

# Metamodeling for Bias Estimation of Biological Reference Points Under Two-Parameter SRRs

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13 August 2024



# Outline

1 Introduction

2 The Schaefer Model

3 The Beverton-Holt Model

4 Delay Differential Growth Extension

5 End

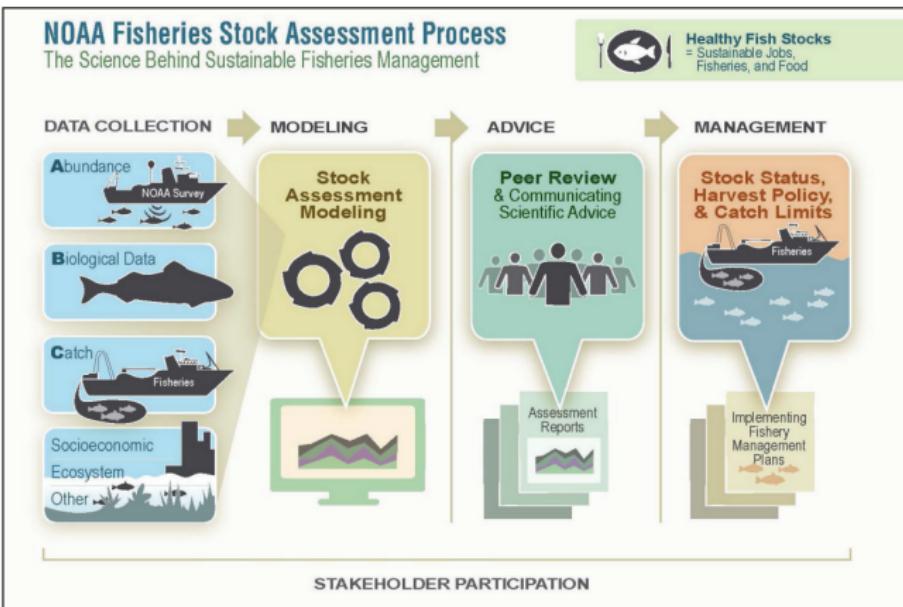


Figure 1: Overview of the stock assessment process from data collection through the provision of scientific advice to fishery managers. Stakeholders and other partners participate in each step of the assessment process. This report captures NOAA Fisheries products associated with the 'Advice' phase of the process.

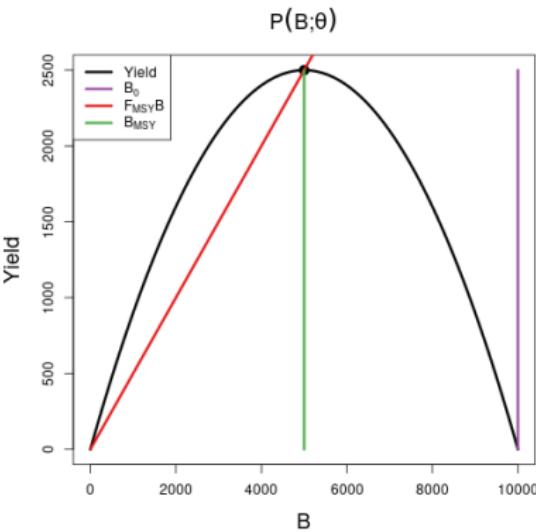
# Surplus Production Model General Structure

$$I_t = qB_t e^\epsilon \quad \epsilon \sim N(0, \sigma^2).$$

$$\frac{dB}{dt} = P(B(t); \theta) - Z(t)B(t).$$

## Reference Points:

- Maximum sustainable Yield (*MSY*)
- $F_{MSY}^a$ : Fishing rate to achieve *MSY*
- $\frac{B_{MSY}}{B_0}$ : Biomass Depletion when at *MSY*
- Driven by the shape of  $P$  as determined by  $\theta$ .

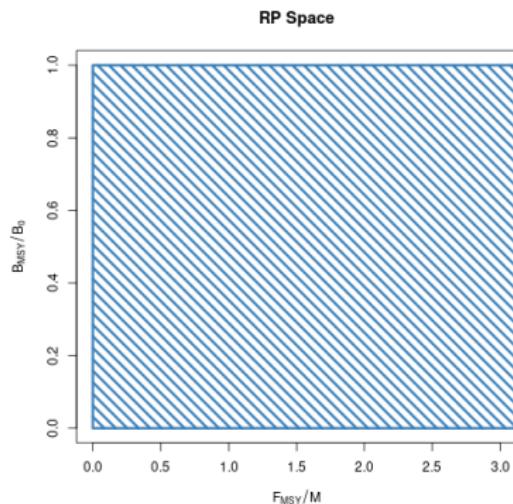


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<sup>a</sup>or  $\frac{F_{MSY}}{M}$

Conceptually:

$$\frac{F_{MSY}}{M} \in \mathbb{R}^+ \quad \frac{B_{MSY}}{B_0} \in (0, 1)$$

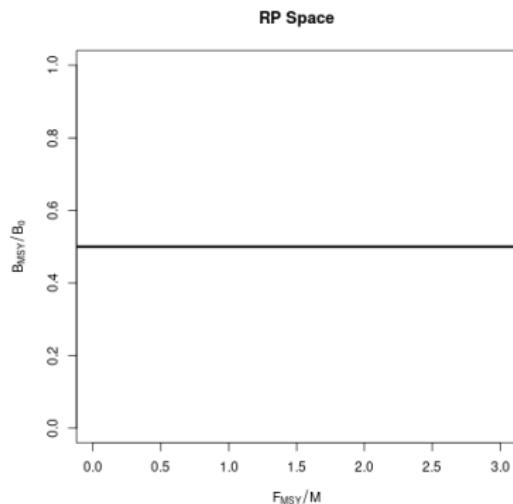


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$$\frac{F_{MSY}}{M} \in \mathbb{R}^+ \quad \frac{B_{MSY}}{B_0} \in (0, 1)$$

■ Schaefer Model:

$$F_{MSY} \in \mathbb{R}^+ \quad \frac{B_{MSY}}{B(0)} = \frac{1}{2}$$



Conceptually:

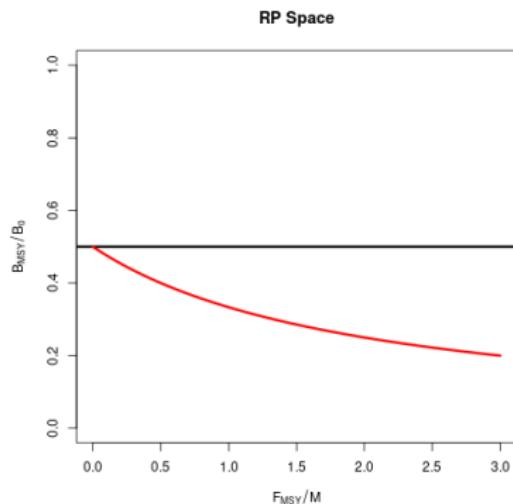
$$\frac{F_{MSY}}{M} \in \mathbb{R}^+ \quad \frac{B_{MSY}}{B_0} \in (0, 1)$$

■ Schaefer Model:

$$F_{MSY} \in \mathbb{R}^+ \quad \frac{B_{MSY}}{B(0)} = \frac{1}{2}$$

■ BH Model:

$$\frac{F_{MSY}}{M} \in \mathbb{R}^+ \quad \frac{B_{MSY}}{B(0)} = \frac{1}{F_{MSY}/M + 2}$$



Conceptually:

$$\frac{F_{MSY}}{M} \in \mathbb{R}^+ \quad \frac{B_{MSY}}{B_0} \in (0, 1)$$

■ Schaefer Model:

$$F_{MSY} \in \mathbb{R}^+ \quad \frac{B_{MSY}}{B(0)} = \frac{1}{2}$$

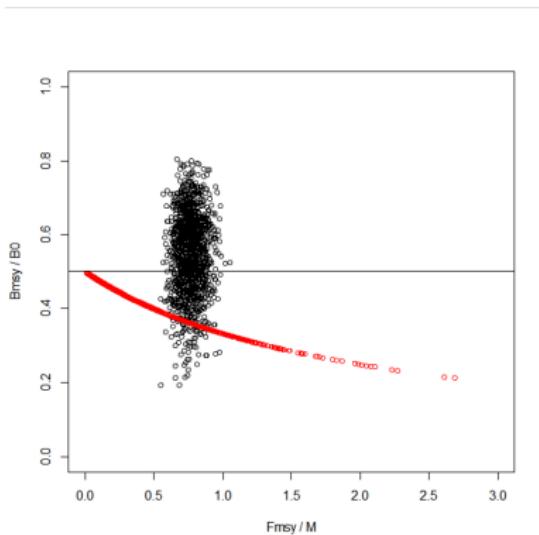
■ BH Model:

$$\frac{F_{MSY}}{M} \in \mathbb{R}^+ \quad \frac{B_{MSY}}{B(0)} = \frac{1}{F_{MSY}/M + 2}$$

■ Similar Constraints for other  
Two-Parameter Models:

Fox, Ricker, etc...

■ Three-Parameter Models Allow  
Independent RP Estimation



<sup>a</sup>Mangel et al. 2013, CJFAS

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- $P_\ell$  is logistic production
  - Logistic map in discrete time
  - Implicit Natural Mortality
  - Explicit Fishing Mortality

$$I_t = qB_t e^\epsilon \quad \epsilon \sim N(0, \sigma^2)$$

$$\frac{dB}{dt} = P_\ell(B(t); \theta) - F(t)B(t)$$

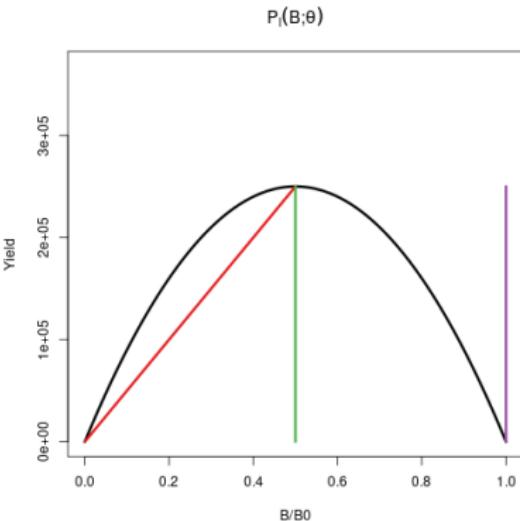
$$P_\ell(B; [r, K]) = rB \left(1 - \left(\frac{B}{K}\right)\right)$$

### Reference Points:

$$F^* = \frac{r}{2}$$

$$B^* = \frac{K}{2} \quad B_0 = K$$

$$\frac{B^*}{B_0} = \frac{1}{2}$$



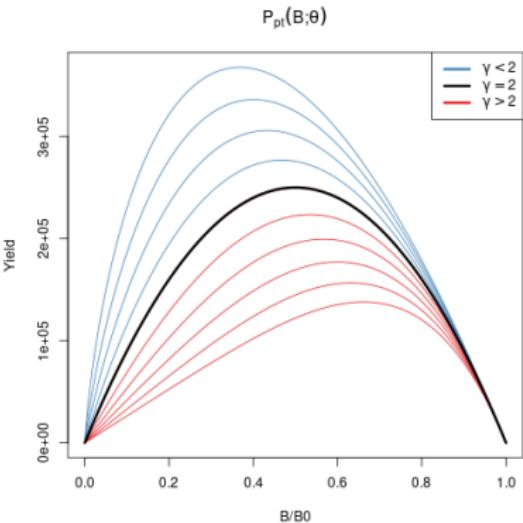
$$P_{pt}(B; [r, K, \gamma]) = \frac{rB}{\gamma - 1} \left( 1 - \left( \frac{B}{K} \right)^{(\gamma-1)} \right)$$

### Reference Points:

$$F^* = \frac{r}{\gamma}$$

$$B^* = K \left( \frac{1}{\gamma} \right)^{\frac{1}{\gamma-1}} \quad B_0 = K$$

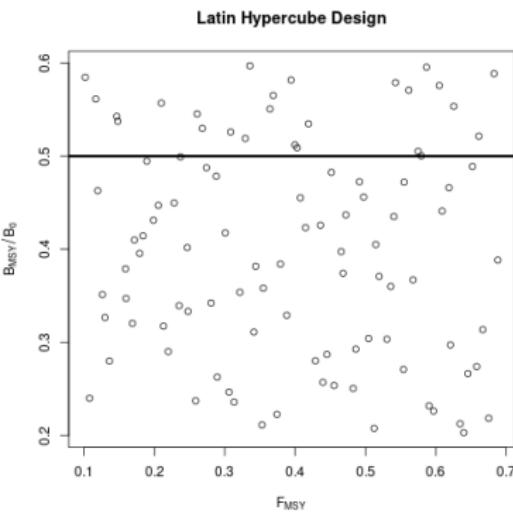
$$\frac{B^*}{B_0} = \left( \frac{1}{\gamma} \right)^{\frac{1}{\gamma-1}}$$



$$F^* = \frac{r}{\gamma} \quad \frac{B^*}{\bar{B}(0)} = \left(\frac{1}{\gamma}\right)^{\frac{1}{\gamma-1}}$$

Closed-Form Inversion

$$r = \gamma F^* \quad \gamma = \frac{W\left(\frac{B^*}{\bar{B}(0)} \log\left(\frac{B^*}{\bar{B}(0)}\right)\right)}{\log\left(\frac{B^*}{\bar{B}(0)}\right)}$$

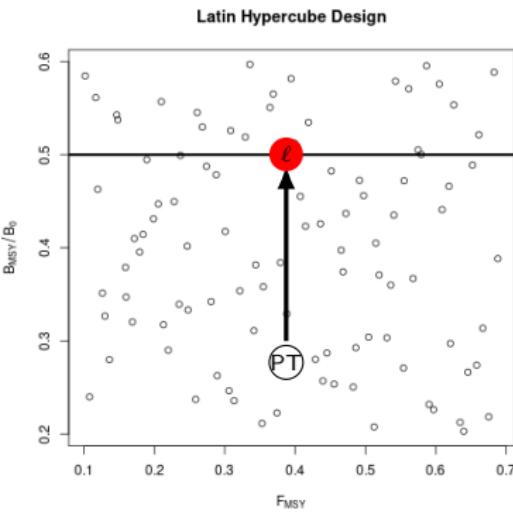


\* Lambert  $W$  function inverts  $xe^x$  s.t.  $W(xe^x) = x$

$$F^* = \frac{r}{\gamma} \quad \frac{B^*}{\bar{B}(0)} = \left(\frac{1}{\gamma}\right)^{\frac{1}{\gamma-1}}$$

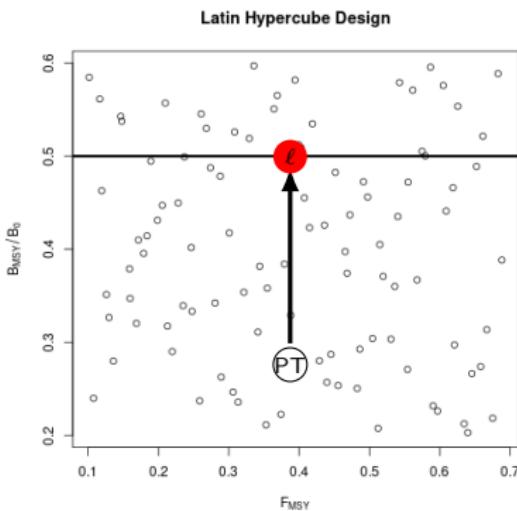
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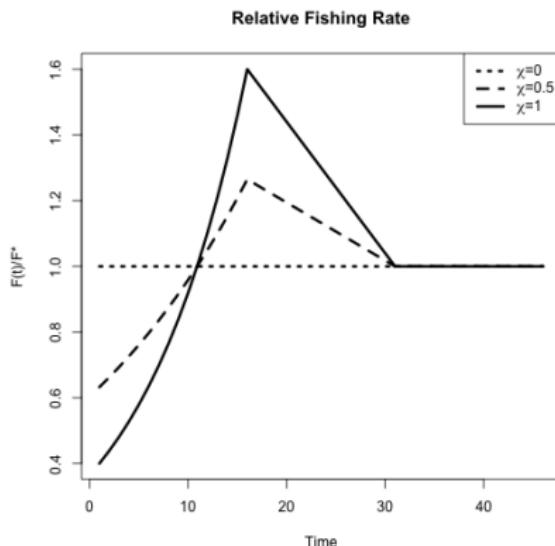


\* Lambert W function inverts  $xe^x$  s.t.  $W(xe^x) = x$

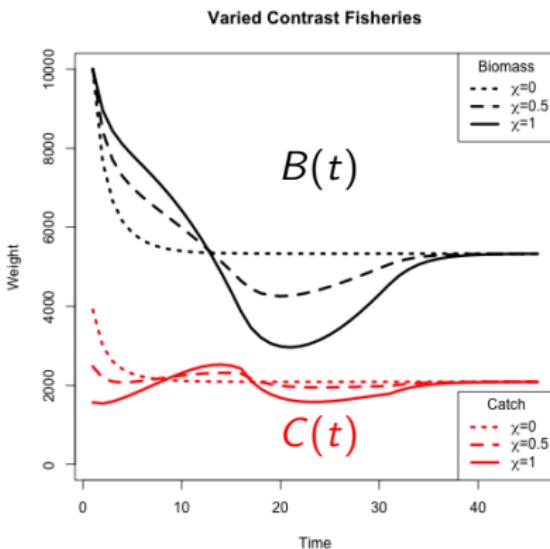
$$\underbrace{\left( F_{MSY}, \frac{B_{MSY}}{\bar{B}(0)} \right)}_{\text{PT Truth}} \xrightarrow{\text{GP}} \underbrace{\left( \hat{F}_{MSY}, \frac{1}{2} \right)}_{\text{Shaefer Estimate}}$$

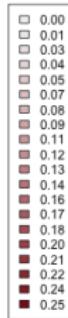
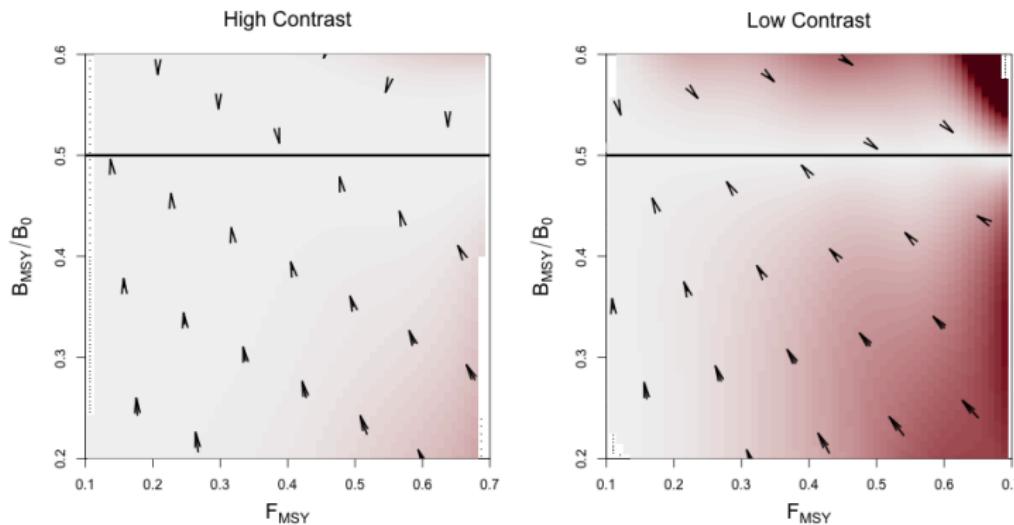


$$\begin{aligned}\frac{dB}{dt} &= P(B(t); \theta) - F(t)B(t) \\ &= P(B(t); \theta) - F_\theta^* \underbrace{\frac{F(t)}{F_\theta^*}}_{\text{Relative Fishing}} B(t)\end{aligned}$$

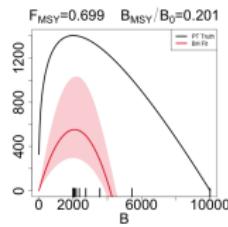
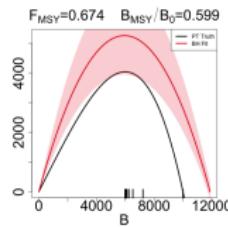
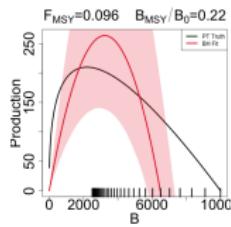
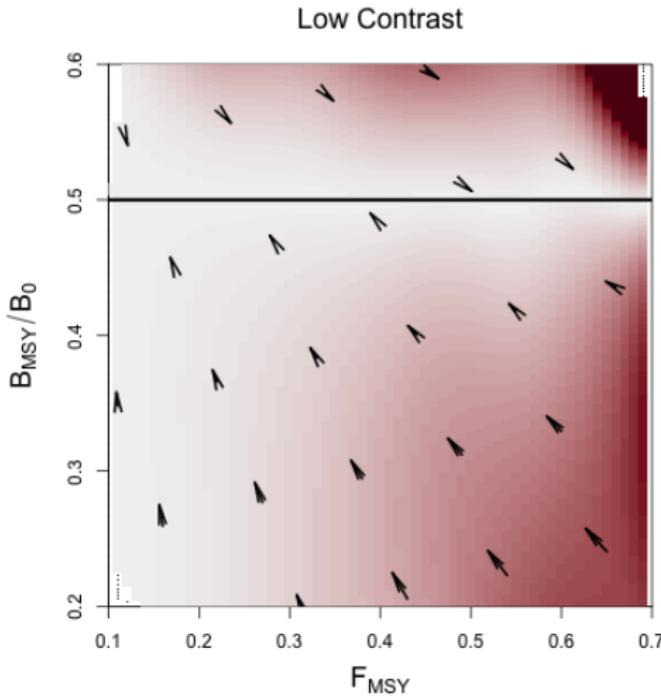
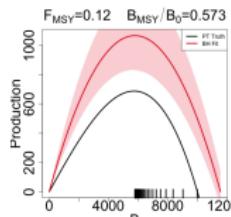


$$\begin{aligned}\frac{dB}{dt} &= P(B(t); \theta) - F(t)B(t) \\ &= P(B(t); \theta) - F_\theta^* \underbrace{\frac{F(t)}{F_\theta^*} B(t)}_{C(t)}\end{aligned}$$



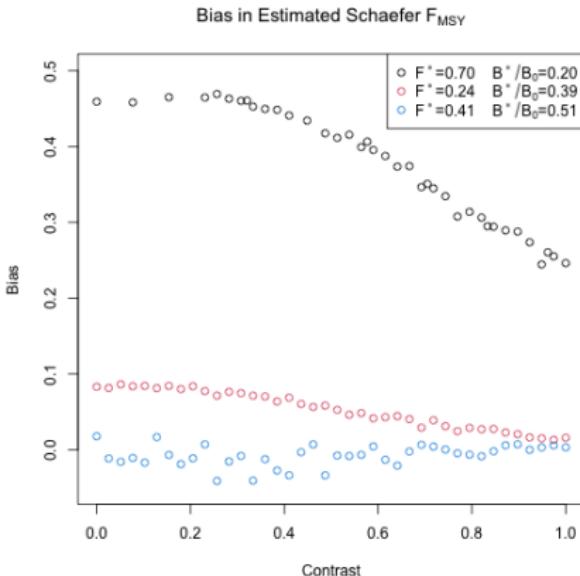


# Mechanism for Bias in $F_{MSY}$ via Contrast



# Mechanism for Bias in $F_{MSY}$ via Contrast

- Only observe upper half
- learn about slope at origin from upper biomass range
- as contrast increases biases perspective diminishes



# Summary

- A quick summary
- just a few points will suffice
- the quicker and more concise the better
- transition to BH

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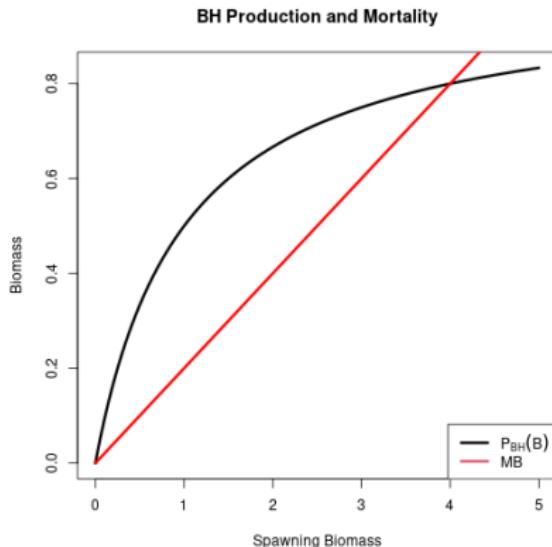
4 Delay Differential Growth Extension

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$$I_t = qB_t e^\epsilon \quad \epsilon \sim N(0, \sigma^2)$$

$$\frac{dB}{dt} = P_{BH}(B; [\alpha, \beta]) - (M + F)B$$

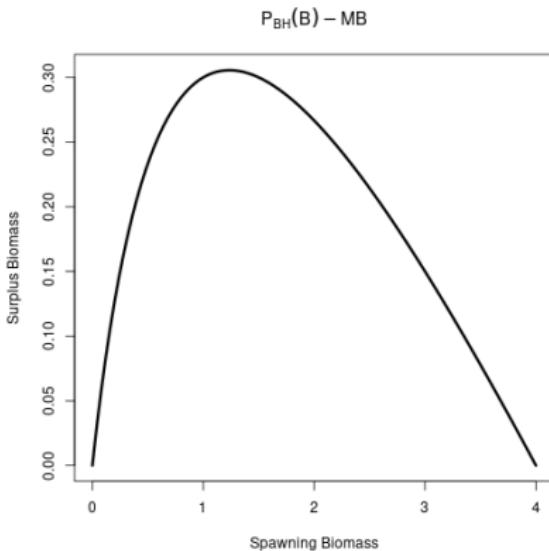
$$P_{BH}(B; [\alpha, \beta]) = \frac{\alpha B}{1 + \beta B}$$



$$I_t = qB_t e^\epsilon \quad \epsilon \sim N(0, \sigma^2)$$

$$\frac{dB}{dt} = \underbrace{P_{BH}(B; [\alpha, \beta]) - MB - FB}_{\text{Surplus Production}}$$

$$P_{BH}(B; [\alpha, \beta]) = \frac{\alpha B}{1 + \beta B}$$



$$P_s(B; [\alpha, \beta, \gamma]) = \alpha B (1 - \beta \gamma B)^{\frac{1}{\gamma}}$$

Logistic

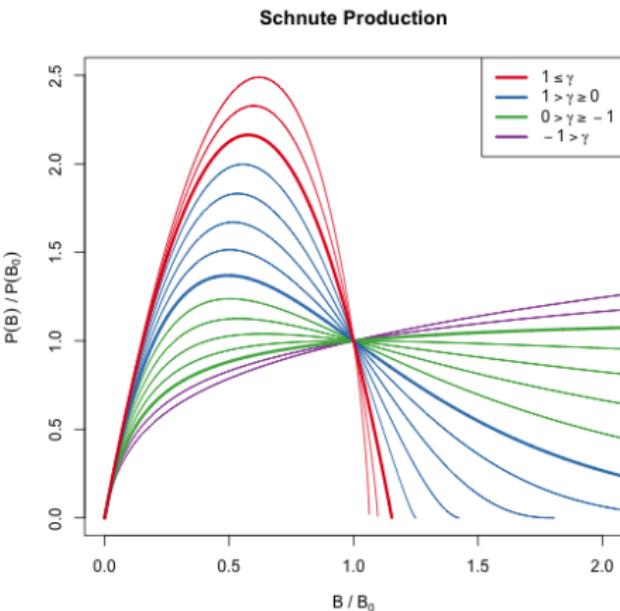
$$\gamma = 1$$

Ricker

$$\gamma \rightarrow 0$$

Beverton-Holt

$$\gamma = -1$$



RPs  $\mapsto \theta$

### Incomplete Inversion

$\theta \mapsto \text{RPs}$

$$\alpha = (M + F^*) \left( 1 + \frac{\gamma F^*}{M + F^*} \right)^{1/\gamma}$$

$$\beta = \frac{1}{\gamma B_0} \left( 1 - \left( \frac{M}{\alpha} \right)^\gamma \right)$$

$$\frac{B^*}{B_0} = \frac{1 - \left(\frac{M+F^*}{\alpha}\right)^\gamma}{1 - \left(\frac{M}{\alpha}\right)^\gamma}.$$

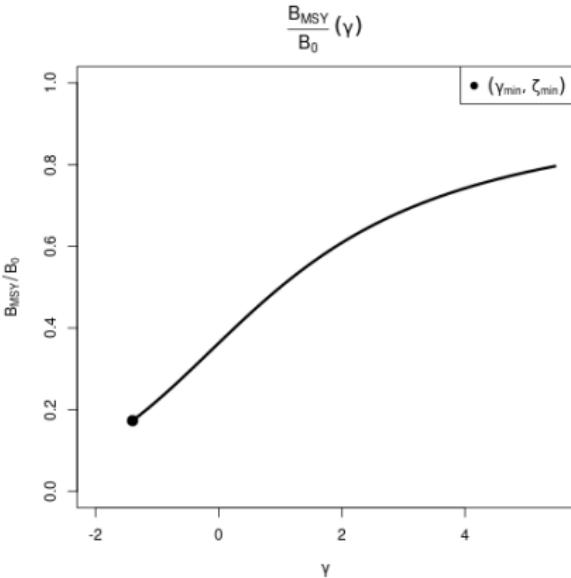
Schnute & Richards (1998). CJFAS.



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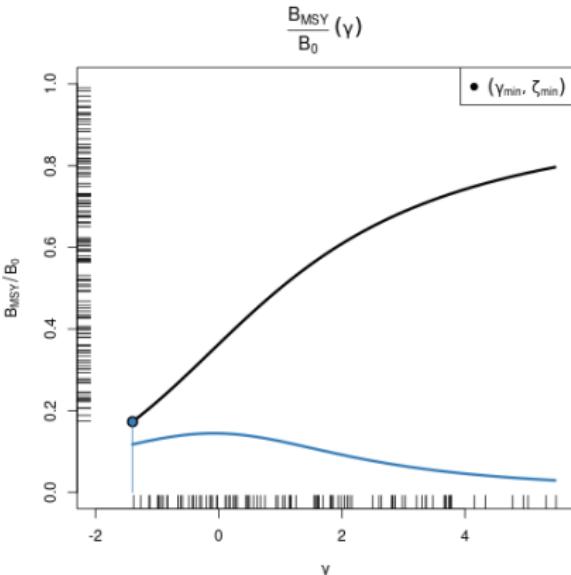
Schnute & Richards (1998). CJFAS.

$$\frac{B^*}{B_0}(\gamma) = \frac{1 - \left(\frac{M+F^*}{\alpha(\gamma)}\right)^\gamma}{1 - \left(\frac{M}{\alpha(\gamma)}\right)^\gamma}$$



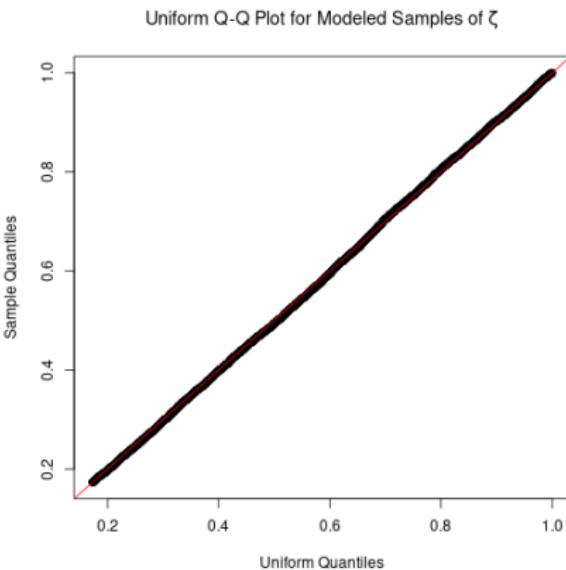
$$\frac{B^*}{B_0}(\gamma) = \frac{1 - \left( \frac{M+F^*}{\alpha(\gamma)} \right)^\gamma}{1 - \left( \frac{M}{\alpha(\gamma)} \right)^\gamma}$$

$$\gamma' \sim \zeta_{min}\delta(\gamma_{min}) + (1 - \zeta_{min})t(\mu, \sigma, \nu)\mathbf{1}_{\gamma > \gamma_{min}}$$



$$\frac{B^*}{B_0}(\gamma) = \frac{1 - \left(\frac{M+F^*}{\alpha(\gamma)}\right)^\gamma}{1 - \left(\frac{M}{\alpha(\gamma)}\right)^\gamma}$$

$$\gamma' \sim \zeta_{min} \delta(\gamma_{min}) + (1 - \zeta_{min}) t(\mu, \sigma, \nu) \mathbf{1}_{\gamma > \gamma_{min}}$$



# Schnute LHS Design

Logistic

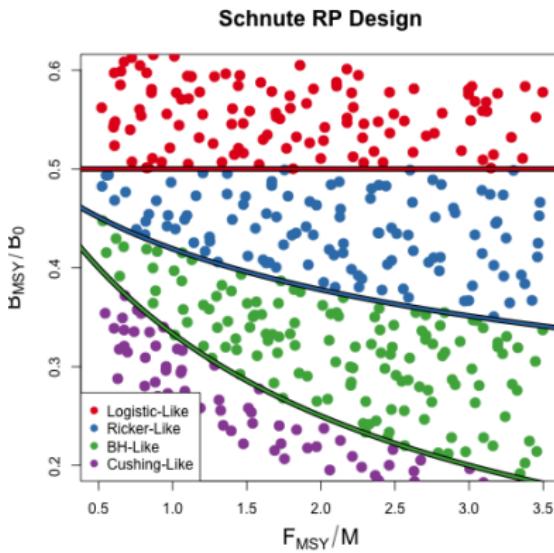
$$\gamma = 1$$

Ricker

$$\gamma \rightarrow 0$$

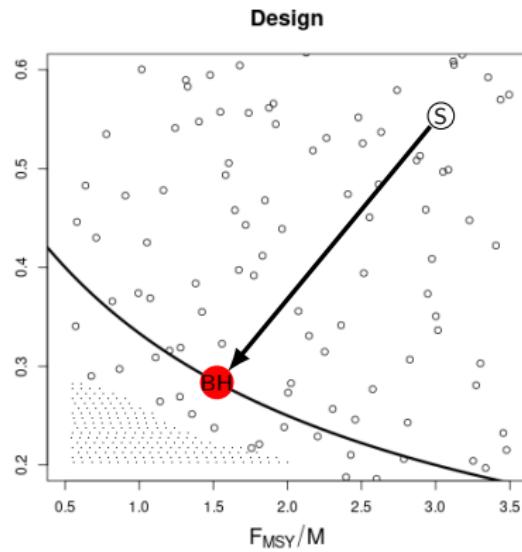
Beverton-Holt

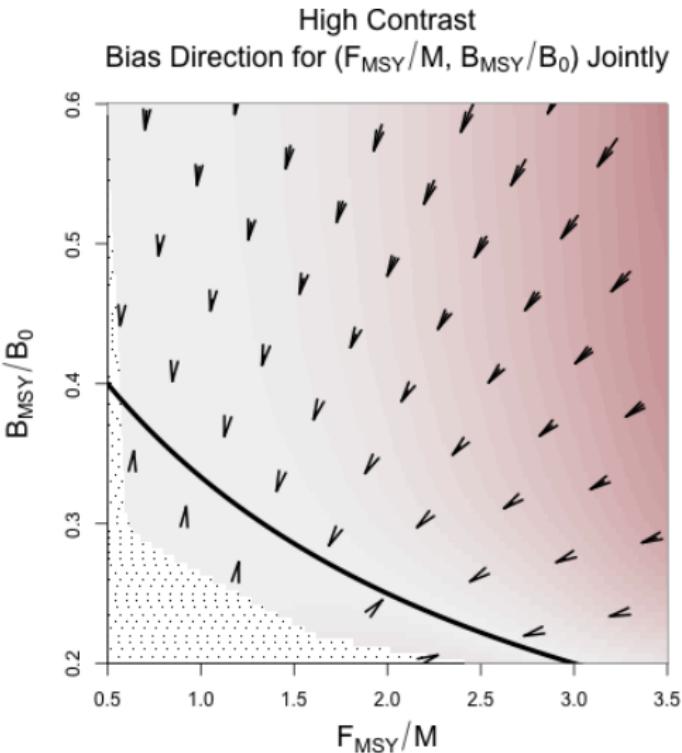
$$\gamma = -1$$

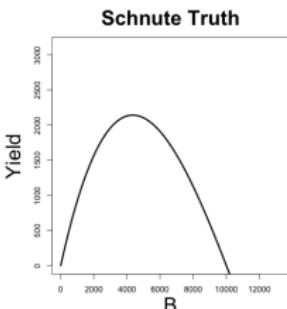
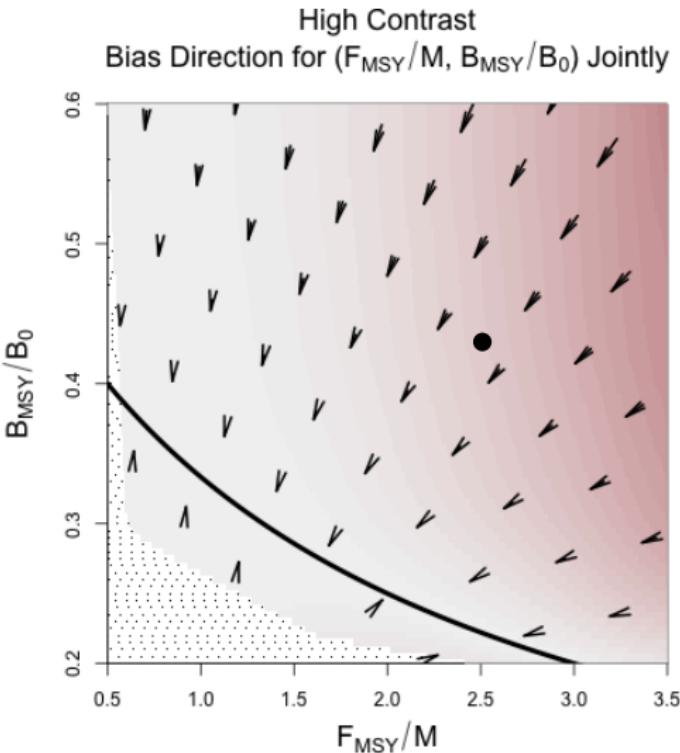


Schnute LHS Design

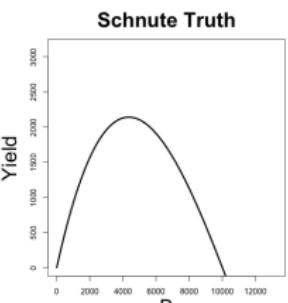
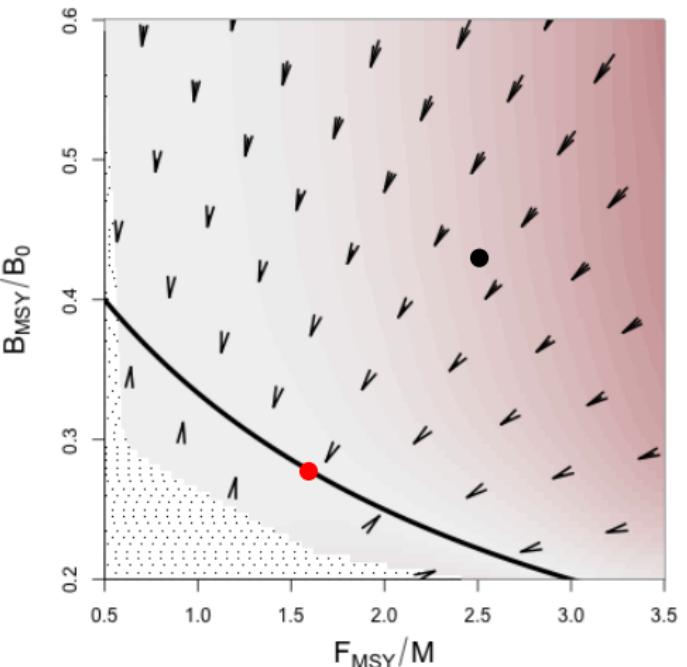
$$\underbrace{\left( \frac{F_{MSY}}{M}, \frac{B_{MSY}}{\bar{B}(0)} \right)}_{\text{Schnute Truth}} \xrightarrow{\text{GP}} \underbrace{\left( \frac{\hat{F}_{MSY}}{M}, \frac{1}{\hat{F}_{MSY}/M + 2} \right)}_{\text{BH Estimate}}$$



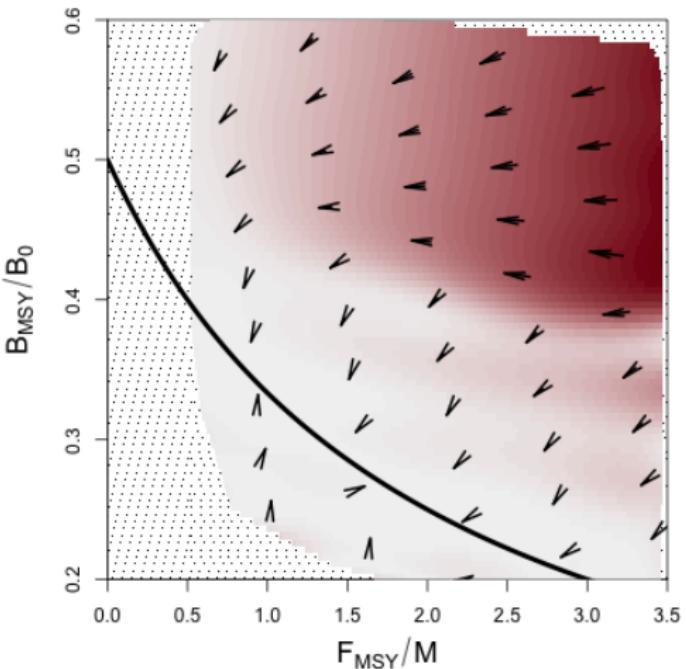




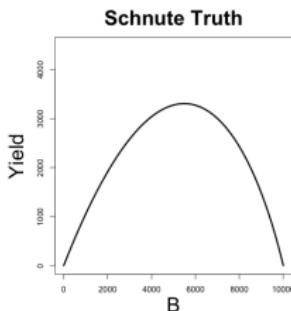
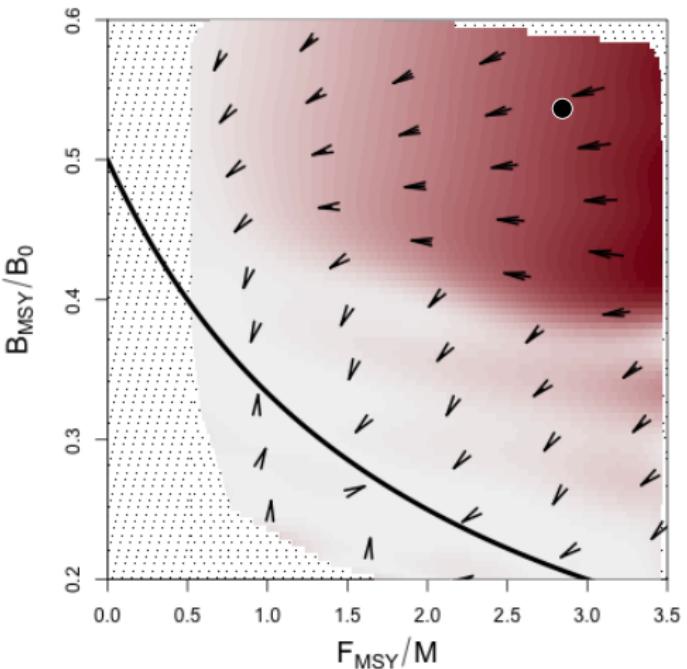
## High Contrast Bias Direction for $(F_{MSY}/M, B_{MSY}/B_0)$ Jointly

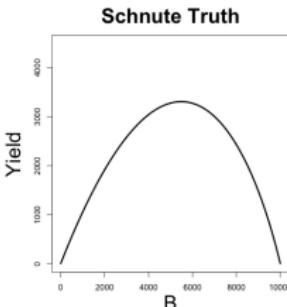
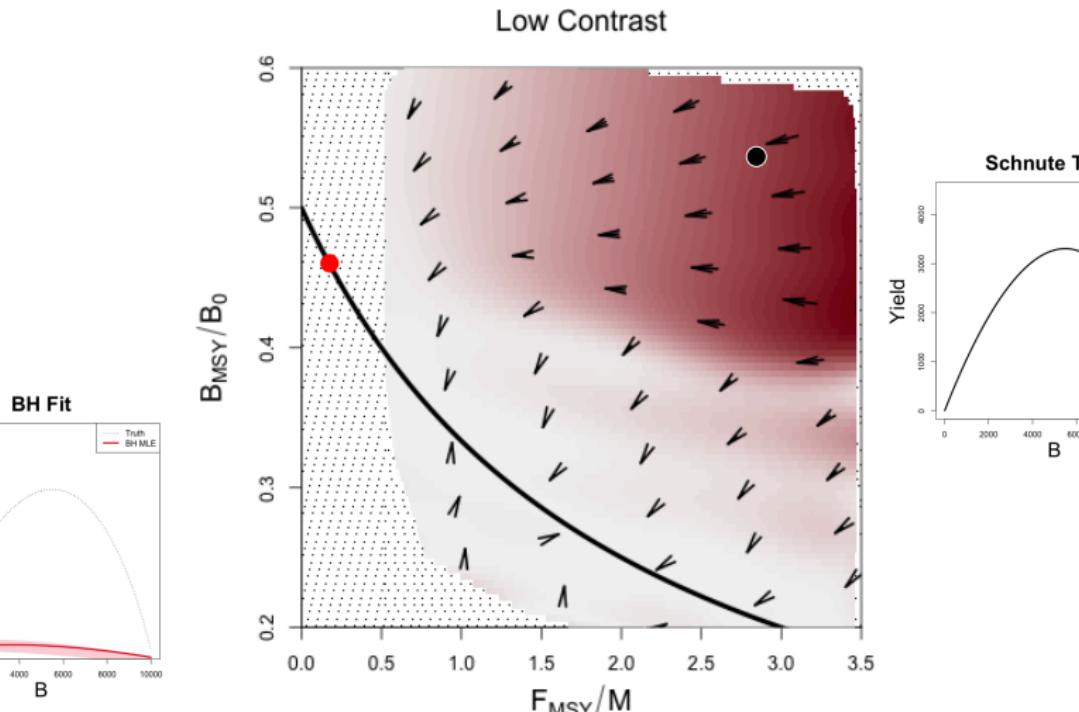


## Low Contrast

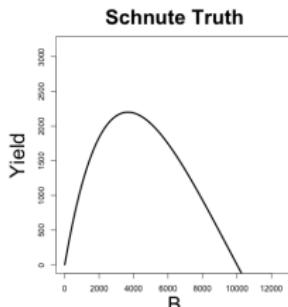
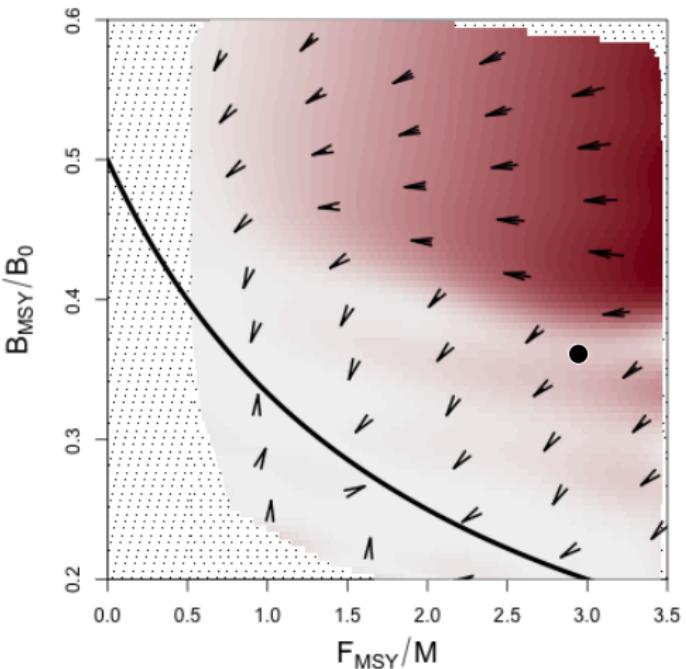


## Low Contrast

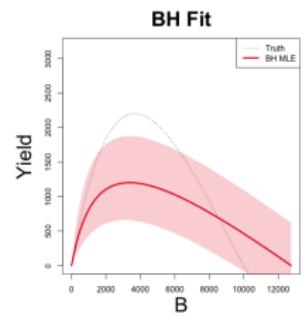
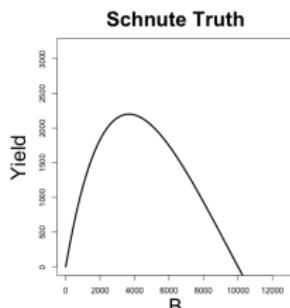
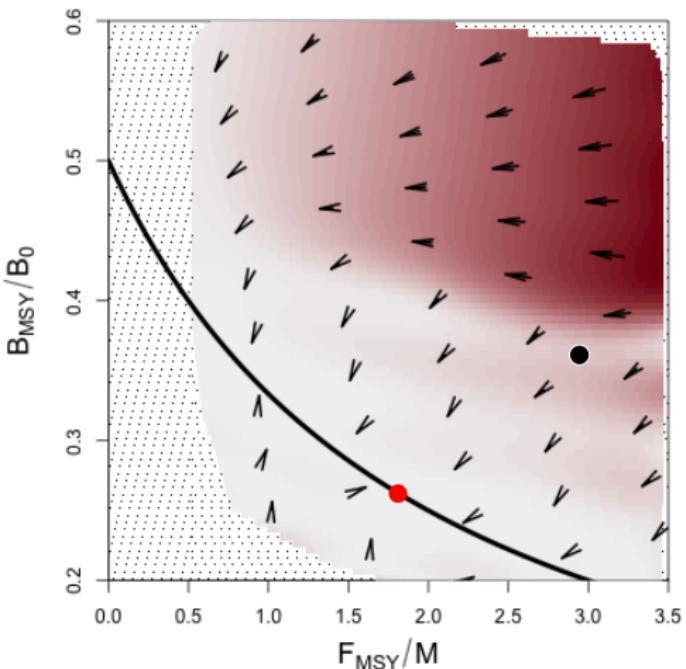




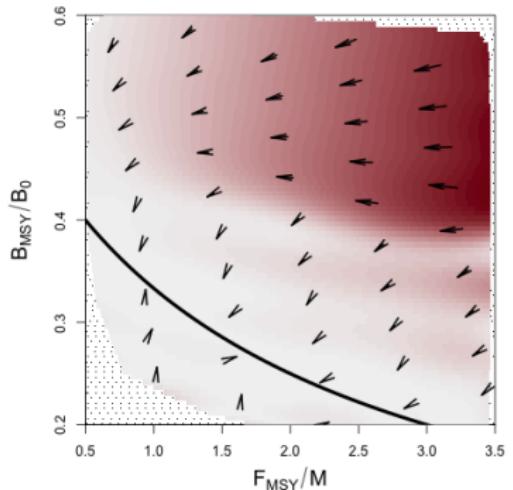
## Low Contrast



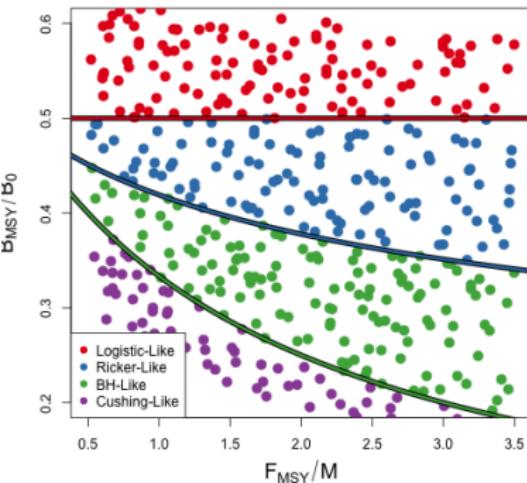
## Low Contrast



## Low Contrast



Schnute RP Design



## Quick Summary and Transition

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# General Modeling Structure

$$I_t = qB_t e^\epsilon \quad \epsilon \sim N(0, \sigma^2)$$

$$\begin{aligned} \frac{dB}{dt} &= \overbrace{w(a_s)R(B; \theta)}^{\text{Recruitment Biomass}} + \overbrace{\kappa [w_\infty N - B]}^{\text{Net Growth}} - \overbrace{(M + F)B}^{\text{Mortality}} \\ \frac{dN}{dt} &= R(B; \theta) - (M + F)N \end{aligned}$$

# Individual Growth

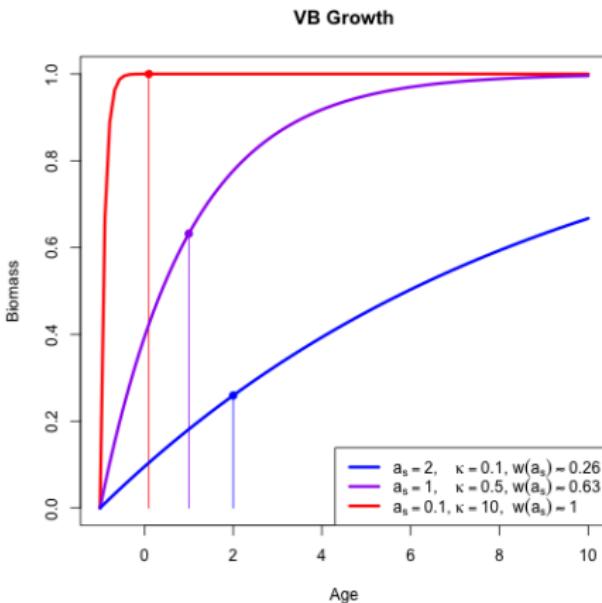
$$w(a) = w_\infty(1 - e^{-\kappa(a-a_0)})$$

$a_s$  : Lagged Maturity &  
Knife-Edge Selectivity

$\kappa$  : Individual Growth

- Instant Growth:  
(Production Model)

$$a_s \rightarrow 0 \quad \kappa \rightarrow \infty$$



# Schnute Recruitment

$$R(B; \alpha, \beta, \gamma) = \alpha B_{t-a_s} (1 - \beta \gamma B_{t-a_s})^{\frac{1}{\gamma}}$$

Logistic

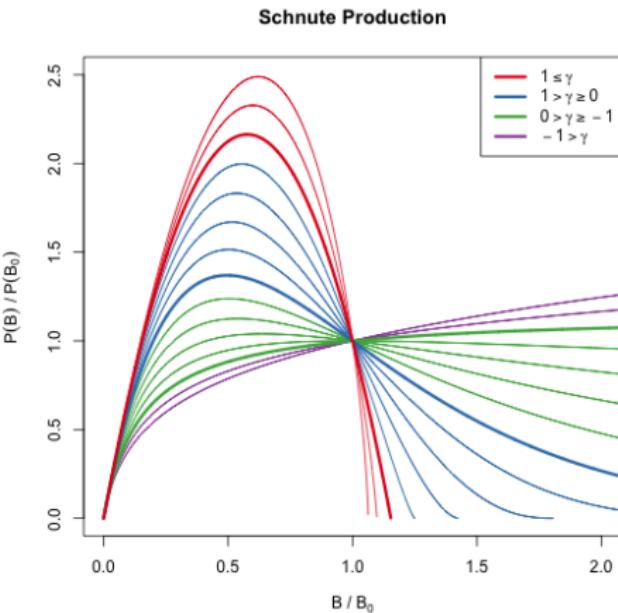
$$\gamma = 1$$

Ricker

$$\gamma \rightarrow 0$$

Beverton-Holt

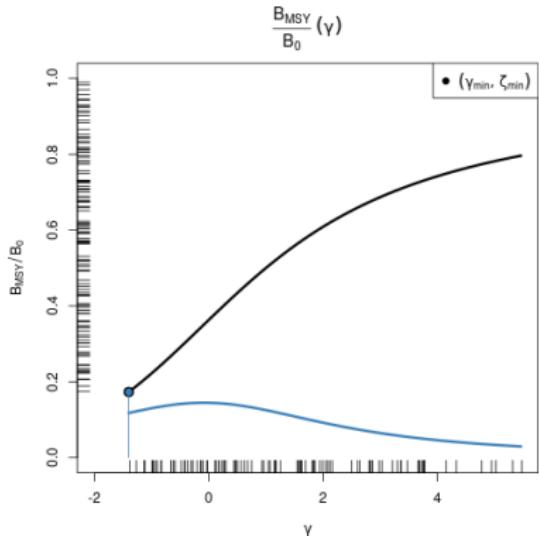
$$\gamma = -1$$

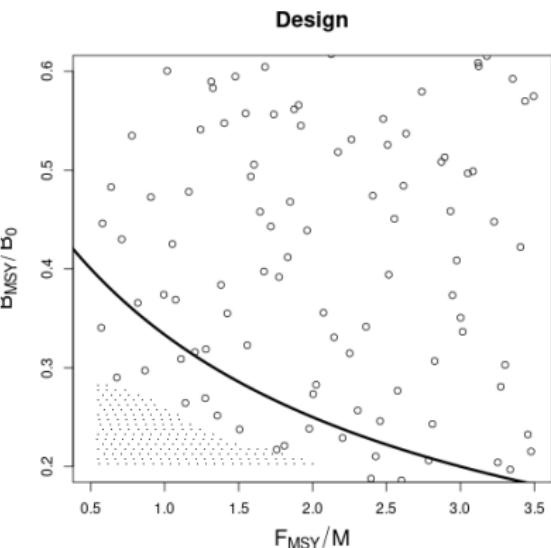
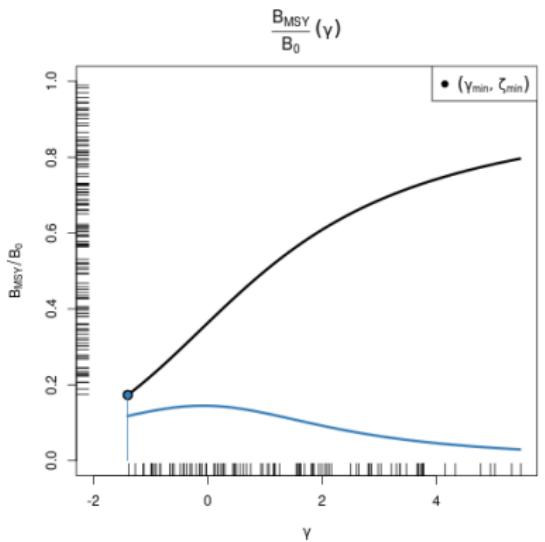


$$\alpha(\gamma) : \alpha = \left[ \left( \frac{Z^*(Z^* + \kappa)}{w(a_s)(Z^* + \frac{\kappa w_\infty}{w(a_s)})} \right)^\gamma + \left( \frac{\gamma F^*}{w(a_s)} \right) \left( \frac{Z^*(Z^* + \kappa)}{w(a_s)(Z^* + \frac{\kappa w_\infty}{w(a_s)})} \right)^{\gamma-1} \left( 1 + \frac{\left( \frac{\kappa w_\infty}{w(a_s)} \right) \left( \kappa - \frac{\kappa w_\infty}{w(a_s)} \right)}{(Z^* + \frac{\kappa w_\infty}{w(a_s)})^2} \right) \right]^{\frac{1}{\gamma}}$$

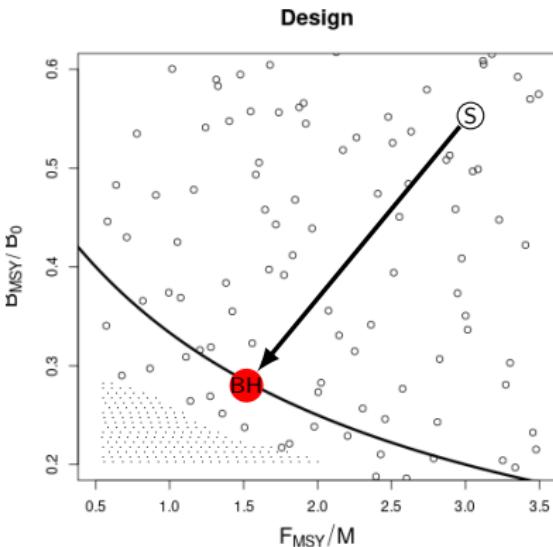
$$\beta(\alpha(\gamma), \gamma) : \beta = \frac{1}{\gamma B_0} \left( 1 - \left( \frac{M(M + \kappa)}{\alpha w(a_s)(M + \frac{\kappa w_\infty}{w(a_s)})} \right)^\gamma \right)$$

$$\frac{B^*}{B_0}(\alpha(\gamma), \gamma) : \frac{B^*}{B_0} = \frac{1 - \left( \frac{(F^* + M)(F^* + M + \kappa)}{\alpha w(a_s)(F^* + M + \frac{\kappa w_\infty}{w(a_s)})} \right)^\gamma}{1 - \left( \frac{M(M + \kappa)}{\alpha w(a_s)(M + \frac{\kappa w_\infty}{w(a_s)})} \right)^\gamma}$$

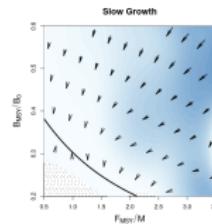
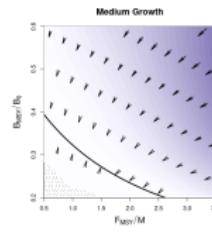
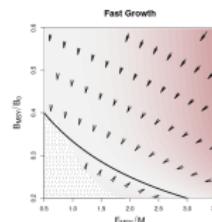
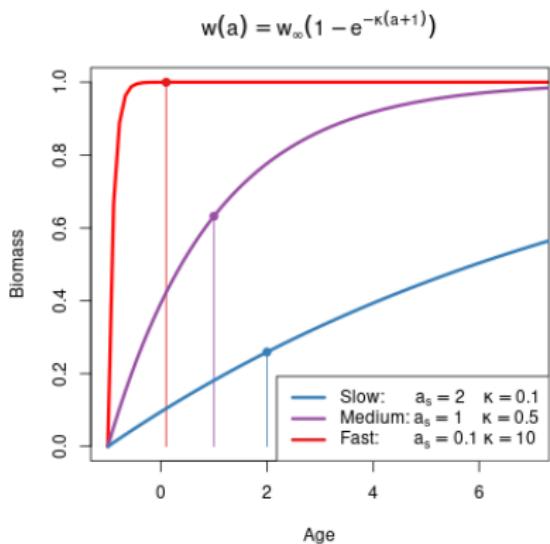




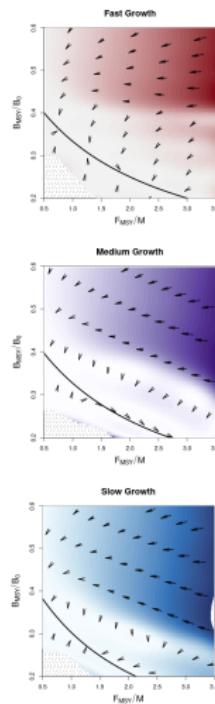
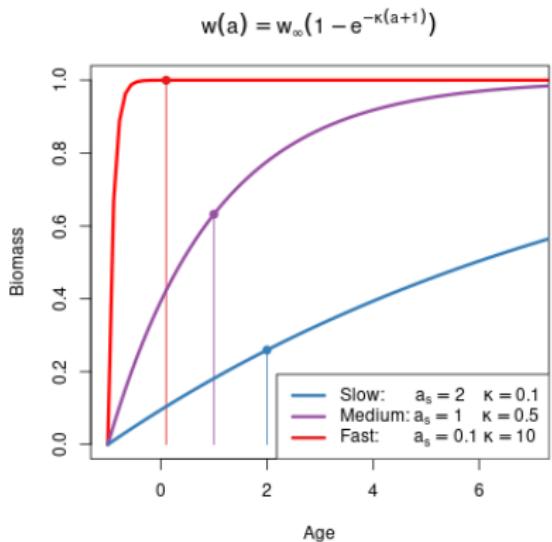
$$\underbrace{\left(\frac{F^*}{M}, \frac{B^*}{\bar{B}(0)}\right)}_{\text{Schnute Truth}} \xrightarrow{\text{GP}} \underbrace{\left(\frac{\hat{F}^*}{M}, \frac{B^*}{B_0}(-1; \hat{F}^*)\right)}_{\text{BH Estimate}}$$



# High Contrast

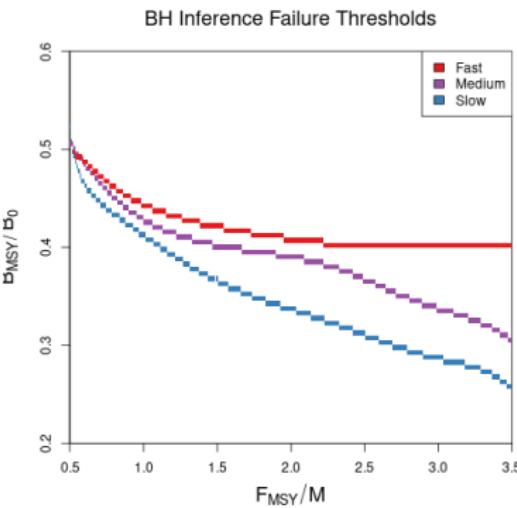
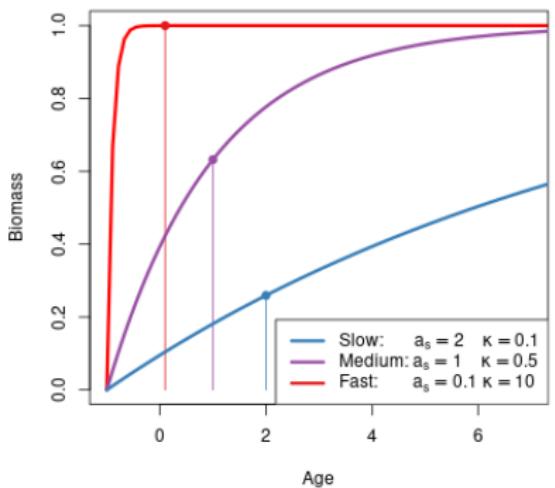


# Low Contrast

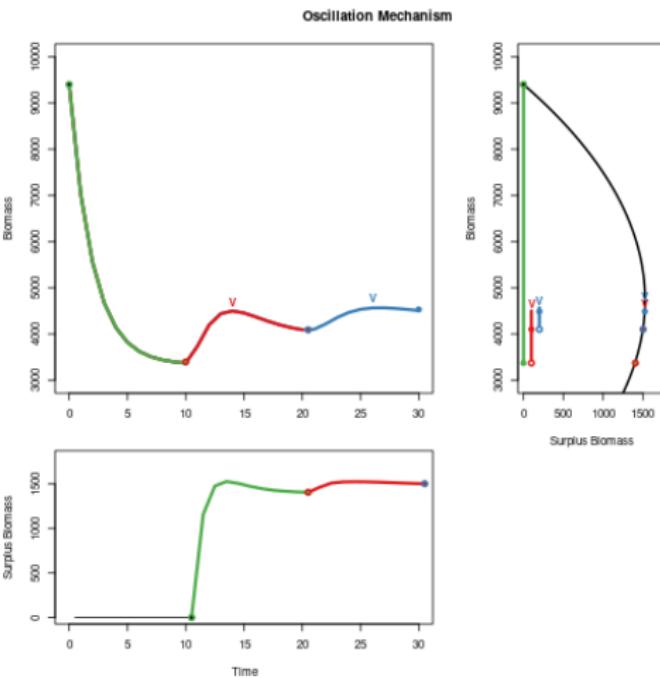


# Low Contrast

$$w(a) = w_\infty(1 - e^{-\kappa(a+1)})$$



## Oscillation



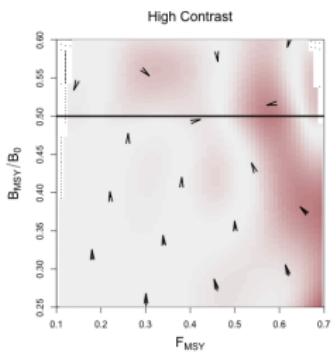
## General Conclusions and future work.

Many Thanks:

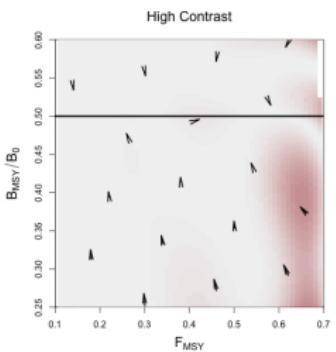
- UCSC Advisors
- Collaborators at NOAA
- NMFS Sea Grant



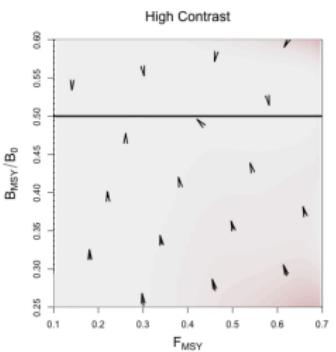
## High Contrast PT $\sigma = 0.12$ Data



1x Samples



## 2x Samples



## 4x Samples

## Metamodel Details

$$\mathbf{y} = \widehat{\log(F_{MSY})} \quad -\text{or}- \quad \mathbf{y} = \widehat{\log(B_0)}$$

$$X = \left( \frac{F_{MSY}}{M}, \frac{B_{MSY}}{\bar{B}(0)} \right)$$

$$\mathbf{y} = \beta_0 + \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\nu} + \epsilon$$

$$\boldsymbol{v} \sim N_n(\mathbf{0}, \tau^2 \boldsymbol{R}_\ell)$$

$$\epsilon \sim N_n(\mathbf{0}, \omega' \mathbf{I})$$

$$R(\mathbf{x}, \mathbf{x}') = \exp \left( \sum_{j=1}^2 \frac{-(x_j - x'_j)^2}{2\ell_j^2} \right)$$

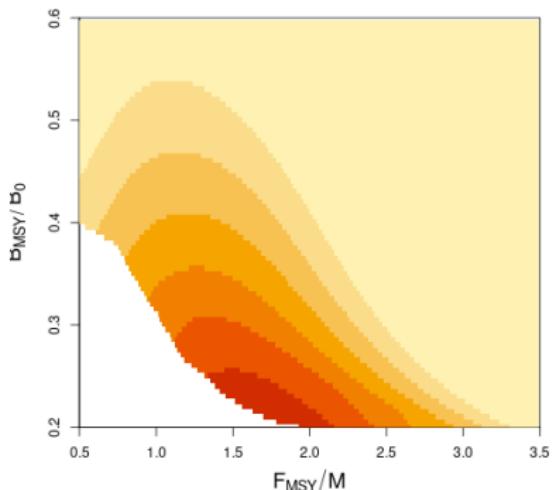
# Prediction

$$\hat{y}(\mathbf{x}^*) = \beta_0 + \mathbf{x}^* \boldsymbol{\beta} + \mathbf{r}(\mathbf{x}^*)' \mathbf{R}_\ell^{-1} \left( \mathbf{y} - (\beta_0 + \mathbf{X}\boldsymbol{\beta}) \right)$$

$$\hat{\sigma}^2(\mathbf{x}^*) = \mathbf{R}(\mathbf{x}^*, \mathbf{x}^*) - \mathbf{r}(\mathbf{x}^*)' R_\ell^{-1} \mathbf{r}(\mathbf{x}^*)$$

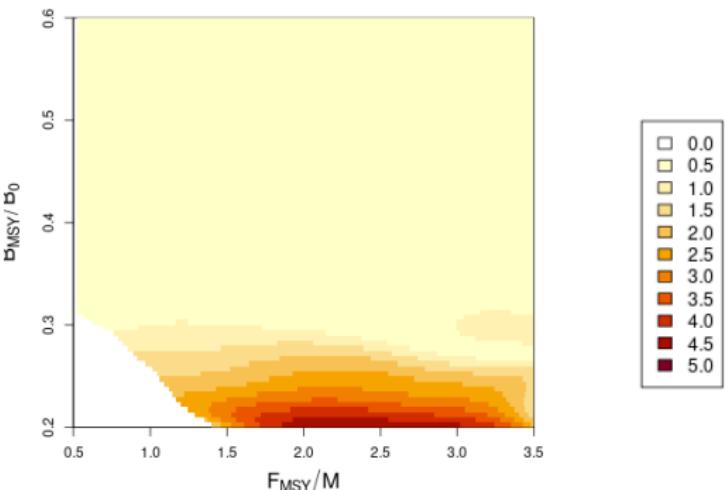
## Contrast

### Size of 1 SD Interval in RP Bias



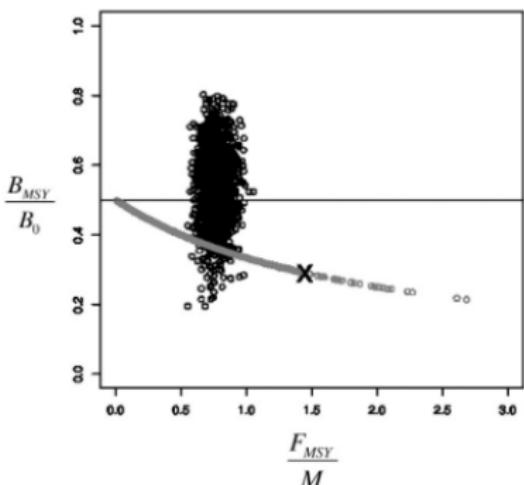
## No Contrast

### Size of 1 SD Interval in RP Bias

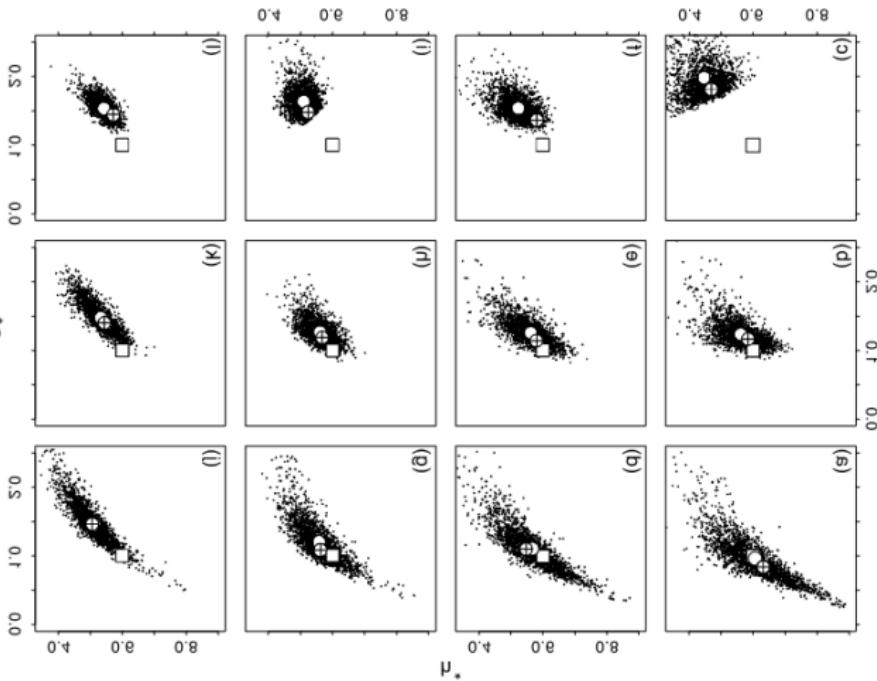


Mangel et al.

**Fig. 4.** DeYoreo et al. (2012) used both a BH-SRR and three-parameter SRR, similar to the S-SRR in a stock assessment of cowcod (*Sebastodes levis*). We show samples from posterior distributions arising from different values of steepness. Unlike most stock assessments, we plot  $B_{MSY}/B_0$  versus  $F_{MSY}/M$ . The grey circles show the results for the BH-SRR. This curve is another way of representing the constraint placed on a stock assessment by using a BH-SRR and specifying steepness — results must lie along this curve. The black circles represent the outcome of the three-parameter SRR. The black X represents the result when steepness is asserted to be 0.6.



## Logistic



Schnute, J. T., & Kronlund, A. R. (2002). Estimating salmon stock recruitment relationships from catch and escape-  
ment data. Canadian Journal of Fisheries and Aquatic Sciences, 59(3), 433–449.

## Space of BH Reference Points

