1)}.$$

Note that

$$\begin{aligned}
&(3j+1+3kj-k^2)^2-(2i+1)(2j+1)(i+1)(j+1)\\
&=i^2j^2\left[\left(\frac{3}{i}+\frac{1}{ij}+3\frac{k}{j}-\frac{k^2}{ij}\right)^2-2\cdot 2+o(1)\right]\\
&=i^2j^2\left[\left(3-(1-o(1))\,\frac{k}{j}+o(1)\right)^2-4+o(1)\right]\\
&>i^2j^2\left[(3-1)^2-4\right]=0
\end{aligned}$$

since $k<j$. Thus $\alpha_{ij}(m,n)>\beta_{ij}(m,n)$ as $c\to\infty$.

□

Lemma 3 *For large b, as* $c\to\infty$, *we have*

$$ARL_0(T_{\mathrm{GLR-L}})\geq\frac{(2\pi)^{\frac{1}{2}}}{b^{\frac{3}{2}}}\frac{e^{c^2/2}}{c}.$$

Proof According to Lemma 2, similar to the proof of Lemma 4 of Han and Tsung (2004), we can prove this lemma. □

Proof of Theorem 1 Applying Lemma 1 and Lemma 3, we can obtain this theorem.

□

We need the following lemma to prove Theorems 2 and 3.

Lemma 4 *Let* $E_\mu(V_n(k))=\mu_{nk}$ *and* $E_\theta(V_n(k))=\theta_{nk}$. *Then*

(i) $\mu_{nk}>\mu_{n(k-1)}$, $\mu_{n(k-1)}=\mu_{(n-1)(k-1)}$;
(ii) $\theta_{nk}>\theta_{n(k-1)}$, $\theta_{n(k-1)}\geq\theta_{(n-1)(k-1)}$.

Proof Note that

$$\mu_{nk}=\mu\sqrt{\frac{3k(k+1)}{2(2k+1)}},\quad \theta_{nk}=\theta\frac{(3n-k+1)\sqrt{k(k+1)}}{\sqrt{6(2k+1)}}. \tag{11}$$

It's trivial to prove this lemma by (11). □

Proof of Theorem 2 Denote $\mu_{nk}=E_\mu(V_n(k))$. It follows from (11) that

$$\mu_{nn}=\mu\sqrt{n}\cdot\sqrt{b_n}\quad b_n=\frac{3(n+1)}{2(2n+1)}$$

Note that $b_n \downarrow b = \frac{3}{4}$ for $n \geq 1$ and $b - b_n = O(\frac{1}{n})$. Putting $N = \frac{1}{\mu^2 b}(c^2 + 4c\sqrt{\ln c})$ and $n = N + k$, we have

$$
\begin{aligned}
c - \mu_{nn} &= -(\mu\sqrt{N+k}\sqrt{b_n} - c) \\
&= -\mu\sqrt{N+k}\sqrt{b_n}\left[1 - \frac{1}{\sqrt{b_n/b + 4b_n\sqrt{\ln c}/(bc) + \mu^2 k b_n/c^2}}\right] \\
&\leq -\mu\sqrt{N+k}\sqrt{b}\left[1 - \frac{1}{\sqrt{1 + 4\sqrt{\ln c}/c}}\right] \\
&= -\mu\sqrt{N+k}B
\end{aligned}
$$

as $c \to \infty$, where $B = \sqrt{b}\left(1 - \frac{1}{\sqrt{1+4\sqrt{\ln c}/c}}\right)$. Let ϕ and Φ be the standard normal density and distribution functions, respectively. Note that $\mu^2 B^2 N = 4\ln c\Big(1 + O(\sqrt{\ln c}/c)\Big)$ for large c. It follows that

$$
\begin{aligned}
\sum_{n=N+1}^{\infty} P_\mu(T_{\mathrm{GLR-L}}(c) > n) &= \sum_{n=N+1}^{\infty} P(V_l(k) < c - \mu_{lk}, 1 \leq k \leq l, 1 \leq l \leq n) \\
&\leq \sum_{n=N+1}^{\infty} \int_{-\infty}^{c-\mu_{nn}} \phi(x)dx \leq \sum_{n=N+1}^{\infty} \int_{\mu B\sqrt{N+k}}^{+\infty} \phi(x)dx \\
&\leq \sum_{k=1}^{\infty} \frac{\exp\{-\frac{1}{2}\mu^2 B^2 (N+k)\}}{\mu\sqrt{2\pi} B\sqrt{N}} \\
&\leq \frac{\exp\{-\frac{1}{2}\mu^2 B^2 N\}}{\mu\sqrt{2\pi} B\sqrt{N}(1 - \exp\{-\frac{1}{2}\mu^2 B^2\})} \\
&= \frac{N\exp\left\{-2\ln c\left(1 + O\left(\sqrt{\ln c}/c\right)\right)\right\}}{4\sqrt{2\pi}\ln c\sqrt{\ln c}} \\
&= \frac{1}{4\sqrt{2\pi} b\mu^2 (\ln c)^{\frac{3}{2}}} + O\left(\frac{1}{c\ln c}\right)
\end{aligned}
$$

for large c. Thus

$$
\begin{aligned}
ARL_\mu(T_{\mathrm{GLR-L}}(c)) &\leq \sum_{n=1}^{N} P_\mu(T_{\mathrm{GLR-L}}(c) > n) + \frac{1}{4\sqrt{2\pi} b\mu^2 (\ln c)^{\frac{3}{2}}} \\
&\leq N + \frac{1}{4\sqrt{2\pi} b\mu^2 (\ln c)^{\frac{3}{2}}} + O\left(\frac{1}{c\ln c}\right) \\
&\leq \frac{1}{\mu^2 b}(c^2 + 4c\sqrt{\ln c}) + o\left(\frac{1}{\ln c}\right) \quad c \to \infty.
\end{aligned}
$$

Now consider the lower bound. Let $m = \frac{1}{\mu^2 b}(c^2 - 4c\sqrt{3\ln c})$. According to Lemma 4 (i), by similar argument to that in the proof of Theorem 2 of Han and Tsung (2004), we can prove

$$ARL_\mu(T_{\text{GLR-L}}(c)) \geq m - O\left(\frac{1}{\sqrt{\ln c}}\right).$$

By Theorem 1, we have $c^2 = 2\ln(ARL_0) + \ln(2\ln(ARL_0)) - 2\ln(\rho_5)$ for large c. Thus this theorem can be obtained immediately. □

Proof of Corollary 1 (i) Han and Tsung (2004) give two approximations for $ARL_\mu(T_{\text{GE}}(c))$ and $ARL_\mu(T_{\text{GLR-S}}(c))$ as follows:

$$\begin{aligned} ARL_\mu(T_{\text{GE}}) &= \frac{2\ln(ARL_0) + \ln(2\ln(ARL_0)) - 2\ln(\rho_2)}{b\mu^2} \\ &\quad \times \left(1 + o\left(\frac{\ln(2\ln(ARL_0))}{[\ln(ARL_0)]^{\frac{1}{2}}}\right)\right), \\ ARL_\mu(T_{\text{GLR-S}}) &= \frac{2\ln(ARL_0) + \ln(2\ln(ARL_0)) - 2\ln(\rho_3)}{\mu^2} \\ &\quad \times \left(1 + o\left(\frac{\ln(2\ln(ARL_0))}{[\ln(ARL_0)]^{\frac{1}{2}}}\right)\right), \end{aligned}$$

where $b = \frac{2(1-e^{-1})}{1+e^{-1}} \approx 0.9242$. Comparing these two equations with (4), we can obtain this corollary immediately.

(ii) From Theorem 1, Proposition 1 and $ARL_0(T_{\text{C}}) = ARL_0(T_{\text{GLR-L}}) \to \infty$, it follows that there exists a positive increasing function $l_1(c)$ such that $l_1(c) = \sqrt{2c + \ln 2c + \varepsilon(c)}$ and $ET_{\text{C}}(c) = ET_{\text{GLR-L}}(l_1(c)) \to \infty$, as $c \to \infty$ where $|\varepsilon(c)| \leq M_1$ and M_1 is a constant. For $\mu \leq \frac{\delta}{2}$, similar to the proof of Theorem 5 of Han and Tsung (2004), we can show that $ARL_\mu(T_{\text{C}}(c)) > ARL_\mu(T_{\text{GLR-L}}(l_1(c))) \to \infty$ as $c \to \infty$. For $\mu > \frac{\delta}{2}$, the approximation for $ARL_\mu(T_{\text{C}}(c))$ is given by Wu (1994) as follows:

$$ARL_\mu(T_{\text{C}}(c)) \approx \frac{2(\mu - \delta/2)(c + 2\rho\delta)/\delta - 1 + e^{-2(\mu-\delta/2)(c+2\rho\delta)/\delta}}{2(\mu - \delta/2)^2},$$

where $\rho \approx 0.583$. Comparing this with (4) we can see that $ARL_\mu(T_{\text{C}}(c)) > ARL_\mu(T_{\text{GLR-L}}(l_1(c)))$ for $\mu > \frac{\delta}{2}$ as $c \to \infty$ if and only if

$$\frac{1}{(\mu - \delta/2)\delta} > \frac{8}{3\mu^2}.$$

This implies that $0 < \mu < \frac{2}{3}\delta$ and $\mu > 2\delta$.

(iii) From Theorem 1, Proposition 1 and $ARL_0(T_{\mathrm{E}}) = ARL_0(T_{\mathrm{GLR-L}}) \to \infty$ it follows that there exists a positive increasing function $l_2(c)$ such that $l_2(c) = c - \varepsilon(c)$ and $ET_{\mathrm{E}}(l_2(c)) = ET_{\mathrm{GLR-L}}(c) \to \infty$ as $c \to \infty$ where $0 < \varepsilon(c) < M_2/c$ and M_2 is a constant. For $\mu \le \sqrt{a^*}\delta$, similar to the proof of Theorem 4 of Han and Tsung (2004), we can show that $ARL_\mu(T_{\mathrm{E}}(l_2(c))) > ARL_\mu(T_{\mathrm{GLR-L}}(c)) \to \infty$ as $c \to \infty$. For $\mu > \sqrt{a^*}\delta$, the approximation for $ARL_\mu(T_{\mathrm{E}}(c))$ is given by Han and Tsung (2004) as follows:

$$ARL_\mu(T_{\mathrm{E}}(l_2(c))) = \frac{-\ln(1-\sqrt{a^*}\delta/\mu)}{2a^*\delta^2}(c-\varepsilon(c))^2\left(1+o\left(\frac{\ln c}{c}\right)\right)$$

Comparing this with (4) we can see that $ARL_\mu(T_{\mathrm{C}}(l_2(c))) > ARL_\mu(T_{\mathrm{GLR-L}}(c))$ for $\mu > \sqrt{a^*}\delta$ as $c \to \infty$ if and only if

$$\frac{-\ln(1-\sqrt{a^*}\delta/\mu)}{2a^*\delta^2} > \frac{4}{3\mu^2}.$$

This implies that $0 < \mu < c_1\delta$ and $\mu > c_2\delta$, where $c_1 = 0.8298$ and $c_2 = 1.3283$ are obtained by a numerical search.

□

Proof of Theorem 3 We prove that

$$\left(\frac{c}{\theta\upsilon_1}\right)^{\frac{2}{3}} + O(1) \le ARL_\theta(T_{\mathrm{GLR-L}}) \le \left(\frac{c}{\theta\upsilon_1}\right)^{\frac{2}{3}} + o(c^{-\frac{1}{3}}). \tag{12}$$

It follows from (11) that

$$\begin{aligned}\theta_{nn} &= \theta\sqrt{\frac{(2n+1)(n+1)n}{6}} = \theta\frac{1}{\sqrt{3}}n^{\frac{3}{2}}(1+\frac{3}{2}n^{-1}+\frac{1}{2}n^{-2})^{\frac{1}{2}} \\ &= (n^{\frac{3}{2}}\upsilon_1 + n^{\frac{1}{2}}\beta_1)\theta + O(\frac{1}{\sqrt{n}})\theta,\end{aligned}$$

where $\beta_1 = \frac{\sqrt{3}}{4}$. Putting $N = \left(\frac{c}{\theta\upsilon_1}\right)^{\frac{2}{3}}$ and $n = N + k$, we have

$$\begin{aligned}c - \theta_{nn} &= -(\theta(N+k)^{\frac{3}{2}}\upsilon_1 - c) - \theta(N+k)^{\frac{1}{2}}\beta_1 + O\left(\frac{1}{\sqrt{n}}\right) \\ &\le -\left[(c^{\frac{2}{3}} + k(\theta\upsilon_1)^{\frac{2}{3}})^{\frac{3}{2}} - c\right] \\ &\le -\left[c\left(1+\frac{3}{2}k(\theta\upsilon_1/c)^{\frac{2}{3}}\right) - c\right] \\ &\le -\frac{3}{2}c^{\frac{1}{3}}(\theta\upsilon_1)^{\frac{2}{3}}\sqrt{k} \triangleq -B_k\end{aligned}$$

as $c \to \infty$. It follows that

$$\begin{aligned}\sum_{n=N+1}^{\infty} P_\theta(T_{\mathrm{GLR-L}}(c) > n) &= \sum_{n=N+1}^{\infty} P(V_l(k) < c - \theta_{lk}, 1 \le k \le l, 1 \le l \le n)\\ &\le \sum_{n=N+1}^{\infty} \int_{-\infty}^{c-\theta_{nn}} \phi(x)dx \le \sum_{n=N+1}^{\infty} \int_{B_k}^{+\infty} \phi(x)dx\\ &\le \sum_{k=1}^{\infty} \frac{\exp\{-\frac{1}{2}B_k^2\}}{\sqrt{2\pi}B_k} \le \frac{\sum_{k=1}^{\infty} \exp\left\{-\frac{k}{2}\left(\frac{3}{2}(\theta\upsilon_1)^{\frac{2}{3}}\right)^2 c^{\frac{2}{3}}\right\}}{\sqrt{2\pi}B_1}\\ &= o\left(c^{-\frac{1}{3}}\right)\end{aligned}$$

for large c. Thus

$$\begin{aligned}ARL_\theta(T_{\mathrm{GLR-L}}(c)) &\le \sum_{n=1}^{N} P_\mu(T_{\mathrm{GLR-L}}(c) > n) + o\left(c^{-\frac{1}{3}}\right)\\ &\le \left(\frac{c}{\theta\upsilon_1}\right)^{\frac{2}{3}} + o\left(c^{-\frac{1}{3}}\right)\end{aligned}$$

for large c. This proves the upward inequality of (12).

Put $m = \left(\frac{c-2\sqrt{\ln c}}{\theta\upsilon_1}\right)^{\frac{2}{3}} - \frac{2}{3}\beta_1/\upsilon_1$. It can be easily shown that $c - \theta_{mm} = 2\sqrt{\ln c} + o(1)$, which results in

$$\begin{aligned}1 - \Phi(c - \theta_{mm}) &= \frac{\phi(c-\theta_{mm})}{c-\theta_{mm}}\left[1 - O\left(\frac{1}{\ln c}\right)\right]\\ &= \frac{1}{2\sqrt{2\pi}c^2\sqrt{\ln c}}\left[1 - O\left(\frac{1}{\ln c}\right)\right]. \qquad (13)\end{aligned}$$

Suppose that Y_{lk}, $1 \le k \le l$, are standard independent normal variables. By Proposition 2 and Lemma 4 (ii), we have

$$\begin{aligned}\sum_{n=1}^{m} P_\theta(T_{\mathrm{GLR-L}}(c) > n) &= \sum_{n=1}^{m} P_\theta\left(\max_{1\le k\le l} V_l(k) < c, 1 \le l \le n\right)\\ &= \sum_{n=1}^{m} P\left(V_l(k) < c - \theta_{lk}, 1 \le k \le l, 1 \le l \le n\right)\\ &\ge \sum_{n=1}^{m} P\left(Y_{lk} < c - \theta_{lk}, 1 \le k \le l, 1 \le l \le n\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{m} \prod_{l=1}^{n} \prod_{k=1}^{l} \Phi(c-\theta_{lk}) \geq \sum_{n=1}^{m} [\Phi(c-\theta_{mm})]^{\frac{(m+1)n}{2}} \qquad (14)\\
&= \frac{[\Phi(c-\theta_{mm})]^{\frac{(m+1)}{2}}\left(1-[\Phi(c-\theta_{mm})]^{\frac{(m+1)n}{2}}\right)}{1-[\Phi(c-\theta_{mm})]^{\frac{(m+1)}{2}}}\\
&= m - O\left(m^2\left(1-[\Phi(c-\theta_{mm})]^{\frac{(m+1)}{2}}\right)\right)\\
&= m - O\left(\frac{1}{\sqrt{\ln c}}\right).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
ARL_\theta(T_{\mathrm{GLR-L}}) &\geq m - O\left(\frac{1}{\sqrt{\ln c}}\right)\\
&= \left(\frac{c-2\sqrt{\ln c}}{\theta \upsilon_1}\right)^{\frac{2}{3}} - \frac{2}{3}\beta_1/\upsilon_1 - O\left(\frac{1}{\sqrt{\ln c}}\right)\\
&= \left(\frac{c}{\theta \upsilon_1}\right)^{\frac{2}{3}} - \frac{2}{3}\beta_1/\upsilon_1 - O\left(\frac{1}{\sqrt{\ln c}}\right) = \left(\frac{c}{\theta \upsilon_1}\right)^{\frac{2}{3}} + O(1)
\end{aligned}
$$

for large c. This is the downward inequality of (12), which completes the proof. □

Proof of Theorem 4 Let $\theta_{nk} = E_\theta \bar{W}_n(\frac{1}{k})$. It follows from (2) that

$$
\theta_{nk} = \theta\sqrt{\frac{2k-1}{1-(1-1/k)^{2n}}}\left\{n-(k-1)\left[1-(1-1/k)^n\right]\right\}.
$$

It can be easily checked that

$$
\theta_{nk} > \theta_{n(k-1)}, \quad \theta_{n(k-1)} \geq \theta_{(n-1)(k-1)}. \qquad (15)
$$

Note that

$$
\begin{aligned}
\theta_{nn} &= \theta\sqrt{\frac{2-1/n}{1-(1-1/n)^{2n}}}\left[\sqrt{n}+\sqrt{n}(n-1)\left(1-\frac{1}{n}\right)^n\right]\\
&= (n^{\frac{3}{2}}\upsilon_2 + n^{\frac{1}{2}}\beta_2)\theta + O\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}
$$

where $\beta_2 = e\upsilon_2 - \frac{7}{4}\upsilon_2 - \frac{1}{4}\upsilon_2^3 > (e-2)\upsilon_2 > 0$. Taking $N = \left(\frac{c}{\theta\upsilon_2}\right)^{\frac{2}{3}}$, $m = \left(\frac{c-2\sqrt{\ln c}}{\theta\upsilon_2}\right)^{\frac{2}{3}} - \frac{2}{3}\beta_2/\upsilon_2$ in the proof of Theorem 3 and using (15), we can prove this theorem. The details are omitted. □

Proof of Theorem 5 Let $\theta_{nk} = E_\theta U_n(k)$. We have

$$\theta_{nk} = \theta \frac{(2n-k+1)k^{\frac{1}{2}}}{2}.$$

Obviously, $\theta_{n(k-1)} \geq \theta_{(n-1)(k-1)}$. For fixed n, it can be easily checked that θ_{nk} attains its maximum value

$$\theta_{nk_n} = (\frac{2}{3}n + \frac{1}{3})(\frac{2}{3}n + \frac{1}{2})^{\frac{1}{2}} = (n^{\frac{3}{2}}\upsilon_3 + n^{\frac{1}{2}}\beta_3)\theta + O\left(\frac{1}{\sqrt{n}}\right)$$

at $k_n = \frac{2}{3}n + \frac{1}{2}$, where $\beta_3 = \frac{7\sqrt{6}}{36}$. Put $N = \left(\frac{c}{\theta\upsilon_3}\right)^{\frac{2}{3}}$ and $n = N + k$. Similar to the proof of Theorem 4, we can check that

$$\begin{aligned} ARL_\theta(T_{\text{GLR}-\text{S}}(c)) &= \sum_{n=1}^{N} P_\mu(T_{\text{GLR}-\text{L}}(c) > n) + \sum_{n=N+1}^{\infty} P_\mu(T_{\text{GLR}-\text{S}}(c) > n) \\ &\leq N + \sum_{n=N+1}^{\infty} \int_{-\infty}^{c-\theta_{nk_n}} \phi(x)dx = \left(\frac{c}{\theta\upsilon_3}\right)^{\frac{2}{3}} + o\left(c^{-\frac{1}{3}}\right) \end{aligned}$$

for large c. Also, let $m = \left(\frac{c-2\sqrt{\ln c}}{\theta\upsilon_3}\right)^{\frac{2}{3}} - \frac{2}{3}\beta_3/\upsilon_3$. The lower bound can be obtained as follows

$$\begin{aligned} ARL_\theta(T_{\text{GLR}-\text{S}}) &\geq \sum_{n=1}^{m} P_\theta(T_{\text{GLR}-\text{S}}(c) > n) = \sum_{n=1}^{m}\prod_{l=1}^{n}\prod_{k=1}^{l} \Phi(c - \theta_{lk}) \\ &\geq \sum_{n=1}^{m} \left[\Phi(c - \theta_{mk_m})\right]^{\frac{(m+1)n}{2}} \\ &\geq m - O\left(\frac{1}{\sqrt{\ln c}}\right) = \left(\frac{c}{\theta\upsilon_3}\right)^{\frac{2}{3}} + O(1) \end{aligned}$$

for large c. This completes the proof. □

Proof of Corollary 2 According to Proposition 1 and Theorem 1, we can prove this corollary. □

Proof of Theorem 6. By Corollary 2, we only need to show that if $ARL_0(T_{\text{GLR}-\text{S}}) = ARL_0(T_{\text{C}}) \to \infty$, then for $\theta > 0$ and $\mu = 0$, $ARL_\theta(T_{\text{GLR}-\text{S}}) < ARL_\theta(T_{\text{C}})$. From Theorem 1 and Proposition 1, it follows that there exists a positive increasing function $l_1(c)$ such that $l_1(c) = \sqrt{2c + \ln 2c + \varepsilon(c)}$ and $ET_{\text{C}}(c) = ET_{\text{GLR}-\text{S}}(l_1(c)) \to \infty$ as $c \to \infty$, where $|\varepsilon(c)| \leq M_1$ and M_1 is a constant. Note that for $1 \leq n \leq m$,

$$
\begin{aligned}
P_\theta\left(T_{\mathrm{C}}(c)>n\right) &= P\left(k^{\frac{1}{2}}U_l(k)+\frac{2l-k+1}{2}k\theta-\frac{k\delta}{2}<\frac{c}{\delta}, 1\le k\le l, 1\le l\le n\right)\\
&\ge P\left(U_l(k)<\frac{ck^{-\frac{1}{2}}}{\delta}-\left(m-\frac{k}{2}+\frac{1}{2}\right)k^{\frac{1}{2}}\theta, 1\le k\le n, 1\le l\le n\right)\\
&\ge \left[\Phi\left(\frac{ck_m^{-\frac{1}{2}}}{\delta}-\left(m-\frac{k_m}{2}+\frac{1}{2}\right)k_m^{\frac{1}{2}}\theta\right)\right]^{\frac{(m+1)n}{2}},
\end{aligned}
$$

since $\frac{ck^{-\frac{1}{2}}}{\delta}-\left(m-\frac{k}{2}+\frac{1}{2}\right)k^{\frac{1}{2}}\theta$ attains its minimum value at $k_m=\frac{m+\frac{1}{2}+\sqrt{(m+\frac{1}{2})^2+6c/(\delta\theta)}}{3}$. Let $m=\sqrt{2}\left(\frac{c}{\delta\theta}\right)^{\frac{1}{2}}-\frac{12}{11\theta}\sqrt{3\ln c}\left(\frac{c}{\delta\theta}\right)^{-\frac{1}{4}}-\frac{1}{2}$. Then we have

$$
\frac{ck_m^{-\frac{1}{2}}}{\delta}-\left(m-\frac{k_m}{2}+\frac{1}{2}\right)k_m^{\frac{1}{2}}\theta=\sqrt{3\ln c}+o\left(\frac{1}{\sqrt{\ln c}}\right).
$$

From this, as (14), we can show that

$$
\begin{aligned}
ARL_\theta(T_{\mathrm{C}}(c)) &\ge m-O\left(\frac{1}{\sqrt{\ln c}}\right)\\
&=\sqrt{2}\left(\frac{c}{\delta\theta}\right)^{\frac{1}{2}}+O\left(\frac{1}{\sqrt{\ln c}}\right).
\end{aligned}
$$

By Theorem 5, we have

$$
ARL_\theta(T_{\mathrm{GLR-S}}(l_2(c)))\sim O(c^{\frac{1}{3}})<ARL_\theta(T_{\mathrm{C}}(c)).
$$

This completes the proof. □

Proof of Theorem 7 From Theorem 1, Proposition 1 and $ARL_0(T_{\mathrm{E}})=ARL_0(T_{\mathrm{GLR-S}})\to\infty$ it follows that there exists a positive increasing function $l_2(c)$ such that $l_2(c)=c-\varepsilon(c)$ and $ET_{\mathrm{E}}(l_2(c))=ET_{\mathrm{GLR-S}}(c)\to\infty$ as $c\to\infty$ where $0<\varepsilon(c)<M_2/c$ and M_2 is a constant. Let $\theta_m=E_\theta W_m(r^*)$. It follows from (1) that

$$
\theta_m=\frac{\sqrt{2-r^*}}{\sqrt{r^*}}\sum_{i=0}^{m-1}r^*(1-r^*)^i(n-i)\theta.
$$

Put $m = \frac{c}{(\theta\sqrt{a^*\delta})^{\frac{1}{2}}}\left(1 - \frac{\sqrt{\ln c}}{c}\right) - \frac{1}{2}$. Note that as $c \to \infty$,

$$
\begin{aligned}
l_2(c) - \theta_m &= c - \varepsilon(c) - \frac{\sqrt{2-r^*}}{\sqrt{r^*}} \sum_{i=0}^{m-1} r^*(1-r^*)^i (n-i)\theta \\
&\geq c - \varepsilon(c) - \frac{c\theta}{\sqrt{a^*\delta}} \left\{ m - \left(\frac{1}{r^*} - 1\right)\left[1 - (1-r^*)^m\right] \right\} \\
&= c\left(1 - \frac{\theta}{\sqrt{a^*\delta}} \left\{ m - \left(\frac{1}{r^*} - 1\right)\left[1 - (1-r^*)^m\right] \right\}\right) + O\left(\frac{1}{c}\right) \\
&= c(1 - \frac{\theta}{2\sqrt{a^*\delta}}(m^2 r^* + m r^*)) + O\left(\frac{1}{c}\right) \\
&= 2\sqrt{\ln c} + O(1).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
ARL_\theta(T_{\mathrm{E}}(l_2(c))) &\geq \sum_{k=1}^{m} P_\theta(T_{\mathrm{E}}(l_2(c)) > k) \geq \sum_{k=1}^{m} \left[\Phi\left(l_2(c) - \theta_m\right)\right]^k \\
&\geq \frac{1 - [\Phi(2\sqrt{\ln c})]^m}{1 - \Phi(2\sqrt{\ln c})} \Phi(2\sqrt{\ln c}) \\
&= m + o(1) = \frac{c}{(\theta\sqrt{a^*\delta})^{\frac{1}{2}}}\left(1 - \frac{\sqrt{\ln c}}{c}\right) + O(1)
\end{aligned}
$$

for large c. By Theorem 5, obviously, $ARL_\theta(T_{\mathrm{GLR-S}}(c)) < ARL_\theta(T_{\mathrm{E}}(l_2(c)))$ as $c \to \infty$. From this result and Corollary 2, we obtain the theorem. □

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