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**A METAMODELING APPROACH FOR BIAS ESTIMATION OF
BIOLOGICAL REFERENCE POINTS**

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Abstract

A Metamodeling Approach for Bias estimation of Biological Reference Points

by

Nicholas Grunloh

Stock assessments often assume a two-parameter functional form (e.g., Beverton-Holt or Ricker) for the expected recruitment produced by a given level of spawning output. Mangel et al. [21] and others have shown that biological reference points such as $\frac{F^*}{M}$ and $\frac{B^*}{\bar{B}(0)}$ are largely determined by a single parameter (steepness) when using two-parameter relationships. These functions introduce strong correlations between reference points (RP) that are pre-determined by the functional form, rather than a biological characteristic of the stock. Mangel et al. note that use of a three-parameter stock-recruitment relationship allows for independent estimation of these reference points. This research seeks to understand the nature of biases in reference points resulting from fitting a two-parameter logistic functional form when the true relationship follows a three-parameter stock-recruitment relationship (SRR). This work demonstrates the useful limits of the misspecified Schaefer model, and the mechanisms of model failure which arise from mapping a three-dimensional parameter space into two dimensions.

To myself,
Perry H. Disdainful,
the only person worthy of my company.

Acknowledgments

I want to “thank” my committee, without whose ridiculous demands, I would have graduated so, so, very much faster.

¹ **Chapter 1**

² **Introduction**

3 *Generalize beyond production model*

4
5 The most fundamental model in modern fisheries management is the surplus-production
6 model. These models focus on modeling population growth via nonlinear paramet-
7 ric ordinary differential equations (ODE). Key management quantities called reference
8 points (RPs) are commonly derived from the ODE equilibrium equations and depend
9 upon the parameterization of biomass production. Two-parameter forms of the pro-
10 duction function have been shown to limit the theoretical domain of RPs [21]. The
11 limited RP-space of two-parameter models makes these models vulnerable to model
12 misspecification with respect to RPs, and thus the limiting structure of two-parameter
13 models may in and of itself induce bias in RP estimation using these models. The be-
14 havior of RP estimation is not well understood and as a result patterns of bias in RP
15 estimation may easily go unnoticed. A metamodeling approach is developed here to
16 describe RP biases and explore mechanisms of model failure under the most common
17 two-parameter models.

18 Data for a typical surplus-production model comes in the form of an index of abun-
19 dance through time which is assumed to be proportional to the reproducing biomass
20 for the modelled population that is vulnerable to fishing. The index is often observed
21 alongside a variety of other known quantities, but at a minimum, each index will be
22 observed in the presence of some known catch for the period. Figure (1.1) shows the
23 classic Namibian Hake dataset [25, 16, 20] exemplifying the form.

Indices are assumed to have multiplicative log-normal errors, and thus the following
observation model arises naturally,

$$I_t = qB_t e^{\varepsilon} \quad \varepsilon \sim N(0, \sigma^2). \quad (1.1)$$

24 Above q is often referred to as the “catchability parameter”; it serves as the propor-

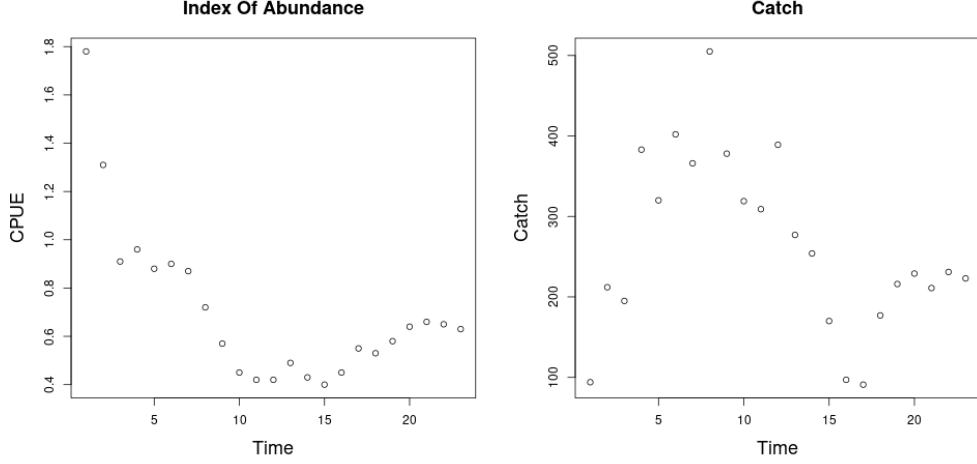


Figure 1.1: *left*: An index of abundance data, catch per unit effort (CPUE), for Namibian Hake from 1965 to 1987. *right*: The associated catch data for Namibian Hake over the same time period.

tionality constant mapping between the observed index of abundance and biomass. σ^2 models residual variation. Biologically speaking q and σ^2 are often treated as nuisance parameters with the “biological parameters” entering the model through a process model on biomass.

Biomass is assumed to evolve as an ODE; in this case I focus on the following form

$$\frac{dB}{dt} = P(B(t); \theta) - Z(t)B(t). \quad (1.2)$$

Here biomass is assumed to change in time by two processes, net production of biomass into the population, $P(B)$, and various sources of biomass removal, Z , from the population.

Firstly, the population grows through a production function, $P(B)$. Production in this setting is defined as the net biomass increase due to all reproduction and maturation processes. The production function is assumed to be a parametric (generally non-linear) function relating the current biomass of the population to an aggregate production of

37 biomass.

38 Secondly, the population decreases as biomass is removed by various sources that
39 are assumed to remove biomass linearly with biomass. Above, $Z(t)$, is an aggregate
40 rate of removal. When the fishing rate, $F(t)$, is the only source of removal $Z(t) = F(t)$,
41 however some models will also included other linear terms in $Z(t)$. Commonly the rate
42 of “natural mortality”, M , is also included as an additional term so that $Z(t) = M + F(t)$.

43 From a management perspective a major goal of modeling is to accurately infer
44 a quantity known as *maximum sustainable yield* (MSY). One could maximize simple
45 yield at a particular moment in time (and only for that moment) by fishing all available
46 biomass in that moment. This strategy is penny-wise but pound-foolish (not to mention
47 ecologically devastating) since it doesn’t leave biomass in the population to reproduce
48 in the future. We seek to fish in a way that allows (or even encourages) future produc-
49 tivity in the population. This is accomplished by maximizing the equilibrium level of
50 catch over time. Equilibrium yield is considered by replacing the steady state biomass
51 (\bar{B}) in the assumed form for catch, so that $\bar{Y} = F\bar{B}(F)$, where $\bar{\cdot}$ indicates a value at
52 steady state. MSY is found by maximizing $\bar{Y}(F)$ with respect to F , and F^* is the fish-
53 ing rate at MSY. Going forward let $*$ decorate any value derived under the condition of
54 MSY.

55 Fisheries are very often managed based upon reference points which serve as sim-
56 plified heuristic measures of population behavior. The mathematical form of RPs de-
57 pends upon the model assumptions through the production function. While a number
58 of different RPs exist which describe the population in different (but related) ways, the
59 most common RPs revolve around the concept of MSY (or robust ways of measuring
60 MSY [15, 26]). Here the focus is primarily on the RPs $\frac{B^*}{\bar{B}(0)}$ and F^* ($\frac{F^*}{M}$ when appropri-
61 ate) for their pervasive use in modern fisheries [27].

62 F^* is the afore mentioned fishing rate which results in MSY. $\frac{B^*}{\bar{B}(0)}$ is the depletion
63 of the stock at MSY. That is to say $\frac{B^*}{\bar{B}(0)}$ describes the fraction of the unfished popu-

64 lation biomass that will remain in the equilibrium at MSY. In general $F^* \in \mathbb{R}^+$ and
65 $\frac{B^*}{B(0)} \in (0, 1)$, however under the assumption of two-parameter production, models will
66 be structurally unable to capture the full theoretical range of RPs.

67 Many of the most commonly used production functions depend only on two-parameters.
68 For example, the Schaefer model depends only on the biological parameters r and
69 K , and limits RP inference so that under the Schaefer model $\left(F^*, \frac{B^*}{B(0)}\right) \in (\mathbb{R}^+, \frac{1}{2})$.
70 The two-parameter Fox model [11] limits $\left(F^*, \frac{B^*}{B(0)}\right) \in (\mathbb{R}^+, \frac{1}{e})$. Similarly the two-
71 parameter Cushing [6], Beverton-Holt [2, BH] and Ricker [31] production functions do
72 not model the full theoretical space of RPs [21, 42].

73 The bias-variance trade-off [29] makes it clear that the addition of a third parameter
74 in the production function will necessarily reduce estimation bias. However the utility
75 of this bias reduction is still under debate because the particular mechanisms and be-
76 havior (direction and magnitude) of these biases for key management quantities are not
77 fully understood or described. Lee et al. [18] provides some evidence that estimation
78 of productivity parameters are dependent on changes in biomass trend through time
79 (i.e. contrast) as well as model specification. Conn et al. [4] comes to similar conclu-
80 sions via calibration modeling techniques. These studies indicate important factors that
81 contribute to inferential failure. However they do not offer mechanisms of model fail-
82 ure, nor do their experimental designs allow for the control of different types of model
83 misspecification.

84 In this study I consider the behavior of inference when index data are simulated
85 from three-parameter PT and Schnute production models, but the simulated data are fit
86 using intentionally misspecified two-parameter logistic or BH production models. The
87 work begins with a derivation of RPs under the three-parameter models. A method is
88 then presented for generating simulation designs based on the parametric form of RPs
89 which serves as a control on the nature of simulated model misspecification. Finally a
90 Gaussian Process (GP) metamodel [12] is constructed for exploration and analysis of

91 RP biases.

92 A key insight of this approach is that bias is considered broadly across RP-space to
93 uncover patterns and correlations between RPs. The GP metamodel is explicit about
94 trade-offs between RPs so as to inform the full utility of reducing bias, as well as to
95 suggest mechanisms for understanding what causes bias. Further, the effect of contrast
96 on estimation is considered together with model misspecification.

⁹⁷ **Chapter 2**

⁹⁸ **Pella-Tomlinson Model**

99 2.1 Introduction

100 2.2 Methods

101 2.2.1 Model

The three-parameter Pella-Tomlinson (PT) family has a convenient form that includes, among others [11, 30], the logistic production function as a special case. PT production function is parameterized so that $\theta = [r, K, \gamma]$ and the family takes the following form,

$$P_p(B; [r, K, \gamma]) = \frac{rB}{\gamma - 1} \left(1 - \left(\frac{B}{K} \right)^{(\gamma-1)} \right). \quad (2.1)$$

102 γ is a parameter which breaks PT out of
 103 the restrictive symmetry of the logistic curve.
 104 In general $\gamma \in (1, \infty)$, with the logistic model
 105 appearing in the special case of $\gamma = 2$, and
 106 the Fox model appearing as a limiting case as
 107 $\gamma \rightarrow 1$. The parameter r controls the maxi-
 108 mum per-capita growth rate of the population
 109 in the absence of competition for resources
 110 (i.e. the slope of production function at the
 111 origin). K is the so called "carrying capac-
 112 ity" of the population. In this context the
 113 carrying capacity can be formally stated as
 114 steady state biomass in the absence of fishing
 115 (i.e. $\bar{B}(0) = K$). In Figure (4.1) PT production
 116 is shown for a range of parameter values so as

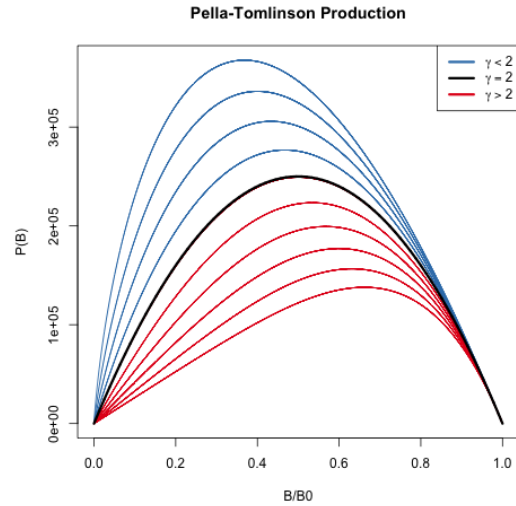


Figure 2.1: The Pella-Tomlinson production function plotted across a variety of parameter values. The special cases of Logistic production is shown in black, and the left-leaning and right-leaning regimes are shown in blue and red respectively.

117 to demonstrate the various productivity shapes that can be achieved under PT.

118 While the form of the PT curve produces some limitations [10], importantly the
 119 introduction of a third parameter allows enough flexibility to fully describe the space of
 120 reference points used in management. To see this, the reference points are analytically
 121 derived for the PT model below.

122 2.2.2 Reference Points

123 With $B(t)$ representing biomass at time t , under PT production, the dynamics of
 124 biomass are defined by the following ODE,

$$\frac{dB}{dt} = \frac{rB}{\gamma-1} \left(1 - \left(\frac{B}{K} \right)^{\gamma-1} \right) - FB. \quad (2.2)$$

An expression for the equilibrium biomass is attained by setting Eq (2.2) equal to zero, and rearranging the resulting equation to solve for B . Thinking of the result as a function of F gives,

$$\bar{B}(F) = K \left(1 - \frac{F(\gamma-1)}{r} \right)^{\frac{1}{\gamma-1}}. \quad (2.3)$$

125 At this point it is convenient to notice that $\bar{B}(0) = K$. The expression for B^* is given
 126 by evaluating Eq (2.3) at F^* . To get an expression for F^* , the equilibrium yield is
 127 maximized with respect to F ,

$$F^* = \operatorname{argmax}_F F \bar{B}(F). \quad (2.4)$$

In the case of PT production this maximization can be done analytically, by differenti-

ating the equilibrium yield with respect to F as follows,

$$\frac{d\bar{Y}}{dF} = \bar{B}(F) + F \frac{d\bar{B}}{dF} \quad (2.5)$$

$$\frac{d\bar{B}}{dF} = -\frac{K}{r} \left(1 - \frac{F(\gamma-1)}{r} \right)^{\frac{1}{\gamma-1}-1}. \quad (2.6)$$

Setting Eq (2.5) equal to 0, substituting $\bar{B}(F)$ and $\frac{d\bar{B}}{dF}$ by Equations (2.3) and (2.6) respectively, and solving for F produces the following expression for the fishing rate required to produce MSY,

$$F^* = \frac{r}{\gamma} \quad (2.7)$$

Plugging the above expression for F^* back into Eq (2.3) gives the following expression for biomass at MSY,

$$B^* = K \left(\frac{1}{\gamma} \right)^{\frac{1}{\gamma-1}}. \quad (2.8)$$

The above derived expressions for $\bar{B}(0)$, B^* , and F^* can then be used to build a specific analytical form for the biological reference points in terms of only productivity parameters.

$$F^* = \frac{r}{\gamma} \quad \frac{B^*}{\bar{B}(0)} = \left(\frac{1}{\gamma} \right)^{\frac{1}{\gamma-1}} \quad (2.9)$$

128 2.2.3 Simulation

Generating simulated indices of abundance from the PT model requires inverting the relationship between $\left(F^*, \frac{B^*}{\bar{B}(0)} \right)$, and (r, γ) . It is not generally possible to analytically invert this relationship for many three-parameter production functions [27, 34]. Most three-parameter production functions lead to RPs that require expensive numeri-

cal methods to invert; more over the numerical inversion procedure can often be unstable. That said, for the case of PT this relationship is analytically invertible, and leads to the following relationship

$$r = \gamma F^* \quad \gamma = \frac{W\left(\frac{B^*}{\bar{B}(0)} \log\left(\frac{B^*}{\bar{B}(0)}\right)\right)}{\log\left(\frac{B^*}{\bar{B}(0)}\right)}. \quad (2.10)$$

129 Above W is the Lambert product logarithm function. More details about this derivation,
130 and the Lambert product logarithm, are given in Appendix (A).

131 Using Eq. (2.10) to obtain production parameters, a PT production model can be
132 fully defined for any combination of the RPs F^* and $\frac{B^*}{\bar{B}(0)}$. Since K does not enter the
133 RP calculation its value is fixed arbitrarily at 10000.

134 Indices of abundance are simulated from the three-parameter PT production model
135 broadly over the space of F^* and $\frac{B^*}{\bar{B}(0)}$ via a space filling design as described in Section
136 (3.2.3). A small amount of residual variation, $\sigma = 0.01$, is added to the simulated index,
137 and these data are then fit with a Schaefer model, at various degrees of misspecification,
138 so as to observe the effect of productivity model misspecification upon RP inference.

139 **2.2.4 Design**

140 Letting \mathcal{F} and \mathcal{B} be regular grids, of size $n = 100$, on $F^* \in (0.1, 0.7)$ and $\frac{B^*}{\bar{B}_0} \in (0.2, 0.6)$
141 respectively, a LHS design of size 100 is collected among the cells produced by $\mathcal{F} \times \mathcal{B}$.

142 Each of the sampled LHS design locations represent a unique PT model with the
143 sampled RP values. Since the relationship mapping RPs analytically to productivity
144 parameters can be found for the PT model, LHS designs the the PT model are computed
145 directly in RP space and Eq. (2.10) is used to map the sampled RP design locations to
146 PT productivity parameters.

147 2.2.5 Gaussian Process Metamodel

148 At its core, a metamodel is simply a model of some mapping of inputs to out-
149 puts (the mapping itself is typically defined by a computer model). By modeling the
150 mapping with a statistical model (that explicitly defines the relevant features of the
151 mapping) a metamodel defines a specific ontology for the mapping. By simulating ex-
152 amples of the mapping, the inferential infrastructure of the statistical model is used to
153 empirically learn an effective emulation of the mapping within the ontology defined by
154 the statistical model. The predictive infrastructure of the statistical model is then use-
155 ful as an approximate abstraction of the system itself to better understand the system
156 through further data collection, cheap approximation of the mapping, and/or study of
157 the mapping itself.

158 In this setting, the aim of metamodeling is to study how well RPs are inferred when
159 typical two-parameter models of productivity (Logistic and BH) are misspecified for
160 populations that are actually driven by more complicated dynamics. The simulation
161 design, \mathbf{X} , provides a sample of different population dynamics that are driven by three-
162 parameter production functions broadly in RP space. By simulating index of abundance
163 data from the three parameter model, and fitting those data with the two-parameter pro-
164 duction model, we observe particular instances of how well RPs are inferred at the
165 given misspecification of the two-parameter model relative to the true three-parameter
166 production model. By gathering all of the simulated instances of how RPs are inferred
167 (under the two-parameter model), we form a set of example mappings to train a meta-
168 model which represents the mapping of true RPs (under the three-parameter model) to
169 estimates of RPs under the misspecified two-parameter production model. The meta-
170 model is essentially a surrogate for inference under the misspecified two-parameter
171 production model that controls for the specific degree of model misspecification.

172 A flexible GP model is assumed for the structure of the metamodel to describe
173 the mapping of RPs under misspecified two-parameter models of productivity. A GP

174 is a stochastic process generalizing the multivariate normal distribution to an infinite
 175 dimensional analog. GP models are often specified primarily through the choice of a
 176 covariance (or correlation) function which defines the relationship between locations
 177 in the input space. Typically correlation functions are specified so that points closely
 178 related in space result in correlated effects in the model. In this setting the inputs to
 179 the GP metamodel are the space of reference points which define the simulated three-
 180 parameter production models.

While index of abundance data are generated from three-parameter models, at each design location of the simulation, fitting the restricted two-parameter model results in a maximum likelihood estimate (MLE; and associated estimation uncertainty) of each of the productivity parameters (i.e. Schaefer:[$\log(r)$, $\log(K)$], BH:[$\log(\alpha)$, $\log(\beta)$]). To simplify the specification of the metamodel, let \mathbf{y} be a vector collecting the fitted MLEs for one of the productivity parameters, and let $\boldsymbol{\omega}$ be a vector of estimates of the estimator variances (via the inverted Fisher information) at each \mathbf{y} . Each of the fitted productivity parameter estimates are then modeled using independent instances of the following GP metamodel.

$$\begin{aligned}\mathbf{y} &= \beta_0 + \mathbf{X}\boldsymbol{\beta} + \mathbf{v} + \boldsymbol{\epsilon} \\ \mathbf{v} &\sim N_n(\mathbf{0}, \tau^2 \mathbf{R}_\ell) \\ \boldsymbol{\epsilon} &\sim N_n(\mathbf{0}, \boldsymbol{\omega}' \mathbf{I})\end{aligned}\tag{2.11}$$

181 \mathbf{X} is the $n \times 2$ LHS design matrix of RPs for each simulated three-parameter data
 182 generating model as described in Section (3.2.4.1). $\boldsymbol{\epsilon}$ models independent normally
 183 distributed error, which provides an ideal mechanism for propagating uncertainty from
 184 inference in the simulation step into the metamodel. By matching each y_i with an ob-
 185 served ω_i variance term, $\boldsymbol{\epsilon}$ serves to down weight the influence of each y_i in proportion
 186 to the inferred production model sampling distribution uncertainty. This has the effect

187 of smoothing the GP model in a way similar to the nugget effect [13], although the
 188 application here models this effect heterogeneously.

The term, \mathbf{v} , contains spatially correlated GP effects. The correlation matrix, \mathbf{R}_ℓ describes how RPs close together in the simulation design are more correlated than those that are far away. This spatial effect is modeled with a squared exponential correlation function,

$$R(\mathbf{x}, \tilde{\mathbf{x}}) = \exp \left(\sum_{i=1}^2 \frac{-(x_i - \tilde{x}_i)^2}{2\ell_j^2} \right). \quad (2.12)$$

189 R has an anisotropic separable form which allows for differing length scales, ℓ_1
 190 and ℓ_2 , in the different RP axes. The flexibility to model correlations separately in
 191 the different RP axes is key due to the differences in the extent of the RP domains
 192 marginally. The metamodel parameters β_0 , β , τ^2 , ℓ_1 and ℓ_2 are fit via MLE against the
 193 observations \mathbf{y} , \mathbf{X} , and ω from simulation fits.

194 Fitting the metamodel allows for a full predictive description of inference under the
 195 misspecified restricted models. Predictive estimates are obtained via kriging [5]

$$\hat{y}(\mathbf{x}) = \beta_0 + \mathbf{x}\beta + \mathbf{r}(\mathbf{x})' \mathbf{R}_\ell^{-1} \left(\mathbf{y} - (\beta_0 + \mathbf{X}\beta) \right) \quad (2.13)$$

196 $\hat{y}(\mathbf{x})$ is the predicted value of the modeled productivity parameter MLE under the
 197 two-parameter production model, when the index of abundance is generated from the
 198 three-parameter production model at RP location \mathbf{x} . $\mathbf{r}(\mathbf{x})$ is a vector-valued function of
 199 correlation function evaluations for the predictive location \mathbf{x} against all observations in
 200 \mathbf{X} (i.e. $\mathbf{r}(\mathbf{x}) = \mathbf{R}(\mathbf{x}, \mathbf{x}_i) \forall \mathbf{x}_i \in \mathbf{X}$).

201 While metamodeling occurs on the inferred productivity parameters of the restricted
 202 production model, the metamodel can also be used to build estimates of major bio-
 203 logical RPs. For the BH model the relevant transformations for relating productivity

parameters with RPs are given in Eqs. (3.5, 3.8) with γ fixed to -1; for the Schaefer model $\hat{B}^* = \frac{\hat{K}}{2}$ and $\hat{F}^* = \frac{\hat{r}}{2}$. Applying the metamodel predictive surfaces on the scale of RP estimates allows for the quantification of estimation bias that is induced by fitting a misspecified two-parameter production model to indices of abundance generated under three-parameter productivity.

2.2.6 Catch

It is known that contrast in the observed index and catch time series can effect inference on the productivity parameters [14]. In this setting contrast refers to changes in the long term trends of index data. Figure (2.2, *right*) demonstrates an example of biomass that includes contrast induced by catch. It is not well understood how contrast may factor into inferential failure induced by model misspecification. Thus catch is parameterized so as to allow for a spectrum of possible contrast simulation settings.

Catch is parameterized so that $F(t)$ can be controlled with respect to F^* . Recall that catch is assumed to be proportional to biomass, so that $C(t) = F(t)B(t)$. To control $F(t)$ with respect to F^* , $C(t)$ is specified by defining the quantity $\frac{F(t)}{F^*}$ as the relative fishing rate. $B(t)$ is defined by the solution of the ODE, and F^* is defined by the biological parameters of the model. By defining $\frac{F(t)}{F^*}$, catch can then be written as $C(t) = F^* \left(\frac{F(t)}{F^*} \right) B(t)$.

Intuitively $\frac{F(t)}{F^*}$ describes the fraction of F^* that $F(t)$ is specified to for the current $B(t)$. When $\frac{F(t)}{F^*} = 1$, $F(t)$ will be held at F^* , and the solution of the ODE brings $B(t)$ into equilibrium at B^* . When $\frac{F(t)}{F^*}$ is held constant in time biomass comes to equilibrium as an exponential decay from K approaching B^* . When $\frac{F(t)}{F^*} < 1$, $F(t)$ is lower than F^* and $B(t)$ is pushed toward $\bar{B} > B^*$. Contrarily, when $\frac{F(t)}{F^*} > 1$, $F(t)$ is higher than F^* and $B(t)$ is pushed toward $\bar{B} < B^*$; the precise values of \bar{B} can be calculated from the steady state biomass equations provided above and depend upon the specific form of the production function.

For the simulations presented here, a family of fishing behaviors are considered where the fishing rate accelerates as technology and fishing techniques improve rapidly until management practices are applied, which ultimately brings fishing into equilibrium at F^* . This is parameterized as three distinct phases, over a total of 45 units of time, with each phase lasting 15 time units. The specific form is given below.

$$\frac{F(t)}{F^*} = ae^{bt}\mathbf{1}_{0 \leq t < 15} + (d - ct)\mathbf{1}_{15 \leq t < 30} + \mathbf{1}_{30 \leq t \leq 45} \quad (2.14)$$

The first term of Eq (2.14) is an exponential increase in fishing, the second term is a linear decline in relative fishing as initial management practices are applied, and the third term, $\mathbf{1}_{30 \leq t \leq 45}$, simply holds the fishing rate at F^* there after. These three phases are controlled by the four parameters a , b , c , and d . By enforcing that the interface of the phases meet at χ_{max} and 1 respectively the relative fishing series is reduced to a two-parameter family.

$$a = e^{\log(\chi_{max}) - 15b} \quad b = \frac{1}{t - 15} \log \left(\frac{\chi_{min}}{\chi_{max}} \right) \quad (2.15)$$

$$c = \frac{\chi_{max} - 1}{15 - 1} \quad d = 15c + \chi_{max} \quad (2.16)$$

230 By further specifying $\chi_{max} = 1.6^\chi$ and $\chi_{min} = 0.4^\chi$ the two-parameters χ_{max} , and χ_{min}
 231 can be reduced to the single parameter χ . The tuning parameter χ then singularly
 232 controls contrast that appears in time series data.

233 When $\chi = 0$, the relative fishing rate is a constant at 1 to create a low contrast
 234 simulation environment. As χ increases Eq (2.14) induces more and more contrast in
 235 the observed index and catch time series until $\chi = 1$ which produces a high contrast
 236 simulation environment. Figure (2.2) demonstrates a spectrum of contrast simulation
 237 environments as well as the time series data they induce in the solution of the production
 238 model ODE.

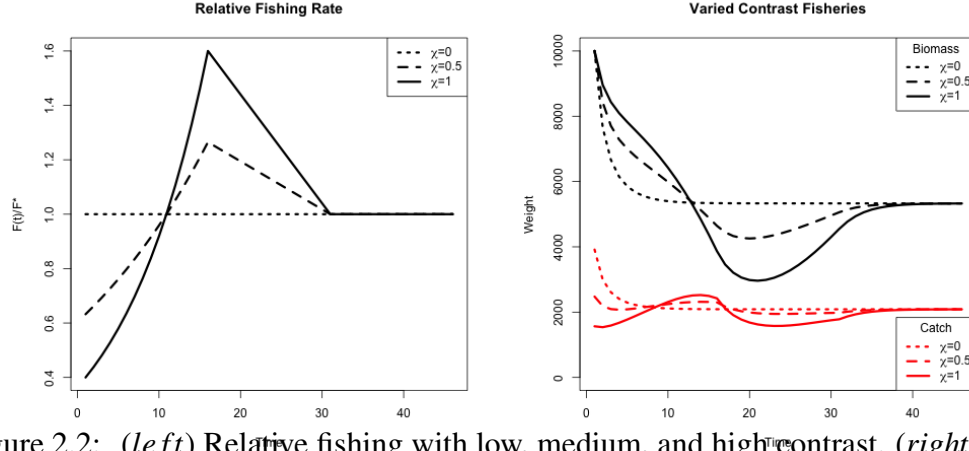


Figure 2.2: (left) Relative fishing with low, medium, and high contrast. (right) Population biomass and catch at each associated level of contrast.

2.2.7 Two-Parameter Production Model Inference

The simulated mapping results from fitting an intentionally misspecified two parameter production model to index of abundance data that are generated from a more complex three-parameter model of productivity. Thus, let I_t be an index of abundance simulated from the three-parameter PT or Schnute production models at time $t \in \{1, 2, 3, \dots, T\}$. However the fitted model is specified to be intentionally misspecified so that the fitted model is driven by a two-parameter Schaefer, or BH production model respectively.

The observation model for the fitted model is log-normal such that,

$$I_t | q, \sigma^2, \theta \sim LN(qB_t(\theta), \sigma^2). \quad (2.17)$$

$B_t(\theta)$ is defined by the solution of the ODEs defined by the Schaefer, or BH models. For the Schaefer model $\theta = [r, K]$, and for the BH model $\theta = [\alpha, \beta]$. From the perspective of the fitted model, the observed I_t are assumed independent conditional on q, σ^2, r, K and the two-parameter ODE model for biomass. Thus the log likelihood can be written

as

$$\log \mathcal{L}(q, \sigma^2, \theta; I) = -\frac{T}{2} \log(\sigma^2) - \frac{1}{2\sigma^2} \sum_t \log \left(\frac{I_t}{qB_t(\theta)} \right)^2. \quad (2.18)$$

247 In this setting, q is fixed at 0.0005 and M is fixed at 0.2, to focus on the inferential
248 effects of model misspecification on biological parameters. σ^2 and θ are reparameter-
249 ized to the log scale and fit via MLE. Reparameterizing the parameters to the log scale
250 improves the reliability of optimization, in addition to facilitating the use of Hessian
251 information for estimating MLE standard errors.

252 Given that the biological parameters enter the likelihood via a nonlinear ODE, and
253 further the parameters themselves are related to each other nonlinearly, the likelihood
254 function can often be difficult to optimize. A hybrid optimization scheme is used to
255 maximize the log likelihood to ensure that a global MLE solution is found. The R pack-
256 age GA [35, 36] is used to run a genetic algorithm to explore parameter space globally.
257 Optimization periodically jumps into the L-BFGS-B local optimizer to refine optima
258 within a local mode. The scheme functions by searching globally, with the genetic al-
259 gorithm, across many initial values for starting the local gradient-based optimizer. The
260 genetic algorithm serves to iteratively improve hot starts for the local gradient-based
261 optimizer. Additionally, optimization is only considered to be converged when the op-
262 timum results in an invertible Hessian at the found MLE.

263 **2.2.8 Continuous model formulation**

264 An important (and often overlooked) implementation detail is the solution to the
265 ODE which defines the progression of biomass through time. As a statistical model it
266 is of paramount importance that this ODE not only have a solution, but also that the
267 solution be unique. Of primary concern, uniqueness of the ODE solution is necessary
268 for well conditioned inference.

269 If the form of $\frac{dB}{dt}$ is at least Lipschitz continuous, then the Cauchy-Lipschitz-Picard
 270 theorem provides local existence and uniqueness of $B(t)$. Recall from Eq(1.2) that $\frac{dB}{dt}$ is
 271 separated into a term for biomass production, $P(B)$, and a term for removals, $Z(t)B(t)$.
 272 For determining Lipschitz continuity of $\frac{dB}{dt}$, the smallest Lipschitz constant of $\frac{dB}{dt}$ will
 273 be the sum of the constants for each of the terms $P(B)$ and $Z(t)B(t)$ separately. Typ-
 274 ically any choice of $P(B)$ will be continuously differentiable, which implies Lipschitz
 275 continuity. At a minimum $Z(t)$ typically contains fishing mortality as a function of time
 276 $F(t)$ to model catch in time as $C(t) = F(t)B(t)$. $Z(t)$ may or may not contain M , but
 277 typically M is modeled as stationary in time and does not pose a continuity issue, unlike
 278 some potential assumptions for $C(t)$.

279 In practice $C(t)$ is determined by a series of observed, assumed known, catches.
 280 Catch observations are typically observed on a quarterly basis, but in practice may not
 281 be complete for every quarter (or year) of the modeled period. It is overwhelmingly
 282 common to discretize the ODE in time via Euler's method with integration step sizes
 283 to match the observation frequency of the modeled data. This is often computationally
 284 convenient when the underlying species dynamics are reasonably well behaved, however
 285 when the dynamics model is used as a statistical model, with the goal of inferring
 286 the behavior of the underlying species dynamics, the regularity of the dynamics are
 287 not guaranteed. An implicit assumption of continuity of catch in time provides the
 288 necessary regularity for the statistical model. Furthermore a continuous handling of
 289 the dynamics provides improved accuracy in evaluating the ODE, particularly when
 290 inferring productivity parameters which largely control the regularity of the dynamics.

291 While there are many ways to handle catch continuity, here I assume that catches
 292 accrue linearly between observed catches. This assumption defines the catch function
 293 as a piecewise linear function of time, with the smallest Lipschitz constant for the catch
 294 term defined by the steepest time segment of the catch function. This assumption rep-
 295 resents one of the simplest ways of handling catch, while retaining Lipschitz continuity

296 overall. Furthermore linearly interpolated catch is adequately parsimonious for the typ-
297 ical handling of catches.

298 **2.2.8.1 Integration and Stiffness**

299 As previously mentioned, the overwhelming majority of implementations of stock
300 assessment models discretized the ODE using Euler’s method with the integration step
301 sized fixed so as to match the observation frequency. In this setting we explore model
302 parameterizations that explore the full extent of biologically relevant reference points.
303 This exercise produces some combinations of parameters that result in numerically stiff
304 ODEs.

305 The concept of stiffness in ODEs is hard to precisely characterize. Hairer and Wan-
306 ner [41, p.2] describe stiffness in the following pragmatic sense, “Stiff equations are
307 problems for which explicit methods don’t work”. It is hard to make this definition
308 more mathematically precise, but this a consistent issue for models of very productive
309 species in the low contrast simulation. Euler’s method, as often implemented, is partic-
310 ularly poorly suited for these stiff regions of parameter space. In these stiff regions it is
311 necessary to integrate the ODE with an implicit integration method.

312 Several of the most common implicit methods were tried including the Livermore
313 Solver for ODEs (lsode), and the Variable Coefficient ODE Solver (vode) as imple-
314 mented in the deSolve package of R [37]. The difference between implicit solvers is
315 negligible, while explicit methods result in wildly varying solutions to the ODE in stiff
316 regions of parameter space. Results shown here are computed using the lsode inte-
317 gration since it runs relatively quickly and has a relatively smaller footprint in system
318 memory.

2.3 Results

2.3.1 An MSY -Optimal Catch History

When $F(t)$ is held constant at F^* , as it is in the “low contrast” simulation setting, $B(t)$ comes to equilibrium as an exponential decay from K to B^* . Understanding model misspecification bias is simplified in this setting due to the relative simplicity that this induces in $B(t)$. However this simplicity is known to poorly inform estimates of r , and thus F^* , due to the limited range of the production function that is observed [14].

Figure (2.3) shows four of the most misspecified example production function fits as compared to the true data generating PT production functions. The rug plots below each set of curves show how the observed biomasses decay exponentially from K to B^* in each case. In particular, notice how observations only exist where the PT biomass is greater than B^* . Due to the leaning of the true PT curves, and the symmetry of the logistic parabola, the logistic curve only observes information about its slope at the origin from data observed on the right portion of the PT curves. The top two panels of Figure (2.3) shows PT data generated such that $\frac{B^*}{B(0)} > 0.5$; in these cases PT is steeper to the right of B^* than it is on the left, and so the the logistic curve over-estimates r , and consequently also over-estimates F^* . The bottom two panels of Figure (2.3) show

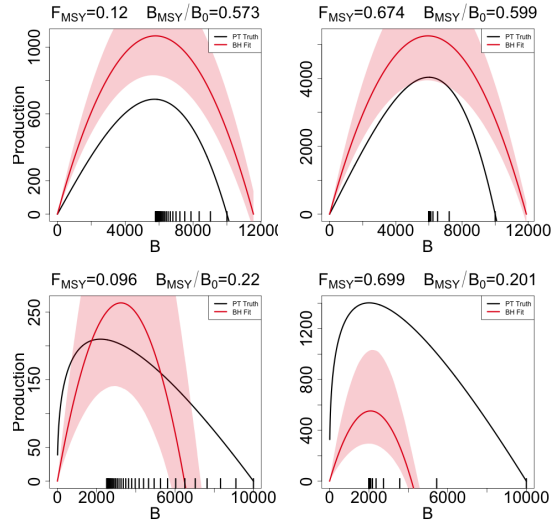


Figure 2.3: A comparison of the true PT production function (in black) and the estimated logistic curve (in red) with 95% CI shown. The examples shown represent the four corners of maximum model misspecification in the simulated RP-space. Observed biomasses are plotted in the rug plots below the curves.

PT data generated with $\frac{B^*}{B(0)} < 0.5$ and where the vice versa phenomena occurs. PT is shallower to the right of B^* than it is on the left and so the logistic parabola estimate tends to under estimate F^* .

2.3.1.1 Metamodeled Trends

Each point in the space of the RPs F^* and $\frac{B^*}{B(0)}$ uniquely identifies a complete PT model with different combinations of parameters values. Recall that when $\gamma = 2$ for the PT model, the PT curve becomes a parabola and is equivalent to the logistic curve of the Schaefer model. Since the logistic curve is symmetric about B^* , the Schaefer model must fix the value of $\frac{B^*}{B(0)}$ at the constant 0.5 for any value of F^* . So the line through RP space defined by $\frac{B^*}{B(0)} = 0.5 \quad \forall \quad F^*$, defines the subset of RP space where $\gamma = 2$ and where the PT model is equivalent to the Schaefer model. For brevity this subset of RP where $\frac{B^*}{B(0)} = 0.5$ will be referred to as the ‘‘Schaefer set’’. Thus simulated data that are generated along the Schaefer set will be the only data that are not misspecified relative to the Schaefer model; as PT data are simulated farther and farther away from this line at $\frac{B^*}{B(0)} = 0.5$ model misspecification of the Schaefer model becomes worse and worse.

While Figure (2.3) demonstrates a real trend in simulation results, individual simulation runs will at best show jittery trends due to the stochastic nature of statistical inference. The GP process metamodel accounts for this stochasticity to focus analysis on the signal in the simulation results. Recall that metamodeling occurs on the scale of the inferred productivity parameters of the restricted production model, by transforming metamodel predictions via Eq. (2.9), metamodeled predictions are obtained for Schaefer RPs. By further subtracting the true data generating PT RPs from the predicted Schaefer RPs at each point in RP space a pattern of inferential RP bias, induced by model misspecification of the Schaefer model, can be seen.

Figure (2.4) shows the pattern of biases the Schaefer model creates when fit to PT data generated at each point of RP space. An equivalent way to think of Figure

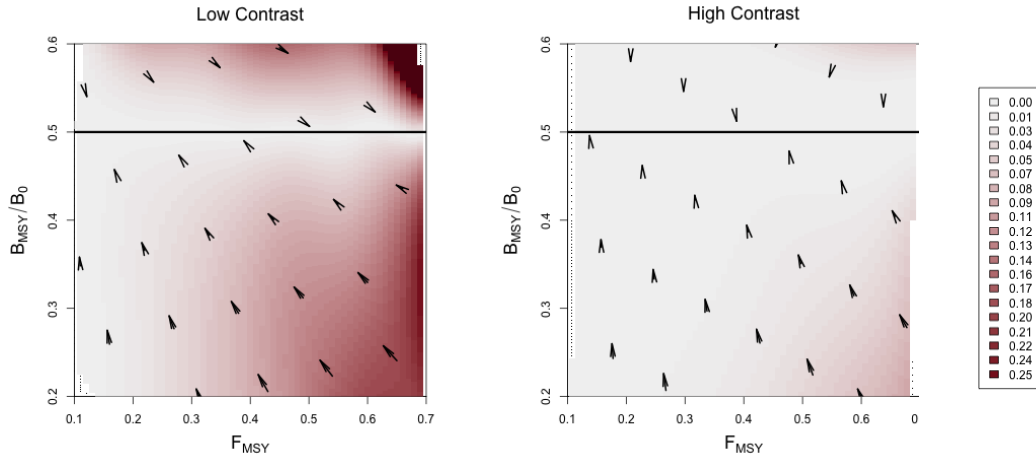


Figure 2.4: Joint bias direction for $(F^*, \frac{B^*}{B_0})$ estimates under the misspecified Schaefer Model. The intensity of color represents the excess bias relative to the shortest possible mapping. Results in the low contrast setting are shown *left*, and the high contrast setting is shown *right*.

(2.4) is that since the Schaefer model must estimate RPs in the Schaefer set, the meta-model arrows indicate the mapping that is created by inferring RPs under a misspecified Schaefer model fit to PT data generated at each point over the pictured region.

Since $\frac{B^*}{B_0}$ must be 0.5 under the Schaefer model, biases in the $\frac{B^*}{B_0}$ direction must simply map vertically onto the Schaefer set. Due to this simplified RP geometry under the Schaefer model, the degree of bias in $\frac{B^*}{B_0}$ estimation is defined solely by the degree of model misspecification irrespective of F^* . Furthermore, the closest possible point along the Schaefer set that Schaefer model inference could map RPs would be the perfectly vertical mapping. This pattern only contains the strictly necessary bias present in $\frac{B^*}{B_0}$, and zero bias in F^* . Any deviation from this minimal bias pattern is necessarily due to added bias in F^* .

The two simulation settings shown in Figure (2.4) are identical except for the amount of contrast present in the simulated index. The left panel of Figure (2.4) shows RP biases in the low contrast setting, while the right panel shows the high contrast setting.

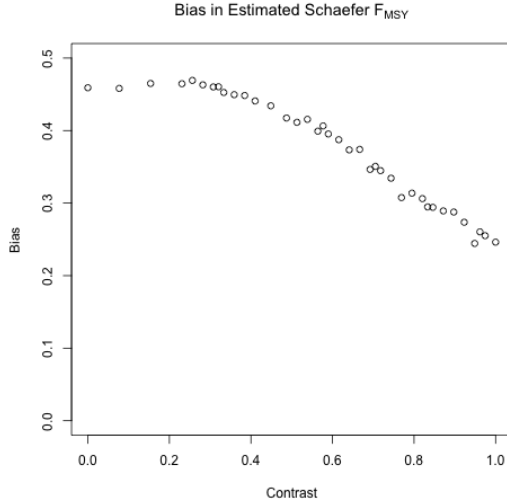


Figure 2.5: Bias in F^* under the Schaefer model when PT data are generated with increasing contrast so that F^* and $\frac{B^*}{B_0}$ are fixed at 0.699 and 0.201 respectively.

Notice that in the low contrast setting the RP bias pattern is far from the minimum distance mapping, however when contrast is added the mapping becomes much closer to a minimal vertical bias mapping. In the low contrast setting the observed bias is consistent with the pattern and mechanism described in Figure (2.3), where F^* is underestimated for data generated below the Schaefer line and overestimated above the Schaefer set. In the high contrast simulation the mapping is nearly minimal distance with the exception of PT data generated with simultaneously low $\frac{B^*}{B_0}$ and high F^* .

Figure (2.5) demonstrates how bias in F^* estimation decreases as contrast is added to PT data as generated in the low $\frac{B^*}{B_0}$ and high F^* regime. By including additional contrast F^* bias is decreased, however parameterizing contrast so as to fully extinguish F^* bias may require a more complex model of fishing.

2.4 Discussion

Tease Out BH

Results presented here generally agree with what is known about estimating popu-

400 lation growth rate parameters [18, 4, 19]. These studies appreciate the role of contrast
 401 for estimating growth rates, however they struggle to make generally extensible conclu-
 402 sions since they focus only on a handful of stocks that fall short of forming a random
 403 sample of the greater population of possible stock behaviors. The LHS design methods
 404 presented here are designed specifically to simulate a representative sample of stocks
 405 broadly across the space of possible RPs. Furthermore, the simulation design, taken
 406 together with the GP metamodel of productivity parameter estimates, allows this study
 407 to control the degree of model misspecification and generalize conclusions about the
 408 behavior of productivity estimation within the production model setting presented.

409 In the presence of contrast, F^* estimation can enjoy very low bias even for a wide
 410 range of poorly specified models; conversely in the absence of contrast F^* estimation
 411 can suffer very large bias even for slightly misspecified models. This pattern is partic-
 412 ularly true for low-contrast inference under the Schaefer model where the geometry of
 413 the restricted RP set isolates estimation failure of F^* from $\frac{B^*}{B(0)}$. While contrast has a
 414 similar impact on F^* estimation under the BH model, the geometry of the BH RP set
 415 correlates estimation bias of F^* and $\frac{B^*}{B(0)}$. The GP metamodeling approach reveals a
 416 more general pattern that highly informative data sets (high contrast) produces a nearly
 417 minimal distance mapping of RPs onto the constrained RP set.

418 In all cases when model misspecification is removed, even with weakly informative
 419 data, RP estimation is unbiased and well estimated. Thus contrast alone is not the
 420 only factor leading to inferential failure. Model misspecification is a necessary but not
 421 sufficient condition for inducing RP estimation bias. The particular RP bias present
 422 depends on the RP geometry of the fitted model and how that geometry is misspecified
 423 relative to the data. The RP mapping is then oriented to the RP geometry of the fitted
 424 model.

425 While the relative fishing rate parameterized in Section (2.2.6) captures a usefully
 426 broad spectrum of relevant fishing behaviors, it is still limiting in the amount of infor-

427 mation that it can induce. Improved methods for quantifying contrast in fisheries data,
 428 and/or methods of discovering more informative fishing behavior, could improve this
 429 analysis. In the absence of a maximally informative dataset simulation methods will not
 430 fully describe how inference fails, but the methods presented here tell the most com-
 431 plete picture yet, with explicit control of the degree model misspecification, contrast,
 432 and a simulation design that allows for uniform representative data generation across
 433 biologically meaningful stocks. The results presented here suggest the conjecture that
 434 under a maximally informative dataset, RP inference with a two parameter production
 435 function will be biased in the direction a shortest distance map from the true RPs onto
 436 restricted set of RPs under the two-parameter model.

437 Given the potential for model misspecification of RPs, a minimal distance map-
 438 ping of RPs represents a best-case scenario where the total bias of RPs, when mea-
 439 sured jointly, is minimized. That said, without recognizing the geometry of how two-
 440 parameter models of productivity limit RP space this may lead to unintuitive implica-
 441 tions in RP estimation. For example, due to the shape of the BH RP set a minimal
 442 distance mapping ensures that if there is bias in one of $\frac{B^*}{B_0}$ or F^* , there will necessarily
 443 be bias in the other RP. However under the Schaefer model, since the RP set is a con-
 444 stant in $\frac{B^*}{B_0}$, bias in F^* is not adulterated in the same way by bias in $\frac{B^*}{B_0}$ estimation. While
 445 models with constant RPs, such as the logistic model $\frac{B^*}{B_0} = \frac{1}{2}$ or the Fox model $\frac{B^*}{B_0} = \frac{1}{e}$,
 446 are extremely limited, they can be valuable tools for developing intuition precisely be-
 447 cause they isolate RP estimation in their free RPs from the correlated RP biases present
 448 in models like the BH or Ricker model.

449 When one considers the implications of RP bias, overestimation of RPs carries the
 450 severe implication of management recommendations potentially leading to overfishing,
 451 while underestimation of RP leads to overly conservative management. In this sense,
 452 when the true model is not known, the geometry of the BH set together with the meta-
 453 modeled bias trends makes the BH model a naturally conservative estimator of RPs for

454 most stocks. For most non-BH populations the BH model is likely to make conservative
455 errors in its estimates of F^* and $\frac{B^*}{B_0}$. The one notable exception to the conservatism of
456 the BH model stands for data generated in the Cushing-like regime of Schnute RPs. In
457 this regime the BH model tends to be fairly unbiased overall, however the bias that is
458 present for these populations tends to be overestimation in both RPs, leading to much
459 more severe management consequences for those populations.

460 The RP bias trends of the Schaefer model demonstrate much less conservatism than
461 the BH overall. For any population with $\frac{B^*}{B_0} < 0.5$, $\frac{B^*}{B_0}$ will be overestimated. When
462 the population comes from the regime where $\frac{B^*}{B_0} > 0.5$, $\frac{B^*}{B_0}$ will be under estimated,
463 but F^* is likely to be overestimated depending on the degree of contrast present in the
464 data. So while the Schaefer model is an intuitive model, it tends to lead to much less
465 conservative RP estimation.

466 While it is important to recognize these limitations of two-parameter models of
467 productivity, we should not solely accept conservatism as a rationale of choosing a
468 BH model of productivity. Increasing the flexibility of the production function by
469 moving toward three-parameter models would release the underlying structural limi-
470 tations [21] that cause these RP biases in the first place. Punt & Cope [27] considers
471 a suite of possible three-parameter curves which could be used instead of current two-
472 parameter curves. For all of their benefits, three parameter production functions have
473 their own complicating factors, and the structure present in the Schnute model explored
474 here makes it an intuitive bridge model for developing three-parameter models going
475 forward.

⁴⁷⁶ **Chapter 3**

⁴⁷⁷ **Schnute Model**

3.1 Introduction

3.2 Methods

3.2.1 Model

The Schnute production function is a three-parameter generalization of many of the most common two-parameter production functions [7, 32]. It can be written in the following form, with parameters α , β , and γ ,

$$P_s(B; [\alpha, \beta, \gamma]) = \alpha B(1 - \beta\gamma B)^{\frac{1}{\gamma}}. \quad (3.1)$$

The BH and Logistic production functions arise when γ is fixed to -1 or 1 respectively. The Ricker model is a limiting case as $\gamma \rightarrow 0$. For $\gamma < -1$ a family of strictly increasing Cushing-like curves arise, culminating in linear production as $\gamma \rightarrow -\infty$. These special cases form natural regimes of similarly behaving production functions as seen in Figure (3.1).

The behavior of RP inference under the BH model is of particular interest due to the overwhelming popularity of the BH assumption in fisheries models. Since

Schnute production models can represent a quantifiably wide variety of possible productivity behaviors, they present an ideal simulation environment for inquiry of the reliability of inference under the BH assumption.

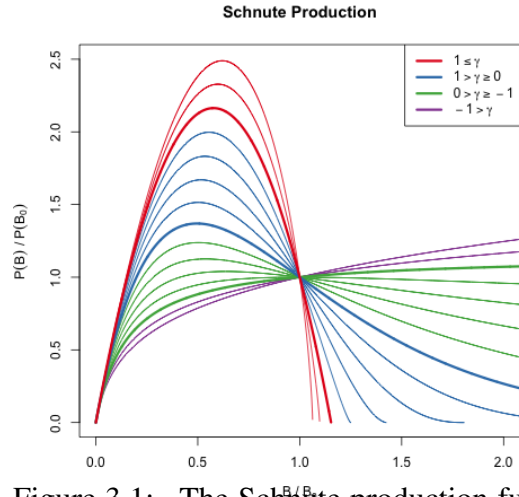


Figure 3.1: The Schnute production function plotted across a variety of parameter values. Regimes of similarly behaving curves are grouped by color.

Under Schnute production, biomass dynamics evolve according to the following ODE,

$$\frac{dB}{dt} = P_s(B; \theta) - (M + F)B. \quad (3.2)$$

497 This equation largely takes the same form as previously described, except that P_s is
 498 the Schnute production function and natural mortality, M , is modeled explicitly here.
 499 Natural mortality models the instantaneous rate of mortality from all causes outside of
 500 fishing. While Eq. (3.2) models M explicitly, natural mortality is implicit to the struc-
 501 ture of the previously described Schaefer, Fox, and PT production models. Explicitly
 502 modeling natural mortality allows for the production function not to approach (or inter-
 503 sect) 0 for large biomasses (e.g. BH production). In turn, the Schunte model requires
 504 the addition of the term $-MB$ to form an interpretable yeild curve and make RPs well
 505 defined over the relevant domain of γ .

The derivation of RPs under Eq. (3.2) follows a similar logic as under the PT model. An expression for equilibrium biomass is attained by setting $\frac{dB}{dt} = 0$ and rearranging the resulting expression to solve for B

$$\bar{B}(F) = \frac{1}{\gamma\beta} \left(1 - \left(\frac{M+F}{\alpha} \right)^\gamma \right). \quad (3.3)$$

The above expression quickly yields B_0 , B^* by evaluation at $F = 0$ and F^* respectively,

$$B_0 = \frac{1}{\gamma\beta} \left(1 - \left(\frac{M}{\alpha} \right)^\gamma \right) \quad (3.4)$$

$$\frac{B^*}{B_0} = \frac{1 - \left(\frac{M+F^*}{\alpha} \right)^\gamma}{1 - \left(\frac{M}{\alpha} \right)^\gamma}. \quad (3.5)$$

Attaining an expression for F^* requires maximization of equilibrium yield, $\bar{Y} = F\bar{B}(F)$,

with respect to F . Analytically maximizing proceeds by differentiating \bar{Y} to produce

$$\frac{d\bar{Y}}{dF} = \bar{B}(F) + F \frac{d\bar{B}}{dF} \quad (3.6)$$

$$\frac{d\bar{B}}{dF} = -\frac{1}{\beta} \left(\frac{\left(\frac{M+F}{\alpha}\right)^\gamma}{F+M} \right). \quad (3.7)$$

Setting $\frac{d\bar{Y}}{dF} = 0$, filling in the expressions for $\bar{B}(F)$ and $\frac{d\bar{B}}{dF}$, then rearranging to solve for F^* is less yielding here than it was in the case of the PT model. This procedure falls short of providing an analytical solution for F^* directly in terms of θ , but rather shows that F^* must respect the following expression,

$$0 = \frac{1}{\gamma} - \left(\frac{1}{\gamma} + \frac{F^*}{F^* + M} \right) \left(\frac{F^* + M}{\alpha} \right)^\gamma. \quad (3.8)$$

506 The lack of an analytical solution here is understood. Schnute & Richards [34,
507 pg. 519] specifically point out that F^* cannot be expressed analytically in terms of
508 productivity parameters, but rather gives a partial analytical expression for the inverse
509 relationship. Although parameterized slightly differently, Schnute & Richards derive
510 expressions for α and β as a function of RPs and γ .

511 Since RPs are left without a closed form expression, computing RPs from produc-
512 tivity parameters amounts to numerically solving the system formed by collecting the
513 expressions (3.8), (3.4), and (3.5).

514 3.2.2 Simulation

515 For the purpose of simulation, it is not necessary to completely know the precise
516 relationships mapping RPs $\mapsto \theta$ or $\theta \mapsto$ RPs. Simulation only requires enough knowl-
517 edge of these mappings to gather a list of (α, β, γ) tuples, for data generation under
518 the Schnute model, and the corresponding RPs in some reasonable space-filling design

519 over RP space.

Similarly to Schnute & Richards [34], expressions (3.8) and (3.4) are solved for α and β respectively. This leads to the partial mapping $(F^*, B_0) \mapsto (\alpha(\cdot, \gamma), \beta(\cdot, \gamma))$ in terms of RPs and γ . By further working with Eq. (3.5), to identify γ , the following system is obtained,

$$\begin{aligned}\alpha &= (M + F^*) \left(1 + \frac{\gamma F^*}{M + F^*} \right)^{1/\gamma} \\ \beta &= \frac{1}{\gamma B_0} \left(1 - \left(\frac{M}{\alpha} \right)^\gamma \right) \\ \frac{B^*}{B_0} &= \frac{1 - \left(\frac{M + F^*}{\alpha} \right)^\gamma}{1 - \left(\frac{M}{\alpha} \right)^\gamma}.\end{aligned}\tag{3.9}$$

520 For a population experiencing natural mortality M , by fixing F^* , B_0 , and $\frac{B^*}{B_0}$ the
521 above system can fully specify α and β for a given γ . Notice for a given γ a cascade
522 of closed form solutions for α and β can be obtained. First $\alpha(\gamma)$ can be computed, and
523 then $\beta(\alpha(\gamma), \gamma)$ can be computed. If $\alpha(\gamma)$ is filled back into the expression for $\frac{B^*}{B_0}$, the
524 system collapses into a single onerous expression for $\frac{B^*}{B_0}(\alpha(\gamma), \gamma)$. For brevity, define
525 the function $\zeta(\gamma) = \frac{B^*}{B_0}(\alpha(\gamma), \gamma, F^*, M)$ based on Eq. (3.5).

526 Inverting $\zeta(\gamma)$ for γ , and computing the cascade of $\alpha(\gamma)$, and then $\beta(\alpha(\gamma), \gamma)$, fully
527 defines the Schnute model for a given $(\frac{F^*}{M}, \frac{B^*}{B_0})$. However inverting ζ accurately is ex-
528 tremely difficult. Inverting ζ analytically is not feasible, and numerical methods for
529 inverting ζ are unstable and can be computationally expensive. Rather than numeri-
530 cally invert precise values of $\zeta(\gamma)$, γ is sampled so that the overall simulation design is
531 space filling as described in Section (3.2.4).

532 Each design location defines a complete Schnute production model with the given
533 RP values. Indices of abundance are simulated from the Schnute model at each design
534 location, a small amount of residual variation, $\sigma = 0.01$, is added to the simulated index,

535 and the data are then fit with a misspecified BH production model. The design at large
536 captures various degrees of model misspecification relative to the BH model, so as to
537 observe the effect of productivity model misspecification upon RP inference.

3.2.3 Latin Hypercube Sampling

The goal of space filling design in this setting is to extend the notion of the random sample (and its desirable parameter estimation properties) across the simulated RP domain so as to represent the simulated space as well as possible [12]. The simple random sample is the classical approach to unbiased parameter estimation, however simple randomness is patchy, often sampling some regions of design space quite densely, while leaving other regions of design space empty. Space filling designs aim to preserve (or enhance) parameter estimation properties across the simulated domain [8, 38], while constraining samples to be spaced in some notion of spread over the entire space. Latin hypercube sampling [22, LHS] is among the most foundational of space filling designs used in computer experiments.

A LHS of size n , in the 2 dimensional space defined by RPs, distributes samples so as to spread points across a design region in a broadly representative way. A LHS design extends the notion of a univariate random uniform sample across multiple dimensions so that each margin of the design space enjoys a uniform distribution.

LHS designs achieve this notion of uniformity by first partitioning each dimension of the design space into regular grids of size n . By intersecting the grids

of each dimension, cells are produced that evenly partition the design space. In two dimensions n^2 cells are produced, from which a total of n samples are taken. Crucially only one point is randomly sampled from a given element of each grid in each

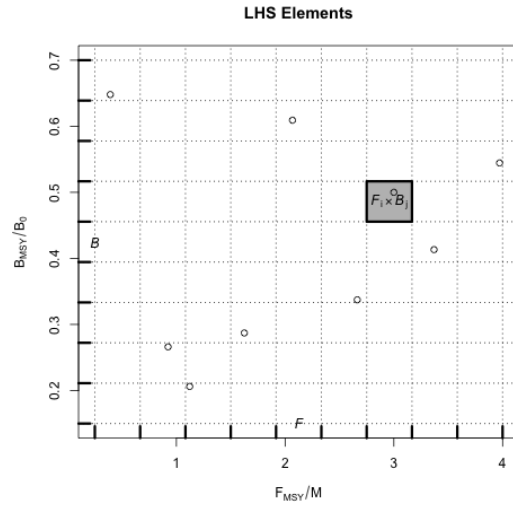


Figure 3.2: LHS grids. Intersecting \mathcal{F} and \mathcal{B} produces n^2 cells; a particular cell $\mathcal{F}_i \times \mathcal{B}_j$ is shown in grey. One point is in each of the marginal \mathcal{F}_i and \mathcal{B}_j grid elements.

565 dimension so as to reduce clumping of the n samples across the design space.

566 3.2.4 Design

567 Due to the lack of an analytical relationship mapping RPs $\mapsto \theta$, analogous to the PT
 568 model's Eq. (2.10), producing a LHS design over Schnute RPs requires a more tactful
 569 approach. The structured relationship between the RPs and productivity parameters,
 570 described in Section (3.2.2), allows an approximate LHS to be obtained by a careful
 571 navigation of the system of equations seen in Eq. (3.9).

572 Under the Schnute model, let \mathcal{F}
 573 and \mathcal{B} represent regular grids on $\frac{F^*}{M} \in (0.25, 4)$ Given B_0 , M , and F^* :
 574 and $\frac{B^*}{B_0} \in (0.15, 0.7)$ respectively which 1) Draw $\gamma^* \sim \gamma|F^*, M$.
 575 can serve as the scaffolding for comput- 2) Compute $\frac{B^*}{B_0} = \zeta(\gamma^*)$
 576 ing an approximate LHS. 3) Compute $\alpha^* = \alpha(\gamma^*, F^*, M)$
 4) Compute $\beta^* = \beta(\alpha^*, \gamma^*, M, B_0)$

Since it is not practical to invert $\zeta(\gamma)$,
 a uniform sample in $\frac{B^*}{B_0}$ can be obtained
 by modeling γ as a random variable, with
 realization γ^* , and thinking of $\zeta(\gamma)$ as its
 cumulative distribution function (CDF).

Figure 3.3: An outline of the sampling procedure for γ given B_0 , M , and F^* .

The aim is to model γ as an easily sampled random variable with a CDF that closely approximates ζ , so that $\zeta(\gamma^*) \sim U(\zeta_{min}, 1)$ as closely as possible. There may be many good models for the distribution of γ , but in this setting the following distribution is very effective,

$$\gamma \sim \zeta_{min} \delta(\gamma_{min}) + t(\mu, \sigma, v) \mathbf{1}_{\gamma > \gamma_{min}}. \quad (3.10)$$

577 Above, t is the density of the three-
 578 parameter location-scale family Student's
 579 t distribution with location μ , scale σ , and
 580 degrees of freedom ν . $\mathbf{1}_{\gamma > \gamma_{min}}$ is an indi-
 581 cator function that serves to truncate the
 582 Student's t distribution at the lower bound
 583 γ_{min} . $\delta(\gamma_{min})$ is the Dirac delta function
 584 evaluated at γ_{min} , which is scaled by the
 585 known value ζ_{min} ; this places probability
 586 mass ζ_{min} at the point γ_{min} . Since sam-
 587 pling from a Student's t distribution is
 588 readily doable, sampling from a truncated
 589 Student's t mixture only requires slight modification.

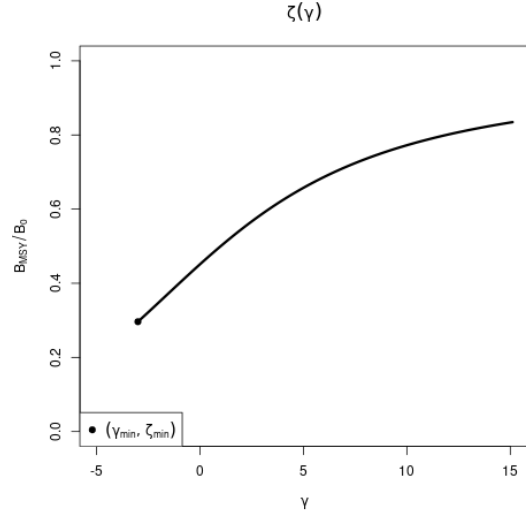


Figure 3.4: $\zeta(\gamma)$ Plotted for $F^* = 0.1$ and $M = 0.2$. The point $(\gamma_{min}, \zeta_{min})$ shows the lowest biologically meaningful value of γ ; below which productivity is negative.

Let T be the CDF of the modeled distribution of γ . Since the point $(\gamma_{min}, \zeta_{min})$ is known from the dynamics of the Schnute model at a given RP, full specification of Eq. (3.10) only requires determining the values for μ , σ , and ν which make T best approximate $\zeta(\gamma)$. Thus, the values of μ , σ , and ν are chosen by minimizing the L^2 distance between $T(\gamma)$ and $\zeta(\gamma)$.

$$[\hat{\mu}, \hat{\sigma}, \hat{\nu}] = \arg \min_{[\mu, \sigma, \nu]} \int_{\Gamma} (T(\gamma; \mu, \sigma, \nu) - \zeta(\gamma))^2 d\gamma \quad (3.11)$$

590 The distribution $T(\gamma|\hat{\mu}, \hat{\sigma}, \hat{\nu})$ is fit for
 591 use in generating γ^* random variates at
 592 a specific F^* and M . This approxima-
 593 tion releases the need to invert ζ w.r.t. γ
 594 by using samples of γ^* values to generate
 595 approximatly uniform samples of $\zeta(\gamma^*)$.
 596 By sampling approximatly uniform $\zeta(\gamma^*)$
 597 random variates in this way, and making
 598 use of the structure in Eq. (3.9), an ap-
 599 proximate LHS sample can be collected
 600 via Algorithm (1).

601 For a given i , $\frac{F^*}{M}$ is drawn uniformly
 602 from within \mathcal{F}_i . Conditioning on the sam-
 603 ple of F^* , and M , $T(\gamma|\hat{\mu}, \hat{\sigma}, \hat{\nu})$ is fit and
 604 γ^* is sampled. ζ^* is then computed and
 605 placed into the appropriate grid element
 606 \mathcal{B}_j . Given γ^* , the cascade $\alpha(\gamma^*)$, and
 607 $\beta(\alpha(\gamma^*), \gamma^*)$, can be computed. The al-
 608 gorithm continues until all of the design elements, $(\frac{F^*}{M}, \zeta^*) \Leftrightarrow (\alpha^*, \beta^*, \gamma^*)$, have been
 609 computed for all $i \in [1, \dots, n]$.

Algorithm 1 LHS of size n on rectangle R .

```

1: procedure  $LHS_n(R)$ 
2:   Define  $n$ -grids  $\mathcal{F}, \mathcal{B} \in R$ 
3:   for each grid element  $i$  do
4:     Draw  $\frac{F^*}{M} \sim Unif(\mathcal{F}_i)$ 
5:     Compute  $[\hat{\mu}, \hat{\sigma}, \hat{\nu}]$  given  $F^*$  &  $M$ 
6:     while  $\mathcal{B}_j$  not sampled do
7:       Draw  $\gamma^* \sim T(\gamma|\hat{\mu}, \hat{\sigma}, \hat{\nu})$ 
8:       Compute  $\zeta^* = \zeta(\gamma^*)$ 
9:       Compute  $j$  such that  $\zeta^* \in$ 
         $\mathcal{B}_j$ 
10:    end while
11:    Compute  $\alpha^* = \alpha(\gamma^*, F^*, M)$ 
12:    Compute  $\beta^* = \beta(\alpha^*, \gamma^*, M, B_0)$ 
13:    Save  $(\frac{F^*}{M}, \zeta^*) \Leftrightarrow (\alpha^*, \beta^*, \gamma^*)$  in  $\mathcal{F}_i \times \mathcal{B}_j$ 
14:  end for
15: end procedure

```

610 3.2.4.1 Design Refinement

611 Since the behavior of RP inference, under misspecified models, will vary in yet-
 612 unknown ways, the exact sampling design density may be hard to know a priori. Sev-
 613 eral factors, including the particular level of observation uncertainty, high variance (i.e.
 614 hard to resolve) features of the response surface, or simply "gappy" instantiations of

the initial LHS design may necessitate adaptive design refinement, to accurately describe RP biases. Given the temperamental relationship between RPs and productivity parameters in the Schnute model, a recursive refinement algorithm that makes use of the previously described LHS routine, is developed.

While LHS ensures uniformity in the design margins, and a certain degree of spread, it is widely recognized that particular LHS instantiations may leave substantive gaps in the simulation design. To correct this, LHS is often paired with design elements of maximin design [24, 8]. Maximin designs sample the design space by maximizing the minimum distance between sampled points. This has the advantage of definitionally filling holes in the design, however because no points are ever drawn outside of the design domain, samples tend to clump around edges (particularly corners) of the design domain. Since LHS ensures uniformity in the margins and maximin designs enjoys a certain sense of optimality in how they define and fill gaps [17], the methods are quite complimentary when combined.

Making use of this complimentary relationship, holes in the existing LHS design of RPs are identified based on maximin design principles. New design points are collected based on areas of the RP design space which maximizes the minimum distance between all pairs of points in the current design, based on the following distance function

$$d(\mathbf{x}, \mathbf{x}') = \sqrt{(\mathbf{x} - \mathbf{x}')^T \mathbf{D}^{-1} (\mathbf{x} - \mathbf{x}')} \quad (3.12)$$

$$\mathbf{D} = \mathbf{diag} \left[(\max(\mathcal{F}) - \min(\mathcal{F}))^2, (\max(\mathcal{B}) - \min(\mathcal{B}))^2 \right].$$

Above, d is a scaled distance function that defines the distance between points in the differing scales of $\frac{B^*}{B_0}$ and $\frac{F^*}{M}$. \mathbf{D} is a diagonal matrix that measures the squared size of the domain in each axis of so as to normalize distances to a common scale.

If \mathbf{X}_n is the initial design, computed on R_{full} , let \mathbf{x}_a be the augmenting point which

maximizes the minimum distance between all of the existing design points,

$$\mathbf{x}_a = \operatorname{argmax}_{\mathbf{x}'} \min\{d(\mathbf{x}_i, \mathbf{x}') : i = 1, \dots, n\}. \quad (3.13)$$

The point \mathbf{x}_a is used as an anchor for augmenting \mathbf{X}_n . An additional $LHS_{n'}$ (via Algorithm (1)) is collected, adding n' design points, centered around \mathbf{x}_a , to the overall design. The augmenting region, $R_{(\mathbf{x}_a, d_a)}$, for collecting $LHS_{n'}$ is defined based on the square centered at \mathbf{x}_a with side length $2d_a$, where $d_a = \min\{d(\mathbf{x}_i, \mathbf{x}_a) : i = 1, \dots, n\}$, in the space defined by the metric d .

Due to the tendency of maximin sampling to cluster augmenting points on the edges of the design space, $R_{(\mathbf{x}_a, d_a)}$ is truncated by the outer most limits of R_{full} so as to focus design augmentation within the specified domain of the simulation. Furthermore, since the design space has a nonlinear constraint at low values of $\frac{B^*}{B_0}$, the calculation of \mathbf{x}_a is further truncated based on a convex hull defined by the existing samples in the overall design.

Design refinement then proceeds as follows. An initial design is computed, $\mathbf{X}_n = LHS_n(R_{full})$, based on an overall simulated region of RPs R_{full} . The maximin augmenting point, \mathbf{x}_a , is computed at a maximin distance of d_a from the existing samples. An augmenting design $\mathbf{X}_{n'} = LHS_{n'}(R_{(\mathbf{x}_a, d_a)})$ is collected and added to \mathbf{X}_n . Design refinement carries on recursively collecting augmenting designs in this way until the maximin distance falls below the desired level.

3.2.5 refer back to GP?

3.3 Results

3.3.1 Design

Algorithm (1) enforces uniform marginals in $\frac{F^*}{M}$ directly, as well as the adherence of the overall design to latin squares. Figure (3.5) shows a uniform Q-Q plot for sampled ζ , using Algorithm (1), against theoretical uniform quantiles. As evidence by the excellent coherence to the theoretical uniform quantiles, the approximation in Section (3.2.4) for sampling γ (and therefore $\zeta(\gamma)$), is very effective. Furthermore since numerical inversion of $\zeta(\gamma)$ is costly and unreliable, the relative speed and accuracy that this approximate LHS sampling method provides is pivotal for the rest of the work presented here.

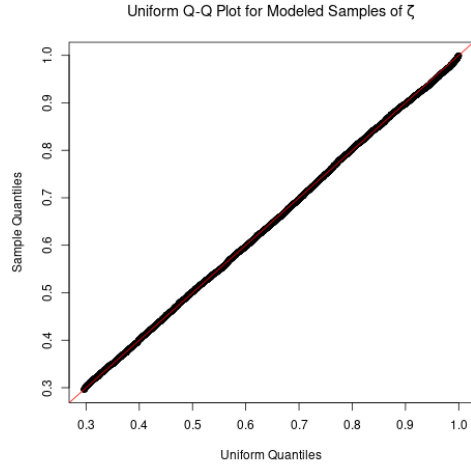


Figure 3.5: Uniform Q-Q plot for ζ plotted for $F^* = 0.1$ and $M = 0.2$.

Similarly to the PT model, the three-parameter Schnute model is uniquely identified by each point in the space of $\frac{F^*}{M}$ and $\frac{B^*}{B_0}$ RPs. As seen in Figure (3.6), Schnute production has different behaviors in different ranges of RPs space, which are entirely defined by the value of γ (shown in Figure (3.1)). When $\gamma \geq 1$ the Schnute model produces a family of Logistic-like curves that are increasingly right leaning as γ increases. For $1 > \gamma \geq 0$, Schnute production takes a family of left leaning Ricker-like curves that all, at least, approach the x-axis. For $0 > \gamma > -1$ there are a family of BH-like curves that do not approach the x-axis but still have decreasing productivity for large biomass stocks.

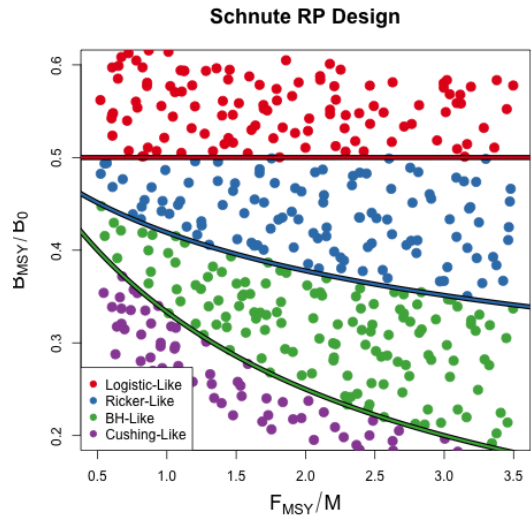


Figure 3.6: A Schnute RP design. Colors indicate different regimes of Schnute production. The black curve shows the BH set.

When γ is exactly -1 Schnute reduces to BH production which has asymptoting production for large biomass. Finally when $-1 > \gamma$ Schnute produces a family of increasing Cushing-like curves that do not asymptote, and produces linear production as $\gamma \rightarrow -\infty$.

Modeling index data that are simulated broadly over the theoretical space of RPs with misspecified BH production greatly limits the range of possible RPs that can be inferred. Under BH production the full theoretical space of RPs are limited to the curve $\frac{B^*}{B_0} = \frac{1}{F^*/M+2}$. Define the “BH set” as the set of RPs defined by this limited space, i.e. the curve $\left\{ \left(\frac{F^*}{M}, \frac{B^*}{B_0} \right) \mid \frac{B^*}{B_0} = \frac{1}{F^*/M+2} \right\}$. as seen in the **black curve** in Figure (3.6). The farther away from this set that Schnute data are simulated, the more the BH model is misspecified for those data.

3.3.2 Metamodeled Trends

Unlike the Schaefer model, the BH set is not a constant in $\frac{B^*}{B_0}$. Under the BH model, bias in $\frac{B^*}{B_0}$ is no longer entirely defined by the degree of model misspecification, but rather the structure of BH RPs allows bias in both $\frac{B^*}{B_0}$ and $\frac{F^*}{M}$ to interact as a function of contrast in the data.

3.3.2.1 High Contrast

Figure (3.7) shows metamodeled RP bias surfaces for inference under the BH model in the high contrast setting. The (*left*) and (*bottom*) panels focus only on the $\frac{B^*}{B_0}$ and $\frac{F^*}{M}$ components of bias respectively. In these panels bias is shown as relative bias, $\frac{\widehat{RP} - RP}{RP}$, similar to a percent error calculation. Where RP represents the true value of the three-parameter RP, and \widehat{RP} refers to the metamodel estimate.

Figure (3.7, *top-right*) combines the components of bias to show the overall map-

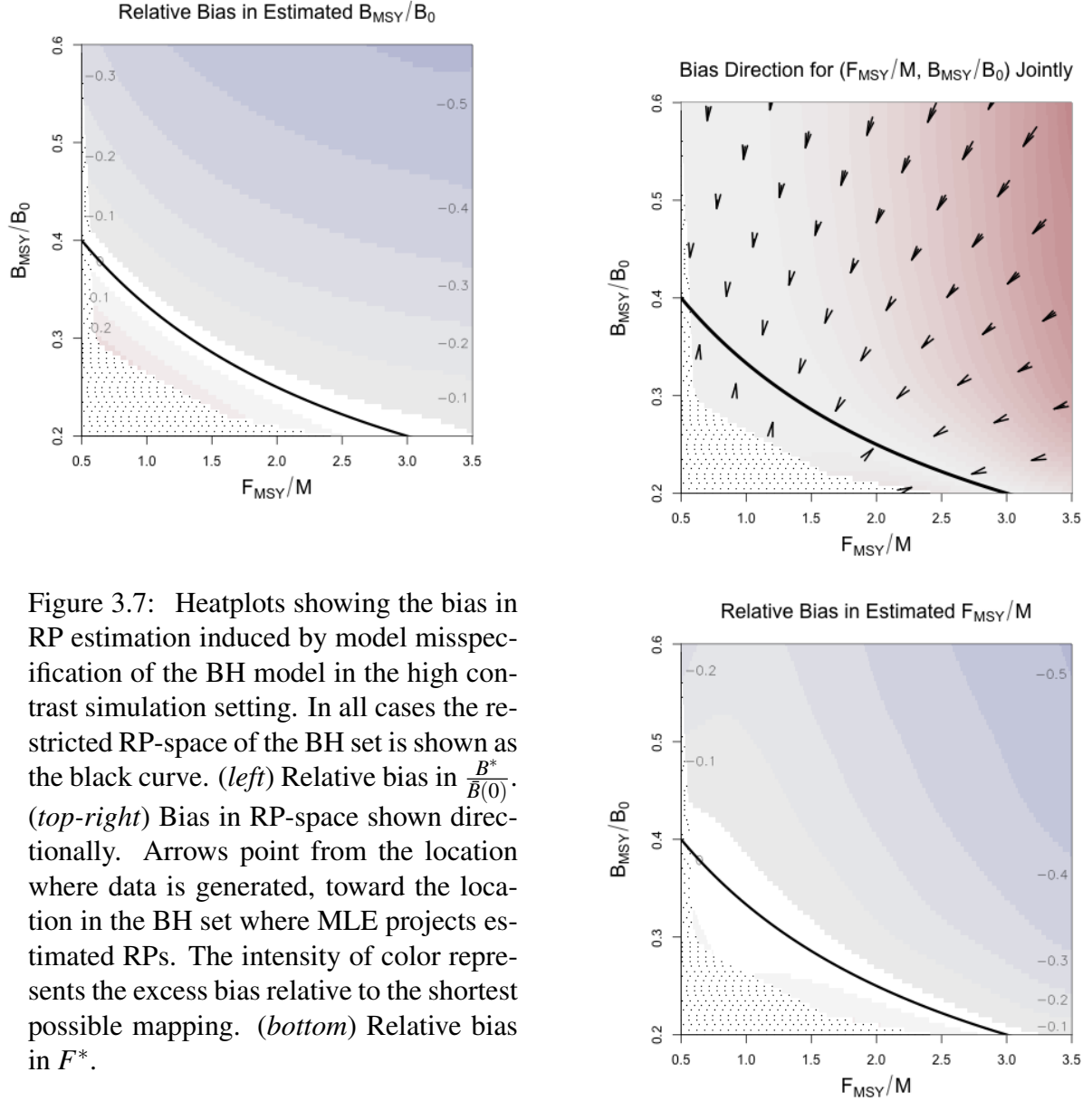


Figure 3.7: Heatplots showing the bias in RP estimation induced by model misspecification of the BH model in the high contrast simulation setting. In all cases the restricted RP-space of the BH set is shown as the black curve. (*left*) Relative bias in $\frac{B^*}{B(0)}$. (*top-right*) Bias in RP-space shown directionally. Arrows point from the location where data is generated, toward the location in the BH set where MLE projects estimated RPs. The intensity of color represents the excess bias relative to the shortest possible mapping. (*bottom*) Relative bias in F^* .

ping of RPs under BH inference in the high contrast simulation setting. Unlike high
 contrast RP inference under the Schaefer model, where mainly bias in $\frac{B^*}{B(0)}$ occurred, the
 BH model does shows bias in both RPs here. Despite the bias in $\frac{B^*}{B(0)}$ and $\frac{F^*}{M}$ these

706 results are similar to that of the Schaefer model in that the overall mapping of RPs is
 707 very nearly a minimal distance mapping onto the constrained set of RPs. The primary
 708 difference between Schaefer model and BH RP inference is the geometry of their lim-
 709 ited RP spaces. Unlike the Schaefer model the BH set encourages bias in both RPs for
 710 misspecified models even in very well informed setting.

711 3.3.2.2 Low Contrast

712 Figure (3.8) shows the mapping of
 713 RPs in the low contrast simulation set-
 714 ting. Figures (3.8) and (3.7, *top-right*)
 715 share a common scale for the intensity
 716 of color to facilitate comparison. In Fig-
 717 ure (3.8) notice that the mildly misspec-
 718 ified area around the BH set produces
 719 mappings onto the BH set which resem-
 720 ble the minimal distance mapping seen
 721 in the high contrast setting. The primary
 722 difference in this low contrast setting, is
 723 the break point around $\frac{B^*}{B(0)} = 0.4$ above
 724 which $\frac{F^*}{M}$ is sharply underestimated.

725 The region of RPs where the BH
 726 model manages to recover the minimal
 727 distance mapping may be considered a “safe regime” of data types that are reason-
 728 ably well modeled by a BH model. By comparison of Figure (3.8), with Figure (3.6),
 729 this safe regime of the BH model occurs for data generated for Cushing-like or BH-like
 730 production. While bias of the RPs can still become concerningly large, this region can
 731 be considered safe in the sense that even for low contrast data RP estimation under the

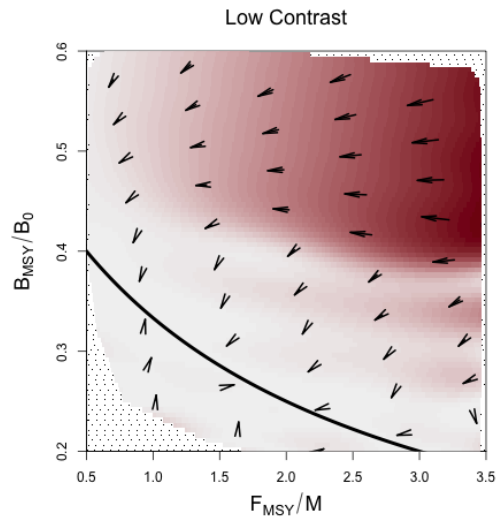


Figure 3.8: Joint bias direction of RP inference in the low contrast simulation setting. The intensity of color represents the excess bias relative to the shortest possible mapping.

the BH model recovers the minimal distance mapping.

Outside of this safe regime, RP estimation breaks from the minimal distance mapping at the interface between BH-Like and Ricker-Like regimes of the Schnute model (again see Figure (3.6)). The Ricker model lies along this regime interface, and represents the first model to approach the x-axis for large biomasses as γ increases. This markedly unBH-like productivity in the low information simulation setting breaks MLE inference from the minimal distance mapping and

instead maps RPs to extremely low values of F^* ; consequently $\frac{B^*}{\bar{B}(0)}$ is estimated near the limiting value under the BH (i.e. $\lim_{F^* \rightarrow 0} \frac{1}{F^*/M+2} = 0.5$). Similarly the set of Ricker RPs (as well as the Schaeffer set) include this trivial limiting point in common ($\frac{F^*}{M} = 0$, $\frac{B^*}{\bar{B}(0)} = 0.5$).

Interestingly, in the high contrast setting this trivial mapping for highly misspecified BH models is not present. This suggests that, under a misspecified BH model, the presence of adequate information in the data to produce reasonable estimates of $\frac{F^*}{M}$, drives $\frac{B^*}{\bar{B}(0)}$ below 0.5 in accordance with $\frac{B^*}{\bar{B}(0)} = \frac{1}{F^*/M+2}$, even when the true $\frac{B^*}{\bar{B}(0)} > 0.5$. This phenomena balances RP estimation within the constrained BH set as mediated by the information content of the data and the degree of model misspecification. When the information content in the data is too small to drive a compromised RP estimate, inference completely disregards accurate estimation of F^* in order to better estimate $\frac{B^*}{\bar{B}(0)}$ by exploiting the common limiting behavior of the BH set and that of Ricker-like and Logistic-like models.

Estimated Yield Curves For Poorly Specified BH

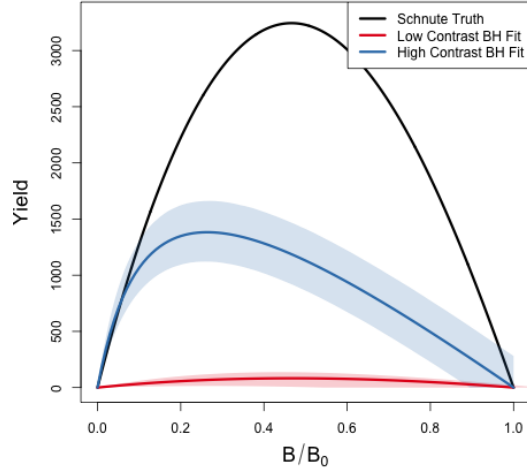


Figure 3.9: Yield curves for data generated with $\frac{F^*}{M} = 3.48$ and $\frac{B^*}{\bar{B}(0)} = 0.48$.

3.4 Discussion

Tease Out BH

Results presented here generally agree with what is known about estimating population growth rate parameters [18, 4, 19]. These studies appreciate the role of contrast for estimating growth rates, however they struggle to make generally extensible conclusions since they focus only on a handful of stocks that fall short of forming a random sample of the greater population of possible stock behaviors. The LHS design methods presented here are designed specifically to simulate a representative sample of stocks broadly across the space of possible RPs. Furthermore, the simulation design, taken together with the GP metamodel of productivity parameter estimates, allows this study to control the degree of model misspecification and generalize conclusions about the behavior of productivity estimation within the production model setting presented.

In the presence of contrast, F^* estimation can enjoy very low bias even for a wide range of poorly specified models; conversely in the absence of contrast F^* estimation can suffer very large bias even for slightly misspecified models. This pattern is particularly true for low-contrast inference under the Schaefer model where the geometry of the restricted RP set isolates estimation failure of F^* from $\frac{B^*}{\bar{B}(0)}$. While contrast has a similar impact on F^* estimation under the BH model, the geometry of the BH RP set correlates estimation bias of F^* and $\frac{B^*}{\bar{B}(0)}$. The GP metamodeling approach reveals a more general pattern that highly informative data sets (high contrast) produces a nearly minimal distance mapping of RPs onto the constrained RP set.

In all cases when model misspecification is removed, even with weakly informative data, RP estimation is unbiased and well estimated. Thus contrast alone is not the only factor leading to inferential failure. Model misspecification is a necessary but not sufficient condition for inducing RP estimation bias. The particular RP bias present

depends on the RP geometry of the fitted model and how that geometry is misspecified relative to the data. The RP mapping is then oriented to the RP geometry of the fitted model.

While the relative fishing rate parameterized in Section (2.2.6) captures a usefully broad spectrum of relevant fishing behaviors, it is still limiting in the amount of information that it can induce. Improved methods for quantifying contrast in fisheries data, and/or methods for discovering more informative fishing behavior, could improve this analysis. In the absence of maximally informative dataset simulation methods will not fully describe how inference fails, but the methods presented here tell the most complete picture yet, with explicit control of the degree model misspecification, contrast, and a simulation design that allows for uniform representative data generation across biologically meaningful stocks. The results presented here suggest the conjecture that under a maximally informative dataset, RP inference with a two parameter production function will be biased in the direction a shortest distance map from the true RPs onto restricted set of RPs under the two-parameter model.

Given the potential for model misspecification of RPs, a minimal distance mapping of RPs represents a best-case scenario where the total bias of RPs, when measured jointly, is minimized. That said, without recognizing the geometry of how two-parameter models of productivity limit RP space this may lead to unintuitive implications in RP estimation. For example, due to the shape of the BH RP set a minimal distance mapping ensures that if there is bias in one of $\frac{B^*}{B_0}$ or F^* , there will necessarily be bias in the other RP. However under the Schaefer model, since the RP set is a constant in $\frac{B^*}{B_0}$, bias in F^* is not adulterated in the same way by bias in $\frac{B^*}{B_0}$ estimation. While models with constant RPs, such as the logistic model $\frac{B^*}{B_0} = \frac{1}{2}$ or the Fox model $\frac{B^*}{B_0} = \frac{1}{e}$, are extremely limited, they can be valuable tools for developing intuition precisely because they isolate RP estimation in their free RPs from the correlated RP biases present in models like the BH or Ricker model.

813 When one considers the implications of RP bias, overestimation of RPs carries the
814 severe implication of management recommendations potentially leading to overfishing,
815 while underestimation of RP leads to overly conservative management. In this sense,
816 when the true model is not known, the geometry of the BH set together with the meta-
817 modeled bias trends makes the BH model a naturally conservative estimator of RPs for
818 most stocks. For most non-BH populations the BH model is likely to make conservative
819 errors in its estimates of F^* and $\frac{B^*}{B_0}$. The one notable exception to the conservatism of
820 the BH model stands for data generated in the Cushing-like regime of Schnute RPs. In
821 this regime the BH model tends to be fairly unbiased overall, however the bias that is
822 present for these populations tends to be overestimation in both RPs, leading to much
823 more severe management consequences for those populations.

824 The RP bias trends of the Schaefer model demonstrate much less conservatism than
825 For any population with $\frac{B^*}{B_0} < 0.5$, $\frac{B^*}{B_0}$ will be overestimated. When the population
826 comes from the regime where $\frac{B^*}{B_0} > 0.5$, $\frac{B^*}{B_0}$ will be under estimated, but F^* is likely
827 to be overestimated depending on the degree of contrast present in the data. So while
828 the Schaefer model is an intuitive model, it tends to lead to much less conservative RP
829 estimation.

830 While it is important to recognize these limitations of two-parameter models of
831 productivity, we should not solely accept conservatism as a rationale of choosing a
832 BH model of productivity. Increasing the flexibility of the production function by
833 moving toward three-parameter models would release the underlying structural limi-
834 tations [21] that cause these RP biases in the first place. Punt & Cope [27] considers
835 a suite of possible three-parameter curves which could be used instead of current two-
836 parameter curves. For all of their benefits, three parameter production functions have
837 their own complicating factors, and the structure present in the Schnute model explored
838 here makes it an intuitive bridge model for developing three-parameter models going
839 forward.

840 **Chapter 4**

841 **A Delay Differential Model**

4.1 Introduction

While the SPM captures the majority of variation in RPs, individual growth is the next most influential dynamic for explaining RP variation [cite or generalize](#). The SPM captures the net effect of biomass production by including all aspects of biomass production (i.e. maturity, recruitment, growth, etc.) together in the production function. While there are a number of approaches to modeling these dynamics [28, 14], for the purpose of fisheries management, in a “data-rich” setting it is currently considered best practice to use an age structured models (ASM) [23]. Due to their complexity, ASMs require a lot of expensive data to make make inferences from. Even when these data do exists, the highly parameterized structures used in these models introduce the potential for identifiability issues [cite](#).

Delay differential models (DDM) are an intermediate approach between SPM and ASM. Simple DDMs may not explicitly model individual growth [9, 1] but even so the delayed structure of DDMs are beneficial in modeling the lag between egg production and recruitment into the reproducing population. Nevertheless, DDMs are capable of exactly representing simple ASMs with explicit representations of simple individual growth and maturity dynamics [7, 32, 33]. When information about growth is available DDMs have enough growth and maturity infrastructure present to make use of the information without requiring the use of heavily parameterized age structured models e.g. [?]. The relatively smaller size, and flexibility, of DDMs makes them ideal for modeling “data-poor” stocks, while entertaining hypotheses that relate growth and maturity.

While DDMs are effective and flexible data-poor models, there use in stock assessments has been limited by the accessibility of RP calculations [?, ?]. While there has been work to develop accessible RPs using DDMs [?, ?], this work expands those efforts, by deriving analytical expressions for DDMs (with growth) under the the schnute modeled of recruitment. Munyandorero2023 gives RP calculations for this

868 model under BH, Ricker, and Shepard models, this work adds to that work by deriving
 869 the results under the Schnute model. The BH and Ricker special cases of the schnute
 870 reiterate [?] calucations, While the schunte model etends thos results by also provid-
 871 ing calucations under the Schafer and Cushing models as dattional special cases. the
 872 schnute model unifes these models with a single analytical result. These calculations
 873 reveild that RPs are largely dtermined by the form of R. While growth does have an
 874 effect on RPs in these models, the effect is secondary to the form of R.

875 4.2 Methods

876 4.2.1 Delay Differential Model

877 Age structured fisheries models typically
 878 assume [39, VB] gorwth in length with
 879 age. To model weight the assumption of
 880 VB growth in length is composed with a
 881 power law relating length to weight, $w = al^b$.
 882 Since b is usually ~ 3 this composition of as-
 883 sumed functional forms typically results in a
 884 monotonically increasing sigmoidal curve of
 885 weight with age. When $b \leq 1$ weight at age
 886 takes a VB-like form with $b = 1$ resulting in
 887 an exact correspondence of simulanious VB-
 888 growth in length and weight.

The delay model slightly abridges these

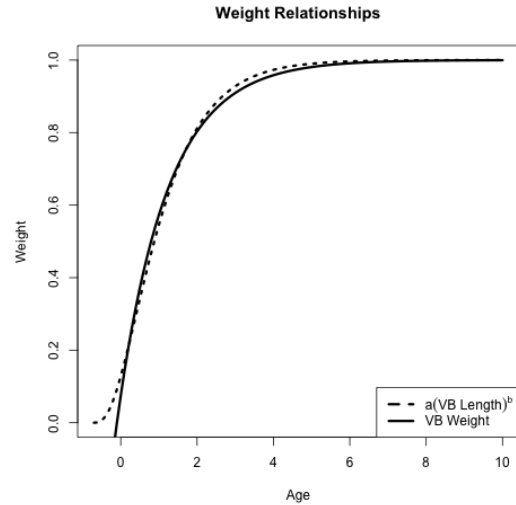


Figure 4.1: The typical composition of allometric weight ($b = 3$) with VB growth in length, as approximated by VB growth in weight directly.

relationships by directly assuming VB growth in weight as follows,

$$w(a) = w_{\infty}(1 - e^{-\kappa(a-a_0)}). \quad (4.1)$$

889 κ is a parameter that controls the instantaneous rate of individual growth (in weight)
890 with age. w_{∞} is the maximum weight of individuals in the population, and $w(a)$ is the
891 average weight of an individual at age a . The parameter a_0 controls the age at which
892 individuals are assumed to have zero weight; by letting $a_0 < 0$ this allows fish of age
893 zero to have positive weight. Rather than taking a sigmoidally increasing function, VB
894 growth directly in weight results in an monotonically inceasing curve that asymptotes
895 with a strictly decreasing growth rate with age. (only a good approximation for older
896 ages where growth begins to decline)

897 Together with VB growth, the delay model is derived from the assumption that
898 both natural mortality and fishing selectivity are separately propotional to a common
899 heavyside step function with age. That is to say, before a threshold age of selectivity,
900 a_s , the population is assumed not to experience any mortality whatsoever, but all fish
901 older then a_s experience the same rate of natural mortality. Simulaneously all fish older
902 than a_s are equally vulnerable to fishing (i.e. knife edge selectivity at age a_s), although
903 fishing effort may vary from through time.

[40] shows that within these assumptions the following delay differential system of equations exactly models the population dynamics of the total exploitable biomass $B(t)$ and number of individuals $N(t)$ through time.

$$\frac{dB}{dt} = w(a_s)R(B; \theta) + \kappa[w_{\infty}N - B] - (M + F)B \quad (4.2)$$

$$\frac{dN}{dt} = R(B; \theta) - (M + F)N \quad (4.3)$$

904 This formulation separates the number of individuals in the population from the

905 biomass of the population. The dynamics of N , as seen in Eq (4.3), are very similar to
 906 that of the **production models previously presented**, however the role of the production
 907 function is now filled by a "recruitment" function, $R(B)$, which describes the number
 908 of new individuals recruiting into the exploitable population as a function of exploitable
 909 biomass. In turn, the biomass dynamics are coupled to the numbers dynamics by the
 910 assumption of VB growth with growth parameters appearing in Eq (4.2), converting
 911 population numbers into biomass and accounting for the growth of biomass with age.

912 Eq (4.2) of the above model expands the notion of biomass production into the
 913 processes of recruitment, individual growth, and maturity. The term $w(a_s)R(B; \theta)$ rep-
 914 resents the biomass of new recruits; with $w(a_s)$ representing the weight of individuals
 915 at the age of maturity, a_s , and $R(B; \theta)$ representing the number of new recruits entering
 916 the exploitable population at time t . The negative term, $(M + F)B$, represents all causes
 917 of mortality as it is applied to biomass. Finally, the term $\kappa[w_\infty N - B]$ accounts for
 918 the net growth of the existing biomass by discounting the limiting maximal individual
 919 growth rate by metabolic weight loss proportional to $B(t)$. This term, together with the
 920 delay structure in R , provides the major computational savings of the delay differential
 921 setting, as compared with full age structured models, by automatically keeping track of
 922 changes in the mean size and growth associated with changes in recruitment as cohorts
 923 mature into the population.

Often a BH functional form is assumed for the stock recruitment relationship, but
 any adequately flexible family of functions may model this relationship. For the sake
 of evaluating the adequacy of assumed BH recruitment the simulation setting below is
 derived for the delay model under the assumption of the generalized three parameter
 Schnute recruitment as follows.

$$R(B; [\alpha, \beta, \gamma]') = \alpha B(t - a_s) (1 - \beta \gamma B(t - a_s))^{\frac{1}{\gamma}} \quad (4.4)$$

924 The parameters $\theta' = [\alpha, \beta, \gamma]$ function similarly in this setting as previously described
 925 in Section (??). That said, since the delay model explicitly parses out growth in it's
 926 dynamics, these parameters only describe the net processes of larval production, and
 927 maturation into the population, where as the production model used these parameters to
 928 also model the net effects of growth on biomass production. The γ parameter general-
 929 izes the family to model varying degrees of decreasing recruitment for large biomasses
 930 as γ increases. The Schnute function is exactly equivalent to BH recruitment at the
 931 special case when $\gamma = -1$, it passes through the Ricker model as $\gamma \rightarrow 0$, and Logistic
 932 recruitment occurs when $\gamma = 1$.

933 Since the delay model assumes knife edge selectivity, at age a_s , the term $B(t -$
 934 $a_s)$ appears in R . That is to say fish recruiting into the exploitable population are the
 935 result of larval production of biomass a_s time units in the past. This is because fishing
 936 selectivity is only assumed to occur for fish that are at least a_s time units old and thus
 937 fish younger than a_s are not exploitable. This waiting period requires that new recruits
 938 be the result of spawning biomass a_s time units in the past. Modeling maturity in this
 939 way results in dynamics equations which are a system of delay differential equations as
 940 opposed to the simple ODEs that arise in the production model setting.

941 \sim interpretation of recruitment (larval production, recruitment) [growth external]
 942 vs. production (larval production, recruitment, growth)

943 • general structure: [40] [14, pg. 334]

944 • growth: [39]

945 • recruitment: [32, 34]

4.2.2 Reference Points

Deriving reference points for the delay model under Schnute recruitment is conceptually similar to the production model setting. The additional nonlinear VB growth assumptions along side Schnute recruitment quickly make the expressions look somewhat unweildy, although analytical solutions can still be derived for most of the same quantities (although complicated by growth parameters).

Starting from Eqs. (4.2) and (4.3), setting both $\frac{dB}{dt}$ and $\frac{dN}{dt}$ simultaneously equal to zero, and solving for B and N as a function of fishing, gives the equilibrium biomass and numbers equations.

$$\bar{B}(F) = \frac{1}{\beta\gamma} \left(1 - \left(\frac{(F+M)(F+M+\kappa)}{\alpha w(a_s)(F+M + \frac{\kappa w_\infty}{w(a_s)})} \right)^\gamma \right) \quad (4.5)$$

$$\bar{N}(F) = \frac{\alpha \bar{B}(F)(1 - \beta\gamma \bar{B}(F))^{1/\gamma}}{F+M} \quad (4.6)$$

Eq. (4.6) is just $\frac{R(\bar{B})}{F+M}$, and is coupled to $\bar{B}(F)$ where most of the dynamics appear. Eq. (4.5) resembles Eq (3.3) from the simple production model setting although the growth parameters κ , w_∞ and $w(a_s)$, make slight adjustments to the balance of the maximum rate of recruitment and mortality rate to give an expression for equilibrium biomass that accounts for the factors of individual growth.

Expressions for B_0 and B^* are attained by evaluating $\bar{B}(F)$ at $F = 0$ and $F = F^*$ respectively. Calculation of F^* typically involves maximization of equilibrium yield, $\bar{Y} = F\bar{B}(F)$. While it was not possible to analytically maximize \bar{Y} , stable numerical solutions for calculating F^* were obtained by numerically solving for the roots of the analytical derivative of equilibrium yield with respect to F . Below a greatly simplified expression for $\frac{d\bar{Y}}{dF}$ is shown; the substitution $Z = F + M$ (total mortality rate) has been made to produce a more compact expression.

$$\frac{d\bar{Y}}{dF} = \frac{1}{\beta\gamma} \left[1 - \left(\frac{Z(Z+\kappa)}{\alpha w(a_s)(Z + \frac{\kappa w_{\infty}}{w(a_s)})} \right)^{\gamma} - \left(\frac{\gamma F}{\alpha w(a_s)} \right) \left(\frac{Z(Z+\kappa)}{\alpha w(a_s)(Z + \frac{\kappa w_{\infty}}{w(a_s)})} \right)^{\gamma-1} \left(1 + \frac{\left(\frac{\kappa w_{\infty}}{w(a_s)} \right) \left(\kappa - \frac{\kappa w_{\infty}}{w(a_s)} \right)}{\left(Z + \frac{\kappa w_{\infty}}{w(a_s)} \right)^2} \right) \right] \quad (4.7)$$

964 F^* is calculated as the numerical root, w.r.t. F , of the above expression. The numerical
 965 root is calculated using the base R uniroot function which employs a derivative free
 966 search given by [3].

967 4.2.2.1 BH Constraint

968 In the simple production model the
 969 BH constrained RPs are fixed to $\frac{1}{x+2}$. In
 970 the delay differential modeling setting the
 971 constrained BH RP set is complicated by
 972 the growth parameters a_s and κ . Under
 973 BH recruitment these parameters of the
 974 delay model slightly influence this rela-
 975 tionship as seen in Figure (4.2). That
 976 said, the influence of a_s and κ on RPs
 977 is still largely limited to a confined region
 978 of reference point space which resembles
 979 the $\frac{1}{x+2}$ form. In fact the confined region
 980 of RPs is bounded above by $\frac{1}{x+2}$. In Fig-
 981 ure (4.2) notice that for values of a_s and κ that result in high $w(a_s)$ (high values of κ
 982 and small values of a_s seen in red) the BH RP space converges to $\frac{1}{x+2}$ as derived in the
 983 simple production model setting. In opposition to the simple production model limit,
 984 when $w(a_s)$ is low (as seen in the more blue region of Figure(4.2)), RPs decrease as the
 985 influence of growth in the dynamics increases.

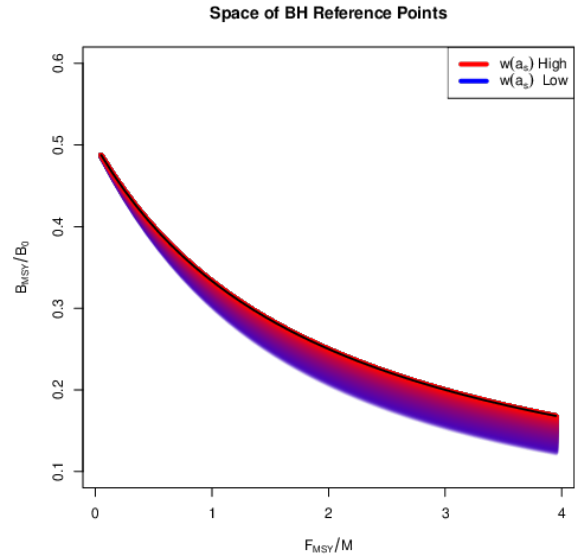


Figure 4.2: The space of BH RPs for the delay model as a function of κ and a_s . The RP space is plotted for 80×80 combinations of $\kappa \in [0.1, 2]$ and $a_s \in [0.1, 10]$. The color drawn is the resulting value of $w(a_s)$ mapped between blue and red. $\frac{1}{x+2}$ is plotted in black for reference.

986 4.2.3 Delay Differential Integration

987 The delay model belongs to a class of differential equations known as delay dif-
988 ferential equations (DDE). The delay arises from the $B(t - a_s)$ terms found in the
989 recruitment function. Solving DDEs require special care which depends on the nature
990 of the time delay. The addition of time-varying delays, many different delays, or very
991 small delays (delays below the step size of the numerical integrator) results in some
992 of the more challenging settings for solving DDEs. However with a single stationary
993 model of the age of selectivity, the delay model in this setting represents one of the most
994 straight forward DDE structures. The most numerically challenging case presented here
995 arises in the case of the limiting production model when $a_s \rightarrow 0$ while $\kappa \rightarrow \infty$. That
996 said the limiting production model can be approximated for values of $a_s \approx 0.1$, and it
997 was straightforward to ensure that the step size of the integrator remained reasonably
998 below 0.1.

999 The DDE presented here is integrated with the initial values fixed at B_0 and N_0 as
1000 given by Eqs. (4.5) and (4.6) with $F = 0$ at any given configuration of θ and growth
1001 parameters. The system given in Eqs. (4.2) and (4.3) are then solved numerically
1002 using the implicit Livermore Solver (lsode) as implemented in the `dede` function of
1003 the R package `deSolve` [37]. The `dede` solver provides many methods for integrating
1004 DDEs, but `lsode` was chosen because it is an implicit method that runs relatively quickly
1005 with a relatively smaller footprint in system memory as compared with other methods.
1006 The `radau` method was also tried in more computationally challenging settings with
1007 good results (albeit running more slowly than `lsode`). Ultimately the simulated parameter
1008 space did not produce DDEs that require the more expensive `radau` integrator to solve
1009 accurately.

4.2.4 Simulation Design

Similarly as previously described in Section (3.2.2) the relationship between RPs $\mapsto \theta$ cannot be fully expressed analytically for the Schnute delay model. However, just as in the production model setting, simulation only requires enough knowledge of these mappings to gather a list of (α, β, γ) tuples and the corresponding RPs in some reasonable space-filling design over RP space.

In the delay model a partial mapping for $(F^*, B_0) \mapsto (\alpha(\cdot, \gamma), \beta(\cdot, \cdot, \gamma))$ can be derived analytically in terms of RPs and γ . The substitution $Z^* = F^* + M$ is made where F^* and M appear together to produce a more compact expression.

$$\alpha = \left[\left(\frac{Z^*(Z^* + \kappa)}{w(a_s)(Z^* + \frac{\kappa w_\infty}{w(a_s)})} \right)^\gamma + \left(\frac{\gamma F^*}{w(a_s)} \right) \left(\frac{Z^*(Z^* + \kappa)}{w(a_s)(Z^* + \frac{\kappa w_\infty}{w(a_s)})} \right)^{\gamma-1} \left(1 + \frac{\left(\frac{\kappa w_\infty}{w(a_s)} \right) \left(\kappa - \frac{\kappa w_\infty}{w(a_s)} \right)}{(Z^* + \frac{\kappa w_\infty}{w(a_s)})^2} \right) \right]^{\frac{1}{\gamma}} \quad (4.8)$$

$$\beta = \frac{1}{\gamma B_0} \left(1 - \left(\frac{M(M + \kappa)}{\alpha w(a_s)(M + \frac{\kappa w_\infty}{w(a_s)})} \right)^\gamma \right) \quad (4.9)$$

Above Eq. (4.8) results from setting Eq. (4.7) equal to zero and solving for α , and Eq. (4.9) results from solving the $\bar{B}(0)$ expression, as derived from Eq. (4.5), for β . The system is completed by further working with the $\frac{\bar{B}(F^*)}{\bar{B}(0)}$ expression, as seen below, to identify γ .

$$\frac{B^*}{B_0} = \frac{1 - \left(\frac{(F^* + M)(F^* + M + \kappa)}{\alpha w(a_s)(F^* + M + \frac{\kappa w_\infty}{w(a_s)})} \right)^\gamma}{1 - \left(\frac{M(M + \kappa)}{\alpha w(a_s)(M + \frac{\kappa w_\infty}{w(a_s)})} \right)^\gamma} \quad (4.10)$$

The system formed by collecting Eqs. (4.8), (4.9), and (4.10) can be navigated similarly to Eq. (3.9) in the Schnute production model setting. For a population experiencing natural mortality M , VB growth with paramters κ and w_∞ , and age of selectivity a_s the above system can fully specify α and β for a given γ , by fixing F^* , B_0 , and $\frac{B^*}{B_0}$. For a given γ a cascade of closed form solutions for α and β can be obtained, just as in

1024 Section (3.2.2). First $\alpha(\gamma)$ can be computed, and then $\beta(\alpha(\gamma), \gamma)$ can be computed. If
 1025 $\alpha(\gamma)$ is filled back into the expression for $\frac{B^*}{B_0}$, the system collapses into a single oner-
 1026 ous expression for $\frac{B^*}{B_0}(\alpha(\gamma), \gamma)$. For brevity, define the function $\zeta(\gamma) = \frac{B^*}{B_0}(\alpha(\gamma), \gamma, F^*, M)$
 1027 based on Eq. (4.10).

1028 Again rather than inverting $\zeta(\gamma)$ for γ , γ is the sampled so that the overall simulation
 1029 design is space filling as described in Section (3.2.4). Given the sampled γ , the cascade
 1030 of $\alpha(\gamma)$, and then $\beta(\alpha(\gamma), \gamma)$, can be computed, and the Schnute delay model is fully
 1031 defined by a given $(\frac{F^*}{M}, \frac{B^*}{B_0})$. While conceputally this framing is similar to the Schnute
 1032 production model, the analytical expressions are more complex, and numerically trech-
 1033 erous, since growth parameters appear explicitly here. Other ways of navigating the
 1034 RPs $\mapsto \theta$ system are possible, but for the sake of numerical stability this strategy has
 1035 proven the most reliably accurate by limiting exposure to numerical error propogation.

1036 Each design location defines a complete Schnute delay differential model with the
 1037 given RP values. Indices of abundance are simulated from the Schnute model at each
 1038 design location, a small amount of residual variation, $\sigma = 0.01$, is added to the simu-
 1039 lated index, and the data are then fit with a misspecified BH model. The design captures
 1040 various degrees of model misspecification relative to the BH model, so as to observe
 1041 the effect of recruitment misspecification upon RP inference.

1042 **point to catch, and LHS design, and Metamodel.**

1043 4.2.5 Parameter Estimation

- 1044 • I use B only here
- 1045 • quick statement of inference, and reference to previous section

Let I_t , $t \in \{1, 2, 3, \dots, T\}$, be a series of indicies of abundance, proportional to biomass, as simulated from the Schnute Delay model. These data are modelled with the following log-normal observation model that has been intentionally constrained to

BH recruitment,

$$I_t \sim LN(qB_t(\boldsymbol{\theta}, \boldsymbol{\phi}), \sigma^2). \quad (4.11)$$

1046 $B_t(\boldsymbol{\theta}, \boldsymbol{\phi})$ is the biomass solution of the BH constrained DDE system. The BH constraint
1047 is implemented by fixing $\gamma = -1$ so that $\boldsymbol{\theta}' = [\alpha, \beta, \gamma = -1]$. $\boldsymbol{\phi}$ is a vector of growth
1048 and maturity parameters, $\boldsymbol{\phi}' = [\kappa, w_\infty, a_0, a_s]$. The nuisance parameter q models the
1049 proportionality constant of the index with process biomass, and σ^2 models residual
1050 variation of the index.

1051 In this setting, $\boldsymbol{\phi}$ and q are fixed to focus on the inferential affects of model misspec-
1052 ification on recruitment parameters and RPs. Without an explicite mechanism for the
1053 delay model to incorporate age data, under the BH model $\boldsymbol{\phi}$ is not well informed and
1054 would typically be estimated externally for data limited stocks. Under BH recruitment $\boldsymbol{\phi}$
1055 can only slightly impact RPs as seen in Figure (4.2).

1056 σ^2 and $\boldsymbol{\theta}$ are reparameterized to the log scale and fit via MLE. Reparameterizing
1057 the parameters to the log scale improves the reliability of optimization, in addition to
1058 facilitating the use of Hessian information for estimating MLE standard errors. Given
1059 that the biological parameters enter the likelihood via a nonlinear differential equation,
1060 and further the parameters themselves are related to each other nonlinearly, the like-
1061 lihood function can often be difficult to optimize. A hybrid optimization scheme is
1062 used to maximize the log likelihood to ensure that a global MLE solution is found.
1063 The R package GA [35, 36] is used to run a genetic algorithm to explore parameter
1064 space globally. Optimization periodically jumps into the L-BFGS-B local optimizer to
1065 refine optima within a local mode. The scheme functions by searching globally, with
1066 the genetic algorithm, across many initial values for starting the local gradient-based
1067 optimizer. The genetic algorithm serves to iteratively improve hot starts for the local
1068 gradient-based optimizer. Additionally, optimization is only considered to be converged

1069 when the optimum results in an invertible Hessian at the found MLE.

- 1070 • fixed $M = 0.2$, $a_0 = -1$, $w_\infty = 1$
- 1071 • play with κ and age of selectivity a_s

1072 4.2.5.1 Numbers Indices

While not utilized here, age structured models may commonly model indices as proportional to numbers rather than (or simultaneously to) biomass. When solving the DDE, Eq. (4.3) points out that the full DDE solution will expose a numbers solution simultaneously with a biomass solution that may be used for these purposes. These solutions are often quite similar since the main driver of process behavior comes from the form of R which is shared among N and B . However, it is common on the west coast of the US that indices derived from commercial fisheries are measured as weights while indices derived from recreational fisheries are often measured as counts. If a numbers index, J_t , is observed alongside the previously mentioned biomass index, the following likelihood component is often added as a conditionally independent component of the likelihood,

$$J_t \sim LN(pN_t(\theta, \phi), \tau^2). \quad (4.12)$$

1073 $N_t(\theta, \phi)$ is the numbers solution of the DDE system. θ and ϕ are the productivity
1074 and growth parameters shared in common with the biomass component. p and τ^2 are
1075 then the analogous proportionality constant and residual variation of the numbers index
1076 respectively.

1077 4.2.6 GP Metamodel

1078 point to catch, and LHS design, and Metamodel.

1079 4.2.7 Clustering Model Failure

1080 Considering the behavior observed in Section (??), where $\frac{F_{MSY}}{M}$ is dramatically un-
1081 derestimated, it is natural to ask where specifically in RP space we might see this catas-
1082 trophic failure of the BH model. The structure of RPs under the BH model suggests sev-
1083 eral natural avenues for forming hypotheses to identify highly misspecified RP regions.
1084 The single clearest feature to identify are cases where $\frac{F_{MSY}}{M}$ is heavily under-estimated.
1085 Here this idea is expressed by a hypothesis testing inspired framework that uses the GP
1086 metamodel to propagate estimate uncertainty across the simulated space of misspec-
1087 ified BH RPs. This allows for a rejection threshold (against the null hypothesis that
1088 BH RP estimates are unbiased) to be derived in terms of the GP predictive structures
1089 to define a classifier for identifying where BH inference breaks down broadly over RP
1090 space.

1091 Recall that the metamodel models MLE estimates of $\log(F_{MSY})$ under the misspec-
 1092 ified BH model. Thus, for a given set of RPs, \mathbf{x} , of the BH metamodeled quantity is
 1093 given by kriging prediction as $N(\hat{y}(\mathbf{x}), \hat{\sigma}^2(\mathbf{x}))$, where $\hat{y}(\mathbf{x})$ is the kriging mean (as pre-
 1094 viously described in Eq. (??)) and $\hat{\sigma}^2(\mathbf{x})$ provides estimate uncertainty via the kriging
 1095 predictive variance given by,

$$\hat{\sigma}^2(\mathbf{x}) = \mathbf{R}(\mathbf{x}, \mathbf{x}) - \mathbf{r}(\mathbf{x})' \mathbf{R}_\ell^{-1} \mathbf{r}(\mathbf{x}). \quad (4.13)$$

1096 Model failure with respect to estimating $\frac{F_{MSY}}{M}$ under the BH model is measured by
 1097 the percent error as previously described in Section (??). When the BH model estimates
 1098 $\frac{F_{MSY}}{M}$ well the percent error is expected to be small in the following sense,

$$\frac{\frac{F_{MSY}}{M} - \frac{\hat{F}_{MSY}}{M}}{\frac{F_{MSY}}{M}} < P. \quad (4.14)$$

1099 P defines the extent of model failure on the scale of percent error. For measuring
 1100 catastrophic model failure P was chosen to be 0.5, but smaller values of P may be
 1101 chosen to emphasize regions of more subtle model failure. Thus when the percent
 1102 error is statistically greater than P the notion that the BH model estimates $\frac{F_{MSY}}{M}$ well (in
 1103 P -sense) is rejected.

1104 For statistical evaluation, it is convenient to rearrange Eq. (4.14) as $\hat{F}_{MSY} > (1 - P)F_{MSY}$.
 1105 \hat{F}_{MSY} is then distributed as $LN(\hat{y}(\mathbf{x}), \hat{\sigma}^2(\mathbf{x}))$, and the rejection region is then defined
 1106 as the RP's for which the 5th percentile from the Log-normal distribution falls below
 1107 $(1 - P)F_{MSY}$.

4.3 Results

Biological Regeim $\text{corr}(a_s, \kappa) < 0$

Figure (4.3) shows three hypothetical individual-growth/maturity curves that span a wide range of RPs. As seen in Figure (4.2), the larger values of $w(a_s)$ correspond to larger recruits relative to maximum size. This leaves little growth to be evaluated by the biomass dynamics equations; the red curve demonstrates the simple (no growth) production model limit ($a_s \rightarrow 0$ and $\kappa \rightarrow \infty$). The cases shown with smaller $w(a_s)$ values (blue and purple curves) correspond to slower growth behaviors. The blue curve, where $a_s = 2$ and $\kappa = 0.1$, emphasizes the effect of growth on the biomass dynamics most among these examples.

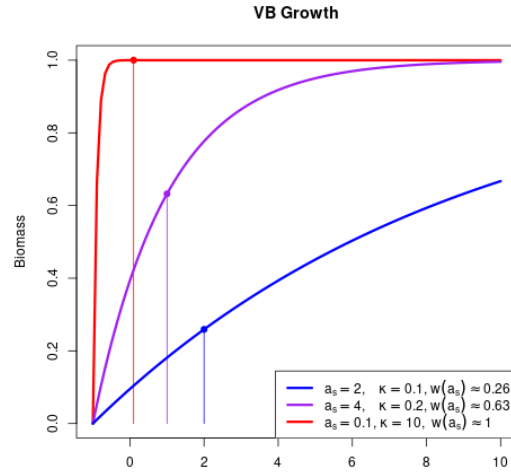


Figure 4.3: Three hypothetical individual-growth curves, showing $w(a_s)$ on each curve.

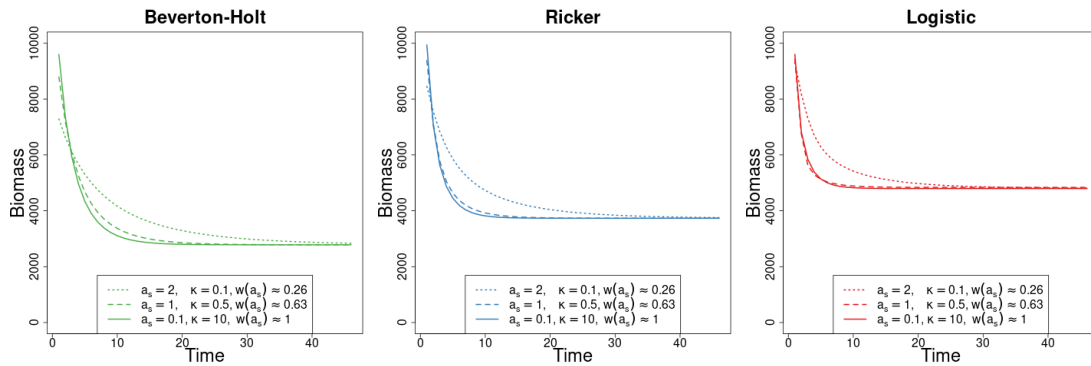


Figure 4.4: Biomass dynamics of BH (*left*), Ricker (*center*), and Logistic (*right*) delay differential models in the low contrast simulation setting. In all cases $\alpha = 1.2$ and β is chosen so that each model shares the same B_{MSY} within each given γ .

Figure (4.4) demonstrates a range of biomass dynamics that the Schnute delay model can display under a spectrum of growth behaviors with fishing held consistent at F_{MSY} . The three special cases of $\gamma = -1$ (BH), $\gamma \rightarrow 0$ (Ricker), and $\gamma = 1$ (Logistic) recruitment are shown in each of the above growth configurations.

Figure (4.5) shows the range of RPs that can be modeled with each of the BH, Ricker, and Logistic recruitments over the spectrum of individual-growth/maturity models simulated here. Notice for smaller values of $w(a_s)$ the further the RP curve lies from the simple production model, and each recruitment model reacts slightly differently under each of the given growth parameters. The Ricker and BH RP-spaces are qualitatively similar in shape with smaller values of $w(a_s)$ decreasing $\frac{B_{MSY}}{B_0}$ relative to the simple production model setting. The Logistic model on the other hand increases $\frac{B_{MSY}}{B_0}$ relative to the simple production model setting as smaller values of $w(a_s)$ decreases. It is also worth noting that the Ricker model's RPs are much less influenced by growth parameters as compared with that of the BH or Logistic model.

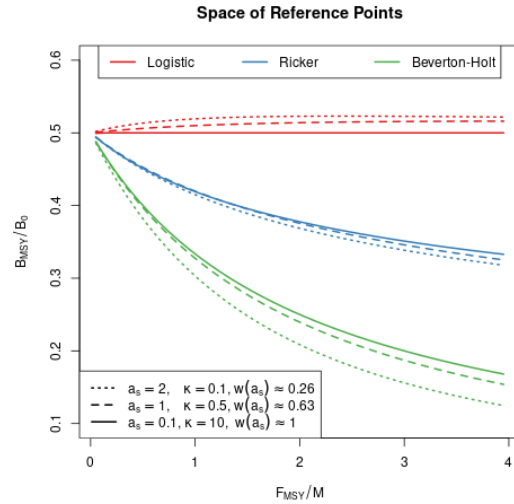


Figure 4.5: Restricted RP-space under each recruitment models, with each growth curve.

4.3.1 Simple Production Model Limit

Under the delay differential's limiting simple production model ($a_s = 0.1$ and $\kappa = 10$), the expectation is that RP inference should be identical to that of the model seen in Chapter (3). By way of verifying this equivalence, Figure (4.6) demonstrates a virtually identical pattern of RP biases as previously seen in Figures (3.7) and (3.8) (under both of the high and low contrast settings).

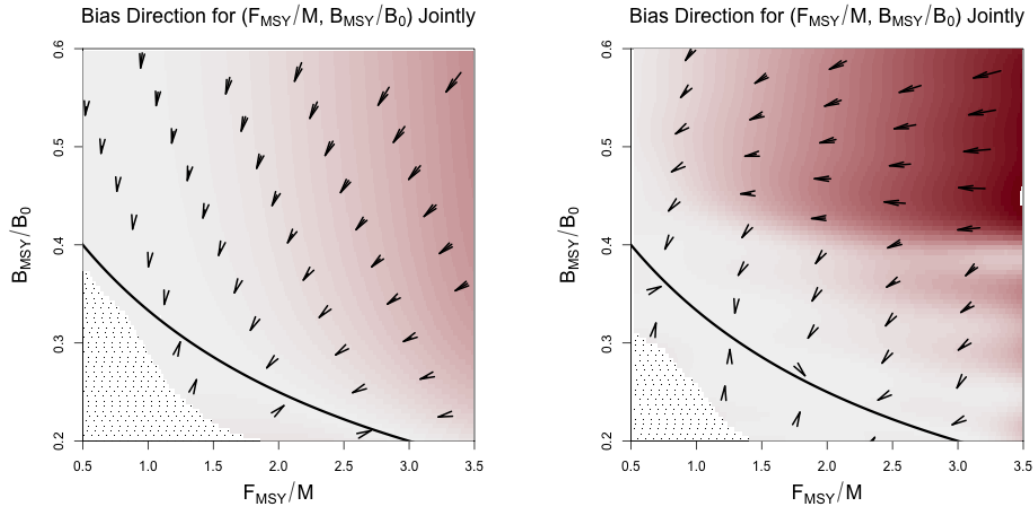


Figure 4.6: RP mapping of BH delay model fit to Schnute delay data under the simple (no growth) production model limit. *left* : High contrast simulation. *Right* : Low contrast simulation.

Indeed in the high contrast setting, Figure (4.6, *left*) shows how the BH model induces the same pattern of bias as seen in Chapter (3). There is bias in both RPs (in accordance with the $\frac{B^*}{B(0)} = \frac{1}{F^*/M+2}$ RP-set) so as to produce a nearly minimal distance mapping of RPs onto the constrained BH set of RPs. Similarly, in the low contrast setting, Figure (4.6, *right*) again shows the same two regimens pattern of RP inference. Firstly, there is a region of relatively small model misspecification where the minimal distance mapping is preserved. Secondly, as model misspecification becomes greater (around the Ricker set) $\frac{F^*}{M}$ begins to be sharply underestimated. Above this break point

1159 in RP estimation inference appears to be driven toward the trivial RP $\frac{F^*}{M} = 0, \frac{B^*}{B(0)} = 0.5$
1160 that is shared in common among all of the two-parameter models described here.

1161 These results confirm that the theoretical limiting dynamics do indeed replicate
1162 expected RP inference patterns as previously observed in Chapter (3).

1163 4.3.2 Moderate Growth

1164 Moving past the simple production model, other values of a_s and κ provide a probe
1165 into the effects individual growth dynamics may have on RP inference. Individual
1166 growth is a multifaceted phenomena that is not easily reduced to a single number, but
1167 for the purposes of this model $w(a_s)$ serves as a decent proxy for the extent of the model
1168 dynamics that are due to individual growth. This follows from the intuition that individ-
1169 uals maturing at a smaller fraction of w_∞ demonstrate the dynamics of growth during
1170 an observable (to the model) phase rather than growth occurring prior to selection.

1171 That said, $w(a_s)$ is not a one-to-one map of κ and a_s . A level curve of $w(a_s; \kappa) =$
1172 c is attained by increasing the value of a_s and decreasing κ correspondingly, or vice
1173 versa. The case where $a_s = 1$ and $\kappa = 0.5$ (resulting in $w(a_s) \approx 0.6$) represents a
1174 reasonable biological example of moderate growth. Similar examples of the $w(a_s) =$
1175 0.6 level curve result in much larger lags (discussed in Section (4.3.5)) or larger κ 's
1176 which quickly tend toward behaviors previously described in the simple production
1177 model setting.

1178 The RP mappings seen in Figure (4.7) show very similar RP mappings to that of
1179 the simple production model, with the biggest differences occurring around the location
1180 of the break point where the low contrast model begins to dramatically underestimate
1181 $\frac{F^*}{M}$. In the high contrast simulation setting Figure (4.7; *left*), the RP mappings again
1182 demonstrate a nearly identical minimal distance mapping of RPs onto the constrained
1183 BH RP set. In the low contrast setting Figure (4.7; *right*) a very similar two regiem
1184 pattern of RP inference is observed, however the location of the break between these

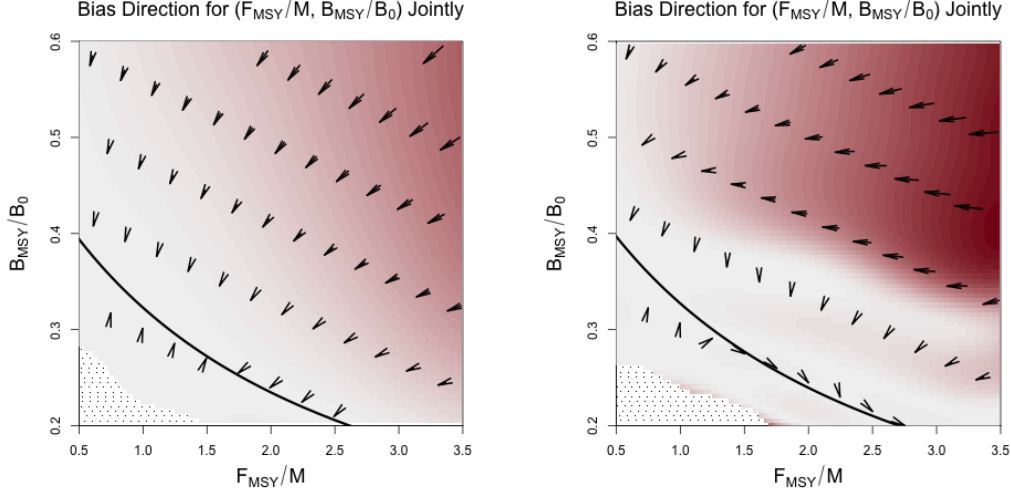


Figure 4.7: RP mapping of BH delay model fit to Schnute delay data under moderate growth ($a_s = 1$ and $\kappa = 0.5$). *Left* : High contrast simulation. *Right* : Low contrast simulation.

regiems appears at lower values of $\frac{B^*}{B(0)}$. In this moderate growth setting the break point
occures around values of $\frac{B^*}{B(0)}$ just below 0.4 as opposed to the simple production model
the break point occurs at $\frac{B^*}{B(0)}$ just above 0.4.

4.3.3 Emphatic Growth Dynamics

The emphatic growth setting simulated here fixes $a_s = 2$ and $\kappa = 0.1$, to simulate a
species that grows quite slowly and matures into the reproducing stock at a relatively
early age. This combination has the effect of exaggerating the components of the model
dynamics which are related to individual growth since individuals recruit at a small size
and slowly grow over the extent of the modeled period.

The slow growth of these dynamics oppose the simple production model setting in
the sense that they move the constrained RP set a large distance (largest among the
spectrum of decreasing $w(a_s)$ populations simulated here) away from the $\frac{1}{x+2}$ limiting
case. It is interesting to note that this is true for all of the two parameter constrained

1198 constrained RP sets as seen in Figure (4.5).

1199 Despite the heavily growth influenced driven biomass dynamics in this setting, the
 1200 RP mappings seen in Figure (4.8) obviously bare a huge resemblance to the previously
 1201 seen RP mappings. Again the biggest differences in the RP mappings occur around
 1202 the location of the break point where the low contrast model begins to dramatically
 underestimate $\frac{F^*}{M}$. In this low contrast setting the break point in RP estimation occurs

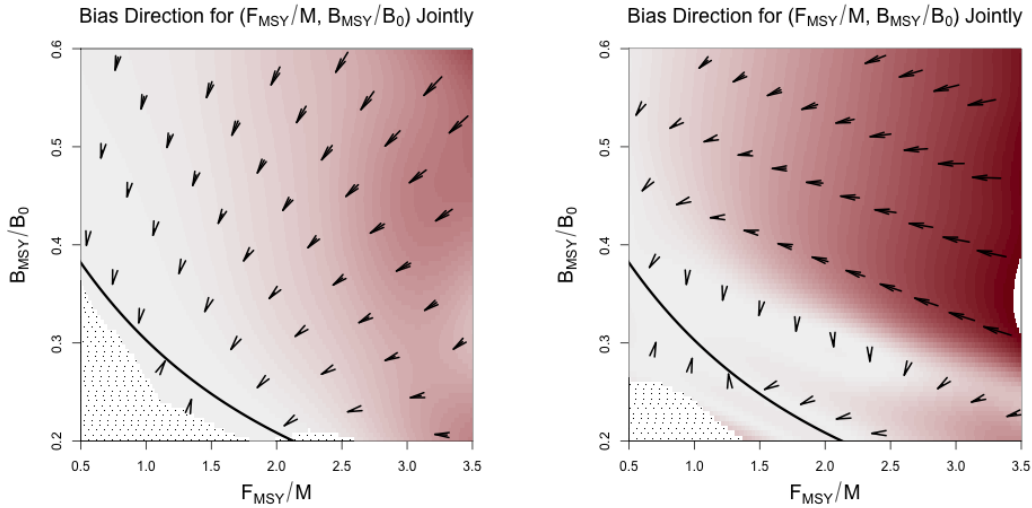


Figure 4.8: RP mapping of BH delay model fit to Schnute delay data under dramatic growth ($a_s = 2$ and $\kappa = 0.1$). *Left* : High contrast simulation. *Right* : Low contrast simulation.

1203 around values of $\frac{B^*}{B(0)}$ well below 0.4 with the behaviour extending as far down as $\frac{B^*}{B(0)} =$
 1204 0.3. This regiem shift occurs well below that of the Ricker set, as initially observed in
 1205 the production model setting. This reduced range of acceptable RP inference indicates
 1206 that under increasingly emphatic growth the model misspecification issue of the BH
 1207 model becomes an increasingly brittle assumption with respect of RPs.

1208 Interestingly this pattern only follows for the low contrast setting. In the high con-
 1209 trast setting inference returns to a pattern resmbleing the minimal distance mapping
 1210 onto BH RP set. Further pointing to the importance of contrast for informing these
 1211

models.

4.3.4 Clustering Catastrophic Model Failure

Figure (4.9) shows the rejection thresholds for the low contrast simulations of each of the emphatic, moderate, and no growth settings. The dark lines represent the rejection threshold with a false positive rate of about 15%, and the light shaded regions show how the rejection threshold changes as the false positive rate ranges from 50% to 2.25%. When applied to the high contrast simulations the rejection threshold falls outside of the simulated RP range as expected by inspection of the high contrast RP mappings.

Notice in Figure (4.9) that the rejection threshold is subject to two axes of sensitivity. Firstly, for each simulated growth the rejection threshold is more sensitive for small values of $\frac{F_{MSY}}{M}$ than for large values. This is a natural result since discerning $\hat{y}(x)$ below the minimum simulated RP becomes more difficult when the data are truly generated near the minimum simulated $\frac{F_{MSY}}{M}$. For large $\frac{F_{MSY}}{M}$ the minimum distance mapping results in $\hat{y}(x)$ well above the minimum simulated RP but for

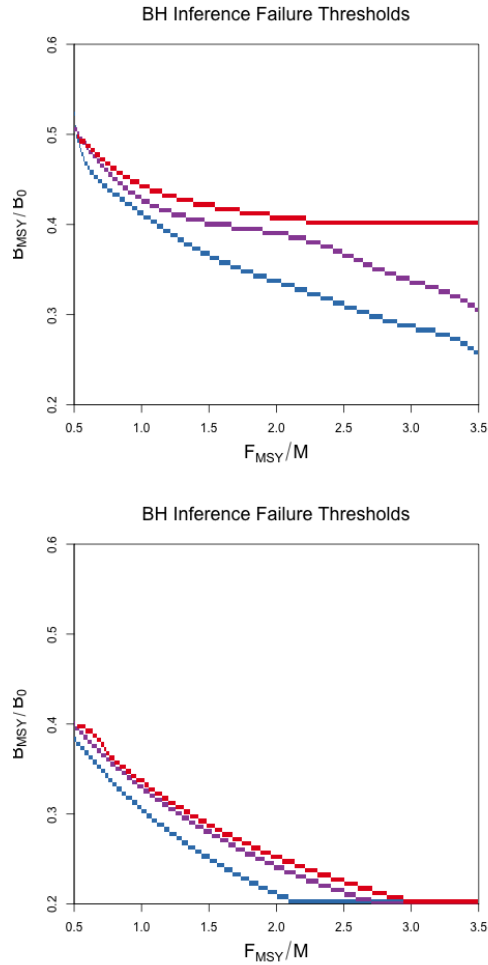


Figure 4.9: BH RP estimation failure thresholds with increasingly emphatic individual growth dynamics.

1238 small $\frac{F_{MSY}}{M}$ even the minimum distance mapping may be close to the rejection threshold.

1239 The second axis of sensitivity is between individual growth simulations. The no
1240 growth setting produces a very clear threshold of model failure, while the failure thresh-
1241 old for emphatic growth is much more varied, especially near the minimum simulated
1242 $\frac{F_{MSY}}{M}$. This is largely due to the increased RP estimate uncertainty as growth becomes
1243 more emphatic in the dynamics.

1244 Model misspecification of the BH model is compounded for the more emphatic
1245 growth settings as recruitment can interact with growth dynamics to produce unique
1246 behaviors as exemplified in Section (4.3.5).

4.3.5 Oscillatory Growth Influence

While the above patterns of RP estimation follow for biological regimens of the $w(a_s; \kappa) = c$ level curve, as a_s increases an oscillatory regimen also exists within these dynamics. While RP estimation behaves similarly in this oscillatory regimen there are unique features in this setting that are not present in the more biological regimens. Below consider the oscillatory example of a logistic delay model with $a_s = 10$ fixing fishing at F_{MSY} .

Figure (4.10) demonstrates the mechanism of how these oscillatory dynamics form. Oscillatory dynamics appear when fishing pushes biomass past B_{MSY} within the lagged a_s window of recruitment. The delay model assumes that biomass is fixed in equilibrium at B_0 , for $t \leq 0$. Therefore in the green region of the biomass series, $0 < t < 10$, the population recruits at $R(B_0)$. Figure (4.10) shows that in this initial period $R(B_0)$ results in zero yield for that period, and biomass falls as a result.

Once t exceeds a_s , the lagged recruitment refers to the integrated biomass series to evaluate recruitment based on $R(B_{t-a_s})$. The red region of the biomass series is the result of yield over the initial green biomasses. Figure (4.10) shows that the yield over the green biomass series first increases, as biomass approaches B_{MSY} and then decreases as biomass passes B_{MSY} . This creates the local maximum in the red biomass series.

Furthermore, the blue region of the biomass series is then based on yield over the red biomasses. Notice that since the red biomasses first increase and then decrease, yield increases as the red biomass increases toward B_{MSY} , and yield subsequently decreases following the descending leg of the red biomass series. This yield pattern carries the oscillation of the red biomass region forward into the blue region.

This process of biomass oscillation carries on in this manner nonetheless approaching equilibrium at B_{MSY} . Equilibrium is reached in an oscillatory manner set off by the green biomass series crossing over from above B_{MSY} to below it. The example shown in Figure (4.10) exemplifies the oscillatory phenomena simulated here, but the mecha-

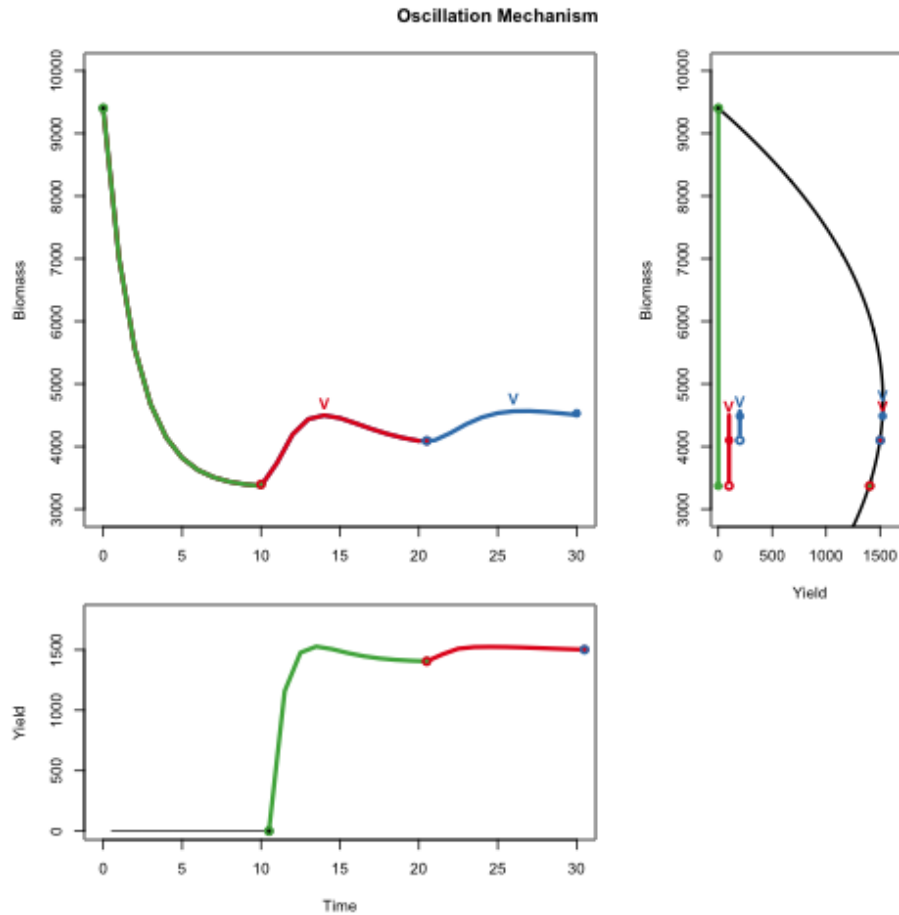


Figure 4.10: *top left* : Logistic biomass over 30 epochs of time with $a_s = 10$. Green, red, and blue colors indicate three 10 epoch long windows of biomass. v indicates local biomass oscillation maxima. *top right* : Yield plotted over the range of biomasses shown. The biomass range of each 10 epoch window is shown in the vertical colored lines. *bottom left* : Yield plotted through time. Colors correspond to the lagged biomass region that results in the evaluated yield. The black horizontal line demonstrates the pre-model assumption of biomass fixed at B_0 .

1274 nism that produces these oscillations may occur with other forms of recruitment outside
 1275 of logistic recruitment whenever fishing cases biomass to cross over B_{MSY} within the
 1276 lagged recruitment window.

4.3.5.1 RP Estimation

Statistical inference in the oscillatory regiem can be challenging. Depending on the parameters inferred, the likelihood can have multiple local modes which require global optimization techniques to distiguish. Furthermore, parameter estimation is more uncertain in this setting as the likelihood may confuse oscillations with residual noise.

Figure (4.11) shows the BH RP mapping fixing $w(10;0.1) \approx 0.6$ in the high contrast simulation setting. This places the dynamics firmly in the ocillatory regiem, but the high contrast setting provides significant information for inferring recruitment parameters.

Interestingly in this high contrast setting, a very similar two regiem pattern of RP inference is observed as previously seen in low contrast settings. That said the boundary between the regiems in this setting is much smoother and the location of the break between these regiems appears around higher values of $\frac{B^*}{\bar{B}(0)}$.

This higher $\frac{B^*}{\bar{B}(0)}$ break point, hovering around 0.5, is consistent with the mecha-

nism which induces ocillation. Starting the biomass at $\bar{B}(0)$ in the ocillatory regiem, increased $\frac{B^*}{\bar{B}(0)}$ will tend to exasterbate oscillatory behavior by increasing B_{MSY} so that biomass is more easily pushed past B_{MSY} within the initial lagged as window of recruitment. This produces more dramatic oscillations in the higher $\frac{B^*}{\bar{B}(0)}$ region of RP space.

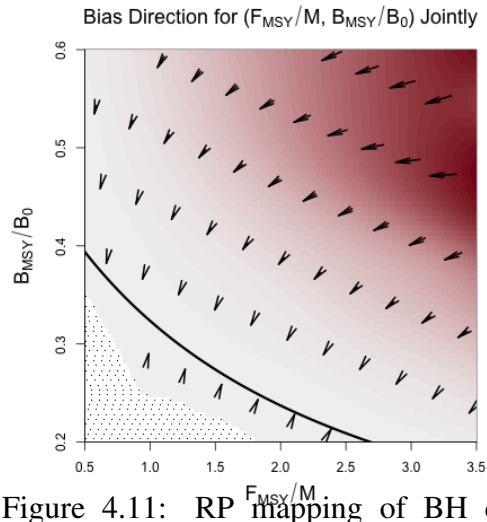


Figure 4.11: RP mapping of BH delay model fit to high contrast Schnute delay data under oscillatory growth ($a_s = 10$ and $\kappa = 0.1$).

1302 The fitted BH model does not pro-
 1303 duce significant oscillations because un-
 1304 der the BH model $\frac{B^*}{B(0)}$ is constrained be-
 1305 low 0.5 with the majority of the simula-
 1306 tion BH $\frac{B^*}{B(0)}$ RPs falling between 0.4 and
 1307 0.2. Therefore, the fitted BH model will
 1308 not tend to push biomass past B_{MSY} and
 1309 thus is incapable of modeling oscillatory
 1310 biomass series. Figure (4.12) shows a
 1311 subset of example BH fits, which demon-
 1312 strates the limited oscillatory capacity of
 1313 the BH fits. Furthermore, since the BH model has a limited oscillatory capacity in this
 1314 setting, the BH model tends to explain the oscillations with artificailly high residual
 1315 variation and artificailly low steepness focusing on overly simplistic trends in the data.

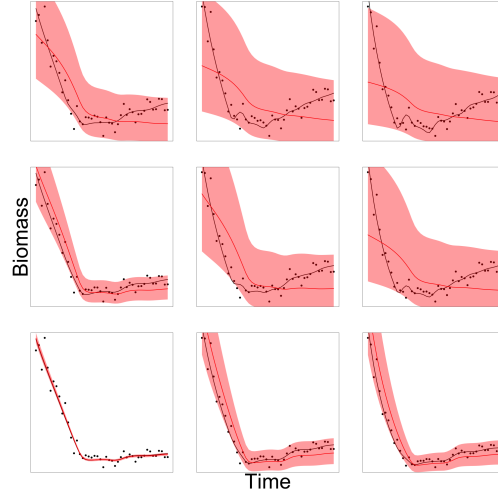


Figure 4.12: Example BH fits (*red*) to Schnute data (*black*). Each example plot is arranged to mirror its location in RP space.

1316 4.3.5.2 Estimating More

1317 Figure (4.13) shows a subset of ex-
 1318 ample model fits broadly over RP space.
 1319 Model fits are shown both under the two-
 1320 parameter BH model as well as under
 1321 the three parameter Schnute model, each
 1322 model estimating all of its recruitment pa-
 1323 rameters as well as the growth and matu-
 1324 rity parameters κ and a_s . Notice that the
 1325 BH model, even when additionally esti-
 1326 mating κ and a_s , does not gain the flexi-
 1327 bility to properly model Schnute data.

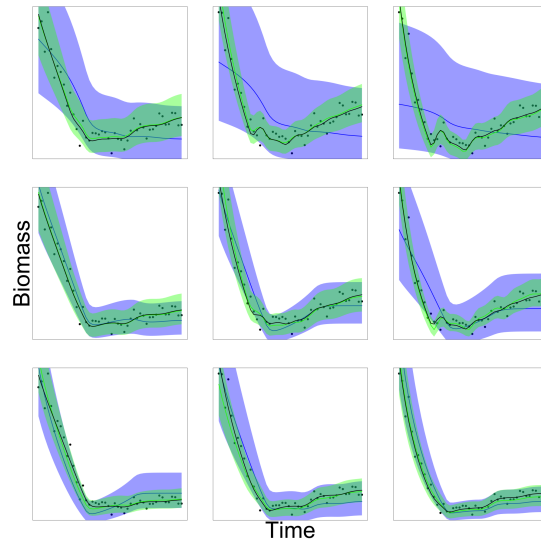


Figure 4.13: κ and a_s estimation under BH (*blue*) and Schnute (*green*) fits to Schnute data (*black*) arranged to mirror RP space.

1328 The lack of oscillatory dynamics pro-
1329 duced by the BH model causes the mis-
1330 specified BH fits in Figure (4.13) to
1331 largely estimates κ and a_s so as to approximate the production model limiting case.
1332 The fitted Schnute model can produce the oscillatory dynamics and thus the informa-
1333 tion in the oscillatory data well inform estimates of κ and a_s under the Schnute model.
1334 Furthermore, the Schnute model has no issue learning its γ parameter.

1335 While Statistical inference in the oscillatory regiem can be challenging in the highly
1336 constrained BH model, the Schnute model can easily estimate its extra γ parameter. The
1337 flexibility of estimating γ simplifies inference by correctly specifying RPs, and also by
1338 opening up the model dynamics to reveal additional information about κ and a_s in the
1339 data.

1340 4.4 Discussion

- 1341 • break point decreases with growth
- 1342 • inference becomes more brittle with more dramatic growth.
 - 1343 – interaction between assumed form of growth and stock recruitment.
 - 1344 – low-side steepness bias masks oscillatory/shock patterns induced by growth
 - 1345 and maturity parameters
- 1346 • misspecified BH prevents learning growth
- 1347 • increasing growth accelerates model misspecification
- 1348 • statistical evidence of minimum distance mapping within acceptable region, al-
- 1349 though float idea of PT-like pattern as BH set flattens. (explaining perturbations)

1350 4.5 old ideas

- 1351 • show production model limit (contrast
 - 1352 – $a_s \rightarrow 0$: instant maturity
 - 1353 – $\kappa \rightarrow \infty$: recruit as an adult ()
- 1354 • describe second order shapes of growth/maturity (and cause)
 - 1355 – weight of recruits \Rightarrow scaling biomass (q , β , and w_∞)
 - 1356 –
- 1357 • describe RP bias
- 1358 • flat

1359 **Chapter 5**

1360 **Conclusion**

1361 **Appendix A**

1362 **Inverting $\frac{B^*}{\bar{B}(0)}$ and γ for the PT Model**

Let $\zeta = \frac{B^*}{B(0)}$.

$$\begin{aligned}\zeta &= \left(\frac{1}{\gamma}\right)^{\frac{1}{\gamma-1}} \\ \zeta &= \gamma \zeta^\gamma \\ \zeta &= \gamma e^{\gamma \log(\zeta)} \\ \zeta \log(\zeta) &= \gamma \log(\zeta) e^{\gamma \log(\zeta)}\end{aligned}$$

The Lambert product logarithm, W , is defined as the inverse function of $z = xe^x$ such that $x = W(z)$. Applying this definition allows for the isolation of γ .

$$\begin{aligned}\gamma \log(\zeta) &= W(\zeta \log(\zeta)) \\ \gamma &= \frac{W(\zeta \log(\zeta))}{\log(\zeta)}\end{aligned}\tag{A.1}$$

1363 The Lambert product logarithm is a multivalued function with a branch point at $-\frac{1}{e}$.
1364 The principal branch, $W_0(z)$, is defined on $z \in (-\frac{1}{e}, \infty)$, and the lower branch, $W_{-1}(z)$,
1365 is defined on $z \in (-\frac{1}{e}, 0)$. Taken individually, each respective branch is analytic, but
1366 cannot be expressed in terms of elementary functions.

1367 When $\zeta \in (0, \frac{1}{e})$ the solution of interest in Eq. (2.10) comes from W_0 . When $\zeta \rightarrow \frac{1}{e}$,
1368 the Fox Model emerges as $\gamma \rightarrow 1$. When $\zeta \in (\frac{1}{e}, 1)$ the solution of interest comes from
1369 W_{-1} . For the use case presented here, Eq. (2.10) is to be interpreted as,

$$\gamma = \begin{cases} \frac{W_0(\zeta \log(\zeta))}{\log(\zeta)} & \zeta \in (0, \frac{1}{e}) \\ \frac{W_{-1}(\zeta \log(\zeta))}{\log(\zeta)} & \zeta \in (\frac{1}{e}, 1) \end{cases}.\tag{A.2}$$

¹³⁷⁰ **Appendix B**

¹³⁷¹ **Relation to F_{SPR} Proxy**

1372 • Define F_{MSY} Proxy F_{SPR_x}

1373 • Define $\frac{B_{MSY}}{B_0}$ Proxy B_y

1374 • List management targets for RF, GF, FF

	B_y	SPR_x	$\left[\frac{B_{MSY}}{B_0}\right]_{BH}$	$\left[\frac{F_{MSY}}{M}\right]_{BH} (F Only)$
RF	$y = 0.40$	$x = 0.50$	0.29	1.45 (0.29)
GF	$y = 0.40$	$x = 0.45$	0.22	2.46 (0.49)
FF	$y = 0.25$	$x = 0.30$	0.21	2.87 (0.57)

1376 • Show BH calculation only hits target for $\frac{\alpha}{M} = 6$; target cannot equal MSY.

1377 • Show general α - γ proxy relation under Schnute.

1378 B_0 given by Eq. (3.4). R_0 is given by evaluating $R(B_0; \theta)$.

$$R_0 = \frac{M}{\gamma\beta} \left(1 - \left(\frac{M}{\alpha} \right)^\gamma \right) \quad (B.1)$$

$$F_{SPR_x} = \frac{R_0}{xB_0} - M = M \left(\frac{1}{x} - 1 \right) \quad (B.2)$$

1379 Evaluating $\frac{B_{MSY}}{B_0}$ Eq.(3.5) at F_{SPR_x} . Solving for the compensation ratio $\frac{\alpha}{M}$ gives,

$$\frac{\alpha}{M} = \left[\frac{\frac{1}{x^\gamma} - y}{1 - y} \right]^{1/\gamma} \quad (B.3)$$

1380 • Show general α - γ MSY relation under Schnute.

1381 Reference α Eq.(3.9) from text in F_{MSY} .

1382 When $F_{MSY} = F_{SPR_x}$ substitute in Eq.(B.2)

$$\frac{\alpha}{M} = \frac{1}{x} \left(1 + \gamma(1 - x) \right)^{1/\gamma} \quad (B.4)$$

1383 • Show single (α, γ) pair to hit both proxy and MSY.

1384

Equate Eqs. (B.3) and (B.4)

$$\left[\frac{\frac{1}{x^\gamma} - y}{1 - y} \right]^{1/\gamma} = \frac{1}{x} \left(1 + \gamma(1 - x) \right)^{1/\gamma} \quad (\text{B.5})$$

$$1 - yx^\gamma = \left(1 + \gamma(1 - x) \right) (1 - y) \quad (\text{B.6})$$

$$1 = \left[1 - \gamma \frac{(1 - x)(1 - y)}{y} \right] x^{-\gamma} \quad (\text{B.7})$$

1385

$$r(x, y) = \frac{y}{(1-x)(1-y)}$$

$$r(x, y) = \left(r(x, y) - \gamma \right) x^{-\gamma} \quad (\text{B.8})$$

1386

Recall the Lambert product logarithm, W , is defined as the inverse function of

1387

$z = xe^x$ such that $x = W(z)$. Isolating γ requires that the above expression be

1388

placed into xe^x form to apply the definition of W .

$$r(x, y)x^{r(x, y)} = \left(r(x, y) - \gamma \right) x^{r(x, y) - \gamma} \quad (\text{B.9})$$

$$r(x, y)x^{r(x, y)} \log(x) = \left(r(x, y) - \gamma \right) \log(x) e^{\left(r(x, y) - \gamma \right) \log(x)} \quad (\text{B.10})$$

$$W_{-1} \left(r(x, y)x^{r(x, y)} \log(x) \right) = \left(r(x, y) - \gamma \right) \log(x) \quad (\text{B.11})$$

$$\gamma = r(x, y) - \frac{W_{-1} \left(r(x, y)x^{r(x, y)} \log(x) \right)}{\log(x)} \quad (\text{B.12})$$

1389

The solution of interest for γ in terms of only the proxy values comes from W_{-1} .

1390

To complete the point (α, γ) in terms of only proxy values α is given by substituting

1391

γ from Eq.(B.12) into either of Eqs. (B.3) or (B.4).

1392

- a few pics demonstrating the result

¹³⁹³ **Appendix C**

¹³⁹⁴ **Delay Differential Replacement Line**

1395 The replacement line is the rate of productivity which exactly balances biomass loss
 1396 in the absence of fishing. In the simple production model, productivity must simply
 1397 balance biomass loss due to M . Thus when $R(B; \theta) > MB$ there will be some surplus
 1398 productivity to enable fishing.

In the delay model, productivity is complicated by biomass changing, both with the recruitment of young into the reproducing population, as well as biomass accumulation due to the growth of existing individuals in the population. To derive the replacement line in the case of the delay model the equilibrium equations in the absence of fishing are considered and the $R(B)$ that this implies is then isolated.

$$0 = \frac{dB}{dt} = w(a_s)R(B) + \kappa[w_\infty N - B] - MB \quad (\text{C.1})$$

$$0 = \frac{dN}{dt} = R(B) - MN. \quad (\text{C.2})$$

Eq(C.2) quickly gives $\bar{N} = R(B)/M$. Substituting this equilibrium value into Eq(C.1) to rewrite N in terms of B ,

$$0 = w(a_s)R(B) + \kappa \left[w_\infty \frac{R(B)}{M} - B \right] - MB. \quad (\text{C.3})$$

Collecting like terms,

$$R(B) \left[w(a_s) + \frac{\kappa w_\infty}{M} \right] = [M + \kappa] B. \quad (\text{C.4})$$

Finally solving for $R(B)$, and simplifying, gives the equation of the replacement line as,

$$R(B) = \left[\frac{M(M + \kappa)}{w(a_s)M + \kappa w_\infty} \right] B. \quad (\text{C.5})$$

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