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Source: *Technometrics*, Vol. 45, No. 3 (Aug., 2003), pp. 199-207
Published by: American Statistical Association and American Society for Quality
Stable URL: http://www.jstor.org/stable/25047047
Accessed: 28/05/2014 17:33

An Adaptive Exponentially Weighted Moving Average Control Chart

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Lucas and Saccucci showed that exponentially weighted moving average (EWMA) control charts can be designed to quickly detect either small or large shifts in the mean of a sequence of independent observations. But a single EWMA chart cannot perform well for small and large shifts simultaneously. Furthermore, in the worst-case situation, this scheme requires a few observations to overcome its initial inertia. The main goal of this article is to suggest an adaptive EWMA (AEWMA) chart that weights the past observations of the monitored process using a suitable function of the current "error." The resulting scheme can be viewed as a smooth combination of a Shewhart chart and an EWMA chart. A design procedure for the new control schemes is suggested. Comparisons of the standard and worst-case average run length profiles of the new scheme with those of different control charts show that AEWMA schemes offer a more balanced protection against shifts of different sizes.

KEY WORDS: Adaptive weighting; Average run length; Control chart; Exponentially weighted moving average.

1. INTRODUCTION

Let y_t, $t = 1, 2, \ldots$, be a sequence of independent normal observations with a common variance, such as single measurements, sample means, or residuals from a particular model. Suppose that (a) before an unknown time, say t_0, the mean of the process is equal to a target value η_0, (b) after t_0, the mean becomes $\eta_1 = \eta_0 + \mu$, where μ is an unknown shift, and (c) we want to detect the occurrence of this shift as soon as possible. Shewhart, cumulative sum (CUSUM), and exponentially weighted moving average (EWMA) control charts are three of the statistical tools used for this purpose (see Shewhart 1931; Page 1954, 1955; Roberts 1959; Rowlands and Whetherill 1991; Yashchin 1993). Statistical properties of a control chart are usually evaluated in terms of the average run length (ARL), that is, the average number of observations required to signal a change for a particular size of the shift.

An EWMA control chart is based on the statistic

$$x_t = (1-\lambda)x_{t-1} + \lambda y_t = (1-\lambda)^t x_0 + \lambda \sum_{i=0}^{t-1}(1-\lambda)^i y_{t-i}, \quad (1)$$

where λ is a suitable constant such that $0 < \lambda \le 1$. The quantity x_0 represents the starting value, often the target value η_0. This scheme signals when $|x_t - \eta_0|$ exceeds a specified action limit, h.

The literature on the efficiency and robustness of EWMA charts indicates that they can be constructed to perform well for either small or large shifts. But because small values of λ must be used to quickly detect small shifts and large values must be used to efficiently signal the occurence of large shifts, a single EWMA scheme cannot have a "nearly minimum" ARL for both small and large shifts (Crowder 1987a; Lucas and Saccucci 1990).

In addition, Yashchin (1987) studied the "inertia problem" for the EWMA scheme. When x_t is close to one of its control limits and a sudden change occurs in the opposite direction (i.e., a worst-case situation), there is a large discrepancy between x_t and the true current level of the process. Hence for small values of λ, the EWMA scheme must overcome inertia to react to shifts in the process mean (Woodall and Maragah 1990). The EWMA's inertia can be counteracted by using the combined Shewhart EWMA chart (Lucas and Saccucci, 1990; Albin, Kang, and Shea 1997). This scheme is achieved by adding Shewhart limits to an EWMA control scheme so that an out-of-control signal is given if the EWMA statistic is outside the control limits or if the current observation is outside the Shewhart limits. However, as noted by Woodall and Maragah (1990), introducing of the Shewhart limits only ameliorates the inertia problem.

In this article we propose a new adaptive exponentially weighted moving average (AEWMA) control chart that essentially tries to combine an EWMA and a Shewhart chart in a smooth way. The underlying idea is to adapt the weight of the past observations, according to the magnitude of the "error," $y_t - x_{t-1}$, to detect, in a more balanced way, shifts of different sizes while diminishing the inertia problem. Adaptive weighting schemes have been already considered within several frameworks, including Kalman filter estimation (Kalman 1960), time series forecasting (West and Harrison 1989; Pantazopoulos and Pappis 1996), and change-detection procedures (Jun and Suh 1999). But we believe that our approach is simpler and easier to understand and apply. In addition, comparisons in Section 5 indicate that the AEWMA control chart has a better performance, in terms of ARL, than other adaptive methods.

To design an AEWMA scheme, we consider a criterion different from the one used by Lucas and Saccucci (1990) for EWMA charts. These authors choose the parameters that, for a given in-control ARL, minimize the out-of-control ARL at a specified shift. Thus parameters depend crucially on the magnitude of the specified shift. In contrast, we recommend a criterion that, although simple, tries to achieve good performance for a wider range of shifts.

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DOI 10.1198/004017003000000023

The new scheme is described in Section 2, an example is given in Section 3, and the recommended design procedure is presented in Section 4. In Section 5 the AEWMA chart is compared with other control schemes. In Appendix A the approximation of the ARL of an AEWMA chart, based on a Markov chain approach, is briefly discussed. The numerical procedure used to design the AEWMA scheme is sketched in Appendix B.

2. THE ADAPTIVE EWMA SCHEME

The class of control charts that we consider attempts to track the current level of the process using a statistic of the type

$$x_t = x_{t-1} + \phi(e_t), \qquad x_0 = \eta_0, \tag{2}$$

where $e_t = y_t - x_{t-1}$ and $\phi(e_t)$ is a "score" function. An alarm is raised when $|x_t - \eta_0| > h$, where η_0 denotes the target value of the process mean and h is a suitable threshold. The h value is mainly determined by ensuring that the desired mean time between false alarms is large. Observe that when $y_t \neq x_{t-1}$, (2) can be rewritten in the form

$$x_t = (1 - w(e_t))x_{t-1} + w(e_t)y_t,$$

where $w(e) = \phi(e)/e$, that is, as an EWMA statistic with varying weights. Shewhart and EWMA charts can be obtained as special cases of (2) for $\phi(e) = e$ and $\phi(e) = \lambda e$. Many score functions could be used. To blend together the best features of EWMA and Shewhart charts, it seems reasonable to use scores within the envelope of the EWMA and Shewhart ϕ-functions and, in particular, to choose $\phi(\cdot)$ such that (a) $\phi(e)$ is monotone increasing in e; (b) $\phi(e) = -\phi(-e)$; (c) $\phi(e) \approx \lambda e$, when $|e|$ is small, for a suitable λ, $0 \le \lambda \le 1$; and (d) $(\phi(e)/e) \approx 1$, when $|e|$ is large. Observe that (c) and (d) have been introduced to obtain a control statistic that behaves like an EWMA chart when $|e_t|$ is small and like a Shewhart chart when $|e_t|$ is large. In addition, note that (d) may cure the inertia problem of the EWMA scheme because it leads to control statistics that can "jump" toward the observed value y_t in the worst-case situation.

In this article we show results for the following three score functions:

$$\phi_{hu}(e) = \begin{cases} e + (1-\lambda)k & \text{if } e < -k \\ \lambda e & \text{if } |e| \le k \\ e - (1-\lambda)k & \text{if } e > k, \end{cases} \tag{3}$$

$$\phi_{bs}(e) = \begin{cases} e\big(1 - (1-\lambda)(1-(e/k)^2)^2\big) & \text{if } |e| \le k \\ e & \text{otherwise,} \end{cases} \tag{4}$$

and

$$\phi_{cb}(e) = \begin{cases} e & \text{if } e \le -p_1 \\ -\tilde{\phi}_{cb}(-e) & \text{if } -p_1 < e < -p_0 \\ \lambda e & \text{if } |e| \le p_0 \\ \tilde{\phi}_{cb}(e) & \text{if } p_0 < e < p_1 \\ e & \text{if } e \ge p_1, \end{cases} \tag{5}$$

where $0 < \lambda \le 1$, $k \ge 0$, and $0 \le p_0 < p_1$ denote suitable constants and $\tilde{\phi}_{cb}(\cdot)$ is the cubic polynomial that makes $\phi_{cb}(\cdot)$ and its first derivative continuous, that is,

$$\tilde{\phi}_{cb}(e) = \lambda e + (1-\lambda)\left(\frac{e-p_0}{p_1-p_0}\right)^2 \times \left(2p_1 + p_0 - (p_0+p_1)\left(\frac{e-p_0}{p_1-p_0}\right)\right).$$

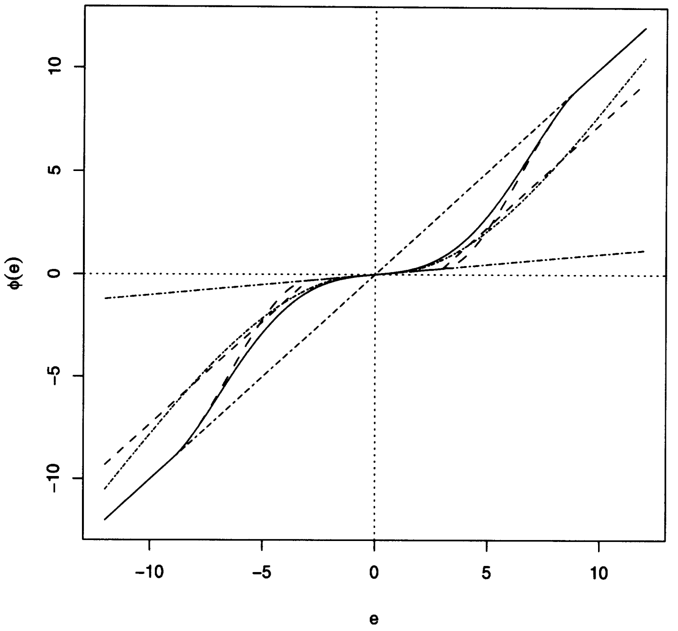

Figure 1. Comparisons of the $\phi(\cdot)$ Functions of Shewhart (– – -); EWMA with $\lambda = .1$ (– · – ·); and AEWMA Schemes $\phi_{hu}(\cdot)$ with $\lambda = .1$ and $k = 3$ (- - -); $\phi_{bs}(\cdot)$ with $\lambda = .1$ and $k = 9$ (—); $\phi^1_{cb}(\cdot)$ with $\lambda = .1$, $p_0 = 1$, and $p_1 = 18$ (- · · · -); and $\phi^2_{cb}(\cdot)$ with $\lambda = .1$, $p_0 = 3$, and $p_1 = 9$ (– – –).

The first two ϕ-functions are inspired by Huber's function (Huber 1981) and Tukey's bisquare function (Beaton and Tukey 1974). The third function has been suggested by an anonymous referee as another simple way to blend together the ϕ-functions of Shewhart and EWMA schemes. Because $\lim_{e\to\infty} \phi_{hu}(e)/e = 1$ but $\phi_{hu}(e) \neq e$ for every nonzero e, schemes based on (3) can almost (but not completely) ignore the past observations of the process. In contrast, schemes based on (4) and (5) completely discard the previous history when y_t is too far from x_{t-1}.

Figure 1 shows the ϕ-functions of the following schemes: Shewhart; an EWMA chart with $\lambda = .1$; a scheme based on $\phi_{hu}(\cdot)$ with $\lambda = .1$ and $k = 3$; a scheme based on $\phi_{bs}(\cdot)$ with $\lambda = .1$ and $k = 9$; a scheme based on $\phi_{cb}(\cdot)$ with $\lambda = .1, p_0 = 1$, and $p_1 = 18$ (denoted by ϕ^1_{cb}); and a scheme based on $\phi_{cb}(\cdot)$ with $\lambda = .1, p_0 = 3$, and $p_1 = 9$ (denoted by ϕ^2_{cb}). The figure illustrates how the suggested AEWMA charts update the control statistic like an EWMA scheme when the current observation y_t is close to x_{t-1} and like a Shewhart scheme when $|y_t - x_{t-1}|$ is large. In addition, note that for different choices of p_0 and p_1, $\phi_{cb}(e)$ can be close to the other two score functions for a wide range of values of e. This shows the greater flexibility of $\phi_{cb}(\cdot)$.

3. EXAMPLE

To illustrate an AEWMA control scheme, we use the capsules weights data given by Wetherill and Brown (1991, table 3.3). The data are a set of 50 measurements (in grams) taken every 30 seconds from a manufacturing process that is working in control. The target value is 5 g, and the standard deviation is $\sigma = 0.3$ g.

We simulate two different out-of-control situations, adding either 1σ or -3σ to the last 41 observations. Figure 2 shows a

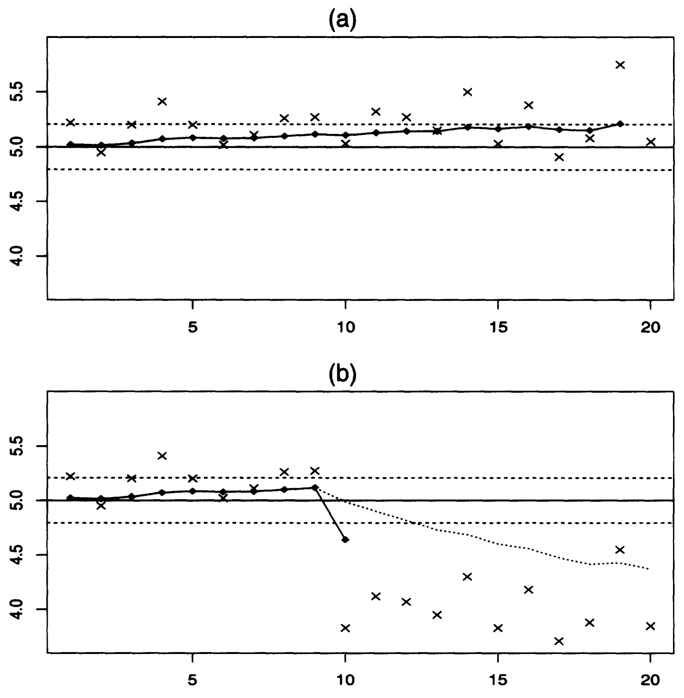

Figure 2. AEWMA Control Scheme Applied to the Capsule Weights. *(a) The value .3 (one standard deviation) is added to the last 41 observations. (b) The value .9 (three standard deviations) is subtracted from the last 41 observations. The original data are represented by X's. The solid line connects the AEWMA values. The dotted line in (b) connects the EWMA values with λ equal to .1. Dashed lines, in (a) and (b) are the control limits.*

plot of an AEWMA control statistic, together with the data. The scheme is based on $\phi_{hu}(\cdot)$ with $\lambda = .1$, $k = 3\sigma$, and $h = .6845\sigma$. The threshold h has been chosen so that the in-control ARL is equal to 500. In the $1\sigma/3\sigma$ shift cases, the scheme raises an alarm at the 10th/1st out-of-control observation; computations for the 3σ shift case are shown in Table 1. Observe in Figure 2(b) the jump of the control statistic at observation 10; the $w(e_t)$'s in Table 1 show that this jump is due to the adaptive nature of the scheme. In particular, y_{10} enters in the computation of x_{10} with a weight equal to 3.7λ.

Table 2 compares the delay in the detection of the change for three schemes: the AEWMA scheme previously considered and two EWMA schemes with the same in-control ARL suggested by Lucas and Saccucci (1990) to detect a 1σ shift and a 3σ shift

Table 1. Calculations of an AEWMA Scheme for the Capsule Weights *Data*

t	y_t	e_t	$\phi(e_t)$	$w(e_t)$	x_t
0					5.000
1	5.22	.220	.022	.10	5.022
2	4.95	−.072	−.007	.10	5.015
3	5.20	.185	.019	.10	5.033
4	5.41	.377	.038	.10	5.071
5	5.20	.129	.013	.10	5.084
6	5.02	−.064	−.006	.10	5.077
7	5.11	.032	.003	.10	5.081
8	5.26	.179	.018	.10	5.099
9	5.27	.171	.017	.10	5.116
10	3.83	−1.286	−.476	.37	4.640*

* Out-of-control signal.

NOTE: The process is in control for the first nine observations. 3σ has been subtracted from the original in-control 10th observation.

Table 2. Number of Out-of-Control Observations Required to Signal a Change for the Capsule Weights *Data*

Scheme	*1σ shift*	*3σ shift*
AEWMA based on $\phi_{hu}(\cdot)$ with $\lambda = .1$, $k = 3\sigma$, and $h = .6845\sigma$	10	1
EWMA with $\lambda = .12$ and $h = 2.8585\sigma\sqrt{\lambda/(2-\lambda)}$	10	3
EWMA with $\lambda = .70$ and $h = 3.0865\sigma\sqrt{\lambda/(2-\lambda)}$	20	1

NOTE: All schemes have an in-control ARL equal to 500. Lucas and Saccucci (1990) suggested the two EWMA schemes for detecting a 1σ shift and a 3σ shift.

in the process. Note that for both shifts, the single AEWMA scheme performs as the standard EWMA scheme designed to optimally detect that shift.

4. DESIGN OF AN AEWMA SCHEME

The usual design strategy is to find the scheme having minimum out-of-control ARL, for a specified shift μ, among the schemes with a desired in-control ARL. The problem is that the "optimal" scheme strongly depends on the specified magnitude of the shift. For example, according to Lucas and Saccucci (1990), if this strategy is used to design EWMA charts with an in-control ARL of 500, then the "optimal" value for λ goes from .05 when $\mu = .05$ to .95 when $\mu = 4$. As a consequence, the ARL of a chart designed for a small shift is quite different from that designed for a large shift.

Let θ be the parameters defining an AEWMA chart. For example, if $\phi_{hu}(\cdot)$ is used, then $\theta = (\lambda, h, k)$. To avoid the described flaw, we devise the following strategy:

1. Choose a desired in-control ARL, say B, and two values of the shift (e.g., a "small" shift, μ_1, and a "large" shift, μ_2).
2. For the specified in-control ARL, find the parameters θ^* having minimum ARL at μ_2; that is, θ^* is the solution of the following problem:

$$\begin{cases} \min_\theta \mathrm{ARL}(\mu_2, \theta) \\ \text{subject to } \mathrm{ARL}(0, \theta) = B, \end{cases}$$

   where $\mathrm{ARL}(\mu, \theta)$ denotes the ARL of a scheme with parameters equal to θ when the shift is μ.
3. Finally, choose a small positive constant α (e.g., $\alpha = .05$) and find the "optimal" θ as the solution of

$$\begin{cases} \min_\theta \mathrm{ARL}(\mu_1, \theta) \\ \text{subject to } \mathrm{ARL}(0, \theta) = B \\ \text{and } \mathrm{ARL}(\mu_2, \theta) \le (1+\alpha)\mathrm{ARL}(\mu_2, \theta^*); \end{cases} \qquad (6)$$

   that is, find the scheme with minimum ARL at μ_1 among those schemes for which the ARL at μ_2 is "nearly minimum."

This approach seems to produce AEWMA charts with reasonable performance for both small and large shifts. But the same approach is rather useless for the design of EWMA or CUSUM schemes. Indeed, when the foregoing strategy is applied to these control charts, the final θ is not substantially different from the intermediate θ^*, at least in terms of ARL. This

Table 3. Some "Optimal" AEWMA Charts Based on $\phi_{hu}(\cdot)$, ($\alpha = .05$)

		In-control ARL equal to 100			In-control ARL equal to 500		
μ_1	μ_2	h	λ	k	h	λ	k
.25	4	.1471	.0162	2.7459	.2017	.0117	3.0326
.50	4	.3927	.0614	2.6306	.4306	.0398	2.8990
1.00	4	.7874	.1813	2.5752	.8238	.1253	2.7765
.25	5	.1457	.0192	3.3249	.1835	.0137	3.4473
.50	5	.3767	.0670	3.2654	.3960	.0437	3.3402
1.00	5	.7688	.1913	3.2907	.7931	.1354	3.2587
.25	6	.1515	.0207	4.2537	.2091	.0175	4.2176
.50	6	.3671	.0657	4.2638	.4052	.0474	4.1490
1.00	6	.7310	.1792	7.9825	.7610	.1305	4.1534

NOTE: h and k are computed assuming that $\sigma = 1$, where σ denotes the standard deviation of the process. In general, these values should be multiplied by σ.

Table 4. Some "Optimal" AEWMA Charts Based on $\phi_{bs}(\cdot)$, ($\alpha = .05$)

		In-control ARL equal to 100			In-control ARL equal to 500		
μ_1	μ_2	h	λ	k	h	λ	k
.25	4	.3542	.0188	12.6145	.4866	.0082	12.5467
.50	4	.4858	.0463	11.4741	.5807	.0256	11.9897
1.00	4	.7697	.1359	10.6114	.8215	.0830	11.1408
.25	5	.2065	.0196	24.0162	.3381	.0094	17.2037
.50	5	.3729	.0520	19.9865	.4763	.0296	15.4062
1.00	5	.6821	.1473	20.1147	.8551	.1199	13.6702
.25	6	.1430	.0168	48.7509	.2283	.0130	31.1230
.50	6	.3484	.0577	40.6934	.4133	.0394	25.8760
1.00	6	.7224	.1736	47.4411	.7659	.1226	25.5065

NOTE: h and k are computed assuming that $\sigma = 1$, where σ denotes the standard deviation of the process. In general, these values should be multiplied by σ.

is probably due to the relative lack of flexibility of EWMA and CUSUM charts compared with the AEWMA schemes.

A Markov chain approximation to the ARL of an AEWMA chart is described in Appendix A. The numerical approach used to find the solution of (6) is sketched in Appendix B. A library of C functions and a driver program are available at *http://sirio.stat.unipd.it/caewma*.

Tables 3–5 show the parameters obtained using the foregoing strategy for some choices of B, μ_1, and μ_2. Observe that the parameters λ and k have a different interpretation in (3) and (4). Hence the design procedure results in different values of λ and k for the schemes based on $\phi_{hu}(\cdot)$ and $\phi_{bs}(\cdot)$.

Tables 6 and 7 report the corresponding ARL profiles. Perusing these tables points to the charts "optimized" for $\mu_1 = .5$, $\mu_1 = 1$, or $\mu_2 = 5$ as the best ones in terms of ARL. Furthermore, observe that charts based on (3) and (5) are slightly better for small shifts, whereas charts based on (4) offer more protection against large shifts. However, differences are not large when the best schemes are considered. In particular, observe that the extra flexibility offered by $\phi_{cb}(\cdot)$ does not give an efficiency gain.

5. COMPARISONS

In this section we compare a single AEWMA chart with various EWMA, CUSUM, combined Shewhart EWMA, Shewhart with supplementary runs rules, and different adaptive charts. The considered AEWMA control scheme is based on the Huber score and is designed for $\mu_1 = 1$, $\mu_2 = 5$, and $\alpha = .05$ ($h = .7931$, $\lambda = .1354$, and $k = 3.2587$). In Figures 3–7, the ARL scale is chosen to be logarithmic.

Figure 3 compares the ARL curves (including the worst-case ones) of the AEWMA and three EWMA charts. For a specified in-control ARL equal to 500, the three EWMA charts were designed to have minimum ARL at shifts .5, 3, and 5.

Figure 4 is similar to Figure 3 but with the EWMA schemes replaced by bilateral CUSUM control charts. Figure 5 compares the AEWMA chart with three combined Shewhart EWMA schemes. As recommended by Lucas and Saccucci (1990), the Shewhart limits were set to ± 4. Then the EWMA parameters were determined minimizing the ARL of the combined scheme at three different values of the shift under a constraint on the in-control ARL.

Figure 6 compares the AEWMA chart with a chart that raises an out-of-control signal when the present observation is larger than 3 or less than -3, two or three of the last three observations are between 2 and 3 or between -2 and -3, or four or five of the last five observations are between 1 and 3 (or between -3 and -1). Results given by Champ and Woodall (1987) suggest that this chart is one of the best Shewhart charts with supplementary runs rules. Both schemes in Figure 6 have an in-control ARL equal to 132.89.

Figure 7 compares the AEWMA scheme with an adaptive method, suggested by Pantazopoulos and Pappis (1996). This method is based on the statistics

$$\hat{x}_{t+1} = (1-\lambda_t)\hat{x}_t + \lambda_t y_t, \tag{7}$$

with $\hat{x}_0 = 0$ and $\lambda_0 = 0$. The value of the smoothing parameter is updated by

$$\lambda_t = \begin{cases} \lambda_{t-1} & \text{if } y_{t-1} - \hat{x}_{t-1} = 0 \\ 1 & \text{if } \lambda_t > 1 \\ |y_t - \hat{x}_{t-1}|/|y_{t-1} - \hat{x}_{t-1}| & \text{otherwise.} \end{cases} \tag{8}$$

Table 5. Some "Optimal" AEWMA Charts Based on $\phi_{cb}(\cdot)$, ($\alpha = 0.05$)

		In-control ARL equal to 100				In-control ARL equal to 500			
μ_1	μ_2	h	λ	p_0	p_1	h	λ	p_0	p_1
.25	4	.1575	.0171	2.6266	4.6070	.1813	.0102	3.0730	4.1439
.50	4	.3566	.0502	2.0990	9.0569	.4369	.0404	2.4084	8.9410
1.00	4	.7817	.1604	1.2707	15.9710	.7690	.1069	1.9423	11.6188
.25	5	.1451	.0191	3.0205	9.6081	.1891	.0143	3.4150	4.9731
.50	5	.3484	.0594	2.4267	18.4671	.3929	.0430	3.1198	6.8573
1.00	5	.7133	.1681	1.7065	40.2725	.7687	.1267	2.4412	12.4915
.25	6	.1581	.0218	3.7129	20.3969	.1950	.0158	4.2436	5.3525
.50	6	.3644	.0650	3.8836	12.3231	.3994	.0463	3.6886	11.1518
1.00	6	.7328	.1798	4.0283	19.0161	.7525	.1281	4.2356	5.2572

NOTE: h, p_0, and p_1 are computed assuming that $\sigma = 1$, where σ denotes the standard deviation of the process. In general, these values should be multiplied by σ.

Table 6. ARLs of Some AEWMA Control Charts (In-Control ARL Equal to 100)

		μ											
μ_1	μ_2	.25	.50	.75	1.00	1.50	2.00	2.50	3.00	3.50	4.00	5.00	6.00
Schemes based on $\phi_{hu}(\cdot)$													
.25	4	41.38	20.30	13.01	9.38	5.67	3.68	2.46	1.73	1.34	1.14	1.02	1.00
.50	4	43.55	19.11	11.42	7.97	4.78	3.21	2.27	1.67	1.33	1.14	1.02	1.00
1.00	4	51.39	21.22	11.34	7.34	4.14	2.81	2.07	1.60	1.31	1.14	1.02	1.00
.25	5	38.18	18.31	11.84	8.72	5.66	4.08	3.02	2.25	1.70	1.35	1.06	1.00
.50	5	39.90	17.39	10.54	7.50	4.74	3.43	2.62	2.04	1.61	1.32	1.06	1.00
1.00	5	47.72	19.45	10.57	6.96	4.06	2.87	2.22	1.80	1.50	1.29	1.06	1.01
.25	6	37.83	18.04	11.68	8.64	5.73	4.32	3.46	2.85	2.34	1.89	1.27	1.05
.50	6	39.40	17.21	10.48	7.49	4.80	3.57	2.86	2.36	1.98	1.69	1.26	1.05
1.00	6	46.48	18.94	10.41	6.92	4.10	2.95	2.35	1.99	1.74	1.52	1.17	1.02
Schemes based on $\phi_{bs}(\cdot)$													
.25	4	45.66	20.36	12.02	8.18	4.60	2.97	2.10	1.60	1.31	1.14	1.02	1.00
.50	4	46.61	19.94	11.41	7.68	4.35	2.86	2.06	1.59	1.31	1.14	1.02	1.00
1.00	4	51.28	21.25	11.37	7.32	4.05	2.71	1.99	1.57	1.30	1.14	1.02	1.00
.25	5	39.43	18.28	11.42	8.13	4.93	3.38	2.48	1.91	1.55	1.30	1.06	1.01
.50	5	40.53	17.64	10.59	7.43	4.52	3.15	2.37	1.87	1.53	1.30	1.06	1.01
1.00	5	45.99	18.81	10.41	6.95	4.08	2.88	2.22	1.80	1.51	1.29	1.06	1.01
.25	6	38.03	18.14	11.66	8.53	5.47	3.95	3.04	2.44	2.01	1.7	1.26	1.05
.50	6	39.26	17.24	10.52	7.51	4.76	3.48	2.74	2.26	1.92	1.67	1.26	1.05
1.00	6	46.36	18.90	10.40	6.92	4.10	2.94	2.33	1.95	1.68	1.46	1.13	1.02
Schemes based on $\phi_{cb}(\cdot)$													
.25	4	41.92	20.53	13.10	9.41	5.63	3.62	2.41	1.7	1.32	1.13	1.01	1.00
.50	4	43.66	19.38	11.63	8.10	4.76	3.14	2.19	1.63	1.31	1.13	1.02	1.00
1.00	4	51.86	21.50	11.46	7.36	4.07	2.71	1.98	1.55	1.29	1.13	1.02	1.00
.25	5	38.19	18.29	11.82	8.69	5.62	4.01	2.96	2.20	1.67	1.34	1.06	1.00
.50	5	39.55	17.39	10.60	7.55	4.73	3.37	2.54	1.97	1.57	1.31	1.06	1.01
1.00	5	46.49	18.97	10.44	6.93	4.07	2.87	2.21	1.79	1.50	1.29	1.06	1.00
.25	6	37.82	17.99	11.62	8.59	5.69	4.27	3.40	2.78	2.28	1.85	1.27	1.05
.50	6	39.36	17.21	10.49	7.50	4.81	3.58	2.86	2.36	1.98	1.69	1.26	1.05
1.00	6	46.53	18.96	10.41	6.92	4.10	2.95	2.35	1.99	1.74	1.52	1.17	1.02

NOTE: The ARL values are computed assuming that $\sigma = 1$, where σ denotes the standard deviation of the process. In general, the same value is attained for a shift of $\mu\sigma$.

Table 7. ARLs of Some AEWMA Control Charts (In-Control ARL Equal to 500)

		μ											
μ_1	μ_2	.25	.50	.75	1.00	1.50	2.00	2.50	3.00	3.50	4.00	5.00	6.00
Schemes based on $\phi_{hu}(\cdot)$													
.25	4	98.51	40.94	25.04	17.59	10.11	6.08	3.66	2.29	1.60	1.26	1.04	1.00
.50	4	114.91	36.40	19.92	13.43	7.71	4.93	3.24	2.19	1.58	1.26	1.04	1.00
1.00	4	168.13	44.86	19.57	11.63	6.13	3.98	2.78	2.03	1.55	1.26	1.04	1.00
.25	5	77.26	33.01	20.68	14.96	9.39	6.44	4.43	2.98	2.04	1.49	1.08	1.01
.50	5	86.03	29.70	16.97	11.77	7.20	5.01	3.62	2.63	1.92	1.47	1.08	1.01
1.00	5	130.6	36.25	16.85	10.38	5.74	3.92	2.92	2.25	1.76	1.42	1.08	1.01
.25	6	74.01	30.88	19.26	13.99	9.07	6.69	5.19	4.04	3.08	2.28	1.33	1.05
.50	6	82.83	28.60	16.36	11.40	7.12	5.19	4.06	3.24	2.58	2.04	1.32	1.05
1.00	6	120.37	33.96	16.19	10.15	5.75	4.04	3.13	2.54	2.09	1.74	1.27	1.05
Schemes based on $\phi_{bs}(\cdot)$													
.25	4	135.01	42.72	21.99	13.91	7.12	4.25	2.80	2.01	1.55	1.28	1.05	1.00
.50	4	139.25	41.21	20.28	12.69	6.59	4.05	2.73	1.99	1.55	1.28	1.05	1.00
1.00	4	163.70	45.52	19.97	11.71	5.94	3.74	2.60	1.94	1.53	1.28	1.05	1.00
.25	5	99.83	35.86	20.20	13.43	7.27	4.50	3.03	2.19	1.69	1.38	1.08	1.01
.50	5	105.87	33.70	17.95	11.78	6.50	4.16	2.89	2.14	1.67	1.37	1.08	1.01
1.00	5	147.68	40.94	18.21	10.79	5.62	3.66	2.65	2.03	1.63	1.36	1.08	1.01
.25	6	78.04	31.56	19.10	13.39	7.95	5.32	3.80	2.84	2.22	1.79	1.27	1.05
.50	6	85.95	29.26	16.47	11.26	6.70	4.61	3.41	2.64	2.12	1.75	1.27	1.05
1.00	6	122.94	34.60	16.37	10.18	5.67	3.90	2.96	2.38	1.98	1.69	1.27	1.05
Schemes based on $\phi_{cub}(\cdot)$													
.25	4	97.03	41.54	25.68	18.16	10.52	6.36	3.80	2.35	1.62	1.27	1.04	1.00
.50	4	112.97	35.81	19.48	13.04	7.38	4.67	3.08	2.12	1.57	1.27	1.04	1.00
1.00	4	155.51	42.24	19.01	11.48	6.06	3.89	2.70	1.98	1.54	1.27	1.05	1.00
.25	5	77.27	32.85	20.54	14.86	9.33	6.41	4.43	2.99	2.04	1.50	1.08	1.01
.50	5	85.99	29.76	17.02	11.80	7.20	4.99	3.59	2.59	1.90	1.46	1.08	1.01
1.00	5	128.25	35.76	16.77	10.39	5.73	3.88	2.84	2.17	1.71	1.39	1.08	1.01
.25	6	73.97	31.29	19.62	14.30	9.29	6.86	5.33	4.15	3.15	2.32	1.34	1.05
.50	6	82.41	28.60	16.41	11.44	7.14	5.19	4.03	3.19	2.52	1.98	1.30	1.05
1.00	6	119.45	33.76	16.16	10.16	5.77	4.06	3.15	2.56	2.10	1.75	1.27	1.05

NOTE: The ARL values are computed assuming that $\sigma = 1$, where σ denotes the standard deviation of the process. In general, the same value is attained for a shift of $\mu\sigma$.

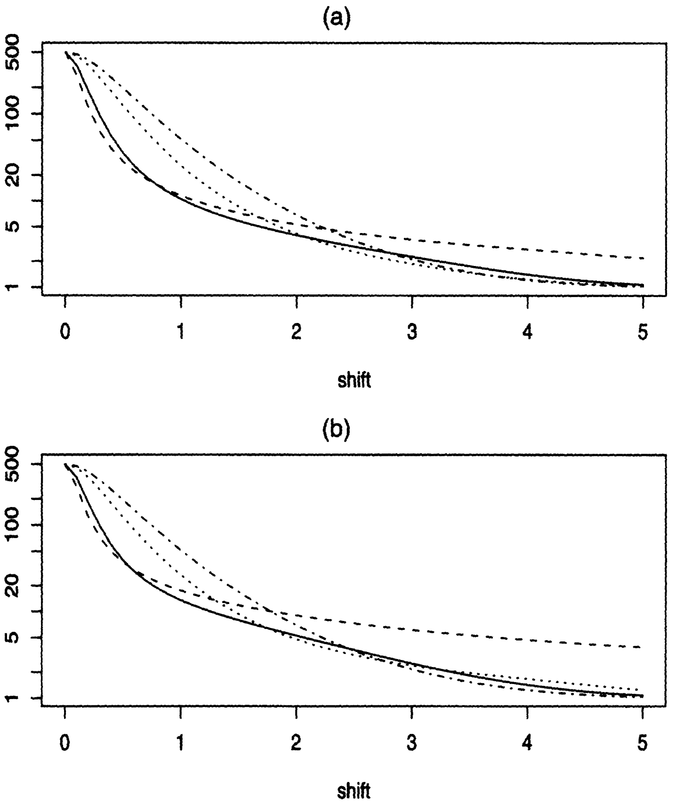

Figure 3. ARL (a) and Worst-Case ARL (b) Comparisons Between AEWMA and EWMA Control Schemes: —, Optimal AEWMA for $\mu_1 = 1$ and $\mu_2 = 5$; - - -, Optimal EWMA for $\mu = .5$; · · · · ·, Optimal EWMA for $\mu = 3$; · — · —, Optimal EWMA for $\mu = 5$.

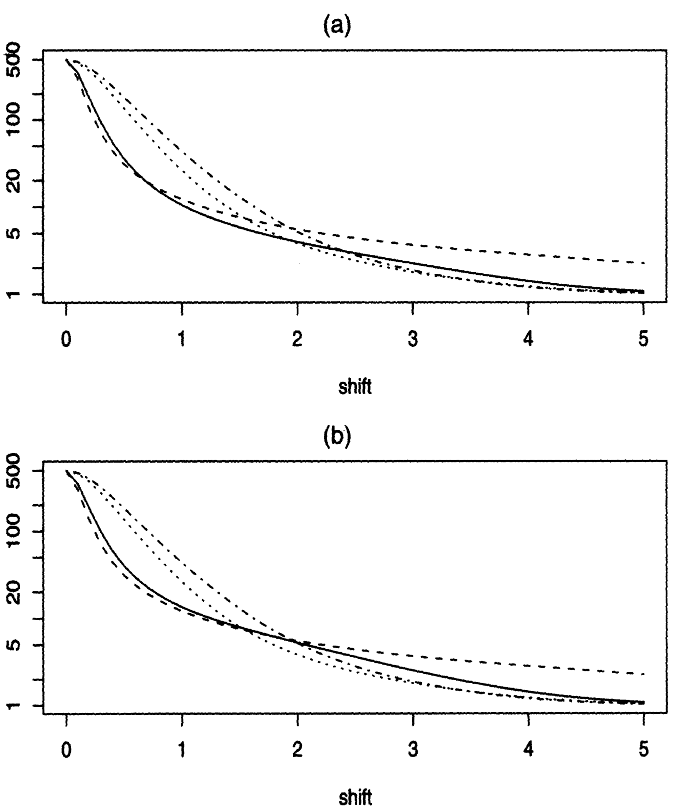

Figure 4. ARL (a) and Worst-Case ARL (b) Comparisons Between AEWMA and CUSUM Control Schemes: —, Optimal AEWMA for $\mu_1 = 1$ and $\mu_2 = 5$; - - -, Optimal CUSUM for $\mu = .5$; · · · · ·, Optimal CUSUM for $\mu = 3$; · — · —, Optimal CUSUM for $\mu = 5$.

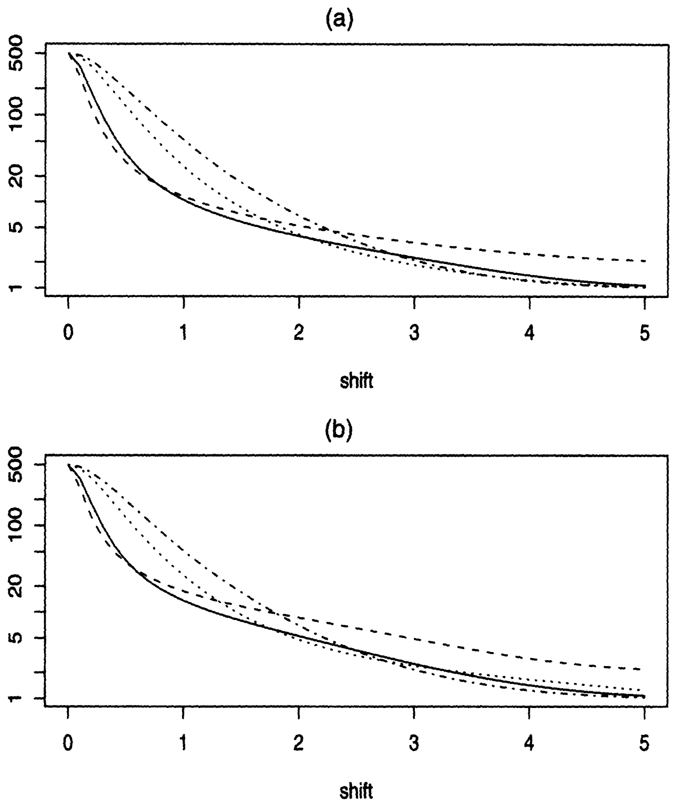

Figure 5. ARL (a) and Worst-Case ARL (b) Comparisons Between AEWMA and Combined Shewhart EWMA Schemes (CSEWMA): —, Optimal AEWMA for $\mu_1 = 1$ and $\mu_2 = 5$; - - -, Optimal CSEWMA for $\mu = .5$; · · · · ·, Optimal CSEWMA for $\mu = 3$; · — · —, Optimal CSEWMA for $\mu = 5$.

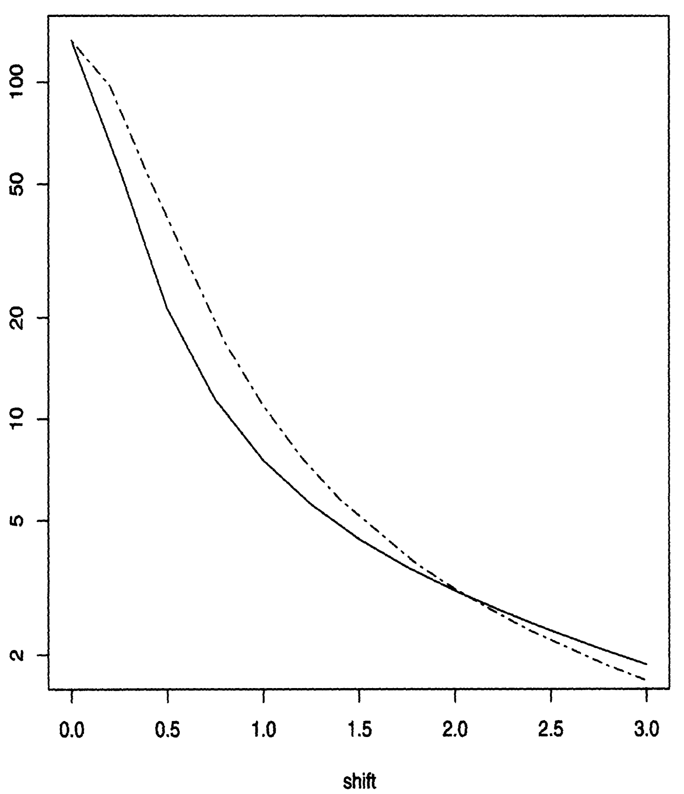

Figure 6. ARL Comparisons Between AEWMA (–) and One Shewhart Chart With Supplementary Run Rules (· — · —).

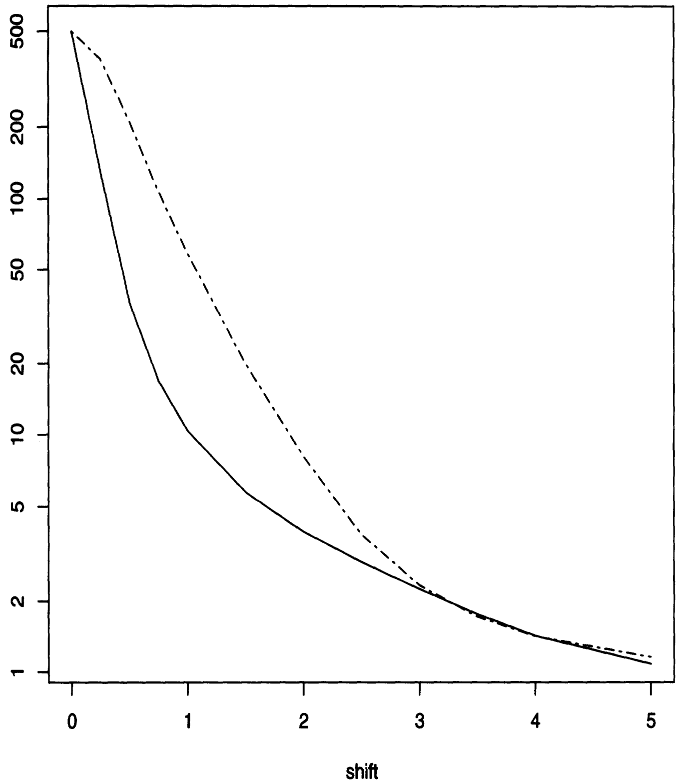

Figure 7. ARL Comparison Between AEWMA (—) and Pantazopoulos and Pappis's (1996) Schemes (· – · –).

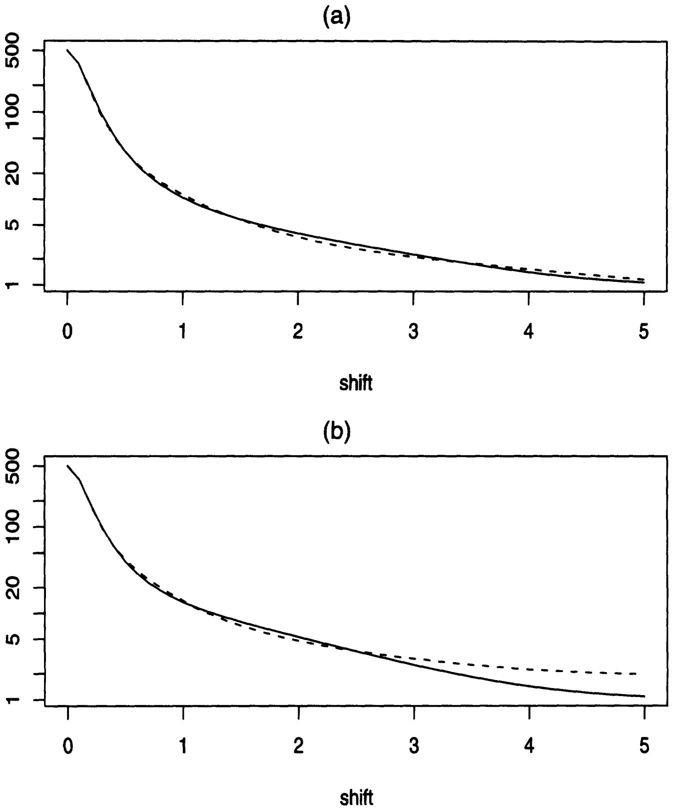

Figure 8. ARL (a) and Worst-Case ARL (b) Comparisons Between AEWMA and Two Simultaneous EWMAs (SEWMA). (—, Optimal AEWMA for $\mu_1 = 1$ and $\mu_2 = 5$; - - -, Simultaneous EWMAs).

Pantazopoulos and Pappis showed that their proposed method gives quite accurate forecasts in comparison with other known methods. If $|\hat{x}_{t+1}| > h$ the scheme declares an alarm. Simulation is used to obtain the value of h that gives an in-control ARL of 500.

Finally, we design a scheme, denoted by SEWMA, based on two EWMA charts: $x_t(\lambda_1) = (1 - \lambda_1)x_{t-1}(\lambda_1) + \lambda_1 y_t$ and $x_t(\lambda_2) = (1 - \lambda_2)x_{t-1}(\lambda_2) + \lambda_2 y_t$. We choose the smoothing constants equal to the optimal values for detecting $\mu = .5$ and $\mu = 2$, that is, $\lambda_1 = .05$ and $\lambda_2 = .36$ (Lucas and Saccucci 1990). A signal is given whenever $|x_t(\lambda_l)| > h_l$, for at least one l, with $h_l = h\sigma\sqrt{\lambda_l/(2 - \lambda_l)}$, $l = 1, 2$. Simulation is used to establish the h value, giving an in-control ARL equal to 500 when the two EWMAs are running simultaneously. The ARL and the worst-case ARL of the SEWMA chart, evaluated by simulating 10,000 run lengths for different values of the shift ranging from 0 to 5, are shown in Figure 8.

Figures 3–8 point to a very good performance of the AEWMA scheme. In particular, observe that the AEWMA ARL and worst-case ARL are either the shortest or near the shortest for every value of the shift, no other scheme offers a comparable protection when both small and large shifts are to be detected; and with the exception of only the SEWMA chart, it seems to be the best chart at detecting intermediate-sized shifts.

6. CONCLUSIONS

We have presented a new class of control charts for monitoring the mean of an independent sequence of random variables. The suggested schemes seem able to combine two properties:

1. Simplicity. The proposed charts require only one plot with a clear-cut explanation. Hence they are easy to explain to users, because their interpretation and operational aspects are similar to those of EWMA and Shewhart charts.
2. Efficiency. It is possible to design a single AEWMA control chart that gives good protection against a wide range of shifts. This adds an extra level of simplicity to the suggested schemes, because it makes it easy to choose a control scheme for a particular application.

In addition, the underlying idea can be used in other cases of practical importance, such as control charts for nonnormal or multivariate data.

Finally, we mention two specific points that seem worthy of further investigation. Varying limits can be used during the initial phase to adjust for the varying variances of the control statistic. Some results (MacGregor and Harris 1990), given for a standard EWMA chart, suggest a possibly large improvement. In addition the problem of the "optimal" choice of the score functions needs additional investigation.

APPENDIX A: COMPUTATION OF THE ARL OF AN AEWMA CHART

The ARL of an AEWMA scheme can be approximated using the approach described by Lucas and Saccucci (1990) for an EWMA control chart; that is, by discretizing the infinite-state transition probability matrix of the continuous-state Markov chain defined by (2) (see also Brook and Evans 1972).

The procedure involves dividing the interval between the upper and the lower control limits in an odd number m of subintervals of width $\delta = 2h/m$. The control statistic, x_t, is said to be in the transient state i, at time t, if $v_i - \delta/2 < x_t \le v_i + \delta/2$, where v_i denotes the midpoint of the ith interval I_i. In addition, x_t falls in an absorbing state when $|x_t - \eta_0| > h$. The transition probability matrix, represented in partitioned matrix form, is given by

$$\Pr = \begin{pmatrix} \mathbf{R} & (\mathbf{I}-\mathbf{R})\mathbf{u} \\ 0 & 1 \end{pmatrix},$$

where the $m \times m$ submatrix $\mathbf{R}$ contains the probabilities r_{ij} of going from one transient state, i, to another, j, $\mathbf{I}$ is the identity matrix, and $\mathbf{u}$ is a column vector of 1. The elements of the vector $(\mathbf{I}-\mathbf{R})\mathbf{u}$ are the probabilities of jumping to the absorbing state. The probabilities r_{ij} are approximated by assuming that the control statistic is equal to v_i whenever it is in state i. This yields

$$\begin{aligned} r_{ij} &= \Pr\{v_i + \phi(y_t - v_i) \in I_j\} \\ &= \Pr\{v_j - v_i - \delta/2 < \phi(y_t - v_i) \le v_j - v_i + \delta/2\}. \end{aligned}$$

If $\phi(e)$ is not decreasing in e, then the transition probabilities are given by

$$r_{ij} = \Pr\{v_i + \phi^{-1}(v_j - v_i - \delta/2) < y_t \le v_i + \phi^{-1}(v_j - v_i + \delta/2)\}.$$

For the functions (3), (4), and (5), the inverse functions are given by

$$\phi_{hu}^{-1}(v) = \begin{cases} v - (1-\lambda)k & \text{if } v < -\lambda k \\ v/\lambda & \text{if } -\lambda k \le v \le \lambda k \\ v + (1-\lambda)k & \text{if } v > \lambda k \end{cases}$$

and

$$\phi_{bs}^{-1}(v) = \begin{cases} \phi_{bs}^{*}(v) & \text{if } |v| \le k \\ v & \text{otherwise,} \end{cases}$$

where $\phi_{bs}^{*}(v)$ is the unique real root, with absolute value less than k, of the polynomial $y - (1-\lambda)y\{1 - (y/k)^2\}^2 - v$ and

$$\phi_{cb}^{-1}(e) = \begin{cases} e & \text{if } e \le -p_1 \\ -\phi_{cb}^{*}(-e) & \text{if } -p_1 < e < -\lambda p_0 \\ e/\lambda & \text{if } |e| \le \lambda p_0 \\ \phi_{cb}^{*}(e) & \text{if } \lambda p_0 < e < -p_1 \\ e & \text{if } e \ge p_1, \end{cases}$$

where $\phi_{cb}^{*}(v)$ is the unique root of $\tilde{\phi}_{cb}(y) - v$ in the interval (p_0, p_1). Because these functions are strictly monotone, it easy to show that a root always exists and it is unique.

Let $z = (\mathbf{I}-\mathbf{R})^{-1}\mathbf{u}$. Then the $(m+1)/2$th element of z can be used as an approximation of the average time spent by $\{x_t\}$ to reach the absorbing state when $x_0 = v_{(m+1)/2} = \eta_0$ (i.e., as an approximation of the ARL) and the maximum of the elements of z can be used to approximate the worst-case ARL. In addition, other properties of the run-length distribution (e.g., higher-order moments) can be easily computed.

Some experiments with different values of m suggest that satisfactory results can be obtained by choosing m to be greater than 50. As an example, the following table shows the approximations to the in-control ARL of an AEWMA scheme computed using different values of m; the scheme considered is based on $\phi_{hu}(\cdot)$ with $\lambda = .1$, $k = 3$, and $h = .5$:

m	5	11	25	51	101
ARL	68.755	87.576	94.112	95.282	95.584
m	151	301	501	1001	
ARL	95.651	95.676	95.683	95.686	

In this article, ARL is computed using $m = 151$. Alternatively, the properties of the schemes can be evaluated by formulating and solving a system of integral equations along the lines of Crowder (1987b).

APPENDIX B: COMPUTATION OF THE OPTIMAL AEWMA PARAMETERS

Approximated solutions of the constrained optimization problem (6) can be obtained minimizing in θ the penalized function

$$\mathrm{ARL}(\mu_1, \theta) + \zeta\left(\left(\frac{\mathrm{ARL}(0,\theta) - B}{B}\right)^2 + I\{\mathrm{ARL}(\mu_2,\theta) \ge C\}\left(\frac{\mathrm{ARL}(\mu_2,\theta) - C}{C}\right)^2\right), \quad (9)$$

where $C = (1-\alpha)\mathrm{ARL}(\mu_2, \theta^*)$, ζ is a large constant (we used $\zeta = 10^8$), and $I\{A\}$ is equal to 1 if A is true and to 0 otherwise. To minimize (9), we used the Nelder–Mead simplex algorithm (Nelder and Mead 1965), initialized using the result of 200,000 iterations of a simulated annealing algorithm (e.g., Aarts and Van Laarhoven 1989) based on a Gaussian Markov kernel and a Boltzmann cooling schedule. The same approach has been used to compute θ^*.

ACKNOWLEDGMENTS

The authors are grateful to the editor, the previous editor, the associate editor, and two anonymous referees for their many comments and suggestions. This work was supported by MURST Cofin2000 grant 13/26.

[Received December 1995. Revised March 2003.]

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