1 Review of Classical Field Theory: Lagrangians, Lorentz Group and its Representations, Noether Theorem

In this book, as I have mentioned, I will assume a knowledge of classical field theory and quantum mechanics, and I will only review a few notions from them, immediately useful, in the first two chapters. In this first chapter, I will start by describing what quantum field theory is, and after that I will review a few things about classical field theory. In the next chapter, a few relevant notions of quantum mechanics, not always taught, will be described.

Conventions I will use theorist's conventions throughout, with $\hbar = c = 1$, which means that, for example, $[E] = [1/x] = 1$. I will also use the *mostly plus* metric, for instance in $3+1$ dimensions with signature $-+++$.

1.1 What is and Why Do We Need Quantum Field Theory?

Quantum mechanics deals with the quantization of particles, and is a nonrelativistic theory: time is treated as special, and for the energy we use nonrelativistic formulas.

On the contrary, we want to apply *quantum field theory*, which is an application of quantum mechanics, to *fields* instead of particles, and this has the property of being *relativistic* as well.

Quantum field theory is often called (when derived from first principles) *second quantization*, the idea being that:

- The *first* quantization is when we have a single particle and we quantize its behavior (its motion) in terms of a wavefunction describing probabilities.
- The *second* quantization is when we quantize the wavefunction itself (instead of a function now we have an operator), the quantum object now being the number of particles the wavefunction describes, which is an arbitrary (variable) quantum number. Therefore, the field is now a description of an arbitrary number of particles (and *antiparticles*), and this number can *change* (i.e. it is not a constant).

People have tried to build a *relativistic quantum mechanics*, but it was quickly observed that if we do that, we cannot describe a single particle:

- First, the relativistic relation $E = mc^2$, together with the existence (experimentally confirmed) of *antiparticles* which annihilate with particles giving only energy (photons), means that if we have an energy $E > m_p c^2 + m_{\bar{p}} c^2$, we can create a particle–antiparticle pair, and therefore the number of particles cannot be a constant in a relativistic theory.

- Second, even if $E < m_p c^2 + m_{\bar{p}} c^2$, the particle–antiparticle pair can still be created for a short time. Indeed, Heisenberg's uncertainty principle in the (E, t) sector (as opposed to the usual (x, p) sector) means that $\Delta E \cdot \Delta t \sim \hbar$, meaning that for a short time $\Delta t \sim \hbar/\Delta E$ we can have an uncertainty in the energy ΔE, for instance such that $E + \Delta E > m_p c^2 + m_{\bar{p}} c^2$. This means that we can create a pair of *virtual particles*, that is particles which are forbidden by energy and momentum conservation to exist as asymptotic particles, but can exist as quantum fluctuations for a short time.
- Third, causality is violated by a single particle propagating via usual quantum mechanics formulas, even with the relativistic formula for the energy, $E = \sqrt{\vec{p}^2 + m^2}$.

The amplitude for propagation from $\vec{x}_0$ to $\vec{x}$ in a time t in quantum mechanics is

$$U(t) = \langle \vec{x} | e^{-iHt} | \vec{x}_0 \rangle, \tag{1.1}$$

and replacing E, the eigenvalue of H, by $\sqrt{\vec{p}^2 + m^2}$, we obtain

$$U(t) = \langle \vec{x} | e^{-it\sqrt{\vec{p}^2+m^2}} | \vec{x}_0 \rangle = \frac{1}{(2\pi)^3} \int d^3\vec{p} e^{-it\sqrt{\vec{p}^2+m^2}} e^{i\vec{p}\cdot(\vec{x}-\vec{x}_0)}. \tag{1.2}$$

But

$$\begin{aligned} \int d^3\vec{p} e^{i\vec{p}\cdot\vec{x}} &= \int p^2 dp \int 2\pi \sin\theta d\theta e^{ipx\cos\theta} \\ &= \int p^2 dp \left[\frac{2\pi}{ipx} (e^{ipx} - e^{-ipx}) \right] = \int p^2 dp \left[\frac{4\pi}{px} \sin(px) \right], \end{aligned} \tag{1.3}$$

and therefore

$$U(t) = \frac{1}{2\pi^2 |\vec{x} - \vec{x}_0|} \int_0^\infty p dp \sin(p|\vec{x} - \vec{x}_0|) e^{-it\sqrt{p^2+m^2}}. \tag{1.4}$$

For $x^2 \gg t^2$, we use a saddle-point approximation, which is the idea that the integral $I = \int dx e^{f(x)}$ can be approximated by the Gaussian around the saddle point (i.e. $I \simeq e^{f(x_0)} \int d\delta x e^{f''(x_0)\delta x^2} \simeq e^{f(x_0)} \sqrt{\pi / f''(x_0)}$, where x_0 is the saddle point) at whose position we have $f'(x_0) = 0$. Generally, if we are interested in leading behavior in some large parameter, the function $e^{f(x_0)}$ dominates $\sqrt{\pi / f''(x_0)}$ and we can just approximate $I \sim e^{f(x_0)}$.

In our case, we obtain

$$\frac{d}{dp} \left(ipx - it\sqrt{p^2 + m^2} \right) = 0 \Rightarrow x = \frac{tp}{\sqrt{p^2 + m^2}} \Rightarrow p = p_0 = \frac{imx}{\sqrt{x^2 - t^2}}. \tag{1.5}$$

Since we are at $x^2 \gg t^2$, we obtain

$$U(t) \propto e^{ip_0 x - it\sqrt{p_0^2 + m^2}} \sim e^{-\sqrt{x^2 - t^2}} \neq 0. \tag{1.6}$$

So we see that even much outside the lightcone, at $x^2 \gg t^2$, we have nonzero probability for propagation, meaning a breakdown of causality.

However, we will see that this problem is fixed in quantum field theory, which will be causal.

In quantum field theory, the fields describe many particles. One example of this fact that is easy to understand is the case of the electromagnetic field, $(\vec{E},\vec{B}) \to F_{\mu\nu}$, which describes many photons. Indeed, we know from the correspondence principle of quantum mechanics that a classical state is equivalent to a state with many photons, and also that the number of photons is not a constant in any sense: we can define a (quantum) average number of photons that is related to the classical intensity of an electromagnetic beam, but the number of photons is not a classically measurable quantity.

We will describe processes involving many particles by *Feynman diagrams*, which will be an important part of this book. In quantum mechanics, a particle propagates forever, so its "Feynman diagram" is always a single line, as in Figure 1.1.

In quantum field theory, however, we will derive the mathematical form of Feynman diagrams, but the simple physical interpretation for which Feynman introduced them is that we can have processes where, for instance, a particle splits into two (or more) (see Figure 1.1(a)), two (or more) particles merge into one (see Figure 1.1(b)), or two (or more) particles of one type disappear and another type is created, like for instance in the annihilation of an e^+ (positron) with an e^- (electron) into a photon (γ) as in Figure 1.1(c), and so on.

Moreover, we can have (as we mentioned) virtual processes, like a photon γ creating an e^+e^- pair, which lives for a short time Δt and then annihilates into a γ, creating an e^+e^- virtual loop inside the propagating γ, as in Figure 1.2. Of course, $E,\vec{p}$ conservation means that $(E,\vec{p})$ is the same for the γ before and after the loop.

Next, we should review a few notions of classical field theory.

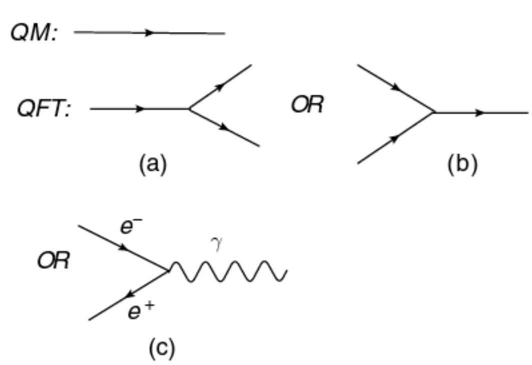

Figure 1.1 Quantum mechanics: particle goes on forever. Quantum field theory: particles can split (a), join (b), and particles of different types can appear and disappear, like in the quantum electrodynamics process (c).

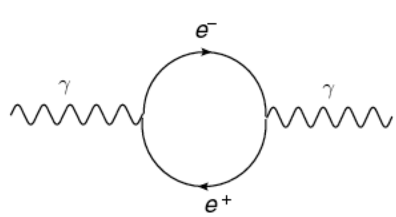

Figure 1.2 Virtual particles can appear for a short time in a loop. Here a photon creates a virtual electron–positron pair, which then annihilates back into the photon.

1.2 Classical Mechanics

Before doing that, however, we begin with an even quicker review of **classical mechanics**.

In classical mechanics, the description of a system is in terms of a Lagrangian $L(q_i, \dot{q}_i)$ for the variables $q_i(t)$, and the corresponding action

$$S = \int_{t_1}^{t_2} dt L(q_i(t), \dot{q}_i(t)). \tag{1.7}$$

By varying the action with fixed boundary values for the variables $q_i(t)$ (i.e. $\delta S = 0$), we obtain the Euler–Lagrange equations (or equations of motion)

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0. \tag{1.8}$$

We can also do a Legendre transformation from the Lagrangian $L(q_i, \dot{q}_i)$ to the Hamiltonian $H(q_i, p_i)$ in the usual way, by

$$H(p, q) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i), \tag{1.9}$$

where

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \tag{1.10}$$

is the momentum canonically conjugate to the coordinate q_i.

Differentiating the Legendre transformation formula, we get the first-order Hamilton equations (instead of the second-order Lagrange equations)

$$\begin{aligned} \frac{\partial H}{\partial p_i} &= \dot{q}_i, \\ \frac{\partial H}{\partial q_i} &= -\frac{\partial L}{\partial q_i} = -\dot{p}_i. \end{aligned} \tag{1.11}$$

1.3 Classical Field Theory

The generalization of classical mechanics to **field theory** is obtained by considering instead of a set $\{q_i(t)\}_i$, which is a collection of given particles, fields $\phi(\vec{x}, t)$, where $\vec{x}$ is a generalization of i, and not a coordinate of a particle.

We will be interested in *local* field theories, which means all objects are integrals over $\vec{x}$ of functions defined at a point, in particular the Lagrangian is written as

$$L(t) = \int d^3\vec{x} \mathcal{L}(\vec{x}, t). \tag{1.12}$$

Here $\mathcal{L}$ is called the *Lagrange density*, but by an abuse of notation, one usually refers to it also as the Lagrangian.

We are also interested in *relativistic field theories*, which means that $\mathcal{L}(\vec{x}, t)$ is a relativistically invariant function of fields and their derivatives:

$$\mathcal{L}(\vec{x}, t) = \mathcal{L}(\phi(\vec{x}, t), \partial_\mu \phi(\vec{x}, t)). \tag{1.13}$$

Considering also several fields ϕ_a, we have an action written as

$$S = \int L dt = \int d^4x \mathcal{L}(\phi_a, \partial_\mu \phi_a), \tag{1.14}$$

where $d^4x = dt d^3\vec{x}$ is the relativistically invariant volume element for spacetime.

The Euler–Lagrange equations are obtained in the same way, as

$$\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right] = 0. \tag{1.15}$$

Note that one could think of $L(q_i)$ as a discretization over $\vec{x}$ of $\int d^3\vec{x} \mathcal{L}(\phi_a)$, but that is not particularly useful.

In the Lagrangian we have relativistic fields, that is fields that have a well-defined transformation property under Lorentz transformations

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \tag{1.16}$$

namely

$$\phi'_i(x') = R_i{}^j \phi_i(x), \tag{1.17}$$

where i is some index for the fields, related to its Lorentz properties. We will come back to this later, but for now let us just observe that for a scalar field there is no i and $R \equiv 1$ (i.e. $\phi'(x') = \phi(x)$).

In this book I will use the convention for the spacetime metric with "mostly plus" on the diagonal, that is the Minkowski metric is

$$\eta_{\mu\nu} = diag(-1, +1, +1, +1). \tag{1.18}$$

Note that this is the convention that is the most natural in order to make heavy use of Euclidean field theory via Wick rotation, as we will do (by just redefining the time t by a factor of i), and so is very useful if we work with the functional formalism, where Euclidean field theory is essential.

On the contrary, for various reasons, people connected with phenomenology and making heavy use of the operator formalism often use the "mostly minus" metric ($\eta_{\mu\nu} = diag(+1, -1, -1, -1)$), for instance the standard textbook of Peskin and Schroeder [1] does so, so one has to be very careful when translating results from one convention to the other.

With this metric, the Lagrangian for a scalar field is generically

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - V(\phi) \\ &= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}|\vec{\nabla}\phi|^2 - \frac{1}{2} m^2 \phi^2 - V(\phi),\end{aligned} \tag{1.19}$$

so is of the general type $\dot{q}^2/2 - \tilde{V}(q)$, as it should be (where the terms $1/2|\vec{\nabla}\phi|^2 + m^2\phi^2/2$ are also part of $\tilde{V}(q)$).

To go to the Hamiltonian formalism, we must first define the momentum canonically conjugate to the field $\phi(\vec{x})$ (remembering that $\vec{x}$ is a label like i):

$$p(\vec{x}) = \frac{\partial L}{\partial \dot{\phi}(\vec{x})} = \frac{\partial}{\partial \dot{\phi}(\vec{x})} \int d^3\vec{y}\mathcal{L}(\phi(\vec{y}), \partial_\mu\phi(\vec{y})) = \pi(\vec{x})d^3\vec{x}, \tag{1.20}$$

where

$$\pi(\vec{x}) = \frac{\delta\mathcal{L}}{\delta\dot{\phi}(\vec{x})} \tag{1.21}$$

is a conjugate momentum density, but by an abuse of notation again will just be called conjugate momentum.

Then the Hamiltonian is

$$\begin{aligned} H &= \sum_{\vec{x}} p(\vec{x})\dot{\phi}(\vec{x}) - L \\ &\to \int d^3\vec{x}[\pi(\vec{x})\dot{\phi}(\vec{x}) - \mathcal{L}] \equiv \int d^3\vec{x}\mathcal{H}, \end{aligned} \tag{1.22}$$

where $\mathcal{H}$ is a Hamiltonian density.

1.4 Noether Theorem

The statement of the Noether theorem is that for every symmetry of the Lagrangian L, there is a corresponding conserved charge.

The best known examples are the time translation $t \to t+a$, corresponding to conserved energy E, and the space translation $\vec{x} \to \vec{x} + \vec{a}$, corresponding to conserved momentum $\vec{p}$, together making the spacetime translation $x^\mu \to x^\mu + a^\mu$, corresponding to conserved 4-momentum P^μ. The *Noether currents* corresponding to these charges form the energy–momentum tensor $T_{\mu\nu}$.

Consider the symmetry $\phi(x) \to \phi'(x) = \phi(x) + \alpha\Delta\phi$ that transforms the Lagrangian density as

$$\mathcal{L} \to \mathcal{L} + \alpha\partial_\mu J^\mu, \tag{1.23}$$

such that the action $S = \int d^4x\mathcal{L}$ is invariant, if the fields vanish on the boundary, usually considered at $t = \pm\infty$, since the boundary term

$$\int d^4x\partial_\mu J^\mu = \oint_{bd} dS_\mu J^\mu = \int d^3\vec{x}J^0|_{t=-\infty}^{t=+\infty} \tag{1.24}$$

is then zero. In this case, there exists a conserved current j^μ, that is

$$\partial_\mu j^\mu(x) = 0, \tag{1.25}$$

where

$$j^{\mu}(x) = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}\Delta\phi - J^{\mu}. \tag{1.26}$$

For linear symmetries (symmetry transformations linear in ϕ), we can define

$$(\alpha\Delta\phi)^{i} \equiv \alpha^{a}(T^{a})^{i}{}_{j}\phi^{j} \tag{1.27}$$

such that, if $J^{\mu} = 0$, we have the Noether current

$$j^{\mu,a} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}(T^{a})^{i}{}_{j}\phi^{j}. \tag{1.28}$$

Applying this general formalism to translations, $x^{\mu} \to x^{\mu} + a^{\mu}$, we obtain, for an infinitesimal parameter a^{μ}:

$$\phi(x) \to \phi(x+a) = \phi(x) + a^{\mu}\partial_{\mu}\phi, \tag{1.29}$$

which are the first terms in the Taylor expansion around x. The corresponding conserved current is therefore

$$T^{\mu}{}_{\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}\partial_{\nu}\phi - \mathcal{L}\delta^{\mu}_{\nu}, \tag{1.30}$$

where we have added a term $J^{\mu}_{(\nu)} = \mathcal{L}\delta^{\mu}_{\nu}$ to get the conventional definition of the *energy–momentum tensor* or *stress–energy tensor*. The conserved charges are integrals of the energy–momentum tensor (i.e. P^{μ}). Note that the above translation can be considered as also giving the term $J^{\mu}_{(\nu)}$ from the general formalism, since we can check that for $\alpha^{\nu} = a^{\nu}$, the Lagrangian changes by $\partial_{\mu}J^{\mu}_{(\nu)}$.

1.5 Fields and Lorentz Representations

The Lorentz group is $SO(1,3)$, that is an orthogonal group that generalizes $SO(3)$, the group of rotations in the (Euclidean) three spatial dimensions.

Its basic objects in the fundamental representation, defined as the representation that acts on coordinates x^{μ} (or rather dx^{μ}), are called $\Lambda^{\mu}{}_{\nu}$, and thus

$$dx'^{\mu} = \Lambda^{\mu}{}_{\nu}dx^{\nu}. \tag{1.31}$$

If η is the matrix $\eta_{\mu\nu}$, the Minkowski metric $diag(-1,+1,+1,+1)$, the orthogonal group $SO(1,3)$ is the group of elements Λ that satisfy

$$\Lambda\eta\Lambda^{T} = \eta. \tag{1.32}$$

Note that the usual rotation group $SO(3)$ is an orthogonal group satisfying

$$\Lambda\Lambda^{T} = \mathbf{1} \Rightarrow \Lambda^{-1} = \Lambda^{T}, \tag{1.33}$$

but we should actually write this as

$$\Lambda\mathbf{1}\Lambda^{T} = \mathbf{1}, \tag{1.34}$$

which admits a generalization to $SO(p,q)$ groups as

$$\Lambda g \Lambda^T = g, \tag{1.35}$$

where $g = diag(-1, \dots, -1, +1, \dots, +1)$ with p minuses and q pluses. In the above, Λ satisfies the group property, namely if Λ_1, Λ_2 belong in the group, then

$$\Lambda_1 \cdot \Lambda_2 \equiv \Lambda \tag{1.36}$$

is also in the group.

General representations are a generalization of (1.31), namely instead of acting on x, the group acts on a vector space ϕ^a by

$$\phi'^a(\Lambda x) = R(\Lambda)^a{}_b \phi^b(x), \tag{1.37}$$

such that it respects the group property, that is

$$R(\Lambda_1)R(\Lambda_2) = R(\Lambda_1 \cdot \Lambda_2). \tag{1.38}$$

Group elements are represented for infinitesimally small parameters β^a as exponentials of the *Lie algebra generators* in the R representation $t_a^{(R)}$, that is

$$R(\beta) = e^{i\beta^a t_a^{(R)}}. \tag{1.39}$$

The statement that $t_a^{(R)}$ form a Lie algebra is the statement that we have a relation

$$[t_a^{(R)}, t_b^{(R)}] = i f_{ab}{}^c t_c^{(R)} \tag{1.40}$$

where $f_{ab}{}^c$ are called the *structure constants*. Note that the factor of i is conventional, with this definition we can have Hermitian generators, for which $\mathrm{Tr}(t_a t_b) = \delta_{ab}$; if we redefine t_a by an i we can remove it from there, but then $\mathrm{Tr}(t_a t_b)$ can be put only to $-\delta_{ab}$ (anti-Hermitian generators).

The representations of the Lorentz group are:

- Bosonic. Scalars ϕ for which $\phi'(x') = \phi(x)$; vectors like the electromagnetic field $A_\mu = (\phi, \vec{A})$ that transform as ∂_μ (covariant) or dx^μ (contravariant), and representations which have products of indices, like for instance the electromagnetic field strength $F_{\mu\nu}$ which transforms as
$$F'_{\mu\nu}(\Lambda x) = \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma F_{\rho\sigma}(x), \tag{1.41}$$
where $\Lambda_\mu{}^\nu = \eta_{\mu\rho}\eta^{\nu\sigma}\Lambda^\rho{}_\sigma$. For fields with more indices, $B^{\nu_1 \dots \nu_j}_{\mu_1 \dots \mu_k}$, it transforms as the appropriate products of Λ.
- Fermionic. Spinors, which will be treated in more detail later on in the book. For now, let us just say that fundamental spinor representations ψ are acted upon by gamma matrices γ^μ.

The Lie algebra of the Lorentz group $SO(1,3)$ is

$$[J_{\mu\nu}, J_{\rho\sigma}] = -i\eta_{\mu\rho}J_{\nu\sigma} + i\eta_{\mu\sigma}J_{\nu\rho} - i\eta_{\nu\sigma}J_{\mu\rho} + i\eta_{\nu\rho}J_{\mu\sigma}. \tag{1.42}$$

Note that if we denote $a \equiv (\mu\nu)$, $b \equiv (\rho\sigma)$, and $c \equiv (\lambda\pi)$, we then have

$$f_{ab}{}^{c} = -\eta_{\mu\rho}\delta^{\lambda}_{[\nu}\delta^{\pi}_{\sigma]} + \eta_{\mu\sigma}\delta^{\lambda}_{[\nu}\delta^{\pi}_{\rho]} - \eta_{\nu\sigma}\delta^{\lambda}_{[\mu}\delta^{\pi}_{\rho]} + \eta_{\nu\rho}\delta^{\lambda}_{[\mu}\delta^{\pi}_{\sigma]}, \tag{1.43}$$

so (1.42) is indeed of the Lie algebra type.

The Lie algebra $SO(1,3)$ is (modulo some global subtleties) the same as the product of two $SU(2)$s (i.e. $SU(2) \times SU(2)$), which can be seen by first defining

$$J_{0i} \equiv K_i; \quad J_{ij} \equiv \epsilon_{ijk}J_k, \tag{1.44}$$

where $i, j, k = 1, 2, 3$, and then redefining

$$M_i \equiv \frac{J_i + iK_i}{2}; \quad N_i \equiv \frac{J_i - iK_i}{2}, \tag{1.45}$$

after which we obtain

$$\begin{aligned} [M_i, M_j] &= i\epsilon_{ijk}M_k, \\ [N_i, N_j] &= i\epsilon_{ijk}N_k, \\ [M_i, N_j] &= 0, \end{aligned} \tag{1.46}$$

which we leave as an exercise to prove.

Important Concepts to Remember

- Quantum field theory is a relativistic quantum mechanics, which necessarily describes an arbitrary number of particles.
- Particle–antiparticle pairs can be created and disappear, both as real (energetically allowed) and virtual (energetically disallowed, only possible due to Heisenberg's uncertainty principle).
- If we use the usual quantum mechanics rules, even with $E = \sqrt{p^2 + m^2}$, we have causality breakdown: the amplitude for propagation is nonzero even much outside the lightcone.
- Feynman diagrams represent the interaction processes of creation and annihilation of particles.
- When generalizing classical mechanics to field theory, the label i is generalized to $\vec{x}$ in $\phi(\vec{x}, t)$, and we have a Lagrangian density $\mathcal{L}(\vec{x}, t)$, conjugate momentum density $\pi(\vec{x}, t)$, and Hamiltonian density $\mathcal{H}(\vec{x}, t)$.
- For relativistic and local theories, $\mathcal{L}$ is a relativistically invariant function defined at a point x^μ.
- The Noether theorem associates a conserved current ($\partial_\mu j^\mu = 0$) with a symmetry of the Lagrangian L, in particular the energy–momentum tensor T^μ_ν with translations $x^\mu \to x^\mu + a^\mu$.
- Lorentz representations act on the fields ϕ^a, and are the exponentials of Lie algebra generators.
- The Lie algebra of $SO(1,3)$ splits into two $SU(2)$s.

Further Reading

See, for instance, sections 2.1 and 2.2 in [1] and chapter 1 in [2].

Exercises

1. Prove that for the Lie algebra of the Lorentz group

$$[J_{\mu\nu}, J_{\rho\sigma}] = -\left(-i\eta_{\mu\rho}J_{\nu\sigma} + i\eta_{\mu\sigma}J_{\nu\rho} - i\eta_{\nu\sigma}J_{\mu\rho} + i\eta_{\nu\rho}J_{\mu\sigma}\right), \tag{1.47}$$

if we define

$$\begin{aligned} J_{0i} &\equiv K_i; \quad J_{ij} \equiv \epsilon_{ijk}J_k, \\ M_i &\equiv \frac{J_i + iK_i}{2}; \quad N_i \equiv \frac{J_i - iK_i}{2}, \end{aligned} \tag{1.48}$$

we obtain that the M_i and N_i satisfy

$$\begin{aligned} [M_i, M_j] &= i\epsilon_{ijk}M_k, \\ [N_i, N_j] &= i\epsilon_{ijk}N_k, \\ [M_i, N_j] &= 0. \end{aligned} \tag{1.49}$$

2. Consider the action in Minkowski space

$$S = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \bar{\psi}(\not{D} + m)\psi - (D_\mu\phi)^* D^\mu\phi\right), \tag{1.50}$$

where $D_\mu = \partial_\mu - ieA_\mu$, $\not{D} = D_\mu\gamma^\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\bar{\psi} = \psi^\dagger i\gamma_0$, ψ is a spinor field and ϕ is a scalar field, and γ_μ are the gamma matrices, satisfying $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. Consider the electromagnetic $U(1)$ transformation

$$\psi'(x) = e^{ie\lambda(x)}\psi(x); \quad \phi'(x) = e^{ie\lambda(x)}\phi(x); \quad A'_\mu(x) = A_\mu(x) + \partial_\mu\lambda(x). \tag{1.51}$$

Calculate the Noether current.

3. Find the invariances of the model for N real scalars Φ^I, with Lagrangian

$$\mathcal{L} = g_{IJ}(\Phi^I\Phi^I)\partial_\mu\Phi^I\partial^\mu\Phi^I \tag{1.52}$$

in the case of a general metric g_{IJ}, and in the particular case of $g_{IJ} = \eta_{IJ}$ (at least in a local neighborhood in scalar space).

4. Calculate the equations of motion of the Dirac–Born–Infeld (DBI) scalar Lagrangian

$$\mathcal{L} = -\frac{1}{L^4}\sqrt{1 + L^4[g(\phi)(\partial_\mu\phi)^2 + m^2\phi^2]}. \tag{1.53}$$

2 Quantum Mechanics: Harmonic Oscillator and Quantum Mechanics in Terms of Path Integrals

The career of a young theoretical physicist consists of treating the harmonic oscillator in ever-increasing levels of abstraction

Sidney Coleman

In this chapter, I will review some facts about the harmonic oscillator in quantum mechanics, and then show how to do quantum mechanics in terms of path integrals, something that should be taught in a standard quantum mechanics course, though it does not always happen.

As the quote above shows, understanding the harmonic oscillator really well is crucial: we understand everything if we understand this simple example really well, such that we can generalize it to more complicated systems. Similarly, most of the issues of quantum field theory in path-integral formalism can be described by using the simple example of the quantum-mechanical path integral.

2.1 The Harmonic Oscillator and its Canonical Quantization

The harmonic oscillator is the simplest possible nontrivial quantum system, with a quadratic potential, that is with the Lagrangian

$$L = \frac{\dot{q}^2}{2} - \omega^2 \frac{q^2}{2}, \tag{2.1}$$

giving the Hamiltonian

$$H = \frac{1}{2}(p^2 + \omega^2 q^2). \tag{2.2}$$

Using the definition

$$\begin{aligned} a &= \frac{1}{\sqrt{2\omega}}(\omega q + ip), \\ a^\dagger &= \frac{1}{\sqrt{2\omega}}(\omega q - ip), \end{aligned} \tag{2.3}$$

inverted as

$$\begin{aligned} p &= -i\sqrt{\frac{\omega}{2}}(a - a^\dagger), \\ q &= \frac{1}{\sqrt{2\omega}}(a + a^\dagger), \end{aligned} \tag{2.4}$$

we can write the Hamiltonian as

$$H = \frac{\omega}{2}(aa^\dagger + a^\dagger a), \tag{2.5}$$

where, even though we are now at the classical level, we have been careful to keep the order of $a, a^\dagger$ as is. Of course, classically we could then write

$$H = \omega a^\dagger a. \tag{2.6}$$

In **classical mechanics**, one can define the Poisson bracket of two functions $f(p,q)$ and $g(p,q)$ as

$$\{f,g\}_{P.B.} \equiv \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right). \tag{2.7}$$

With this definition, we can immediately check that

$$\{p_i, q_j\}_{P.B.} = -\delta_{ij}. \tag{2.8}$$

The Hamilton equations of motion then become

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} = \{q_i, H\}, \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} = \{p_i, H\}. \end{aligned} \tag{2.9}$$

Then, **canonical quantization** is simply the procedure of replacing the c-number variables (q,p) with the operators $(\hat{q},\hat{p})$, and replacing the Poisson brackets $\{,\}_{P.B.}$ with $1/(i\hbar)[,]$ (commutator).

In this way, in theoretical physicist's units, with $\hbar = 1$, we have

$$[\hat{p},\hat{q}] = -i. \tag{2.10}$$

Substituting in p, q the definition of $a, a^\dagger$, we find also

$$[\hat{a},\hat{a}^\dagger] = 1. \tag{2.11}$$

One thing that is not obvious from the above is the picture we are in. We know that we can describe quantum mechanics in the Schrödinger picture, with operators independent of time, or in the Heisenberg picture, where operators depend on time. There are other pictures, in particular the interaction picture that will be relevant for us later, but these are not important at this time.

In the Heisenberg picture, we can translate the classical Hamilton equations in terms of Poisson brackets into equations for the time evolution of the Heisenberg picture operators, obtaining

$$\begin{aligned} i\hbar\frac{d\hat{q}_i}{dt} &= [\hat{q}_i, H], \\ i\hbar\frac{d\hat{p}_i}{dt} &= [\hat{p}_i, H]. \end{aligned} \tag{2.12}$$

For the quantum Hamiltonian of the harmonic oscillator, we write, from (2.5):

$$\hat{H}_{qu} = \frac{\hbar\omega}{2}\left(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \tag{2.13}$$

where we have reintroduced $\hbar$ just so we remember that $\hbar\omega$ is an energy. The operators $\hat{a}$ and $\hat{a}^\dagger$ are called destruction (annihilation) or lowering, and creation or raising operators, since the eigenstates of the harmonic oscillator Hamiltonian are defined by an occupation number n, such that

$$\hat{a}^\dagger|n\rangle \propto |n+1\rangle; \quad \hat{a}|n\rangle \propto |n-1\rangle, \tag{2.14}$$

meaning that

$$\hat{a}^\dagger\hat{a}|n\rangle \equiv \hat{N}|n\rangle = n|n\rangle. \tag{2.15}$$

This means that in the vacuum, for occupation number $n = 0$, we still have an energy

$$E_0 = \frac{\hbar\omega}{2}, \tag{2.16}$$

called the vacuum energy or zero-point energy. On the contrary, the remainder is called the *normal ordered Hamiltonian*

$$:\hat{H}:=\hbar\omega\hat{a}^\dagger\hat{a}, \tag{2.17}$$

where we define the *normal order* such that $\hat{a}^\dagger$ is to the left of $\hat{a}$, that is

$$:\hat{a}^\dagger\hat{a}:=\hat{a}^\dagger\hat{a}; \quad :\hat{a}\hat{a}^\dagger:=\hat{a}^\dagger\hat{a}. \tag{2.18}$$

2.2 The Feynman Path Integral in Quantum Mechanics in Phase Space

We now turn to the discussion of the Feynman path integral, not always taught in quantum mechanics classes, though essential for the generalization to field theory.

Given a position q at time t, an important quantity is the amplitude for the probability of finding the particle at point q' and time t':

$$M(q', t'; q, t) = {}_H\langle q', t'|q, t\rangle_H, \tag{2.19}$$

where $|q, t\rangle_H$ is the state, eigenstate of $\hat{q}(t)$ at time t, in the Heisenberg picture.

Let us remember a bit about pictures in quantum mechanics. There are more pictures, but for now we will be interested in the two basic ones, the Schrödinger picture and the Heisenberg picture. In the Heisenberg picture, operators depend on time, in particular we have $\hat{q}_H(t)$, and the state $|q, t\rangle_H$ is independent of time, and t is just a label. This means that the state is an eigenstate of $\hat{q}_H(T)$ at time $T = t$, that is

$$\hat{q}_H(T = t)|q, t\rangle_H = q|q, t\rangle_H, \tag{2.20}$$

and it is *not* an eigenstate for $T \neq t$. The operator in the Heisenberg picture $\hat{q}_H(t)$ is related to that in the Schrödinger picture $\hat{q}_S$ by

$$\hat{q}_H(t) = e^{i\hat{H}t}\hat{q}_S e^{-i\hat{H}t}, \tag{2.21}$$

and the Schrödinger picture state is related to the Heisenberg picture state by

$$|q\rangle = e^{-i\hat{H}t}|q,t\rangle_H, \tag{2.22}$$

and is an eigenstate of $\hat{q}_S$, that is

$$\hat{q}_S|q\rangle = q|q\rangle. \tag{2.23}$$

In terms of the Schrödinger picture, we then have the probability amplitude

$$M(q',t';q,t) = \langle q'|e^{-i\hat{H}(t'-t)}|q\rangle. \tag{2.24}$$

From now on we will drop the index H and S for states, since it is obvious, if we write $|q,t\rangle$ we are in the Heisenberg picture, if we write $|q\rangle$ we are in the Schrödinger picture.

Let us now derive the path-integral representation.

Divide the time interval between t and t' into a large number $n+1$ of equal intervals, and denote

$$\epsilon \equiv \frac{t'-t}{n+1}; \qquad t_0 = t, t_1 = t+\epsilon, \ldots, t_{n+1} = t'. \tag{2.25}$$

But at any fixed t_i, the set $\{|q_i,t_i\rangle_H | q_i \in \mathbb{R}\}$ is a complete set, meaning that we have the completeness relation

$$\int dq_i |q_i,t_i\rangle\langle q_i,t_i| = \mathbf{1}. \tag{2.26}$$

We then introduce n factors of $\mathbf{1}$, one for each t_i, $i = 1,\ldots,n$, in the amplitude $M(q',t';q,t)$ in (2.19), obtaining

$$M(q,t';q,t) = \int dq_1 \ldots dq_n \langle q',t'|q_n,t_n\rangle\langle q_n,t_n|q_{n-1},t_{n-1}\rangle \ldots |q_1,t_1\rangle\langle q_1,t_1|q,t\rangle, \tag{2.27}$$

where the discrete positions $q_i \equiv q(t_i)$ give us a *regularized path* between q and q'.

But note that this is not a classical path, since at any of the times t_i, the position q can be anything ($q_i \in \mathbb{R}$), independent of q_{i-1}, and independent of how small ϵ is, whereas classically we have a continuous path, meaning as ϵ gets smaller, $q_i - q_{i-1}$ can only be smaller and smaller. But integrating over all $q_i = q(t_i)$ means we integrate over these quantum paths, where q_i is arbitrary (independent on q_{i-1}), as in Figure 2.1. Therefore, we denote

$$\mathcal{D}q(t) \equiv \prod_{i=1}^{n} dq(t_i), \tag{2.28}$$

and this is an "integral over all paths," or "path integral."

In contrast, considering that

$$|q\rangle = \int \frac{dp}{2\pi}|p\rangle\langle p|q\rangle,$$
$$|p\rangle = \int dq|q\rangle\langle q|p\rangle \tag{2.29}$$

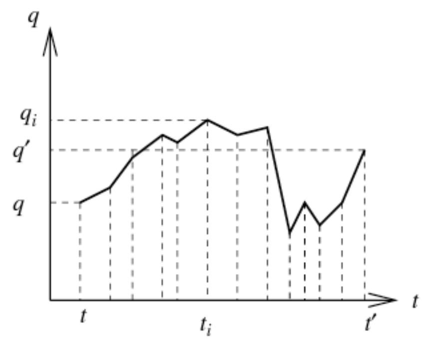

Figure 2.1 In the quantum-mechanical path integral, we integrate over discretized paths. The paths are not necessarily smooth, as classical paths are: we divide the path into a large number of discrete points, and then integrate over the positions of these points.

(note the factor of 2π, which is necessary, since $\langle q|p\rangle = e^{ipq}$ and $\int dq e^{iq(p-p')} = 2\pi\delta(p - p')$), we have

$$
{}_H\langle q(t_i), t_i|q(t_{i-1}), t_{i-1}\rangle_H = \langle q(t_i)|e^{-i\epsilon\hat{H}}|q(t_{i-1})\rangle = \int \frac{dp(t_i)}{2\pi}\langle q(t_i)|p(t_i)\rangle\langle p(t_i)|e^{-i\epsilon\hat{H}}|q(t_{i-1})\rangle. \tag{2.30}
$$

Now we need a technical requirement on the quantum Hamiltonian: it has to be ordered such that all the $\hat{p}$s are to the left of the $\hat{q}$s.

Then, to first order in ϵ, we can write

$$
\langle p(t_i)|e^{-i\epsilon\hat{H}(\hat{p},\hat{q})}|q(t_{i-1})\rangle = e^{-i\epsilon H(p(t_i),q(t_{i-1}))}\langle p(t_i)|q(t_{i-1})\rangle = e^{-i\epsilon H(p(t_i),q(t_{i-1}))}e^{-ip(t_i)q(t_{i-1})}, \tag{2.31}
$$

since $\hat{p}$ will act on the left on $\langle p(t_i)|$ and $\hat{q}$ will act on the right on $|q(t_{i-1})\rangle$. Of course, to higher order in ϵ, we have $\hat{H}(\hat{p},\hat{q})\hat{H}(\hat{p},\hat{q})$, which will have terms like $\hat{p}\hat{q}\hat{p}\hat{q}$, which are more complicated. But since we have $\epsilon \to 0$, we only need the first order in ϵ.

Then we get

$$
\begin{aligned}
M(q',t';q,t) = &\int \prod_{i=1}^{n}\frac{dp(t_i)}{2\pi}\prod_{j=1}^{n}dq(t_j)\langle q(t_{n+1}|p(t_{n+1})\rangle\langle p(t_{n+1})|e^{-i\epsilon\hat{H}}|q(t_n)\rangle\ldots\\
&\ldots\langle q(t_1)|p(t_1)\rangle\langle p(t_1)|e^{-i\epsilon\hat{H}}|q(t_0)\rangle\\
=&\,\mathcal{D}p(t)\mathcal{D}q(t)\exp\left\{i\left[p(t_{n+1})(q(t_{n+1})-q(t_n))+\ldots+p(t_1)(q(t_1)-q(t_0))\right.\right.\\
&\left.\left.-\epsilon(H(p(t_{n+1}),q(t_n))+\ldots+H(p(t_1),q(t_0)))\right]\right\}\\
=&\,\mathcal{D}p(t)\mathcal{D}q(t)\exp\left\{i\int_{t_0}^{t_{n+1}}dt[p(t)\dot{q}(t)-H(p(t),q(t))]\right\},
\end{aligned} \tag{2.32}
$$

where we have used the fact that $q(t_{i+1}) - q(t_i) \to dt\dot{q}(t_i)$. The above expression is called the *path integral in phase space*.

We note that this was derived rigorously (rigorously for a physicist, of course... a mathematician would disagree). But we would like a path integral in configuration space. For that, however, we need one more technical requirement: we need the Hamiltonian to be quadratic in momenta, that is

$$H(p,q) = \frac{p^2}{2} + V(q). \tag{2.33}$$

If this is not true, we have to start in phase space and see what we get in configuration space. But if we have only quadratic terms in momenta, we can use Gaussian integration to derive the path integral in configuration space. Therefore we will make a mathematical interlude to define some Gaussian integration formulas that will be useful here and later.

2.3 Gaussian Integration

The basic Gaussian integral is

$$I = \int_{-\infty}^{+\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}. \tag{2.34}$$

Squaring this integral formula, we obtain also

$$I^2 = \int dxdye^{-(x^2+y^2)} = \int_0^{2\pi} d\phi \int_0^{\infty} rdre^{-r^2} = \pi. \tag{2.35}$$

We can generalize this as

$$\int d^n x e^{-\frac{1}{2}x_i A_{ij} x_j} = (2\pi)^{n/2} (\det A)^{-1/2}, \tag{2.36}$$

which can be proven, for instance, by diagonalizing the matrix A, and since $\det A = \prod_i \alpha_i$, with α_i the eigenvalues of A, we get the above.

Finally, consider the object

$$S = \frac{1}{2} x^T A x + b^T x \tag{2.37}$$

(which later on in the book will be used as a discretized form for an action, hence the name S). Considering it as an action, the classical solution will be $\partial S/\partial x_i = 0$:

$$x_c = -A^{-1} b, \tag{2.38}$$

and then

$$S(x_c) = -\frac{1}{2} b^T A^{-1} b, \tag{2.39}$$

which means that we can write

$$S = \frac{1}{2}(x - x_c)^T A (x - x_c) - \frac{1}{2} b^T A^{-1} b, \tag{2.40}$$

and thus we find

$$\int d^n x e^{-S(x)} = (2\pi)^{n/2} (\det A)^{-1/2} e^{-S(x_c)} = (2\pi)^{n/2} (\det A)^{-1/2} e^{+\frac{1}{2} b^T A^{-1} b}. \tag{2.41}$$

2.4 Path Integral in Configuration Space

We are now ready to go to configuration space. The Gaussian integration we need to do is then the one over $\mathcal{D}p(t)$, which is

$$\int \mathcal{D}p(\tau)e^{i\int_t^{t'} d\tau[p(\tau)\dot{q}(\tau)-\frac{1}{2}p^2(\tau)]}, \tag{2.42}$$

which when discretized becomes

$$\prod_i \frac{dp(t_i)}{2\pi} \exp\left[i\Delta\tau\left(p(t_i)\dot{q}(t_i)-\frac{1}{2}p^2(\tau_i)\right)\right], \tag{2.43}$$

and therefore in our mathematical notation for the Gaussian integral, we have $x_i = p(t_i)$, $A_{ij} = i\Delta\tau\delta_{ij}$, $b = -i\Delta\tau\dot{q}(t_i)$, giving

$$\int \mathcal{D}p(\tau)e^{i\int_t^{t'} d\tau[p(\tau)\dot{q}(\tau)-\frac{1}{2}p^2(\tau)]} = \mathcal{N}e^{i\int_t^{t'} d\tau\frac{\dot{q}(\tau)^2}{2}}, \tag{2.44}$$

where $\mathcal{N}$ contains constant factors of $2, \pi, i, \Delta\tau$, which we will see are irrelevant.

Then the probability amplitude in configuration space is

$$\begin{aligned} M(q',t';q,t) &= \mathcal{N}\int \mathcal{D}q \exp\left\{i\int_t^{t'} d\tau\left[\frac{\dot{q}^2(\tau)}{2}-V(q)\right]\right\} \\ &= \mathcal{N}\int \mathcal{D}q \exp\left\{i\int_t^{t'} d\tau L(q(\tau),\dot{q}(\tau))\right\} \\ &= \mathcal{N}\int \mathcal{D}q e^{iS[q]}. \end{aligned} \tag{2.45}$$

This is the path integral in configuration space that we were seeking. But we have to remember that this formula is valid only if the Hamiltonian is quadratic in momenta, otherwise we need to redo the calculation starting from phase space.

2.5 Correlation Functions

We have found how to write the probability of transition between (q,t) and (q',t'), and that is good. But there are other observables that we can construct which are of interest, for instance the correlation functions.

The simplest one is the one-point function

$$\langle q',t'|\hat{q}(t_a)|q,t\rangle, \tag{2.46}$$

where we can make it such that t_a coincides with one of the t_is in the discretization of the time interval. The calculation now proceeds as before, but in the step (2.27) we introduce the $\mathbf{1}$s such that we have (besides the usual products) also the expectation value

$$\langle q_{i+1},t_{i+1}|\hat{q}(t_a)|q_i,t_i\rangle = q(t_a)\langle q_{i+1},t_{i+1}|q_i,t_i\rangle, \tag{2.47}$$

since $t_a = t_i \Rightarrow q(t_a) = q_i$. Then the calculation proceeds as before, since the only new thing is the appearance of $q(t_a)$, leading to

$$\langle q', t'|\hat{q}(t_a)|q, t\rangle = \int \mathcal{D}q e^{iS[q]} q(t_a). \tag{2.48}$$

Consider next the two-point function

$$\langle q', t'|\hat{q}(t_b)\hat{q}(t_a)|q, t\rangle. \tag{2.49}$$

If we have $t_a < t_b$, we can proceed as before. Indeed, remember that in (2.27) the **1**s were introduced in time order, such that we can do the rest of the calculation and get the path integral. Therefore, if $t_a < t_b$, we can choose $t_a = t_i, t_b = t_j$, such that $j > i$, and then we have

$$\begin{aligned}&\ldots\langle q_{j+1}, t_{j+1}|\hat{q}(t_b)|q_j, t_j\rangle \ldots \langle q_{i+1}, t_{i+1}|\hat{q}(t_a)|q_i, t_i\rangle \ldots \\ &= \ldots q(t_b)\langle q_{j+1}, t_{j+1}|q_j, t_j\rangle \ldots q(t_a)\langle q_{i+1}, t_{i+1}|q_i, t_i\rangle \ldots,\end{aligned} \tag{2.50}$$

besides the usual products, leading to

$$\int \mathcal{D}q e^{iS[q]} q(t_b)q(t_a). \tag{2.51}$$

Then, reversely, the path integral leads to the two-point function where the $q(t)$ are ordered according to time (time ordering), that is

$$\int \mathcal{D}q e^{iS[q]} q(t_a)q(t_b) = \langle q', t'|T\{\hat{q}(t_a)\hat{q}(t_b)\}|q, t\rangle, \tag{2.52}$$

where time ordering is defined as

$$\begin{aligned}T\{\hat{q}(t_a)\hat{q}(t_b)\} &= \hat{q}(t_a)\hat{q}(t_b) \;\text{ if }\; t_a > t_b, \\ &= \hat{q}(t_b)\hat{q}(t_a) \;\text{ if }\; t_b > t_a,\end{aligned} \tag{2.53}$$

which has an obvious generalization to

$$T\{\hat{q}(t_{a_1}) \ldots \hat{q}(t_{a_N})\} = \hat{q}(t_{a_1}) \ldots \hat{q}(t_{a_N}) \;\text{ if }\; t_{a_1} > t_{a_2} > \ldots > t_{a_N}, \tag{2.54}$$

and otherwise they are ordered in the order of their times.

Then we similarly find that the *n-point function* or *correlation function* is

$$G_n(t_{a_1}, \ldots, t_{a_n}) \equiv \langle q', t'|T\{\hat{q}(t_{a_1}) \ldots q(t_{a_n})\}|q, t\rangle = \int \mathcal{D}q e^{iS[q]} q(t_{a_1}) \ldots q(t_{a_n}). \tag{2.55}$$

In mathematics, for a set $\{a_n\}_n$, we can define a *generating function*

$$f(x) \equiv \sum_n \frac{1}{n!} a_n x^n, \tag{2.56}$$

such that we can find a_n from its derivatives:

$$a_n = \left.\frac{d^n}{dx^n} f(x)\right|_{x=0}. \tag{2.57}$$

Similarly, we can now define a generating functional

$$Z[J] = \sum_{N\geq 0} \int dt_1 \dots \int dt_N \frac{i^N}{N!} G_N(t_1, \dots, t_N) J(t_1) \dots J(t_N). \tag{2.58}$$

As we see, the difference is that now we have $G_N(t_1, \dots, t_N)$ instead of a_N, so we needed to integrate over dt, and instead of x, introduce $J(t)$, and i was conventional. Using (2.55), the integrals factorize, and we obtain just a product of the same integral:

$$Z[J] = \int \mathcal{D}q e^{iS[q]} \sum_{N\geq 0} \frac{1}{N!} \left[\int dt iq(t) J(t)\right]^N, \tag{2.59}$$

so finally

$$Z[J] = \int \mathcal{D}q e^{iS[q,J]} = \int \mathcal{D}q e^{iS[q] + i\int dt J(t) q(t)}. \tag{2.60}$$

We then find that this object indeed generates the correlation functions by

$$\left.\frac{\delta^N}{i\delta J(t_1) \dots iJ(t_N)} Z[J]\right|_{J=0} = \int \mathcal{D}q e^{iS[q]} q(t_1) \dots q(t_N) = G_N(t_1, \dots, t_N). \tag{2.61}$$

Important Concepts to Remember

- For the harmonic oscillator, the Hamiltonian in terms of $a, a^\dagger$ is $H = \omega/2(aa^\dagger + a^\dagger a)$.
- Canonical quantization replaces classical functions with quantum operators, and Poisson brackets with commutators, $\{,\}_{P.B.} \to 1/(i\hbar)[,]$.
- At the quantum level, the harmonic oscillator Hamiltonian is the sum of a normal ordered part and a zero-point energy part.
- The transition probability from (q,t) to (q',t') is a path integral in phase space, $F(q',t';q,t) = \int \mathcal{D}q\mathcal{D}p e^{i\int(p\dot{q}-H)}$.
- If the Hamiltonian is quadratic in momenta, we can go to the path integral in configuration space and find $F(q',t';q,t) = \int \mathcal{D}q e^{iS}$.
- The n-point functions or correlation functions, with insertions of $q(t_i)$ in the path integral, give the expectation values of time-ordered $\hat{q}(t)$s.
- The n-point functions can be found from the derivatives of the generating function $Z[J]$.

Further Reading

See section 1.3 in [3] and chapter 2 in [4].

Exercises

1. Consider the Lagrangian
$$L(q,\dot{q}) = \frac{\dot{q}^2}{2} - \frac{\lambda}{4!}q^4. \tag{2.62}$$
Write down the *Hamiltonian* equations of motion and the path integral *in phase space* for this model.
2. Consider the generating functional
$$\ln Z[J] = \int dt \frac{J^2(t)}{2} f(t) + \lambda \int dt \frac{J^3(t)}{3!} + \tilde{\lambda} \int dt \frac{J^4(t)}{4!}. \tag{2.63}$$
Calculate the three-point function and the four-point function.
3. Repeat the change from phase-space path integral to configuration-space path integral for a Hamiltonian of the type
$$H = \frac{p^2}{2} + \alpha p + \beta + V(q). \tag{2.64}$$
4. Calculate the generating functional $Z[J]$ for the case of a quadratic action, with $V[q] = \omega^2 q^2/2$, that is for the harmonic oscillator
$$L = \frac{\dot{q}^2}{2} - \omega^2 \frac{q^2}{2}. \tag{2.65}$$

3 Canonical Quantization of Scalar Fields

In this chapter, we will learn how to quantize classical scalar fields, similarly to the procedure in the quantum mechanics of a finite number of degrees of freedom.

As we saw, in quantum mechanics, for a particle with Hamiltonian $H(p,q)$, we replace the Poisson bracket

$$\{f,g\}_{P.B.} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right) \tag{3.1}$$

with $\{p_i, q_j\}_{P.B.} = -\delta_{ij}$, with the commutator, $\{,\}_{P.B.} \to \frac{1}{i\hbar}[,]$, and all functions of (p,q) become quantum operators, in particular $[\hat{p}_i, \hat{q}_j] = -i\hbar$, and for the harmonic oscillator we have

$$[\hat{a}, \hat{a}^\dagger] = 1. \tag{3.2}$$

3.1 Quantizing Scalar Fields: Kinematics

Then, in order to generalize to field theory, we must first define the Poisson brackets. As we already have a definition in the case of a set of particles, we will discretize *space*, in order to use that definition. Therefore we consider the coordinates, and their canonically conjugate momenta

$$\begin{aligned} q_i(t) &= \sqrt{\Delta V}\phi_i(t), \\ p_i(t) &= \sqrt{\Delta V}\pi_i(t), \end{aligned} \tag{3.3}$$

where $\phi_i(t) \equiv \phi(\vec{x}_i, t)$ and similarly for $\pi_i(t)$. We should also define how to go from the derivatives in the Poisson brackets to functional derivatives. The recipe is

$$\begin{aligned} &\frac{1}{\Delta V}\frac{\partial f_i(t)}{\partial \phi_j(t)} \to \frac{\delta f(\phi(\vec{x}_i,t), \pi(\vec{x}_i,t))}{\delta\phi(\vec{x}_j,t)}, \\ &\Delta V \to d^3x, \end{aligned} \tag{3.4}$$

where *functional derivatives* are defined such that, for instance

$$H(t) = \int d^3x \frac{\phi^2(\vec{x},t)}{2} \Rightarrow \frac{\delta H(t)}{\delta\phi(\vec{x},t)} = \phi(\vec{x},t), \tag{3.5}$$

that is by dropping the integral sign and then taking normal derivatives.

Replacing these definitions in the Poisson brackets (3.1), we get

$$\{f,g\}_{P.B.} = \int d^3x \left[\frac{\delta f}{\delta\phi(\vec{x},t)} \frac{\delta g}{\delta\pi(\vec{x},t)} - \frac{\delta f}{\delta\pi(\vec{x},t)} \frac{\delta g}{\delta\phi(\vec{x},t)} \right], \tag{3.6}$$

and then we immediately find

$$\begin{aligned} &\{\phi(\vec{x},t), \pi(\vec{x}',t)\}_{P.B.} = \delta^3(\vec{x}-\vec{x}'), \\ &\{\phi(\vec{x},t), \phi(\vec{x}',t)\}_{P.B.} = \{\pi(\vec{x},t), \pi(\vec{x}',t)\} = 0, \end{aligned} \tag{3.7}$$

where we note that these are *equal time commutation relations*, in the same way that we had in quantum mechanics, more precisely, $\{q_i(t), p_j(t)\}_{P.B.} = \delta_{ij}$.

We can now easily do *canonical quantization* of this scalar field. We just replace classical fields $\phi(\vec{x},t)$ with quantum Heisenberg operators $\phi_H(\vec{x},t)$ (we will drop the H, understanding that if there is time dependence we are in the Heisenberg picture and if we don't have time dependence we are in the Schrödinger picture), and $\{,\}_{P.B.} \rightarrow 1/(i\hbar)[,]$, obtaining the fundamental equal time commutation relations

$$\begin{aligned} &[\phi(\vec{x},t), \pi(\vec{x}',t)] = i\hbar\delta^3(\vec{x}-\vec{x}'), \\ &[\phi(\vec{x},t), \phi(\vec{x}',t)] = [\pi(\vec{x},t), \pi(\vec{x}',t)] = 0. \end{aligned} \tag{3.8}$$

We further define the Fourier transforms

$$\begin{aligned} \phi(\vec{x},t) &= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{x}\cdot\vec{p}} \phi(\vec{p},t) \\ &\Rightarrow \phi(\vec{p},t) = \int d^3x e^{-i\vec{p}\cdot\vec{x}} \phi(\vec{x},t), \\ \pi(\vec{x},t) &= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{x}\cdot\vec{p}} \pi(\vec{p},t) \\ &\Rightarrow \pi(\vec{p},t) = \int d^3x e^{-i\vec{p}\cdot\vec{x}} \pi(\vec{x},t). \end{aligned} \tag{3.9}$$

We also define, using the same formulas used for the harmonic oscillator, but now for each of the momentum modes of the fields:

$$\begin{aligned} a(\vec{k},t) &= \sqrt{\frac{\omega_k}{2}} \phi(\vec{k},t) + \frac{i}{\sqrt{2\omega_k}} \pi(\vec{k},t), \\ a^\dagger(\vec{k},t) &= \sqrt{\frac{\omega_k}{2}} \phi^\dagger(\vec{k},t) - \frac{i}{\sqrt{2\omega_k}} \pi^\dagger(\vec{k},t). \end{aligned} \tag{3.10}$$

We will see later that we have $\omega_k = \sqrt{\vec{k}^2 + m^2}$, but for the moment we will only need that $\omega_k = \omega(|\vec{k}|)$. Then, replacing these definitions in ϕ and π, we obtain

$$\begin{aligned} \phi(\vec{x},t) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a(\vec{p},t)e^{i\vec{p}\cdot\vec{x}} + a^\dagger(\vec{p},t)e^{-i\vec{p}\cdot\vec{x}}) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} e^{i\vec{p}\cdot\vec{x}} (a(\vec{p},t) + a^\dagger(-\vec{p},t)), \\ \pi(\vec{x},t) &= \int \frac{d^3p}{(2\pi)^3} \left(-i\sqrt{\frac{\omega_p}{2}}\right) (a(\vec{p},t)e^{i\vec{p}\cdot\vec{x}} - a^\dagger(\vec{p},t)e^{-i\vec{p}\cdot\vec{x}}) \end{aligned}$$

$$= \int \frac{d^3p}{(2\pi)^3}\left(-i\sqrt{\frac{\omega_p}{2}}\right)(a(\vec{p},t) - a^\dagger(-\vec{p},t))e^{i\vec{p}\cdot\vec{x}}. \tag{3.11}$$

In terms of $a(\vec{p},t)$ and $a^\dagger(\vec{p},t)$, we obtain the commutators

$$\begin{aligned}
&[a(\vec{p},t), a^\dagger(\vec{p}',t)] = (2\pi)^3\delta^3(\vec{p}-\vec{p}'),\\
&[a(\vec{p},t), a(\vec{p}',t)] = [a^\dagger(\vec{p},t), a^\dagger(\vec{p}',t)] = 0.
\end{aligned} \tag{3.12}$$

As a consistency check, we can check, for instance, that the $[\phi, \pi]$ commutator gives the correct result, given the above $a, a^\dagger$ commutators:

$$\begin{aligned}
[\phi(\vec{x},t), \pi(\vec{x}',t)] = &\int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\left(-\frac{i}{2}\sqrt{\frac{\omega_{p'}}{\omega_p}}\right)([a^\dagger(-\vec{p},t), a(\vec{p}',t)]\\
&-[a(\vec{p},t), a^\dagger(-\vec{p}',t)])e^{i(\vec{p}\cdot\vec{x}+\vec{p}'\cdot\vec{x}')}\\
= &\, i\delta^3(\vec{x}-\vec{x}').
\end{aligned} \tag{3.13}$$

We now note that the calculation above was independent of the form of ω_p ($= \sqrt{p^2+m^2}$), we only used the fact that $\omega_p = \omega(|\vec{p}|)$, but otherwise we just used the definitions adapted from the harmonic oscillator. We have also not written any explicit time dependence, it was left implicit through $a(\vec{p},t), a^\dagger(\vec{p},t)$. Yet, we obtained the same formulas as for the harmonic oscillator.

3.2 Quantizing Scalar Fields: Dynamics and Time Evolution

We should now understand the dynamics, which will give us the formula for ω_p.

We therefore go back, and start systematically. We work with a *free scalar field* (i.e. one with $V = 0$) with Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 \tag{3.14}$$

and action $S = \int d^4x\mathcal{L}$. Partially integrating $-1/2\int \partial_\mu\phi\partial^\mu\phi = +1/2\int \phi\partial_\mu\partial^\mu\phi$, we obtain the *Klein–Gordon (KG)* equation of motion

$$(\partial_\mu\partial^\mu - m^2)\phi = 0 \Rightarrow (-\partial_t^2 + \partial_{\vec{x}}^2 - m^2)\phi = 0. \tag{3.15}$$

Going to momentum ($\vec{p}$) space via a Fourier transform, we obtain

$$\left[\frac{\partial^2}{\partial t^2} + (\vec{p}^2 + m^2)\right]\phi(\vec{p},t) = 0, \tag{3.16}$$

which is the equation of motion for a harmonic oscillator with $\omega = \omega_p = \sqrt{p^2+m^2}$.

This means that the Hamiltonian is

$$H = \frac{1}{2}(p^2 + \omega_p^2\phi^2) \tag{3.17}$$

and we can use the transformation

$$\phi = \frac{1}{\sqrt{2\omega}}(a + a^\dagger); \quad p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger), \tag{3.18}$$

after which we obtain as usual $[a, a^\dagger] = 1$. Therefore, we can now justify *a posteriori* the transformations that we did on the field.

We should also calculate the Hamiltonian. As explained in Chapter 1, using the discretization of space, we write

$$\begin{aligned} H &= \sum_{\vec{x}} p(\vec{x}, t)\dot{\phi}(\vec{x}, t) - L \\ &= \int d^3x[\pi(\vec{x}, t)\dot{\phi}(\vec{x}, t) - \mathcal{L}] \equiv d^3x\mathcal{H}, \end{aligned} \tag{3.19}$$

and from the Lagrangian (3.14) we obtain

$$\begin{aligned} &\pi(\vec{x}, t) = \dot{\phi}(\vec{x}, t) \Rightarrow \\ &\quad \mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2. \end{aligned} \tag{3.20}$$

Substituting ϕ, π inside the Hamiltonian, we obtain

$$\begin{aligned} H = &\int d^3x \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3} e^{i(\vec{p}+\vec{p}')\vec{x}} \left\{ -\frac{\sqrt{\omega_p\omega_{p'}}}{4}(a(\vec{p}, t) - a^\dagger(-\vec{p}, t))(a(\vec{p}', t) - a^\dagger(-\vec{p}', t)) \right. \\ &\left. + \frac{-\vec{p}\cdot\vec{p}' + m^2}{4\sqrt{\omega_p\omega_{p'}}}(a(\vec{p}, t) + a^\dagger(-\vec{p}, t))(a(\vec{p}', t) + a^\dagger(-\vec{p}', t)) \right\} \\ = &\int \frac{d^3p}{(2\pi)^3}\frac{\omega_p}{2}(a^\dagger(\vec{p}, t)a(\vec{p}, t) + a(\vec{p}, t)a^\dagger(\vec{p}, t)), \end{aligned} \tag{3.21}$$

where in the last line we have first integrated over $\vec{x}$, obtaining $\delta^3(\vec{p} + \vec{p}')$, and then integrated over $\vec{p}'$, obtaining $\vec{p}' = -\vec{p}$. We have finally reduced the Hamiltonian to an infinite (continuum, even) sum over harmonic oscillators.

We have dealt with the first issue stated earlier, about the dynamics of the theory. We now address the second, of the explicit time dependence.

We have Heisenberg operators, for which the time evolution is

$$i\frac{d}{dt}a(\vec{p}, t) = [a(\vec{p}, t), H]. \tag{3.22}$$

Calculating the commutator from the above Hamiltonian (using the fact that $[a, aa^\dagger] = [a, a^\dagger a] = a$), we obtain

$$i\frac{d}{dt}a(\vec{p}, t) = \omega_p a(\vec{p}, t). \tag{3.23}$$

More generally, the time evolution of the Heisenberg operators in field theories is given by

$$\mathcal{O}(x) = \mathcal{O}_H(\vec{x}, t) = e^{iHt}\mathcal{O}(\vec{x})e^{-iHt}, \tag{3.24}$$

which is equivalent to

$$i\frac{\partial}{\partial t}\mathcal{O}_H = [\mathcal{O}, H] \tag{3.25}$$

via

$$i\frac{d}{dt}(e^{iAt}Be^{-iAt}) = [B, A]. \tag{3.26}$$

The solution of (3.23) is

$$\begin{aligned} a(\vec{p}, t) &= a_{\vec{p}}e^{-i\omega_p t}, \\ a^\dagger(\vec{p}, t) &= a^\dagger_{\vec{p}}e^{+i\omega_p t}. \end{aligned} \tag{3.27}$$

Replacing in $\phi(\vec{x}, t)$ and $\pi(\vec{x}, t)$, we obtain

$$\begin{aligned} \phi(\vec{x}, t) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a_{\vec{p}}e^{ipx} + a^\dagger_{\vec{p}}e^{-ipx})|_{p^0=E_p}, \\ \pi(\vec{x}, t) &= \frac{\partial}{\partial t}\phi(\vec{x}, t), \end{aligned} \tag{3.28}$$

so we have formed the Lorentz invariants $e^{\pm ipx}$, though we haven't written an explicitly Lorentz-invariant formula. We will do this in Chapter 4. Here we have denoted $E_p = \sqrt{\vec{p}^2 + m^2}$, remembering that it is the relativistic energy of a particle of momentum $\vec{p}$ and mass m.

Finally, of course, if we want the Schrödinger picture operators, we have to remember the relation between the Heisenberg and Schrödinger pictures:

$$\phi_H(\vec{x}, t) = e^{iHt}\phi(\vec{x})e^{-iHt}. \tag{3.29}$$

3.3 Discretization

Continuous systems are hard to understand, so it would be better if we could find a rigorous way to discretize the system. Luckily, there is such a method, namely we consider a space of finite volume V (i.e. we "put the system in a box"). Obviously, this doesn't discretize space, but it does discretize *momenta*, since in a direction z of length L_z, allowed momenta will be only $k_n = 2\pi n/L_z$.

Then, the discretization is defined, as always, by

$$\begin{aligned} \int d^3k &\to \frac{1}{V}\sum_{\vec{k}}, \\ \delta^3(\vec{k} - \vec{k}') &\to V\delta_{\vec{k}\vec{k}'}, \end{aligned} \tag{3.30}$$

to which we add the redefinition

$$a_{\vec{k}} \to \sqrt{V(2\pi)^3}\alpha_{\vec{k}}, \tag{3.31}$$

which allows us to keep the usual orthonormality condition in the discrete limit, $[\alpha_{\vec{k}}, \alpha^\dagger_{\vec{k}'}] = \delta_{\vec{k}\vec{k}'}$.

Using these relations, and replacing the time dependence, which cancels out of the Hamiltonian, we get the Hamiltonian of the free scalar field in a box of volume V:

$$H = \sum_{\vec{k}} \frac{(\hbar)\omega_k}{2}(\alpha^\dagger_{\vec{k}}\alpha_{\vec{k}} + \alpha_{\vec{k}}\alpha^\dagger_{\vec{k}}) = \sum_{\vec{k}} h_{\vec{k}}, \tag{3.32}$$

where $h_{\vec{k}}$ is the Hamiltonian of a single harmonic oscillator:

$$h_{\vec{k}} = \omega_{\vec{k}}\left(N_{\vec{k}} + \frac{1}{2}\right), \tag{3.33}$$

$N_{\vec{k}} = \alpha^\dagger_{\vec{k}}\alpha_{\vec{k}}$ is the number operator for mode $\vec{k}$, with eigenstates $|n_{\vec{k}} >$:

$$N_{\vec{k}}|n_{\vec{k}}\rangle = n_{\vec{k}}|n_{\vec{k}}\rangle, \tag{3.34}$$

and the orthonormal eigenstates are

$$|n\rangle = \frac{1}{\sqrt{n!}}(\alpha^\dagger)^n|0\rangle; \quad \langle n|m\rangle = \delta_{mn}. \tag{3.35}$$

Here, $\alpha^\dagger_{\vec{k}}$ = raising/creation operator and $\alpha_{\vec{k}}$ = lowering/annihilation (destruction) operator, named since they create and annihilate a particle, respectively, that is

$$\begin{aligned}
\alpha_{\vec{k}}|n_{\vec{k}}\rangle &= \sqrt{n_{\vec{k}}}|n_{\vec{k}} - 1\rangle, \\
\alpha^\dagger_{\vec{k}}|n_{\vec{k}}\rangle &= \sqrt{n_{\vec{k}} + 1}|n_{\vec{k}} + 1\rangle, \\
h_{\vec{k}}|n\rangle &= \omega_{\vec{k}}\left(n + \frac{1}{2}\right)|n\rangle.
\end{aligned} \tag{3.36}$$

Therefore, as we know from quantum mechanics, there is a ground state $|0\rangle$, $\Leftrightarrow n_{\vec{k}} \in \mathbf{N}_+$, in which case $n_{\vec{k}}$ is called the occupation number, or number of particles in the state $\vec{k}$.

3.4 Fock Space and Normal Ordering for Bosons

3.4.1 Fock Space

The Hilbert space of states in terms of eigenstates of the number operator is called *Fock space*, or *Fock space representation.* The Fock space representation for the states of a single harmonic oscillator is $\mathcal{H}_{\vec{k}} = \{|n_{\vec{k}} >\}$.

Since the total Hamiltonian is the sum of the Hamiltonians for each mode, the total Hilbert space is the direct product of the Hilbert spaces of the Hamiltonians for each mode, $\mathcal{H} = \otimes_{\vec{k}}\mathcal{H}_{\vec{k}}$. Its states are then

$$|\{n_{\vec{k}}\}\rangle = \prod_{\vec{k}} |n_{\vec{k}}\rangle = \left(\prod_{\vec{k}} \frac{1}{\sqrt{n_{\vec{k}}!}}(\alpha^\dagger_{\vec{k}})^{n_{\vec{k}}}\right)|0\rangle. \tag{3.37}$$

Note that we have defined a unique vacuum for all the Hamiltonians, $|0\rangle$, such that $a_{\vec{k}}|0\rangle = 0, \forall \vec{k}$, instead of denoting it as $\prod_{\vec{k}} |0\rangle_{\vec{k}}$.

3.4.2 Normal Ordering

For a single harmonic oscillator mode, the ground-state energy, or zero-point energy, is $\hbar\omega_{\vec{k}}/2$, which we may think could have some physical significance. But for a free scalar field, even one in a box ("discretized"), the total ground-state energy is $\sum_{\vec{k}} \hbar\omega_{\vec{k}}/2 = \infty$, and since an observable of infinite value doesn't make sense, we have to consider that it is *unobservable*, and put it to zero.

In this simple model, that's no problem, but consider the case where this free scalar field is coupled to gravity. In a gravitational theory, energy is equivalent to mass, and gravitates (i.e. it can be measured by its gravitational effects). So how can we drop a constant piece from the energy? Are we allowed to do that? In fact, this is part of one of the biggest problems of modern theoretical physics, the *cosmological constant problem*, and the answer to this question is far from obvious. At this level, however, we will not bother with this question anymore, and drop the infinite constant.

But it is also worth mentioning that while infinities are of course unobservable, the finite difference between two infinite quantities might be observable, and in fact one such case was already measured. If we consider the difference in the zero-point energies between fields in two different boxes, one of volume V_1 and another of volume V_2, that *is* measurable, and leads to the so-called Casimir effect, which we will discuss at the beginning of Part II.

We are then led to define the *normal ordered Hamiltonian*

$$: H := H - \frac{1}{2}\sum_{\vec{k}} \hbar\omega_{\vec{k}} = \sum_{\vec{k}} \hbar\omega_{\vec{k}} N_{\vec{k}} \tag{3.38}$$

by dropping the infinite constant. The *normal order* is to always have $a^\dagger$ before a, that is $: a^\dagger a := a^\dagger a$, $: a a^\dagger := a^\dagger a$. Since as commute among themselves, as do $a^\dagger$s, and operators from different modes, in case these appear, we don't need to bother with their order. For instance then, $: a a^\dagger a^\dagger a a a a^\dagger := a^\dagger a^\dagger a^\dagger a a a$.

We then consider that only normal ordered operators have physical expectation values, for instance

$$\langle 0| : \mathcal{O} : |0\rangle \tag{3.39}$$

is measurable.

One more observation to make is that in the expansion of ϕ we have

$$(a_{\vec{p}} e^{ipx} + a^\dagger_{\vec{p}} e^{-ipx})_{p^0 = E_p} \tag{3.40}$$

and here $E_p = +\sqrt{\vec{p}^2 + m^2}$, but we note that the second term has *positive frequency* (energy), $a^\dagger_{\vec{p}} e^{-iE_p t}$, whereas the first has *negative frequency* (energy), $a e^{+iE_p t}$, that is we create $E > 0$ and destroy $E < 0$, which means that in this context we have only positive energy excitations. But we will see in Chapter 4 that in the case of the complex scalar field, we create both $E > 0$ and $E < 0$ and similarly destroy, leading to the concept of *antiparticles*. At this time, however, we don't have that.

3.4.3 Bose–Einstein Statistics

Since $[a^\dagger_{\vec{k}}, a^\dagger_{\vec{k}'}] = 0$, for a general state defined by a wavefunction $\psi(\vec{k}_1, \vec{k}_2)$:

$$\begin{aligned} |\psi\rangle &= \sum_{\vec{k}_1,\vec{k}_2} \psi(\vec{k}_1, \vec{k}_2)\alpha^\dagger_{\vec{k}_1}\alpha^\dagger_{\vec{k}_2}|0\rangle \\ &= \sum_{\vec{k}_1,\vec{k}_2} \psi(\vec{k}_2, \vec{k}_1)\alpha^\dagger_{\vec{k}_1}\alpha^\dagger_{\vec{k}_2}|0\rangle, \end{aligned} \tag{3.41}$$

where in the second line we have commuted the two $\alpha^\dagger$s and then renamed $\vec{k}_1 \leftrightarrow \vec{k}_2$.

We then obtain the *Bose–Einstein statistics*

$$\psi(\vec{k}_1, \vec{k}_2) = \psi(\vec{k}_2, \vec{k}_1), \tag{3.42}$$

that is for indistinguishable particles (permuting them, we obtain the same state).

As an aside, note that the Hamiltonian of the free (bosonic) oscillator is $1/2(aa^\dagger + a^\dagger a)$ (and of the free fermionic oscillator is $1/2(b^\dagger b - bb^\dagger)$), so in order to have a well-defined system we must have $[a, a^\dagger] = 1, \{b, b^\dagger\} = 1$. In turn, $[a, a^\dagger] = 1$ leads to Bose–Einstein statistics, as above.

Important Concepts to Remember

- The commutation relations for scalar fields are defined at equal time. The Poisson brackets are defined in terms of integrals of functional derivatives.
- The canonical commutation relations for the free scalar field imply that we can use the same redefinitions as for the harmonic oscillator, for the momentum modes, to obtain the $[a, a^\dagger] = 1$ relations.
- The Klein–Gordon equation for the free scalar field implies the Hamiltonian of the free harmonic oscillator for each of the momentum modes.
- Putting the system in a box, we find a sum over discrete momenta of harmonic oscillators, each with a Fock space.
- The Fock space for the free scalar field is the direct product of the Fock space for each mode.
- We must use normal ordered operators, for physical observables, in particular for the Hamiltonian, in order to avoid unphysical infinities.
- The scalar field is quantized in terms of the Bose–Einstein statistics.

Further Reading

See section 2.3 in [1] and sections 2.1 and 2.3 in [2].

Exercises

1. Consider the classical Hamiltonian

$$H = \int d^3x \left\{ \frac{\pi^2(\vec{x},t)}{2} + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{\lambda}{3!}\phi^3(\vec{x},t) + \frac{\tilde{\lambda}}{4!}\phi^4(\vec{x},t) \right\}. \tag{3.43}$$

Using the Poisson brackets, write the Hamiltonian equations of motion. Quantize canonically the *free* system and compute the equations of motion for the Heisenberg operators.

2. Write down the Hamiltonian above in terms of $a(\vec{p},t)$ and $a^\dagger(\vec{p},t)$ *at the quantum level*, and then write down the *normal ordered* Hamiltonian.

3. Consider the scalar Dirac–Born–Infeld (DBI) Lagrangian

$$\mathcal{L} = -\frac{1}{L^4}\sqrt{1 + L^4(\partial_\mu\phi)^2}. \tag{3.44}$$

Calculate the Hamiltonian, and the Hamiltonian equations of motion, using the Poisson brackets.

4. For the model in Exercise 3, quantize canonically using the fundamental Poisson brackets, and write the Hamiltonian in terms of a and $a^\dagger$ defined as in (3.10).

4 Propagators for Free Scalar Fields

After having defined canonical quantization, in this chapter we will learn how to construct propagators, fundamental objects in quantum field theory, that solve the Klein–Gordon (KG) equation with delta function source, and describe propagation of the field. In order to define that, however, we will first construct a relativistically invariant version of canonical quantization, and then learn to quantize a *complex* scalar field.

4.1 Relativistic Invariant Canonical Quantization

We need to understand the relativistic invariance of the quantization of the free scalar field, which was described in Chapter 3 in nonrelativistic form. The first issue is the normalization of the states. We saw that in the discrete version of the quantized scalar field, we had in each mode state

$$|n_{\vec{k}}\rangle = \frac{1}{\sqrt{n_k!}}(\alpha^\dagger_{\vec{k}})^{n_{\vec{k}}}|0\rangle, \tag{4.1}$$

normalized as $\langle m|n\rangle = \delta_{mn}$. In discretizing, we had $a_{\vec{k}} \to \sqrt{V}\alpha_{\vec{k}}$ and $\delta^3(\vec{k}-\vec{k}') \to V\delta_{\vec{k}\vec{k}'}$.

However, we want to have a *relativistic normalization*

$$\langle \vec{p}|\vec{q}\rangle = 2E_{\vec{p}}(2\pi)^3\delta^3(\vec{p}-\vec{q}), \tag{4.2}$$

or, in general, for occupation numbers in all momentum modes

$$\langle\{\vec{k}_i\}|\{\vec{q}_j\}\rangle = \sum_{\pi(j)}\prod_i 2\omega_{\vec{k}_i}(2\pi)^3\delta^3(\vec{k}_i - \vec{q}_{\pi(j)}), \tag{4.3}$$

where $\{\pi(j)\}$ are permutations of the $\{j\}$ indices. We see that we are missing a factor of $2\omega_k V$ in each mode in order to get $2\omega_k\delta^3(\vec{k}-\vec{k}')$ instead of $\delta_{\vec{k}\vec{k}'}$. We therefore take the normalized states

$$\prod_{\vec{k}}\frac{1}{\sqrt{n_{\vec{k}}!}}\left(\sqrt{2\omega_{\vec{k}}}\sqrt{V(2\pi)^3}\alpha_{\vec{k}}\right)^{n_{\vec{k}}}|0\rangle \to \prod_{\vec{k}}\frac{1}{\sqrt{n_{\vec{k}}!}}[a^\dagger_{\vec{k}}\sqrt{2\omega_{\vec{k}}}]^{n_{\vec{k}}}|0\rangle \equiv |\{\vec{k}_i\}\rangle. \tag{4.4}$$

We now prove that we have a *relativistically invariant* formula.

First, we look at the normalization. It is obviously invariant under rotations, so we need only look at boosts:

$$p_3' = \gamma(p_3+\beta E);\quad E' = \gamma(E+\beta p_3). \tag{4.5}$$

Since

$$\delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|}\delta(x - x_0), \tag{4.6}$$

and a boost acts only on p_3, but not on p_1, p_2, we have

$$\begin{aligned}\delta^3(\vec{p} - \vec{q}) &= \delta^3(\vec{p}\,' - \vec{q}\,')\frac{dp_3'}{dp_3} = \delta(\vec{p}\,' - \vec{q}\,')\gamma\left(1 + \beta\frac{dE_3}{dp_3}\right) = \delta^3(\vec{p}\,' - \vec{q}\,')\frac{\gamma}{E}(E + \beta p_3) \\ &= \delta^3(\vec{p}\,' - \vec{q}\,')\frac{E'}{E}.\end{aligned} \tag{4.7}$$

This means that $E\delta^3(\vec{p} - \vec{q})$ is relativistically invariant, as we wanted.

Also, the expansion of the scalar field

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(a_{\vec{p}}e^{ipx} + a_{\vec{p}}^\dagger e^{-ipx})|_{p^0=E_p} \tag{4.8}$$

contains the relativistic invariants $e^{\pm ipx}$ and $\sqrt{2E_p}a_{\vec{p}}$ (since they create a relativistically invariant normalization, these operators are relativistically invariant), but we also have a relativistically invariant measure

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} = \int \frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2 + m^2)|_{p^0>0} \tag{4.9}$$

(since $\delta(p^2 + m^2) = \delta(-(p^0)^2 + E_p^2)$, and then we use (4.6), allowing us to write

$$\phi(x) \equiv \phi(\vec{x}, t) = \int \frac{d^4p}{(2\pi)^4}(2\pi)\delta(p^2 + m^2)|_{p^0>0}\left(\sqrt{2E_p}a_{\vec{p}}e^{ipx} + \sqrt{2E_p}a_{\vec{p}}^\dagger e^{-ipx}\right)|_{p^0=E_p}. \tag{4.10}$$

4.2 Canonical Quantization of the Complex Scalar Field

We now turn to the quantization of the complex scalar field, needed in order to understand the physics of propagation in quantum field theory.

The Lagrangian for the complex scalar field is

$$\mathcal{L} = -\partial_\mu\phi\partial^\mu\phi^* - m^2|\phi|^2 - U(|\phi|^2). \tag{4.11}$$

This Lagrangian has a $U(1)$ global symmetry $\phi \to \phi e^{i\alpha}$, or in other words ϕ is *charged under the* $U(1)$ *symmetry*.

Note the absence of the factor $1/2$ in the kinetic term for ϕ with respect to the real scalar field. The reason is that we treat ϕ and ϕ^* as independent fields. Then the equation of motion of ϕ is $(\partial_\mu\partial^\mu - m^2)\phi^* = \partial U/\partial\phi$. We could write the Lagrangian as a sum of two real scalars, but then with a factor of 1/2, $-\partial_\mu\phi_1\partial^\mu\phi_1/2 - \partial_\mu\phi_2\partial^\mu\phi_2/2$, since then we get $(\partial_\mu\partial^\mu - m^2)\phi_1 = \partial U/\partial\phi_1$.

Exactly paralleling the discussion of the real scalar field, we obtain an expansion in terms of a and $a^\dagger$ operators, just that now we have complex fields, with twice as many degrees of freedom, so we have $a_\pm$ and $a^\dagger_\pm$, with half the degrees of freedom in ϕ and half in $\phi^\dagger$:

$$\begin{aligned}
\phi(\vec{x},t) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a_+(\vec{p},t)e^{i\vec{p}\cdot\vec{x}} + a^\dagger_-(\vec{p},t)e^{-i\vec{p}\cdot\vec{x}}\right),\\
\phi^\dagger(\vec{x},t) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\left(a^\dagger_+(\vec{p},t)e^{-i\vec{p}\cdot\vec{x}} + a_-(\vec{p},t)e^{i\vec{p}\cdot\vec{x}}\right),\\
\pi(\vec{x},t) &= \int \frac{d^3p}{(2\pi)^3}\left(-i\sqrt{\frac{\omega_p}{2}}\right)\left(a_-(\vec{p},t)e^{i\vec{p}\cdot\vec{x}} - a^\dagger_+(\vec{p},t)e^{-i\vec{p}\cdot\vec{x}}\right),\\
\pi^\dagger(\vec{x},t) &= \int \frac{d^3p}{(2\pi)^3}\left(i\sqrt{\frac{\omega_p}{2}}\right)\left(a^\dagger_-(\vec{p},t)e^{-i\vec{p}\cdot\vec{x}} - a_+(\vec{p},t)e^{i\vec{p}\cdot\vec{x}}\right).
\end{aligned} \tag{4.12}$$

As before, this ansatz is based on the harmonic oscillator, whereas the form of ω_p comes out of the Klein–Gordon (KG) equation. Substituting this ansatz inside the canonical quantization commutator

$$[\phi(\vec{x},t),\pi(\vec{x}',t)] = i\hbar\delta^3(\vec{x}-\vec{x}') \tag{4.13}$$

and its complex conjugate, we find as before

$$[a_\pm(\vec{p},t),a^\dagger_\pm(\vec{p}',t)] = (2\pi)^3\delta^3(\vec{p}-\vec{p}'), \tag{4.14}$$

and the rest being zero (the proof is left as an exercise). Again, we note the equal time for the commutators. Also, the time dependence is the same as before.

We can calculate the $U(1)$ charge operator (left as an exercise), obtaining

$$Q = \int \frac{d^3k}{(2\pi)^3}\left[a^\dagger_{+\vec{k}}a_{+\vec{k}} - a^\dagger_{-\vec{k}}a_{-\vec{k}}\right]. \tag{4.15}$$

Thus, as expected from the notation used, a_+ has charge $+$ and a_- has charge $-$, and therefore we have

$$Q = \int \frac{d^3k}{(2\pi)^3}[N_{+\vec{k}} - N_{-\vec{k}}] \tag{4.16}$$

(the number of $+$ charges minus the number of $-$ charges).

We then see that ϕ creates $-$ charge and annihilates $+$ charge, and $\phi^\dagger$ creates $+$ charge and annihilates $-$ charge.

Since in this simple example there are no other charges, we see that $+$ and $-$ particles are *particle–antiparticle pairs* (i.e. pairs which are equal in everything, except they have opposite charges). As promised in Chapter 3, we have now introduced the concept of an antiparticle, and it is related to the existence of positive and negative frequency modes.

We also see now that for a real field, the particle is its own antiparticle, since $\phi = \phi^\dagger$ identifies "creating $+$ charge with creating $-$ charge" (really, there is no charge now).

4.3 Two-Point Functions and Propagators

In this section, we consider the object

$$\langle 0|\phi^\dagger(x)\phi(y)|0\rangle \tag{4.17}$$

corresponding to propagation from $y = (t_y, \vec{y})$ to $x = (t_x, \vec{x})$, the same way that in quantum mechanics $\langle q', t'|q, t\rangle$ corresponds to propagation from (q, t) to (q', t'). This object corresponds to a measurement of the field ϕ at y, then of $\phi^\dagger$ at x.

For simplicity, we will analyze the real scalar field, and we will use the complex scalar only for interpretation. Substituting the expansion of $\phi(x)$, since $a|0\rangle = \langle 0|a^\dagger = 0$, and for $\phi = \phi^\dagger$, we have

$$\langle 0|\phi^\dagger\phi|0\rangle \sim \langle 0|(a + a^\dagger)(a + a^\dagger)|0\rangle, \tag{4.18}$$

only

$$\langle 0|a_{\vec{p}}a^\dagger_{\vec{q}}|0\rangle e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})} \tag{4.19}$$

survives in the sum, and, as $\langle 0|aa^\dagger|0\rangle = \langle 0|a^\dagger a|0\rangle + [a, a^\dagger]\langle 0|0\rangle$, we get $(2\pi)^3\delta(\vec{p} - \vec{q})$ from the expectation value. Then finally we obtain, for the scalar propagation from y to x:

$$D(x - y) \equiv \langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{ip(x-y)}. \tag{4.20}$$

We now analyze what happens for varying $x - y$. By Lorentz transformations, we have only two cases to analyze.

(a) For timelike separation, we can put $t_x - t_y = t$ and $\vec{x} - \vec{y} = 0$. In this case, using $d^3p = d\Omega p^2 dp$ and $dE/dp = p/\sqrt{p^2 + m^2}$:

$$\begin{aligned} D(x - y) &= 4\pi \int_0^\infty \frac{p^2dp}{(2\pi)^3}\frac{1}{2\sqrt{p^2 + m^2}}e^{-i\sqrt{p^2+m^2}t} \\ &= \frac{1}{4\pi^2}\int_m^\infty dE\sqrt{E^2 - m^2}e^{-iEt} \overset{t\to\infty}{\propto} e^{-imt}, \end{aligned} \tag{4.21}$$

which is oscillatory (i.e. it doesn't vanish). But that's fine, since in this case, we remain in the same point as time passes, so the probability should be large.

(b) For spacelike separation, $t_x = t_y$ and $\vec{x} - \vec{y} = \vec{r}$, we obtain

$$\begin{aligned} D(\vec{x} - \vec{y}) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{i\vec{p}\cdot\vec{r}} = 2\pi \int_0^\infty \frac{p^2dp}{2E_p(2\pi)^3}\int_{-1}^1 d(\cos\theta)e^{ipr\cos\theta} \\ &= 2\pi \int_0^\infty \frac{p^2dp}{2E_p(2\pi)^3}\left[\frac{e^{ipr} - e^{-ipr}}{ipr}\right] = \frac{-i}{(2\pi)^2 2r}\int_{-\infty}^{+\infty} pdp\frac{e^{ipr}}{\sqrt{p^2 + m^2}}, \end{aligned} \tag{4.22}$$

where in the last line we have redefined in the second term $p \to -p$, and then added up $\int_0^\infty$ to $\int_{-\infty}^0$.

In the last form, we have e^{ipr} multiplying a function with poles at $p = \pm im$, so we know, by a theorem from complex analysis, that we can consider the integral in the complex p plane, and add for free the integral on an infinite semicircle in the upper half plane, since

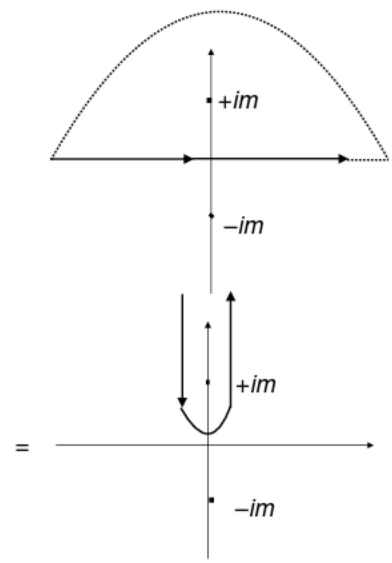

Figure 4.1 By closing the contour in the upper half plane, we pick up the residue of the pole in the upper half plane, at $+im$.

then $e^{ipr} \propto e^{-\mathrm{Im}(p)r} \to 0$. Thus, closing the contour, we can use the residue theorem and say that our integral equals the residue in the upper half plane (i.e. at $+im$), see Figure 4.1. Looking at the residue for $r \to \infty$, its leading behavior is

$$D(\vec{x}-\vec{y}) \propto e^{i(im)r} = e^{-mr}. \tag{4.23}$$

But for spacelike separation, at $r \to \infty$, we are much outside the lightcone (in no time, we move in space), and yet, propagation gives a small but nonzero amplitude, which is not fine.

But the relevant question is, will measurements be affected? We will see later why, but the only relevant issue is whether the commutator $[\phi(x), \phi(y)]$ is nonzero for spacelike separation. We thus compute

$$\begin{aligned}
[\phi(x),\phi(y)] &= \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_p 2E_q}}[(a_{\vec{p}}e^{ipx}+a^\dagger_{\vec{p}}e^{-ipx}),(a_{\vec{q}}e^{iqy}+a^\dagger_{\vec{q}}e^{-iqy})]\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}(e^{ip(x-y)}-e^{ip(y-x)})\\
&= D(x-y)-D(y-x).
\end{aligned} \tag{4.24}$$

But if $(x-y)^2 > 0$ (spacelike), $(x-y) = (0, \vec{x}-\vec{y})$ and we can make a Lorentz transformation (a rotation, really) $(\vec{x}-\vec{y}) \to -(\vec{x}-\vec{y})$, leading to $(x-y) \to -(x-y)$. But since $D(x-y)$ is Lorentz invariant, it follows that for spacelike separation we have $D(x-y) = D(y-x)$, and therefore

$$[\phi(x), \phi(y)] = 0, \tag{4.25}$$

and we have causality. Note that this is due to the existence of negative frequency states (e^{ipx}) in the scalar field expansion. In contrast, we should also check that for timelike separation we have a nonzero result. Indeed, for $(x-y)^2 < 0$, we can set $(x-y) =$

$(t_x - t_y, 0)$ and so $-(x-y) = (-(t_x - t_y), 0)$ corresponds to time reversal, so is not a Lorentz transformation, therefore we have $D(-(x-y)) \neq D(x-y)$, and so $[\phi(x), \phi(y)] \neq 0$.

4.4 Propagators: Retarded and Feynman

4.4.1 Klein–Gordon Propagators

We are finally ready to describe the propagator.

Consider the c-number

$$
\begin{aligned}
[\phi(x), \phi(y)] = \langle 0 | [\phi(x), \phi(y)] | 0 \rangle &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (e^{ip(x-y)} - e^{-ip(x-y)}) \\
&= \int \frac{d^3p}{(2\pi)^3} \left[\frac{1}{2E_p} e^{ip(x-y)}|_{p^0=E_p} + \frac{1}{-2E_p} e^{ip(x-y)}|_{p^0=-E_p} \right].
\end{aligned}
\tag{4.26}
$$

For $x^0 > y^0$, we can write it as

$$
\int \frac{d^3p}{(2\pi)^3} \int_C \frac{dp^0}{2\pi i} \frac{1}{p^2 + m^2} e^{ip(x-y)}, \tag{4.27}
$$

where the contour C is on the real line, except it avoids slightly above the two poles at $p^0 = \pm E_p$, and then, in order to select both poles, we need to close the contour below, with an infinite semicircle in the lower half plane, as in Figure 4.2. Closing the contour below works, since for $x^0 - y^0 > 0$, we have $e^{-ip^0(x^0-y^0)} = e^{Im(p^0)(x^0-y^0)} \to 0$. Note that this way we get a contour closed clockwise, hence its result is minus the residue (plus the residue is for a contour closed anticlockwise), giving the extra minus sign for the contour integral to reproduce the right result.

In contrast, for this contour, if we have $x^0 < y^0$ instead, the same reason above says we need to close the contour above (with an infinite semicircle in the upper half plane). In this case, there are no poles inside the contour, therefore the integral is zero.

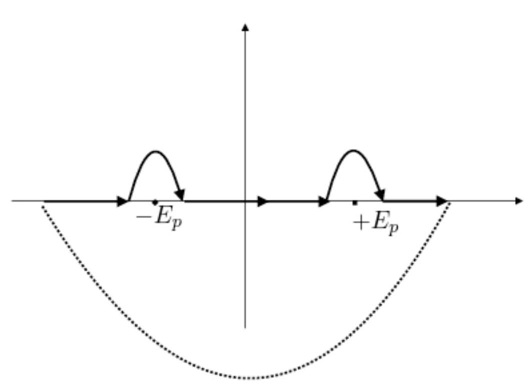

Figure 4.2 For the retarded propagator, the contour is such that closing it in the lower half plane picks up both poles at $\pm E_p$.

4.4.2 Retarded Propagator

We can then finally write for the *retarded propagator* (i.e. one that vanishes for $x^0 < y^0$)

$$D_R(x-y) \equiv \theta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{dp^0}{2\pi i}\frac{1}{p^2+m^2}e^{ip(x-y)}$$
$$= \int \frac{d^4p}{(2\pi)^4}\frac{-i}{p^2+m^2}e^{ip(x-y)}, \tag{4.28}$$

where $\theta(x)$ is the Heaviside function.

The object above is a Green's function for the KG operator. This is easier to see in momentum space. Indeed, making a Fourier transform

$$D_R(x-y) = \int \frac{d^4p}{(2\pi)^4}e^{ip(x-y)}D_R(p), \tag{4.29}$$

we obtain

$$D_R(p) = \frac{-i}{p^2+m^2} \Rightarrow (p^2+m^2)D_R(p) = -i, \tag{4.30}$$

which means that in x space (the Fourier transform of 1 is $\delta(x)$, and the Fourier transform of p^2 is $-\partial^2$):

$$(\partial^2-m^2)D_R(x-y) = i\delta^4(x-y) \leftrightarrow -i(\partial^2-m^2)D_R(x-y) = \delta^4(x-y). \tag{4.31}$$

4.4.3 Feynman Propagator

Consider now a different "$i\epsilon$ prescription" for the contour of integration C. Consider a contour that avoids slightly below the $-E_p$ pole and avoids slightly above the $+E_p$ pole, as in Figure 4.3. This is equivalent to the Feynman prescription for the propagator, changing D_R into D_F, defined as

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{-i}{p^2+m^2-i\epsilon}e^{ip(x-y)}. \tag{4.32}$$

Since $p^2+m^2-i\epsilon = -(p^0)^2+E_p^2-i\epsilon = -(p^0+E_p-i\epsilon/2)(p^0-E_p+i\epsilon/2)$, we have poles at $p^0 = \pm(E_p-i\epsilon/2)$, so in this form we have a contour along the real line, but the poles are modified instead ($+E_p$ is moved below, and $-E_p$ is moved above the contour).

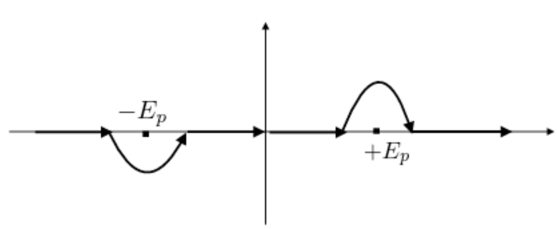

Figure 4.3 The contour for the Feynman propagator avoids $-E_p$ from below and $+E_p$ from above.

For this contour, as for D_R (for the same reasons), for $x^0 > y^0$ we close the contour below (with an infinite semicircle in the lower half plane). The result for the integration is then the residue inside the closed contour (i.e. the residue at $+E_p$). But we saw that the clockwise residue at $+E_p$ is $D(x-y)$ (and the clockwise residue at $-E_p$ is $-D(y-x)$).

For $x^0 < y^0$, we need to close the contour above (with an infinite semicircle in the upper half plane), and then we get the anticlockwise residue at $-E_p$, therefore $+D(y-x)$. The final result is then

$$D_F(x-y) = \theta(x^0 - y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0 - x^0)\langle 0|\phi(y)\phi(x)|0\rangle \equiv \langle 0|T(\phi(x)\phi(y))|0\rangle. \tag{4.33}$$

This is then the Feynman propagator, which again is a Green's function for the KG operator, with the *time ordering operator* T in the two-point function, the same that we defined in the n-point functions in quantum mechanics (e.g. $\langle q', t'|T[\hat{q}(t_1)\hat{q}(t_2)]|q, t\rangle$).

As suggested by the quantum-mechanical case, the Feynman propagator will appear in the Feynman rules, and has a physical interpretation as propagation of the particle excitations of the quantum field.

Important Concepts to Remember

- The expansion of the free scalar field in quantum fields is relativistically invariant, as is the relativistic normalization.
- The complex scalar field is quantized in terms of $a_\pm$ and $a_\pm^\dagger$, which correspond to $U(1)$ charge ± 1. ϕ creates minus particles and destroys plus particles, and $\phi^\dagger$ creates plus particles and destroys minus particles.
- The plus and minus particles are particle–antiparticle pairs, since they only differ by their charge.
- The object $D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$ is nonzero much outside the lightcone, however $[\phi(x), \phi(y)]$ is zero outside the lightcone, and since only this object leads to measurable quantities, quantum field theory is causal.
- $D_R(x-y)$ is a retarded propagator and corresponds to a contour of integration that avoids the poles $\pm E_p$ from slightly above, and is a Green's function for the KG operator.
- $D_F(x-y) = \langle 0|T[\phi(x)\phi(y)]|0\rangle$, the Feynman propagator, corresponds to the $-i\epsilon$ prescription (i.e. avoids $-E_p$ from below and $+E_p$ from above), is also a Green's function for the KG operator, and will appear in Feynman diagrams. It has the physical interpretation of propagation of particle excitations of the quantum field.

Further Reading

See section 2.4 in [1] and section 2.4 in [2].

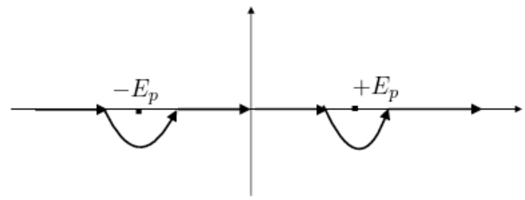

Figure 4.4 The contour for the advanced propagator avoids both poles at $\pm E_p$ from below.

Exercises

1. For the Lagrangian
$$\mathcal{L} = -\partial_\mu \phi \partial^\mu \phi^* - m^2|\phi|^2 - U(|\phi|^2), \tag{4.34}$$
calculate the Noether current for the $U(1)$ symmetry $\phi \to \phi e^{i\alpha}$ in terms of $\phi(\vec{x}, t)$ and show that it then reduces to the expression in the text:
$$Q = \int \frac{d^3k}{(2\pi)^3}[a^\dagger_{+\vec{k}} a_{+\vec{k}} - a^\dagger_{-\vec{k}} a_{-\vec{k}}]. \tag{4.35}$$
2. Calculate the *advanced* propagator $D_A(x-y)$ by using the integration contour that avoids the $\pm E_p$ poles from below, instead of the contours for D_R, D_F, as in Figure 4.4.
3. Show that the canonical quantization equal time commutation relations for the complex scalar field give the same $a, a^\dagger$ relations
$$[a_\pm(\vec{p}, t), a^\dagger_\pm(\vec{p}\,', t)] = (2\pi)^3 \delta^3(\vec{p} - \vec{p}\,'), \tag{4.36}$$
and the rest are zero.
4. Show that the Feynman propagator, and the advanced propagator defined in Exercise 2, are also Green's functions for the KG equation.