# Narrative

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# 1 Objectives

Over the reporting period the primary objective has been to further apply and extend our code base for working with production and Gaussian process (GP) models. The focus is to further explore how model misspecification in a variety of commonly used forms of the stock recruitment relationship (SRR) can affect inference on key reference points. In particular, if data are generated under a three parameter true SRR but fit using a two parameter analog how does this affect inference on $\frac{F^*}{M}$ and $\frac{B^*}{B_0}$. What bias is induced by the choice of SRR?

## 2 Methods

Under both the Beverton-Holt and Schaefer production models, $\frac{F^*}{M}$ and $\frac{B^*}{B_0}$ are known to be fixed when steepness and natural mortality are a'priori fixed constants (?, ?, ?). An additional degree of freedom can be added to these models by adding a third parameter ($\gamma$) to the SRR of these models respectively. Several formulations of these expanded three parameter SRR have been studied (?, ?).

The focus here is on three parameter SRRs which have the Beverton-Holt or Schaefer models as a special case (?, ?, ?). Exploration continues across many different choices of three parameter SRRs, although the new results presented here focus on the Shepherd and Pella-Tomlinson SRRs

$$R_1 = \frac{\alpha B}{1 + \beta B^{\frac{1}{\gamma}}} \qquad R_2 = \frac{rB}{\gamma - 1}\left(1 - \frac{B}{K}\right)^{\gamma - 1}. \tag{1}$$

$R_1$ is equivalent to the Beverton-Holt SRR when $\gamma = 1$, and $R_2$ is equivalent to the Schaefer model's logistic SRR when $\gamma = 2$. Given an informative series of observed catches, and natural mortality fixed at $M = 0.2$, these production models are integrated forward based upon,

$$\frac{dB}{dt} = R - (M + F)B. \tag{2}$$

Above $R$ is recruitment, $F$ is fishing mortality, and $B$ represents biomass. The decorator $^*$ is added to indicate values at maximum sustainable yield (MSY) and $B_0$ is virgin biomass. Biomass and mortality reference point inference is monitored by a GP metamodel. The following variables are defined for ease of use working with these metamodels.

$$\xi = \frac{F^*}{M} \qquad \zeta = \frac{B^*}{B_0} \tag{3}$$

Let $\tilde{}$ decorate any quantity that is derived under the restricted two parameter case, so that $\tilde{\xi}$ and $\tilde{\zeta}$ represent the above reference points under the assumption of a Beverton-Holt or Schaefer model. The construction of a metamodel around these reference points is based upon the restricted relationship in $(\tilde{\xi}, \tilde{\zeta})$ for the two parameters models. First under the Beverton-Holt, and subsequently under the Schaefer model, these structured relationships can be used to show that the reference points are restricted to the following curves respectively,

$$\tilde{\zeta}_1 = \frac{1}{\tilde{\xi} + 2} \qquad \tilde{\zeta}_2 = \frac{\tilde{\xi}}{2\tilde{\xi} + 1}. \tag{4}$$

### 2.1 Simulation

Indices of abundance are simulated from each three parameter SRR over an unrestricted grid of $\xi$ and $\zeta$ values. After data are generated, $\gamma$ is then fixed so that the SRR reduces to the special cases previously described under each model, and the restricted model is subsequently fit to the simulated indices.

By working with the resulting models parameterized in terms of $\log\left(\tilde{F}^*\right)$ it tends to improve optimization convergence. Furthermore, the normality this induces on the log scale,

via the Laplace approximation, yields Log-Normality on $\tilde{F}^*$. Let $\hat{\mu}$ be the maximum likelihood estimate (MLE) of $\log\left(\tilde{F}^*\right)$. Additionally let $\hat{\sigma}^2$ be the inverted Hessian information of the log likelihood evaluated at $\hat{\mu}$.

## 2.2 Gaussian Process Model

A GP is a stochastic process generalizing the normal distribution to an infinite dimensional analog. GPs are often specified primarily through the choice of a covariance function which defines the relationship between locations in an index set. Typically the index set is spatial for GPs, and in this setting the model is in reference point space $(\xi, \zeta)$. A GP model implies an $n$ dimensional multivariate normal distribution on the observations of the model and the covariance function fills out the covariance matrix for the observations.

Each iteration of the simulation produces a single fitted $\hat{\mu}_i$ at an associate $(\xi_i, \zeta_i)$ location with $i \in \{1, ..., n\}$. $\hat{\boldsymbol{\mu}}$ is jointly modeled over the space of reference points as the following GP,

$$\begin{aligned} \boldsymbol{x} &= (\xi, \zeta) \\ \hat{\boldsymbol{\mu}} &= \beta_0 + \boldsymbol{\beta}'\boldsymbol{x} + f(\boldsymbol{x}) + \boldsymbol{\epsilon} \\ f(\boldsymbol{x}) &\sim \mathrm{GP}(0, \tau^2 \mathbf{K}(\boldsymbol{x}, \boldsymbol{x'})) \\ \epsilon_i &\sim \mathrm{N}(0, \hat{\sigma}_i^2). \end{aligned} \tag{5}$$

$\hat{\sigma}_i^2$ is the observed variance for $\hat{\mu}_i$ from inference in the simulation step. This model allows for the full propagation of inferred information from the simulation step to be propagated into the reference point metamodel.

Here $\mathbf{K}$ has been extended to account for the possibility of geometric anisotropy as well as to model the smoothness of the relationship. The previously used squared exponential correlation function has been replaced with the Matern correlation function (?, ?). The updated correlation structure for filling out $\mathbf{K}$ can be summarized as follows,

$$\mathbf{K}(\boldsymbol{x}, \boldsymbol{x'}) = Matern(\|\boldsymbol{x} - \boldsymbol{x'}\|_{\boldsymbol{R}}; \nu) \tag{6}$$

$$\|\boldsymbol{x} - \boldsymbol{x'}\|_{\boldsymbol{R}} = \sqrt{(\boldsymbol{x} - \boldsymbol{x'})^\top \boldsymbol{R}^{-1} (\boldsymbol{x} - \boldsymbol{x'})} \tag{7}$$

$$\boldsymbol{R} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^\top \quad \boldsymbol{P} = \begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix} \quad \boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{8}$$

The full GP model has linear predictor parameters $(\beta_0, \boldsymbol{\beta})$, a process variance parameter $(\tau^2)$, kernel length scale parameters $(\lambda_1, \lambda_2)$, a kernel rotation parameter $(\theta)$, and the Matern smoothness parameter $(\nu)$. All of these parameters are estimated by maximization of the posterior (MAP) inference.

# 3 Results

Let $\check{}$ decorate values which are predictions from the GP metamodel as opposed to predictions from the base-level model. In the posterior, the GP metamodel produces a predictive surface for the estimated $\check{\mu}$ values across the reference point domain. This $\check{\mu}$ surface is used to back out a predictive surface for $\check{F}^*$ under each reduced model. The bias in estimating $F^*$ across the reference point domain is the predicted $\check{F}^*$ minus the true $F^*$ at a given spatial location. Similarly the bias in estimating $\xi$ is given by $\frac{\check{F}^*}{M} - \xi$; using Eq. (4) the bias in estimating $\zeta$ is given by $\tilde{\zeta}(\check{F}^*) - \zeta$. Individually these bias measures indicate a magnitude of bias in each of the reference point directions; together they form a vector field of biases.

The above figures show the bias surfaces for the Beverton-Holt and Schaefer models respectively. Red colors indicate over estimation of the reference points and blue colors indicate underestimation of the reference points respectively. The black curves plotted above show the restricted reference point space as defined by Eq. (4) in each case.

## 3.1 Future Results

The results presented above are generated with limited simulation runs due to issues introduced by the mapping of reference point values to SRR parameters. We are currently developing novel adaptive sampling space-filling methods which are capable of avoiding these issues and will allow for a broader and more stable result to be presented across a wide variety of SRRs as well as age structured models.