TMA4180 Project: Truss optimization

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Abstract

In this paper, you asked for poop. Poop was provided (like, much poop).

1 Introduction

In the simplest of terms, a truss is a collection of things bound together. The truss structure itself, is typically comprised of triangular units, with straight members, whose ends are connected at nodes. As such, they are prime candidates for mathematical modeling, and we introduce some basic physical laws governing their behavior.

Given a straight bar j, between nodes i_1 and i_2 with coordinates $\boldsymbol{v}_{i1}$,$\boldsymbol{v}_{i2} \in \mathbb{R}^3$, we denote its length by $l_j = ||\boldsymbol{v}_{i2} - \boldsymbol{v}_{i1}||_2 > 0$ and choose its spatial orientation as the directional vector $\boldsymbol{\tau}_j = (\boldsymbol{v}_{i2} - \boldsymbol{v}_{i1})/l_j$. Moreover, letting $q_j \in \mathbb{R}$ be the axial force acting on the bar j such that $q_j < 0$ describes a compressing force and $q_j > 0$ corresponds to a tensile force, then we have by Hook's law:

$$\frac{q_j}{A_j} = E_j \frac{(\boldsymbol{u}_{i2} - \boldsymbol{u}_{i1})\tau_j}{l_j}, \tag{1}$$

where $\boldsymbol{u}_{i2}, \boldsymbol{u}_{i1}$ are the displacements of the nodes i_2 and i_1, $A_j > 0$ the area of the bar j, and $E_j > 0$ is young's modulus of the material of which the bar j is made. In deriving this result, it has been assumed that the displacements are small in comparison to the length of the bar, that is: $||\boldsymbol{u}_{i2}||_2/l_j \approx 0$ and $||\boldsymbol{u}_{i1}||_2/l_j \approx 0$.

Upon examining the individual nodes, we have by Newtons third law that:

$$\sum_{j \in \mathcal{J}_i^{\text{out}}} q_j \boldsymbol{\tau}_j - \sum_{j \in \mathcal{J}_i^{\text{in}}} q_j \boldsymbol{\tau}_j = \boldsymbol{f}_i, \tag{2}$$

where $\mathcal{J}_i^{\text{out}}$ are the indices of the bars originating originating in node i, and $\mathcal{J}_i^{\text{in}}$ is for the bars ending in this node. $\boldsymbol{f}_i \in \mathcal{R}^3$ is the vector of external forces acting at point i, which is of unknown reactionary forces $\boldsymbol{f}_i^{\text{supp}}$, produced by the foundation whenever node i is fixed ($\boldsymbol{u}_i = 0$), or external prescribed forces $\boldsymbol{f}_i^{\text{ext}}$ such as weight added to the truss.

2 Optimization problems

2.1 Problem 1

Given the above mathematical model of a truss, we wish to minimize the elastic strain on its structure, with $\boldsymbol{q}$ and $\boldsymbol{f}^{\mathrm{supp}}$, under the physical constraint of Newton's third law. The optimization problem may be formulated as:

$$\min_{(\boldsymbol{q},\boldsymbol{f}^{\mathrm{supp}})} \quad g(\boldsymbol{q},\boldsymbol{f}^{\mathrm{supp}}) = \frac{1}{2}\sum_{j=1}^{m} \frac{l_j q_j^2}{E_j A_j}, \tag{3}$$

$$\text{subject to} \quad B\boldsymbol{q} = I_{\mathrm{supp}}\boldsymbol{f}^{supp} + I_{\mathrm{ext}}\boldsymbol{f}^{ext}. \tag{4}$$

We apply the Theory of Constrained Optimization and introduce the *Lagrangian function* for this system:

$$\mathcal{L}\left(\boldsymbol{q},\boldsymbol{f}^{\mathrm{supp}},\boldsymbol{\lambda}\right) = \frac{1}{2}\sum_{j=1}^{m}\alpha_j q_j^2 - \sum_{i=1}^{3n}\lambda_i c_i\left(\boldsymbol{q},\boldsymbol{f}^{supp}\right),$$

where $\lambda_i \in \mathbb{R}$ is the *Lagrange multipliers*, and

$$\alpha_j = \frac{l_j}{E_j A_j},$$
$$c_i\left(\boldsymbol{q},\boldsymbol{f}^{\mathrm{supp}}\right) = \left[B\boldsymbol{q} - I_{\mathrm{supp}}\boldsymbol{f}^{\mathrm{supp}} - I_{\mathrm{ext}}\boldsymbol{f}^{\mathrm{ext}}\right]_i,$$

such that each constraint, c_i, is the i'th row of the system (4). We will look for a solution, $\left(\boldsymbol{q}^*,\boldsymbol{f}^{\mathrm{supp}^*}\right)$, to the Optimization Problem in the *feasible set* given by :

$$\Omega = \{\boldsymbol{q}\times\boldsymbol{f}^{\mathrm{supp}} \subset \mathbb{R}^m\times\mathbb{R}^{3n} \mid c_i\left(\boldsymbol{q},\boldsymbol{f}^{\mathrm{supp}}\right) = 0,\ \ i\in\mathcal{E},\ \ c_i\left(\boldsymbol{q},\boldsymbol{f}^{\mathrm{supp}}\right)\geq 0,\ \ i\in\mathcal{I}\}.$$

In our particular case, $\mathcal{I} = \varnothing$. Note that Ω is closed set, as it is a hyperplane in $\mathbb{R}^m \times \mathbb{R}^{3n}$.

2.1.1 Necessary conditions

The First-Order Necessary Conditions for the solution is given by the *KKT conditions* such that the following is satisfied:

$$\nabla_{\boldsymbol{q}}\mathcal{L}\left(\boldsymbol{q}^*,\boldsymbol{f}^{\mathrm{supp}^*},\boldsymbol{\lambda}^*\right) = 0 \tag{5}$$

$$\nabla_{\boldsymbol{f}^{\mathrm{supp}}}\mathcal{L}\left(\boldsymbol{q}^*,\boldsymbol{f}^{\mathrm{supp}^*},\boldsymbol{\lambda}^*\right) = 0 \tag{6}$$

$$c_i\left(\boldsymbol{q}^*,\boldsymbol{f}^{\mathrm{supp}^*}\right) = 0, \quad \forall i\in\mathcal{E}. \tag{7}$$

We write out the conditions in the terms of the Optimization Problem. By component-wise derivation, one can easily show that

$$\begin{aligned}(5) &\Rightarrow D\boldsymbol{q}^* - \sum_{i=1}^{3n} \lambda_i^* \left(B\right)_i^T = 0 \\ &\Rightarrow D\boldsymbol{q}^* = B^T\boldsymbol{\lambda}^* \\ (6) &\Rightarrow - \sum_{i=1}^{3n} \lambda_i^* \left(I_{\text{supp}}\right)_i^T = 0 \\ &\Rightarrow I_{\text{supp}}^T\boldsymbol{\lambda}^* = 0\end{aligned}$$

Thus, the First-Order Necessary Conditions of the Optimization Problem can be compactly stated as

$$\begin{aligned}D\boldsymbol{q}^* &= B^T\boldsymbol{\lambda}^* \\ B\boldsymbol{q}^* &= I_{\text{supp}}\boldsymbol{f}^{\text{supp}^*} + I_{\text{ext}}\boldsymbol{f}^{\text{ext}} \\ I_{\text{supp}}^T\boldsymbol{\lambda}^* &= 0\end{aligned}$$

where D is $m \times m$, $\boldsymbol{q}^*$ is $m \times 1$, B^T is $m \times 3n$ and $\boldsymbol{\lambda}^*$ is $3n \times 1$. Letting $\boldsymbol{u} = \boldsymbol{\lambda}^*$, this compact form is equivalent to the system stated in (1) and (2), so the system at least constitutes the necessary conditions. However, it remains to prove that they are sufficient.

2.1.2 Sufficient conditions

It can be shown that the KKT conditions for minimization are sufficient under convexity if the cost function is convex, and the inequality and equality constraints are concave and affine, respectively, assuming a KKT point exists[1]. Obviously, the equality constraints are affine, and with no inequality constraints, we only need to prove that (3) is convex.

A function is convex if its corresponding hessian is positive semi-definite on the interior of the convex set. The hessian of the cost function g in (3) is easily shown to be:

$$H(g)_{(m+3n_s)\times(m+3n_s)} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \tag{8}$$

which by the definition of D implies that the hessian is positive semi-definite. Hence, g is convex in Ω. Thus the system (1) and (2) constitutes the necessary and sufficient conditions for optimality in (3) and (4).

2.1.3 Existence and uniqueness of solutions

Assuming that the feasible set is non-empty, we need to show that our optimization problem admits at least one solution. We use theorem 2[2] which states that

[1]Theorem: KKT conditions are sufficient under convexity, TMA4180 Notes, URL: `http://www.math.ntnu.no/emner/TMA4180/2014v/Notes/KKT_suff.pdf`

[2]Theorem 2, TMA4180 Notes, URL: `http://www.math.ntnu.no/emner/TMA4180/2014v/Notes/note01.pdf`

if Ω is a non-empty and closed set, and the cost function g is coersive and lower semi-continuous, then the optimization problem admits at least one solution. It is obvious that $g(\boldsymbol{p}, \boldsymbol{f}^{\text{supp}}) \to +\infty$ as $||\boldsymbol{q}||_2 \to +\infty$, and thus it is a coersive function. To see that it lower semi-continuous, we construct the lower level set $\Omega_\alpha = \{\Omega \mid g(\boldsymbol{q}, \boldsymbol{f}^{\text{supp}}) \leq \alpha\}$. We then have that:

$$\Omega_\alpha = \begin{cases} \varnothing & \alpha < 0 \\ \Omega & \alpha \geq 0, \end{cases} \tag{9}$$

all of which are closed sets in Ω. Thus, by definition, g is l.s.c.

To show uniqueness of this solution, we first consider the case $\boldsymbol{f}^{\text{ext}} = 0$. We then have that for some $\boldsymbol{q}, \boldsymbol{f}^{\text{supp}} \in \Omega$:

$$\boldsymbol{q}^T D \boldsymbol{q} = \boldsymbol{u}^T I_{\text{supp}} = 0,$$

which by the KKT conditions with $\boldsymbol{\lambda} = \boldsymbol{u}$ implies that both $\boldsymbol{q} = 0$ and $\boldsymbol{f}^{\text{supp}} = 0$. Thus the system of equations:

$$\begin{pmatrix} D & -B^T & 0 \\ B & 0 & -I_{\text{supp}} \\ 0 & I_{\text{supp}}^T & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{\lambda} \\ \boldsymbol{f}_{\text{supp}} \end{pmatrix} = \begin{pmatrix} 0 \\ I_{\text{ext}} \boldsymbol{f}_{\text{ext}} \\ 0 \end{pmatrix} \tag{10}$$

only contains the zero vectors in its null space, given that the row vectors B are linearly independent (LICQ condition). This in turn implies that the solution to the optimization problem is unique.

2.2 Additional constraints

We introduce the additional constrains on Problem 1 that,

$$\sum_{j=1}^{M} \rho_j l_j A_j \leq M, \tag{11}$$

$$\underline{A}_j \leq A_j \leq \overline{A}_j \quad j = 1, ..., m, \tag{12}$$

where $M > 0$ is the maximal allowable weight on the truss structure, $\rho_j > 0$ is the density of the material in the bar j, and $0 < \underline{A}_j < \overline{A}_j \leq +\infty$ are lower and upper bounds on the bar cross-sectional areas. With the addition of these constraints, we now consider the problem of minimizing $g(\boldsymbol{q}, \boldsymbol{f}^{supp}, \boldsymbol{A})$ over the area as well.

2.2.1 Existence and uniqueness of solutions

The objective function is still coersive. However, the feasible set has changed, since $\mathcal{I} \neq \varnothing$, and so we need to check if g is still lower semi-continuous on the new Ω. Since Ω is still a closed set, because the set of $\boldsymbol{f}^{supp}, \boldsymbol{q}$ and now also $\boldsymbol{A}$ which satisfy the inequality and equality constraints is still closed, we find that g is lower semi-continuous as before. This is easily seen upon the construction of Ω_α. Then, by theorem 2, there exists at least one optimal solution, under the assumption that the feasible set is non-empty.

Suppose that the lower bound $\underline{A}_j = 0$ for some j, we then adopt the convention that $q_j^2/0 = +\infty$, if $q_j \neq 0$, and 0 whenever $q_j = 0$. Then g is still coersive, and to see if its still lower semi-continuous at this point we look at the pointwise definition for some function $f(x)$:

$$\liminf_{x \to x_0} f(x) \geq f(x_0). \tag{13}$$

Clearly, for any neighborhood U of $q_j = 0$, we must have that $g(q_j \neq 0, ...) \geq g(q_j = 0, ...) - \epsilon$ for any $\epsilon > 0$. Likewise, starting in some $q_j \neq 0$ we also have that $g(q_j \neq 0, ...) \to +\infty$ as $q_j \to q_j^\dagger$ where $q_j^\dagger \in U$, since $g(q_j^\dagger \neq 0, ...) = +\infty$ Thus, g is l.s.c.

2.2.2 Necessary Conditions

Letting $\overline{A}_j = +\infty$ and $\underline{A}_j = 0$ for all j. Then, whenever $A_j = +\infty$ or $A_j = 0$, its respective constraint in (12) is active. However, if indeed $A_j = +\infty$ then the inequality (11) cannot possibly be satisfied, which in turn means that the feasible set is empty whenever the upper bound constraint on A is active. Additionally, whenever $A_j = 0$, we must fix $q_j = 0$ or else the objective function is infinitely large, in which case, it makes little sense to minimize it[3]...

The First-Order Necessary Conditions of Problem 1 will then be modified to:

$$\nabla_{\boldsymbol{q}} \mathcal{L}\left(\boldsymbol{q}^*, \boldsymbol{f}^{\text{supp}^*}, \boldsymbol{A}^*, \boldsymbol{\lambda}^*\right) = 0, \tag{14}$$

$$\nabla_{\boldsymbol{f}^{\text{supp}}} \mathcal{L}\left(\boldsymbol{q}^*, \boldsymbol{f}^{\text{supp}^*}, \boldsymbol{A}^*, \boldsymbol{\lambda}^*\right) = 0, \tag{15}$$

$$\nabla_{\boldsymbol{A}} \mathcal{L}\left(\boldsymbol{q}^*, \boldsymbol{f}^{\text{supp}^*}, \boldsymbol{A}^*, \boldsymbol{\lambda}^*\right) = 0, \tag{16}$$

$$c_i\left(\boldsymbol{q}^*, \boldsymbol{f}^{\text{supp}^*}, \boldsymbol{A}^*\right) = 0, \quad \forall i \in \mathcal{E}, \tag{17}$$

$$\lambda_i^* \geq 0, \quad \forall i \in \mathcal{I}, \tag{18}$$

$$\lambda_i^* c_i(\boldsymbol{q}^*, \boldsymbol{f}^*_{\text{supp}}, \boldsymbol{A}^*) = 0, \quad \forall i \in \mathcal{I} \cup \mathcal{E}, \tag{19}$$

$$c_i(\boldsymbol{q}^*, \boldsymbol{f}^*_{\text{supp}}, \boldsymbol{A}^*) \geq 0, \quad \forall i \in \mathcal{I}. \tag{20}$$

2.2.3 Sufficient Conditions

As before, we use that the KKT conditions for minimization are sufficient under convexity, and define that the linear inequality constraints as concave. Then we only need to show that the objective function is convex.

[3] Infinity Doge, URL: `http://4.bp.blogspot.com/-2RxHjA9d71U/Unk6iKQU6kI/AAAAAAAAZus/55bG95S2Cqs/s1600/doge+gif+infinity+dr+heckle+funny+wtf+gifs.gif`

The hessian for g will be:

$$H(g) = \begin{pmatrix} D & 0 & \cdots & -\frac{l_1 q_1}{E_1 A_1^2} & 0 & \cdots \\ 0 & 0 & & 0 & \ddots & \\ \vdots & & \ddots & \vdots & & -\frac{l_m q_m}{E_m A_m^2} \\ -\frac{l_1 q_1}{E_1 A_1^2} & 0 & \cdots & \frac{l_1 q_1}{E_1 A_1^3} & & \\ 0 & \ddots & & & \ddots & \\ \vdots & & -\frac{l_m q_m}{E_m A_m^2} & & & \frac{l_m q_m}{E_m A_m^3} \end{pmatrix}, \tag{21}$$

which is symmetric.

References

[1] Brynjulf Owren, *TMA4212: Numerical solution of partial differential equations*, NTNU, 24 Feb, 2012.