# Metamodeling for Bias Estimation of Biological Reference Points Under Two-Parameter SRRs

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# Outline


1 Introduction

2 The Schaefer Model

3 The Beverton-Holt Model

4 Delay Differential Growth Extension

5 End

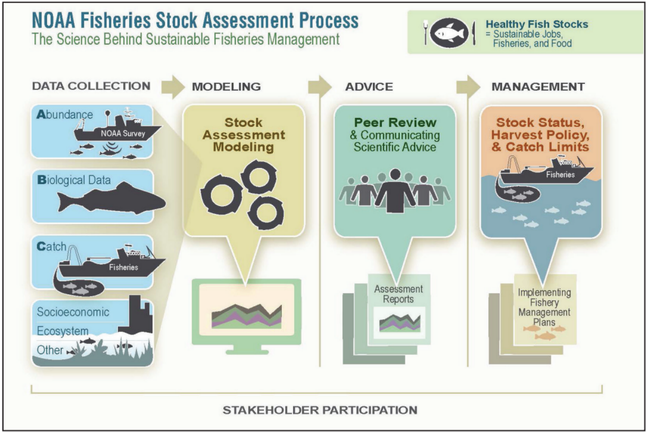



Figure 1: Overview of the stock assessment process from data collection through the provision of scientific advice to fishery managers. Stakeholders and other partners participate in each step of the assessment process. This report captures NOAA Fisheries products associated with the 'Advice' phase of the process. [1]

[1] NOAA Fisheries Annual Stock Assessment Report 2020

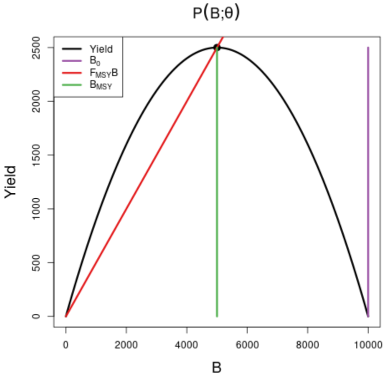
# Surplus Production Model General Structure

$$I_t = qB_te^{\epsilon} \quad \epsilon \sim N(0, \sigma^2).$$

$$\frac{dB}{dt} = P(B(t); \boldsymbol{\theta}) - Z(t)B(t).$$

Reference Points:

- Maximum sustainable Yield ($MSY$)
- $F_{MSY}$[a]: Fishing rate to achieve $MSY$
- $\frac{B_{MSY}}{B_0}$: Biomass Depletion when at $MSY$
- Driven by the shape of $P$ as determined by $\theta$.

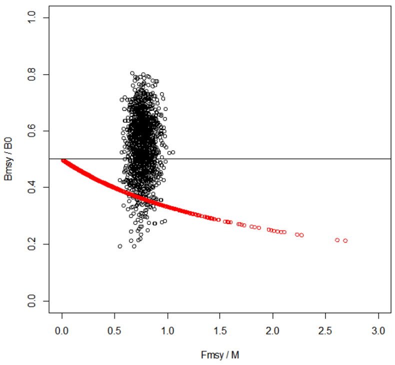



[a] or $\frac{F_{MSY}}{M}$

Conceptually:

$$\frac{F_{MSY}}{M} \in \mathbb{R}^+ \qquad \frac{B_{MSY}}{B_0} \in (0,1)$$

- Schaefer Model:
  $F_{MSY} \in \mathbb{R}^+ \qquad \frac{B_{MSY}}{\bar{B}(0)} = \frac{1}{2}$
- BH Model:
  $\frac{F_{MSY}}{M} \in \mathbb{R}^+ \qquad \frac{B_{MSY}}{\bar{B}(0)} = \frac{1}{F_{MSY}/M+2}$
- Similar Constraints for other Two-Parameter Models:
  Fox, Ricker, etc...
- Three-Parameter Models Allow Independent RP Estimation



[a]

[a] Mangel et al. 2013, CJFAS

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- $P_\ell$ is logistic production
- Logistic map in discrete time
- Implicit Natural Mortality
- Explicit Fishing Mortality

$$I_t = qB_te^{\epsilon} \quad \epsilon \sim N(0, \sigma^2)$$

$$\frac{dB}{dt} = P_\ell(B(t); \boldsymbol{\theta}) - F(t)B(t)$$

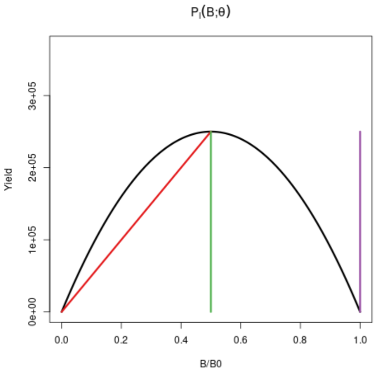
$$P_\ell(B;[r,K]) = rB\left(1-\left(\frac{B}{K}\right)\right)$$

Reference Points:

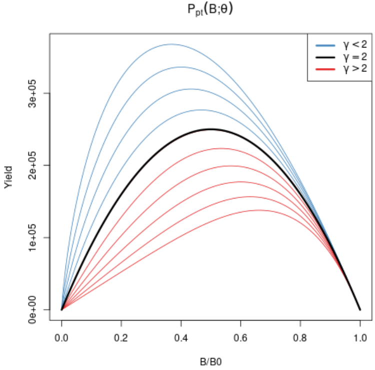
$$F^* = \frac{r}{2}$$

$$B^* = \frac{K}{2} \qquad B_0 = K$$

$$\frac{B^*}{B_0} = \frac{1}{2}$$



$$P_{pt}(B;[r,K,\gamma]) = \frac{rB}{\gamma-1}\left(1-\left(\frac{B}{K}\right)^{(\gamma-1)}\right)$$




Reference Points:

$$F^* = \frac{r}{\gamma}$$

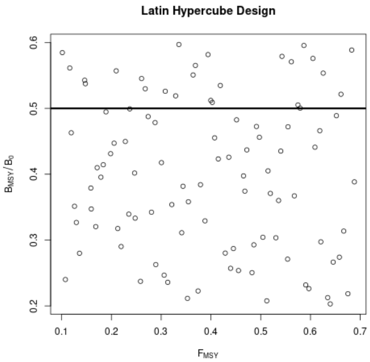
$$B^* = K\left(\frac{1}{\gamma}\right)^{\frac{1}{\gamma-1}} \quad B_0 = K$$

$$\frac{B^*}{B_0} = \left(\frac{1}{\gamma}\right)^{\frac{1}{\gamma-1}}$$

$$F^* = \frac{r}{\gamma} \qquad \frac{B^*}{\bar{B}(0)} = \left(\frac{1}{\gamma}\right)^{\frac{1}{\gamma-1}}$$

Closed Form Inversion

$$r = \gamma F^* \quad \gamma = \frac{W\left(\frac{B^*}{\bar{B}(0)} \log\left(\frac{B^*}{\bar{B}(0)}\right)\right)}{\log\left(\frac{B^*}{\bar{B}(0)}\right)}$$




* Lambert $W$ function inverts $xe^x$ $s.t.$ $W(xe^x) = x$

$$F^* = \frac{r}{\gamma} \qquad \frac{B^*}{\bar{B}(0)} = \left(\frac{1}{\gamma}\right)^{\frac{1}{\gamma-1}}$$

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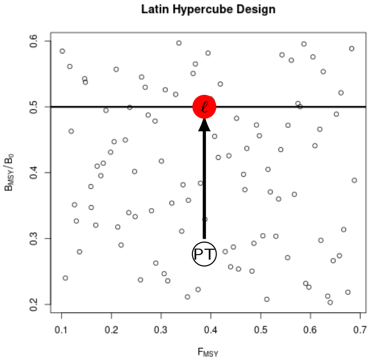
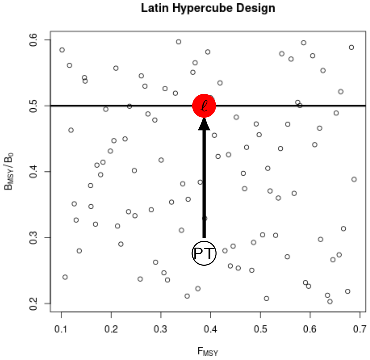



* Lambert $W$ function inverts $xe^x$ $s.t.$ $W(xe^x) = x$

$$\underbrace{\left(F_{MSY}, \frac{B_{MSY}}{\bar{B}(0)}\right)}_{\text{PT Truth}} \overset{\text{GP}}{\mapsto} \underbrace{\left(\hat{F}_{MSY}, \frac{1}{2}\right)}_{\text{Shaefer Estimate}}$$



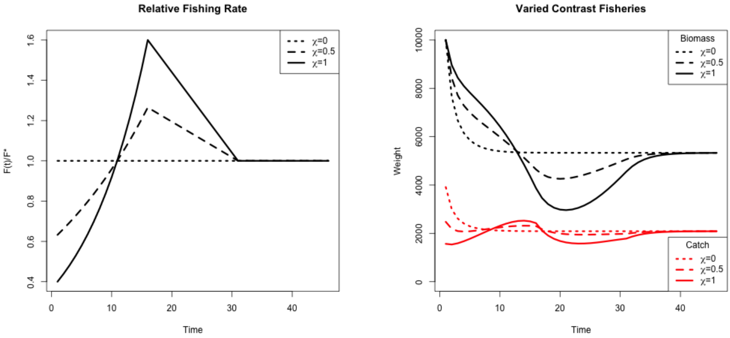



Figure: (*left*) Relative fishing with low, medium, and high contrast. (*right*) Population biomass and catch at each associated level of contrast.

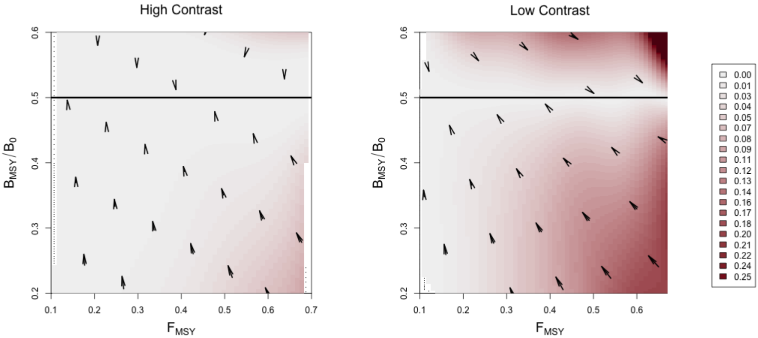
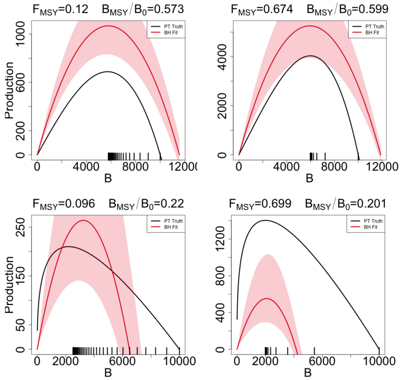



Figure: Joint bias direction for $(F^*, \frac{B^*}{B_0})$ estimates under the misspecified Schaefer Model. The intensity of color represents the excess bias relative to the shortest possible mapping. Results in the low contrast setting are shown *right*, and the high contrast setting is shown *left*.

# Mechanism for Bias in $F_{MSY}$ via Contrast

- Only observe upper half
- learn about slope at origin from upper biomass range



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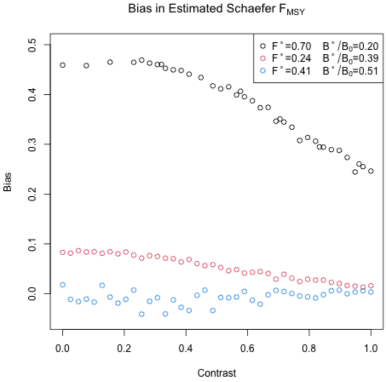

a quick summary

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- Beverton-Holt production
- Production asymptotes
- Explicit Natural Mortality
- Explicit Fishing Mortality

$$I_t = qB_t e^{\epsilon} \quad \epsilon \sim N(0, \sigma^2)$$

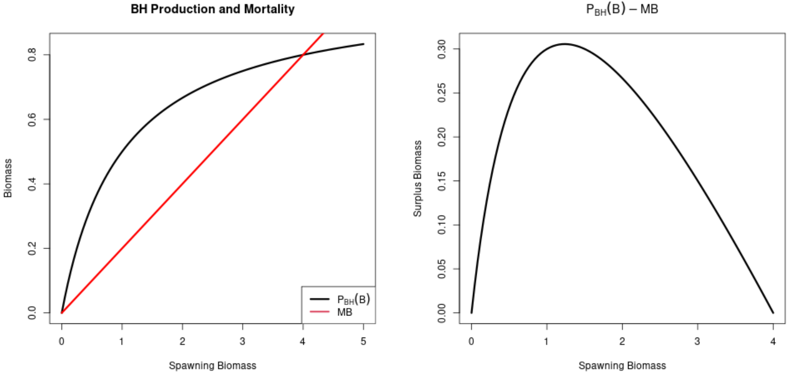
$$\frac{dB}{dt} = P_{BH}(B(t); \boldsymbol{\theta}) - (M + F(t))B(t)$$

$$P_{BH}(B) = \frac{\alpha B}{1 + \beta B}.$$

$$F^* = \qquad (1)$$

$$P_{BH}(B; [\alpha, \beta]) = \frac{\alpha B}{1 + \beta B}.$$

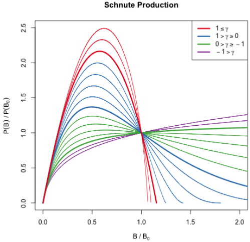
$$\frac{dB}{dt} = P_{BH}(B; [\alpha, \beta]) - MB - FB$$



$$P_s(B;[\alpha,\beta,\gamma]) = \alpha B(1-\beta\gamma B)^{\frac{1}{\gamma}}$$

$$0 = \frac{1}{\gamma} - \left(\frac{1}{\gamma} + \frac{F^*}{F^*+M}\right)\left(\frac{F^*+M}{\alpha}\right)^{\gamma}$$

$$\frac{B^*}{B_0} = \frac{1-\left(\frac{M+F^*}{\alpha}\right)^{\gamma}}{1-\left(\frac{M}{\alpha}\right)^{\gamma}}$$



$$\bar{B}(F) = \frac{1}{\gamma\beta}\left(1 - \left(\frac{M+F}{\alpha}\right)^{\gamma}\right). \tag{2}$$

$$B_0 = \frac{1}{\gamma\beta}\left(1 - \left(\frac{M}{\alpha}\right)^{\gamma}\right) \tag{3}$$

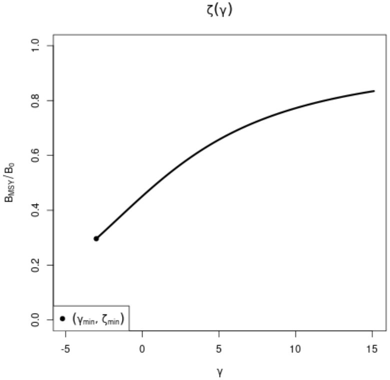
$$\frac{B^*}{B_0} = \frac{1 - \left(\frac{M+F^*}{\alpha}\right)^{\gamma}}{1 - \left(\frac{M}{\alpha}\right)^{\gamma}}. \tag{4}$$

$$\frac{d\bar{Y}}{dF} = \bar{B}(F) + F\frac{d\bar{B}}{dF} \tag{5}$$

$$\frac{d\bar{B}}{dF} = -\frac{1}{\beta}\left(\frac{\left(\frac{M+F}{\alpha}\right)^{\gamma}}{F+M}\right). \tag{6}$$

$$
\begin{aligned}
\alpha &= (M + F^*)\left(1 + \frac{\gamma F^*}{M + F^*}\right)^{1/\gamma} \\
\beta &= \frac{1}{\gamma B_0}\left(1 - \left(\frac{M}{\alpha}\right)^{\gamma}\right) \\
\frac{B^*}{B_0} &= \frac{1 - \left(\frac{M+F^*}{\alpha}\right)^{\gamma}}{1 - \left(\frac{M}{\alpha}\right)^{\gamma}}.
\end{aligned} \tag{8}
$$

$$\gamma' \sim \zeta_{min}\delta(\gamma_{min}) + t(\mu, \sigma, \nu)\mathbf{1}_{\gamma > \gamma_{min}} \tag{9}$$



# Schnute Recruitment

$$R(B;\alpha,\beta,\gamma) = \alpha B_{t-a_s}(1-\beta\gamma B_{t-a_s})^{\frac{1}{\gamma}}$$

Logistic

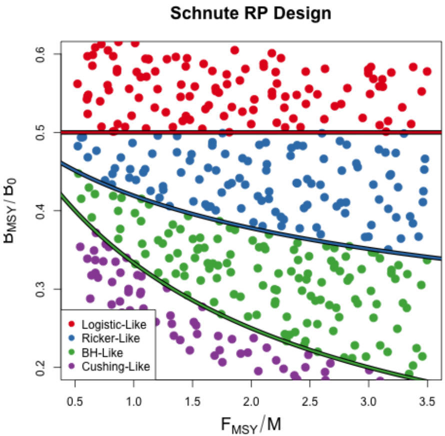
$\gamma = 1$

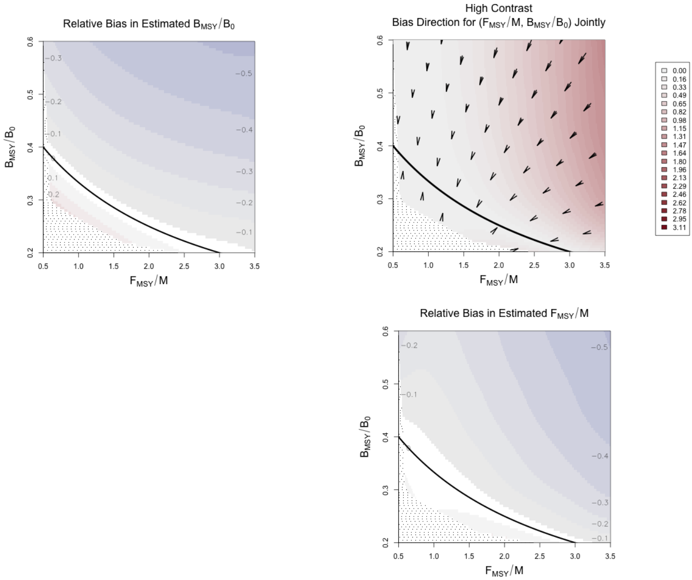
Ricker

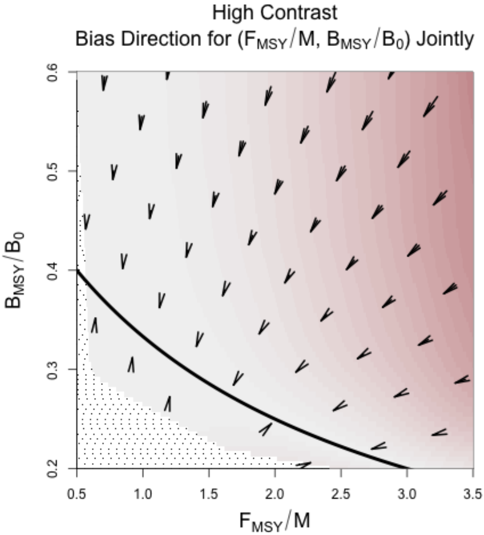
$\gamma \to 0$

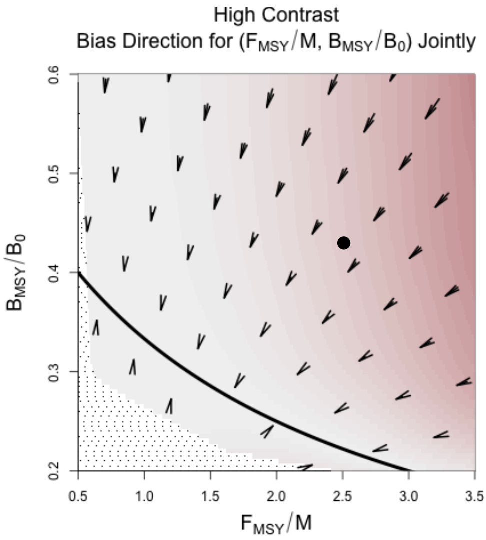
Beverton-Holt

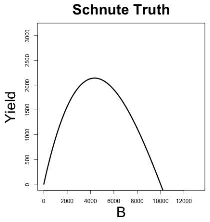
$\gamma = -1$

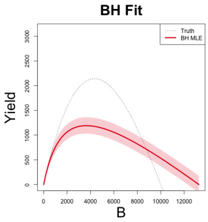



Relative Bias in Estimated $B_{MSY}/B_0$
High Contrast
Bias Direction for ($F_{MSY}/M$, $B_{MSY}/B_0$) Jointly
Relative Bias in Estimated $F_{MSY}/M$
$F_{MSY}/M$
$B_{MSY}/B_0$
0.00 0.16 0.33 0.49 0.65 0.82 0.98 1.15 1.31 1.47 1.64 1.80 1.96 2.13 2.29 2.46 2.62 2.78 2.95 3.11


High Contrast
Bias Direction for ($F_{MSY}/M$, $B_{MSY}/B_0$) Jointly
$B_{MSY}/B_0$
0.2
0.3
0.4
0.5
0.6
$F_{MSY}/M$
0.5
1.0
1.5
2.0
2.5
3.0
3.5


High Contrast
Bias Direction for ($F_{MSY}/M$, $B_{MSY}/B_0$) Jointly
$B_{MSY}/B_0$
0.2
0.3
0.4
0.5
0.6
0.5
1.0
1.5
2.0
2.5
3.0
3.5
$F_{MSY}/M$


Schnute Truth
Yield
0
500
1000
1500
2000
2500
3000
0
2000
4000
6000
8000
10000
12000
B


BH Fit
Truth
BH MLE
Yield
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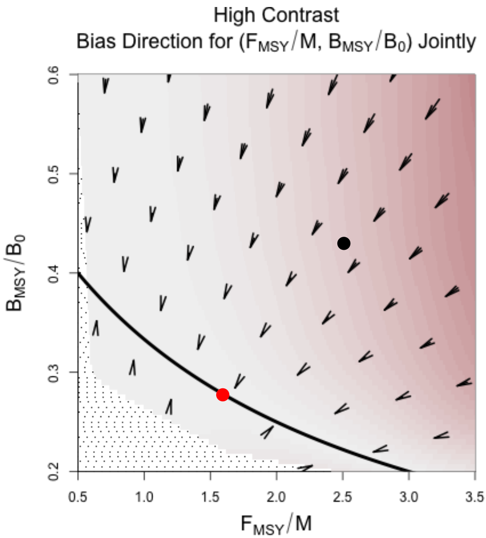
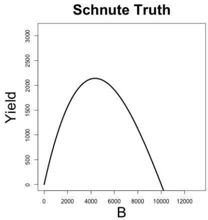

High Contrast
Bias Direction for ($F_{MSY}/M$, $B_{MSY}/B_0$) Jointly
$B_{MSY}/B_0$
$F_{MSY}/M$
0.2
0.3
0.4
0.5
0.6
0.5
1.0
1.5
2.0
2.5
3.0
3.5


Schnute Truth
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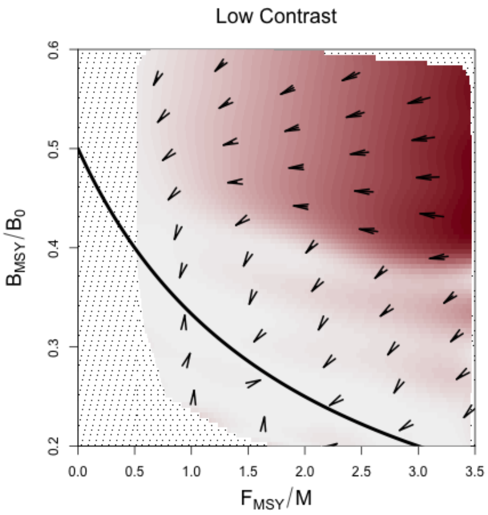
Low Contrast
$B_{MSY}/B_0$
$F_{MSY}/M$
0.2
0.3
0.4
0.5
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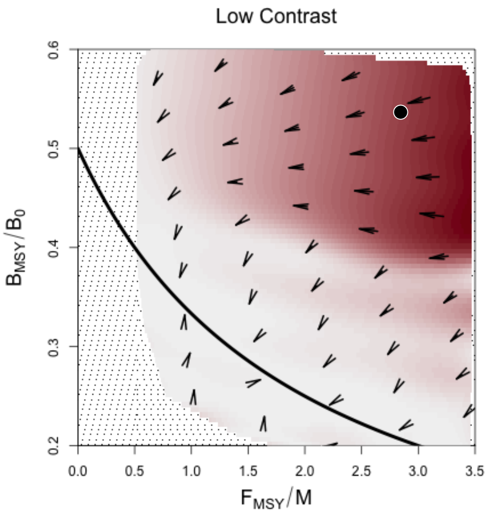
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5


Low Contrast
0.6
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0.2
$B_{MSY}/B_0$
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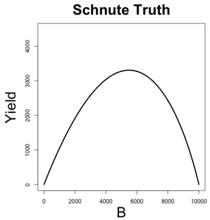
0.5
1.0
1.5
2.0
2.5
3.0
3.5
$F_{MSY}/M$


Schnute Truth
Yield
B
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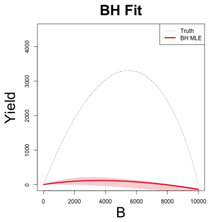
BH Fit
Truth
BH MLE
Yield
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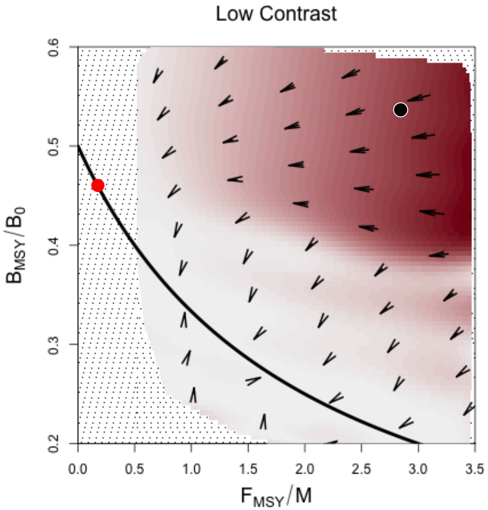
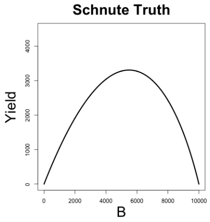

Low Contrast
$B_{MSY}/B_0$
0.2
0.3
0.4
0.5
0.6
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
$F_{MSY}/M$


Schnute Truth
Yield
0
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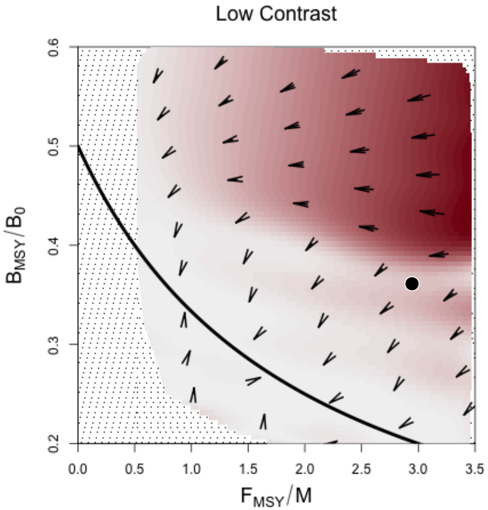
6000
8000
10000
B


Low Contrast
0.2
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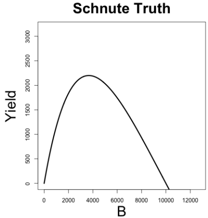
0.5
0.6
$B_{MSY}/B_0$
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
$F_{MSY}/M$


Schnute Truth
Yield
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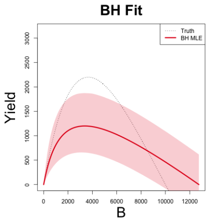
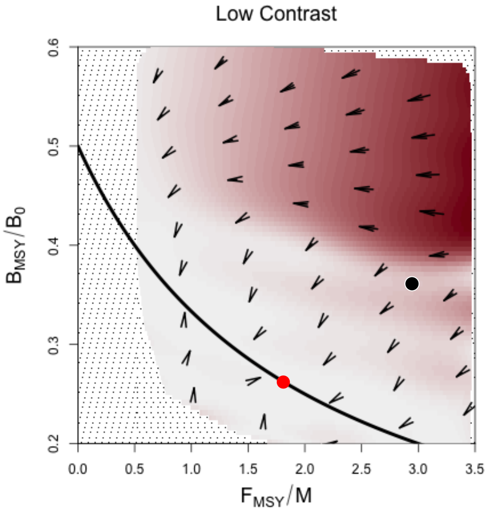
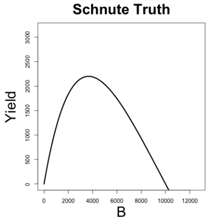
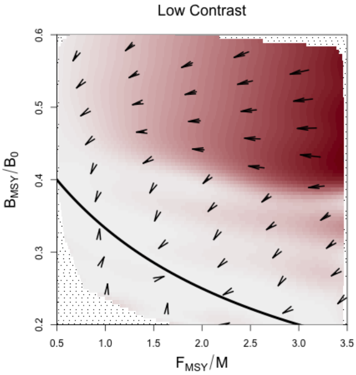
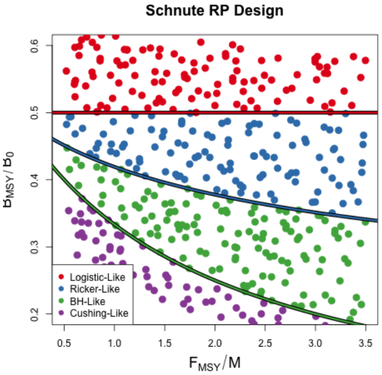
500
1000
1500
2000
2500
3000
0
2000
4000
6000
8000
10000
12000
B


BH Fit
Truth
BH MLE
Yield
B


Low Contrast
$B_{MSY}/B_0$
$F_{MSY}/M$


Schnute Truth
Yield
B


Low Contrast
$B_{MSY}/B_0$
$F_{MSY}/M$
0.5 1.0 1.5 2.0 2.5 3.0 3.5
0.2 0.3 0.4 0.5 0.6


Schnute RP Design
$B_{MSY}/B_0$
$F_{MSY}/M$
0.5 1.0 1.5 2.0 2.5 3.0 3.5
0.2 0.3 0.4 0.5 0.6
Logistic-Like
Ricker-Like
BH-Like
Cushing-Like

## Quick Summary and Transition

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# General Modeling Structure

$$\frac{dB}{dt} = \overbrace{w(a_s)R(B;\theta)}^{\text{Recruitment Biomass}} + \overbrace{\kappa\left[w_\infty N - B\right]}^{\text{Net Growth}} - \overbrace{(M+F)B}^{\text{Mortality}}$$
$$\frac{dN}{dt} = R(B;\theta) - (M+F)N$$

$R(B_{t-a_s};\alpha,\beta,\gamma)$ : Schnute Three Parameter Recruitment

$w(a)$ : Von Bertalanffy Individual Growth

Knife-Edge Selectivity at age $a_s$

# Schnute Recruitment

$$R(B;\alpha,\beta,\gamma) = \alpha B_{t-a_s}(1-\beta\gamma B_{t-a_s})^{\frac{1}{\gamma}}$$

Logistic

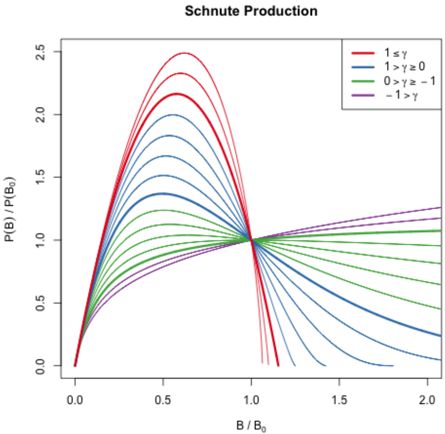
$\gamma = 1$

Ricker

$\gamma \to 0$

Beverton-Holt

$\gamma = -1$



# Schnute Recruitment

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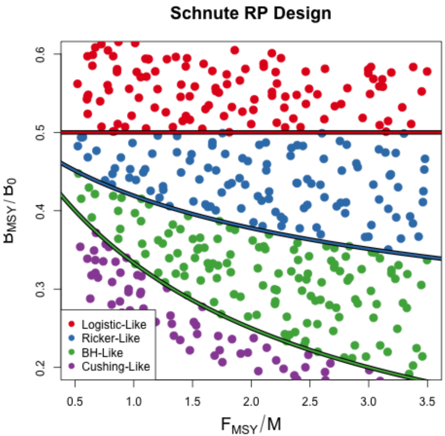
Logistic
$\gamma = 1$

Ricker
$\gamma \to 0$

Beverton-Holt
$\gamma = -1$

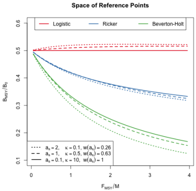


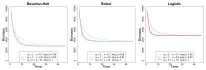
# Von Bertalanffy Individual Growth

$$w(a) = w_\infty(1 - e^{-\kappa(a-a_0)})$$

- Instant Growth:
  (Production Model)
  $a_s \to 0 \quad \kappa \to \infty$

- Dynamic Growth:
  $a_s$ : Not Instant
  $\kappa$ : Peakness of Dynamics



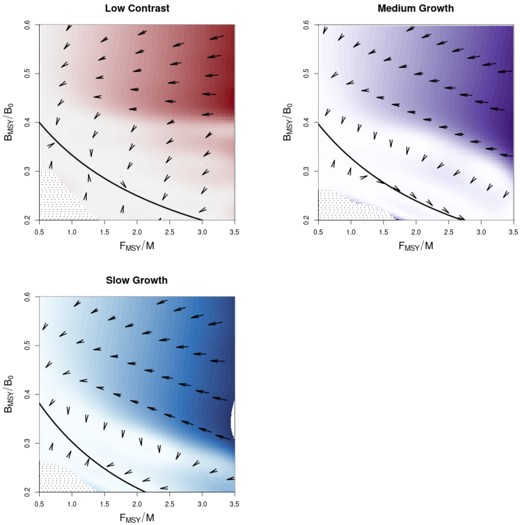
$$\bar{B}(F) = \frac{1}{\beta\gamma}\left(1 - \left(\frac{(F+M)(F+M+\kappa)}{\alpha w(a_s)(F+M+\frac{\kappa w_\infty}{w(a_s)})}\right)^{\gamma}\right) \quad (10)$$

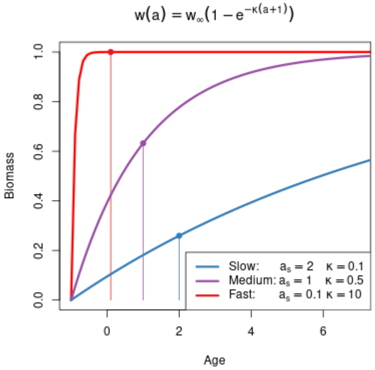
$$\bar{N}(F) = \frac{\alpha\bar{B}(F)(1-\beta\gamma\bar{B}(F))^{1/\gamma}}{F+M} \quad (11)$$

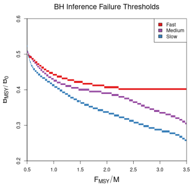
$$\alpha = \left[\left(\frac{Z^*(Z^*+\kappa)}{w(a_s)(Z^*+\frac{\kappa w_\infty}{w(a_s)})}\right)^{\gamma} + \left(\frac{\gamma F^*}{w(a_s)}\right)\left(\frac{Z^*(Z^*+\kappa)}{w(a_s)(Z^*+\frac{\kappa w_\infty}{w(a_s)})}\right)^{\gamma-1}\left(1+\frac{\left(\frac{\kappa w_\infty}{w(a_s)}\right)\left(\kappa - \frac{\kappa w_\infty}{w(a_s)}\right)}{(Z^*+\frac{\kappa w_\infty}{w(a_s)})^2}\right)\right]^{\frac{1}{\gamma}} \quad (12)$$

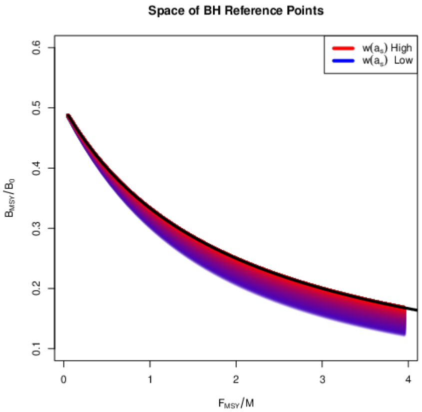
$$\beta = \frac{1}{\gamma B_0}\left(1 - \left(\frac{M(M+\kappa)}{\alpha w(a_s)(M+\frac{\kappa w_\infty}{w(a_s)})}\right)^{\gamma}\right) \quad (13)$$

$$\frac{B^*}{B_0} = \frac{1 - \left(\frac{(F^*+M)(F^*+M+\kappa)}{\alpha w(a_s)(F^*+M+\frac{\kappa w_\infty}{w(a_s)})}\right)^{\gamma}}{1 - \left(\frac{M(M+\kappa)}{\alpha w(a_s)(M+\frac{\kappa w_\infty}{w(a_s)})}\right)^{\gamma}} \quad (14)$$


Low Contrast
Medium Growth
Slow Growth
$B_{MSY}/B_0$
$F_{MSY}/M$
0.2
0.3
0.4
0.5
0.6
0.5
1.0
1.5
2.0
2.5
3.0
3.5


$w(a) = w_\infty(1 - e^{-\kappa(a+1)})$
Biomass
Age
Slow: $a_s = 2$ $\kappa = 0.1$
Medium: $a_s = 1$ $\kappa = 0.5$
Fast: $a_s = 0.1$ $\kappa = 10$


BH Inference Failure Thresholds
Fast
Medium
Slow
$F_{MSY}/M$


Space of BH Reference Points
w(a_s) High
w(a_s) Low
B_MSY/B_0
0.1
0.2
0.3
0.4
0.5
0.6
0
1
2
3
4
F_MSY/M

General Conclusions and future work.

Many Thanks:

- Dr. Marc Mangel
- Collaborators at NOAA
- NMFS Sea Grant

# Metamodel Details

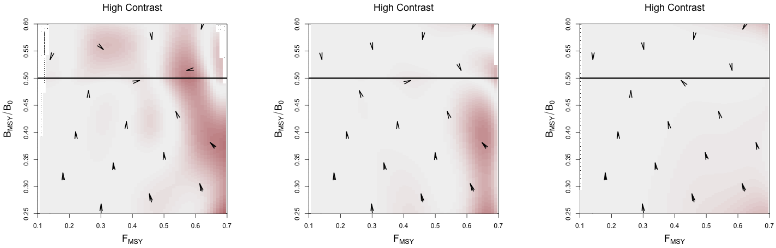
$$\hat{\mu} = \widehat{log(r)} \quad - or - \quad \hat{\mu} = \widehat{log(K)}$$

$$\boldsymbol{x} = \left(F_{MSY}, \frac{B_{MSY}}{\bar{B}(0)}\right)$$

$$\begin{aligned} \hat{\mu} &= \beta_0 + \boldsymbol{\beta}'\boldsymbol{x} + f(\boldsymbol{x}) + \epsilon \\ f(\boldsymbol{x}) &\sim \text{GP}(0, \tau^2 R(\boldsymbol{x}, \boldsymbol{x}')) \\ \epsilon_i &\sim \text{N}(0, \hat{\omega}_i). \end{aligned}$$

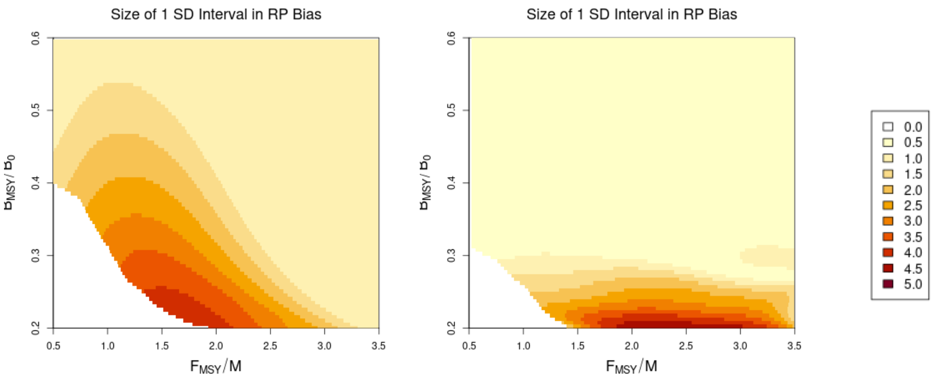
$$R(\boldsymbol{x}, \boldsymbol{x}') = \exp\left(\sum_{j=1}^{2} \frac{-(x_j - x_j')^2}{2\ell_j^2}\right)$$

# High Contrast PT $\sigma = 0.12$ Data



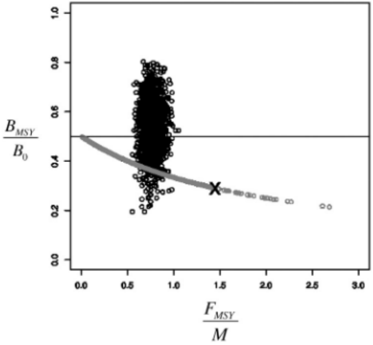

1x Samples 2x Samples 4x Samples

Contrast | No Contrast



Mangel et al.

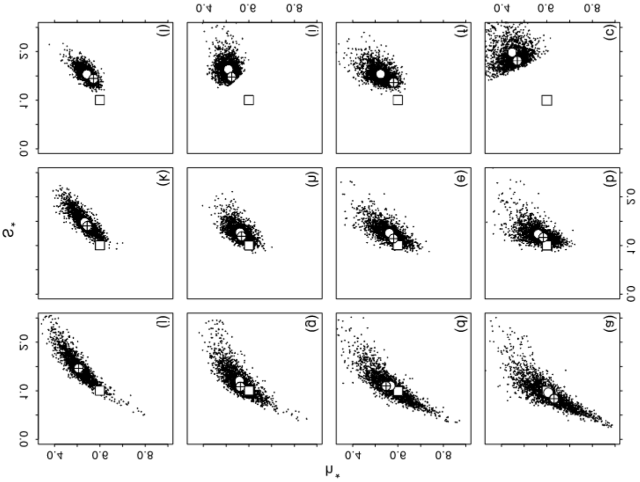
**Fig. 4.** DeYoreo et al. (2012) used both a BH-SRR and three-parameter SRR, similar to the S-SRR in a stock assessment of cowcod (*Sebastes levis*). We show samples from posterior distributions arising from different values of steepness. Unlike most stock assessments, we plot $B_{MSY}/B_0$ versus $F_{MSY}/M$. The grey circles show the results for the BH-SRR. This curve is another way of representing the constraint placed on a stock assessment by using a BH-SRR and specifying steepness — results must lie along this curve. The black circles represent the outcome of the three-parameter SRR. The black X represents the result when steepness is asserted to be 0.6.



Logistic

Ricker

BH



Schnute, J. T., & Kronlund, A. R. (2002). Estimating salmon stock recruitment relationships from catch and escapement data. Canadian Journal of Fisheries and Aquatic Sciences, 59(3), 433–449.