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Frequency domain of image and its applications

Linear Algebra second interim report

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Abstract

Image denoising is a component of digital image processing, aiming to enhance the visual quality of images by reducing noise originating from various sources such as environmental conditions or camera sensor issues. The primary aim of our project is to research different denoising methods primary Fourier-based to denosise images while preserving and restoring important information from noisy input.

1 Introduction

The research area of our project lies within the domain of image processing. During image acquisition, transmission, quantization and discrete sources of radiation susceptibility to noise corruption in digital images arises[3, 5]. Noise appears as undesirable grainy areas, leading to a degradation of image quality and loss of important information. Hence, the task of image denoising emerges, with the primary objective being the elimination of noise from an image, thus introducing a new challenge: preserving critical information such as edges and textures while preventing artifacts and blurring[2]. The aim of our project is to compare the efficacy of Fourier-based denoising methods with other techniques.

2 Spacial and frequency domain of an image

Images can be represented in two basis domains: spatial and frequency.

The spatial domain represents an image in terms of its pixel values, where each pixel corresponds to a specific location and intensity value in the image. In contrast, the frequency domain represents an image in terms of its frequency components, a transformation accomplished through the Fourier Transform.

Figure 1: Image in Spatial Domain

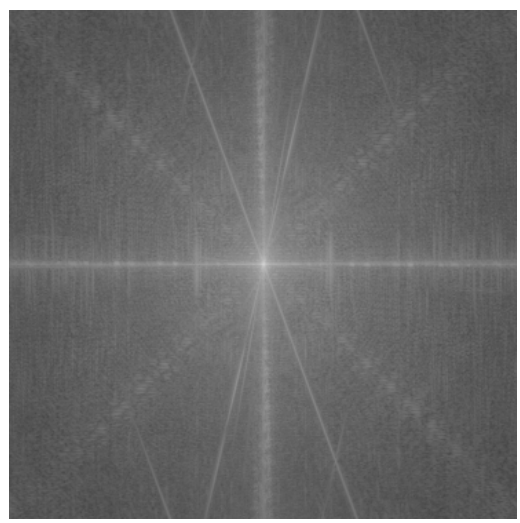

Figure 2: Image in Frequency Domain

In spatial domain an image is commonly represented as a two-dimensional function, denoted as $f(x, y)$, where x and y represent spatial coordinates within the image. The value of f at any given pair of coordinates (x, y) corresponds to the intensity of the image at that point [3].

In practical scenarios, however, images are often corrupted by various sources of noise, degrading the quality of the captured data.

The resulting degraded image $g(x, y)$ is formed by the addition of the true pixel value $f(x, y)$ to an additive noise value $\eta(x, y)$.

$$g(x, y) = f(x, y) + \eta(x, y) \tag{1}$$

3 What is noise?

Gaussian noise, commonly known as normal noise, is the most prevalent type of additive noise in the context of natural images.[1]

The probability distribution function of Gaussian noise is given by:

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{2}$$

where μ denotes the mean and σ^2 denotes the variance.

4 Fourier series and transform

Joseph Fourier claimed that any function of a variable, whether continuous or discontinuous, can be expanded in a series of sines of multiples of the variable.

4.1 One-dimensional Fourier analysis

The Fourier transform $g(x) \overset{\text{FT}}{\rightarrow} G(u)$ converts a function in the spatial domain to its frequency domain function:

$$\mathcal{F}\{g(x)\} = G(u) = \int_{-\infty}^{\infty} g(x) e^{-i2\pi ux}\, dx \tag{3}$$

where u denotes the frequency parameter along the x-axis.

The Inverse Fourier transform $G(u) \overset{\text{IFT}}{\rightarrow} g(x)$ converts a function frequency domain function to its spatial domain:

$$\mathcal{F}^{-1}\{G(u)\} = g(x) = \int_{-\infty}^{\infty} G(u) e^{i2\pi ux}\, du \tag{4}$$

4.2 Two-dimensional Fourier analysis

As images are two-dimentional we need to extend Fourier Transform from one-dimention to two-dimention.

The Fourier transform $g(x, y) \overset{\text{FT}}{\rightarrow} G(u, v)$:

$$\mathcal{F}\{g(x, y)\} = G(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{-i2\pi(ux+vy)}\, dxdy \tag{5}$$

The Inverse Fourier transform $G(u, v) \overset{\text{IFT}}{\rightarrow} g(x, y)$:

$$\mathcal{F}^{-1}\{G(u, v)\} = g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(u, v) e^{i2\pi(xu+yv)}\, dudv \tag{6}$$

The Fourier Transform is used to analyze the frequency content of continuous signals, while the Discrete Fourier Transform (DFT) for discrete and finite signals, as commonly encountered in image processing.

Discrete Fourier Transform $f[m,n] \overset{\text{DFT}}{\to} F[p,q]$:

$$\mathcal{F}\{f[m,n])\} = F[p,q] = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-i2\pi\left(\frac{pm}{M}+\frac{qn}{N}\right)} \tag{7}$$

where $M \times N$ is the dimension of an image.

Inverse Discrete Fourier Transform $F[p,q] \overset{\text{IDFT}}{\to} f[m,n]$:

$$\mathcal{F}^{-1}\{F[p,q]\} = f[m,n] = \frac{1}{MN} \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} F[p,q] e^{i2\pi\left(\frac{pm}{M}+\frac{qn}{N}\right)} \tag{8}$$

5 Overview of approaches

Two primary approaches for image denoising are spatial and transform domain filtering methods [5].

5.1 Spatial domain filtering methods

Spatial filtering involves the direct manipulation of pixel values within the image and encompasses both linear and non-linear techniques. Linear filters, such as Mean filtering [3] implemented through convolution operations. In contrast, non-linear filters, such as Median filtering and weighted median filtering, replace each pixel value with a non-linear function of its surrounding pixels. Spatial filters, despite their efficacy in reducing noise by employing low pass filtering techniques, often introduce a trade-off: while they effectively reduce noise, they tend to induce edge corruption and image blurring [2, 4].

5.2 Transform domain filtering methods

Transform domain filtering, on the other hand, is expected to overcome these challenges by utilizing various types of thresholds to differentiate between edges and noise, thereby offering a more refined approach to denoising [4].

In transform domain filtering, we consider frequency domain. Image denoising in the frequency domain is the process of transforming an image from the spatial domain to the frequency domain, which involves linear transformation from the Euclidean basis to the Fourier basis, and designing a frequency domain filter, by which we can target and filter out high-frequency values, commonly associated with image noise [5].

According to Joseph Fourier, function representation of an image $f(x,y)$ can be expanded in a series of sine waves, also called wave superposition.

Alternatively, in the frequency domain, Equation (1) can be represented as:

$$G(u,v) = F(u,v) + N(u,v) \tag{9}$$

where $G(u,v)$ denotes the Fourier transform of the observed image, $F(u,v)$ the Fourier transform of the original image, and $N(u,v)$ the Fourier transform of the noise component.

One possible approach to obtain the denoised image is by subtracting the N(u, v) term, see Equation (9) [3]. However, since this term is generally unknown due to the purely random nature of noise, an alternative approach should be considered. Referring to the frequency domain, the estimation of $N(u, v)$ can be obtained by analyzing the spectrum of $G(u, v)$.

6 Implementation Pipeline

The denoising process proceeds as follows:

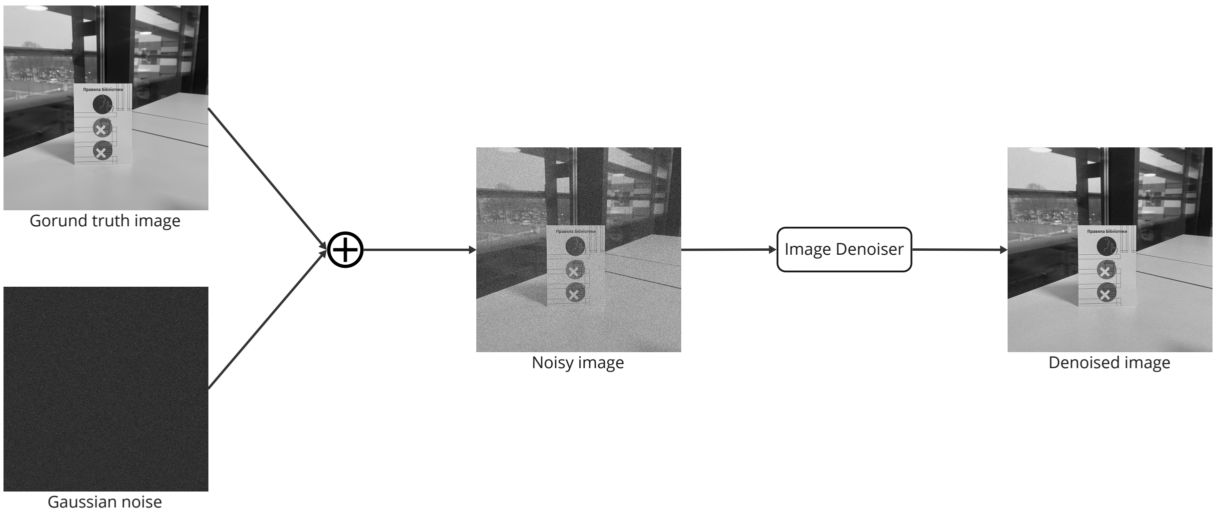

Figure 3: Denoising concept

Initially, the image is transformed from its native Euclidean basis to the Fourier basis via Fourier transformation. This conversion enables the representation of the image in terms of frequency components. Following the transformation, we can observe the image in the form of frequency with high and low frequency components, where high-frequency components are typically indicative of noise present within the image. The next step is to reduce this noise by filtering out the high-frequency components. After filtering out the high-value frequencies, the next step is to perform an inverse Fourier transformation to convert the image back to its spatial domain. This step restores the image to its original form but with reduced noise, resulting in a denoised representation.

However, it's essential to strike a balance between noise reduction and preservation of image details. Over-filtering can lead to loss of important information and result in a blurred image. Therefore careful adjustment of the filter parameters is required.

7 Testing

To test our implementation, we use the CIFAR-10 dataset with Gaussian noise added to these images to simulate real-world conditions. For evaluation purposes, we devise a metric to compare the denoised images with the ground truth images. This metric includes measures such as mean squared error (MSE) and peak signal-to-noise ratio (PSNR).

7.1 Mean Square Error

The Mean Squared Error (MSE), quantifies the cumulative square error between the denoised and ground truth images, is defined by:

$$\mathrm{MSE} = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \left(f[m,n] - \hat{f}[m,n] \right)^2 \tag{10}$$

7.2 Peak Signal-to-Noise Ratio

Peak Signal-to-Noise Ratio (PSNR) is a ratio of signal power to noise power and is defined by:

$$\mathrm{PSNR} = 10 \cdot \log_{10} \left(\frac{255^2}{\mathrm{MSE}} \right) \tag{11}$$

8 Plan of Future Research

Our project timeline is as follows:

Until April 24, 2024:

- Refine the algorithm implementation based on the insights gained.
- Explore other alorithms.
- Prepare the final project report.

9 Challenges and Potential Limitations

One significant challenge we anticipate is striking a balance between image denoising and preserving valuable image information. Following the Fourier transform, the resulting high frequencies, typically associated with noise, can also encode essential image features such as edges and textures. The challenge lies in accurately distinguishing between high frequencies representing noise and those embodying meaningful image information.

10 Conclusions

In conclusion, we have laid the groundwork for implementing image denoising using Fourier-based methods. We have conducted thorough research into the problem of image noise, explored various denoising algorithms and their theoretical background with the stress on linear algebra methods. Furthermore, we have begun the initial implementation of the algorithm.

References

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