Artificial Intelligent Systems

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Fuzzy Logic

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Lecture Topics

- Fundamentals of fuzzy sets
- Fuzzy relations
- Fuzzy linguistic terms and variables
- Fuzzy rules
- Fuzzy reasoning
- Fuzzy systems

Introduction

- The word "fuzzy" means something that is indistinct, vague, hazy, blurred, confused or not clear.
- Fuzziness occurs when the boundary of a piece of information is not clear-cut.
- In classical set theory, an element either belongs or does not belong to the set.
- Fuzzy set theory introduces the gradual membership of elements in a set that is described using a membership function valued in the real unit interval [0, 1].

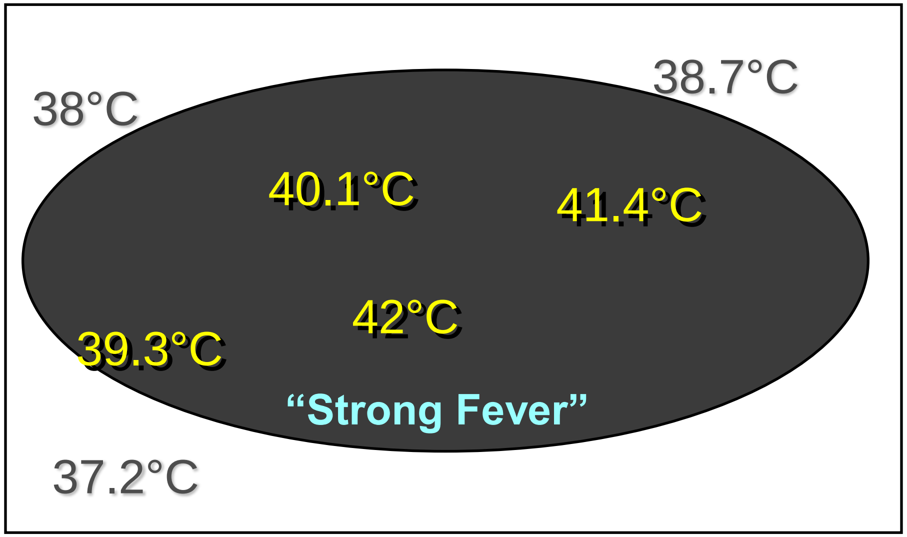
INTRODUCTION

- The terms such as hot, old, tall and fast etc are fuzzy.
- For example, there is no single quantitative value which defines the term hot.
- For some people, 30 degrees Celsius is hot, and for others 30 degrees Celsius is only warm.
- On the other hand, the temperature of 0 degrees Celsius is definitely not hot and the temperature of 100 is definitely hot.
- The term hot, thus, has no clear boundary and it depends on the context in which it is being considered.

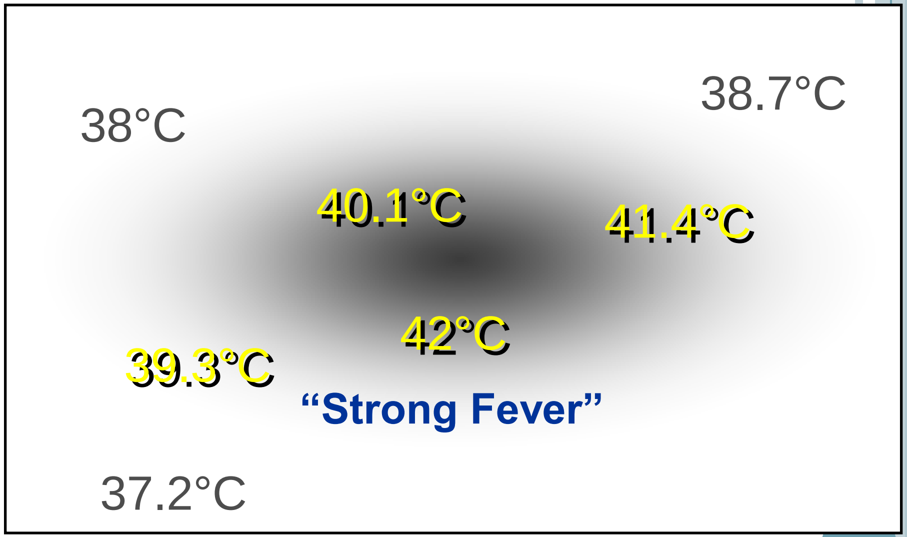
Fuzzy Set Theory

- Fuzzy set theory is an extension of classical set theory, where elements have degrees of membership.

Classical set theory



Fuzzy set theory



Fuzzy Logic

- Fuzzy logic handles the concept of partial truth, where the truth value may range between completely true and completely false.
- Furthermore, when linguistic (fuzzy) variables are used, these degrees may be managed by specific functions.
- The fuzzy logic for a simple temperature regulator that uses a fan might look like this

```
IF temperature IS very cold THEN stop fan
IF temperature IS cold THEN turn down fan
IF temperature IS normal THEN maintain fan
IF temperature IS hot THEN speed up fan
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History of Fuzzy Logic

1965 Seminal Paper “Fuzzy Logic” by Prof. Lotfi Zadeh, Faculty in Electrical Engineering, U.C. Berkeley, Sets the Foundation of the “Fuzzy Set Theory”

1970 First Application of Fuzzy Logic in Control Engineering (Europe)

1975 Introduction of Fuzzy Logic in Japan

1980 Empirical Verification of Fuzzy Logic in Europe

1985 Broad Application of Fuzzy Logic in Japan

1990 Broad Application of Fuzzy Logic in Europe

1995 Broad Application of Fuzzy Logic in the U.S.

2000+ Fuzzy Logic Becomes a Standard Technology and Is Also Applied in Data and Sensor Signal Analysis.

2014+ IEEE Transactions On Fuzzy Systems achieved the highest journal impact factor in the field of artificial intelligence.

FUNDAMENTALS OF FUZZY SETS

- A conventional (crisp) set is a set of objects that satisfy a predicate.
- If, for an object, the predicate is true then the object is an element of the set, and vise-versa if the predicate is false.
- Let X be a finite universal set of elements, then a conventional set A in X is a set of ordered pairs

$$A = \{(x, \psi_A(x))\}, \forall x \in X$$

where $\psi_A(x) \in \{0, 1\}$ is a binary-valued predicate

$$\psi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

- A conventional set A is usually denoted as a list, $A = \{x\}$, of the elements which predicates are true, i.e., $\psi_A(x) = 1$.

Fundamentals of Fuzzy Sets

- A fuzzy set is obtained if, instead of a predicate, a membership function is used that provide, for each element of a universal set X, a degree of membership to a set A.

- The membership function takes the value between 0 and 1, where a higher value means a higher degree of membership.

- If X is a finite universal (crisp) set of elements, then a fuzzy set A in X is a set of ordered pairs

$$A = \{(x,\ \mu_A(x))\}, \forall x \in X$$

where $\mu_A(x) \in [0, 1]$ is a real-valued membership function that provide a degree of membership of an element x to a set A.

- A fuzzy set is often written in a more readable form as

$$A = \bigcup_{\forall x \in X} \frac{\mu_A(x)}{x} .$$

EXAMPLE

- Let a universal set X be

$$X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

- Prime numbers in the set X are uniformly defined and, thus, a set of prime numbers A can be defined as a crisp set

$$A = \{(1,0), (2,1), (3,1), (4,0), (5,1), (6,0), (7,1), (8,0), (9,0), (10,0)\}$$

or shortly $A = \{2, 3, 5, 7\}$.

- On the other hand, "small numbers" in the set X are not uniformly defined and it is thus more appropriate to define a fuzzy set A of small numbers, e.g.,

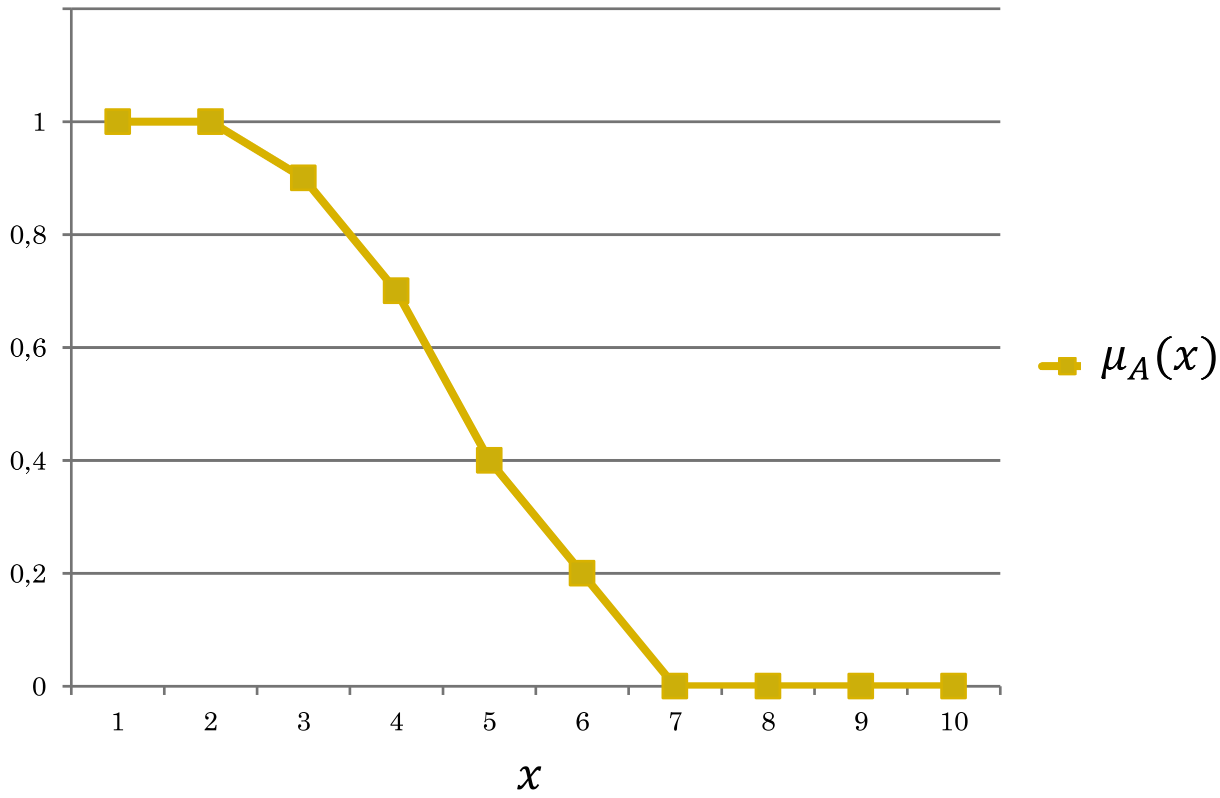
$$A = \{(1,1), (2,1), (3,0.9), (4,0.7), (5,0.4), (6,0.2), (7,0), (8,0), (9,0), (10,0)\}$$

or shortly

$$A = \left\{\frac{1}{1}, \frac{1}{2}, \frac{0.9}{3}, \frac{0.7}{4}, \frac{0.4}{5}, \frac{0.2}{6}, \frac{0}{7}, \frac{0}{8}, \frac{0}{9}, \frac{0}{10}\right\}.$$

EXAMPLE

- The membership function $\mu_A(x)$ of the given example of a fuzzy set A of small numbers in X can be represented graphically as



Definitions

- The "power" (cardinality) of a fuzzy set A in X is defined as

$$\mathrm{card}(A) = \sum_{x \in X} \mu_A(x)$$

and a relative cardinality as

$$\mathrm{rcard}(A) = \frac{\mathrm{card}(A)}{\mathrm{card}(\mathrm{X})}$$

- A fuzzy set A in X is normalized if

$$\mu_A(x) = 1, \text{for at least one } x \in X$$

- The α-cut of a fuzzy set A in X is a crisp set of all $x \in X$ that satisfy the predicate

$$\mu_A(x) \geq \alpha$$

Definitions

- The strong α-cut of a fuzzy set A in X is a crisp set of all $x \in X$ that satisfy the predicate

$$\mu_A(x) > \alpha.$$

- The core of a fuzzy set A in X is a crisp set of all $x \in X$ that satisfy the predicate

$$\mu_A(x) = 1.$$

  The core of a fuzzy set A is thus its α-cut, where $\alpha = 1$.

- The support of a fuzzy set A in X is a crisp set of all $x \in X$ that satisfy the predicate

$$\mu_A(x) > 0.$$

  The core of a fuzzy set A is thus its strong α-cut, where $\alpha = 0$.

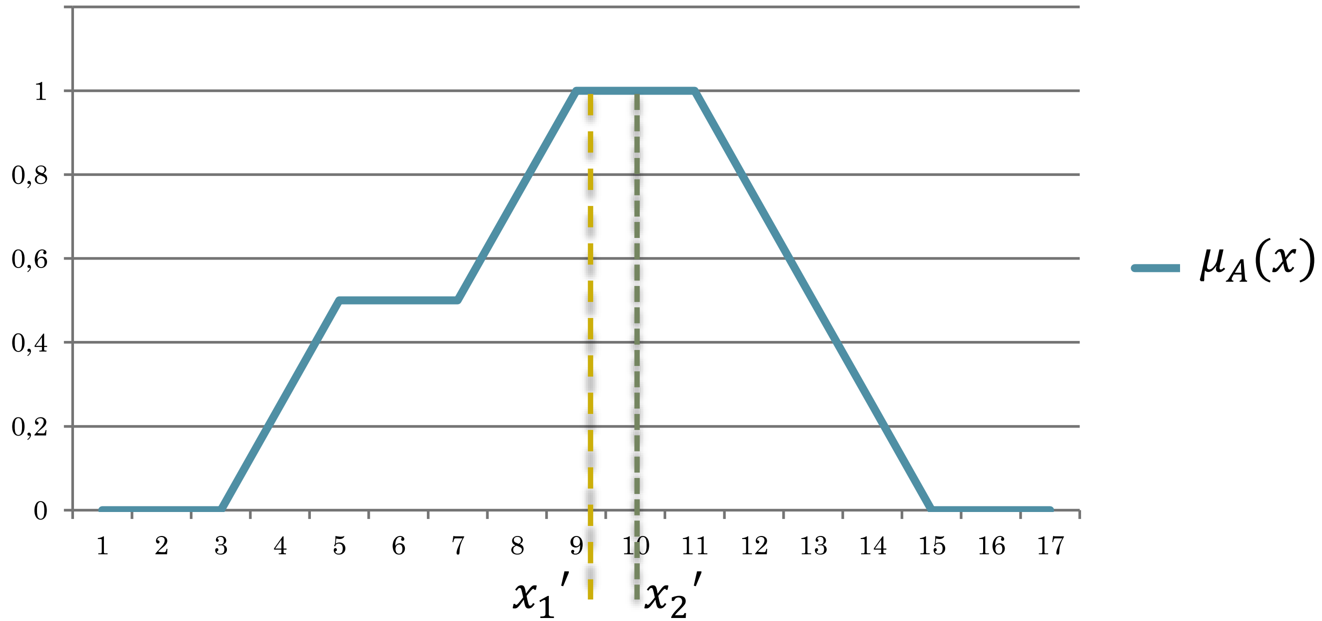
Definitions

- The crisp value of a fuzzy set A in X is a fuzzy set that has support with only one element.
- Conversion of a fuzzy set into a crisp value is called defuzzification.
- Defuzzification is necessary, since controllers of physical systems usually require real-valued discrete signals.
- There are many defuzzification methods, and two of the more common techniques are the centroid and maximum methods.
- In the center of gravity (COG) method, the crisp value x' of a fuzzy A in X is computed by finding the value of the center of gravity of the membership function $\mu_A(x)$, i.e.,

$$x' = \frac{\sum_{i=1}^{\mathrm{card}(X)} x_i \mu_A(x_i)}{\sum_{i=1}^{\mathrm{card}(X)} \mu_A(x_i)}$$

Definitions

- In the (mean of) maximum (MOM) method the value at which the membership function $\mu_A(x)$ has its maximum is chosen as the crisp value.
- If the membership function $\mu_A(x)$ has the same maximum value for several values of x, then the crisp value is defined as the average of the values at which $\mu_A(x)$ is maximal.



Example

- Given the following fuzzy set A, define the crisp values using the two presented methods.

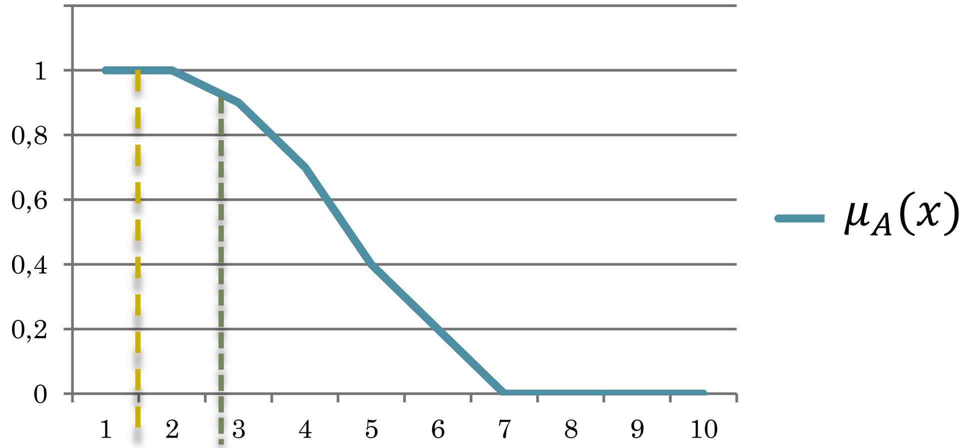
$$A = \{(1,1),(2,1),(3,0.9),(4,0.7),(5,0.4),(6,0.2),(7,0),(8,0),(9,0),(10,0)\}$$

- The COG crisp value is

$$x' = \frac{1 \cdot 1 + 2 \cdot 1 + 3 \cdot 0.9 + 4 \cdot 0.7 + 5 \cdot 0.4 + 6 \cdot 0.2}{1+1+0.9+0.7+0.4+0.2} = 2.67$$

- The MOM crisp value is

$$x' = 1/2\,(1+2) = 1.50$$



Definitions

- A fuzzy set A is convex in X if

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq min\{\mu_A(x_1), \mu_A(x_2)\}, \quad x_1, x_2 \in X, \lambda \in [0,1].$$

- The complement of a fuzzy set A in X is a fuzzy set $\bar{A}$ with the membership function

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x), \qquad \forall x \in X, \qquad \text{where } \overline{(\bar{A})} = A.$$

- The product of a fuzzy set A in X with the real value r is a fuzzy set with the membership function

$$\mu_{rA}(x) = r \cdot \mu_A(x), \qquad \forall x \in X.$$

- The m-th power of a fuzzy set A in X is a fuzzy set with the membership function

$$\mu_{A^m}(x) = \left(\mu_A(x)\right)^m, \qquad \forall x \in X.$$

Binary Operations on Two Fuzzy Sets

- Fuzzy sets A and B are equal in X $(A = B)$ if and only if

$$\mu_A(x) = \mu_B(x), \qquad \forall x \in X.$$

- A fuzzy set A is a subset of a fuzzy set B in X $(A \subset B)$ if and only if

$$\mu_A(x) \leq \mu_B(x), \qquad \forall x \in X.$$

- The algebraic product of fuzzy sets A and B in X $(A \cdot B)$ is a fuzzy set with the member function

$$\mu_{A \cdot B}(x) = \mu_A(x) \cdot \mu_B(x), \qquad \forall x \in X.$$

- The algebraic mean of fuzzy sets A and B in X $\text{mean}(A \cdot B)$ is a fuzzy set with the member function

$$\mu_{\text{mean}(A \cdot B)}(x) = \frac{1}{2}\big(\mu_A(x) + \mu_B(x)\big), \qquad \forall x \in X.$$

Binary Operations on Two Fuzzy Sets

- The intersection between fuzzy sets A and B in X $(A \cap B)$ is a fuzzy set with the membership function

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, \qquad \forall x \in X.$$

- The union between fuzzy sets A and B in X $(A \cup B)$ is a fuzzy set with the membership function

$$\mu_{A\cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}, \qquad \forall x \in X.$$

- The α operation on fuzzy sets A and B in X $(A\alpha B)$ is a fuzzy set with the member function

$$\mu_{A\alpha B}(x) = \begin{cases} 1 & \mu_A(x) \le \mu_B(x) \\ \mu_B(x) & \mu_A(x) > \mu_B(x) \end{cases}, \qquad \forall x \in X.$$

EXAMPLE

- Let X be a universal set

$$X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

- Let then A be a fuzzy set of small numbers and B a fuzzy set of large numbers

$$A = \{(1,1), (2,1), (3,0.9), (4,0.7), (5,0.4), (6,0.2), (7,0), (8,0), (9,0), (10,0)\}.$$

$$B = \{(1,0), (2,0), (3,0), (4,0), (5,0.3), (6,0.5), (7,0.9), (8,0.9), (9,1), (10,1)\}.$$

- The algebraic product of the two fuzzy sets is then

$$C = \{(1, 1 \cdot 0), (2, 1 \cdot 0), (3, 0.9 \cdot 0), (4, 0.7 \cdot 0), (5, 0.4 \cdot 0.3), (6, 0.2 \cdot 0.5), (7, 0 \cdot 0.9), (8, 0 \cdot 0.9), (9, 0 \cdot 1), (10, 0 \cdot 1)\}.$$

or shortly

$$C = \{(1,0), (2,0), (3,0), (4,0), (5,0.12), (6,0.1), (7,0), (8,0), (9,0), (10,0)\}.$$

Example

- The algebraic mean of the fuzzy sets A and B is then

$$D = \{(1,\frac{1}{2}[1+0]),(2,\frac{1}{2}[1+0]),(3,\frac{1}{2}[0.9+0]),(4,\frac{1}{2}[0.7+0]),(5,\frac{1}{2}[0.4+0.3]),$$
$$(6,\frac{1}{2}[0.2+0.5]),(7,\frac{1}{2}[0+0.9]),(8,\frac{1}{2}[0+0.9]),(9,\frac{1}{2}[0+1]),(10,\frac{1}{2}[0+1])\}.$$

or shortly

$$D = \{(1,0.5),(2,0.5),(3,0.45),(4,0.35),(5,0.35),(6,0.35),(7,0.45),(8,0.45),(9,0.5),(10,0.5)\}$$

- The intersection of the fuzzy sets A and B is

$$E = \{(1,\min\{1,0\}),(2,\min\{1,0\}),(3,\min\{0.9,0\}),(4,\min\{0.7,0\}),(5,\min\{0.4,0.3\}),$$
$$(6,\min\{0.2,0.5\}),(7,\min\{0,0.9\}),(8,\min\{0,0.9\}),(9,\min\{0,1\}),(10,\min\{0,1\})\}.$$

or shortly

$$E = A \cap B = \{(1,0),(2,0),(3,0),(4,0),(5,0.3),(6,0.2),(7,0),(8,0),(9,0),(10,0)\}.$$

Example

- The union of the fuzzy sets A and B is

$$F = \{(1, \max\{1,0\}), (2, \max\{1,0\}), (3, \max\{0.9,0\}), (4, \max\{0.7,0\}), (5, \max\{0.4,0.3\}), (6, \max\{0.2,0.5\}), (7, \max\{0,0.9\}), (8, \max\{0,0.9\}), (9, \max\{0,1\}), (10, \max\{0,1\})\}.$$

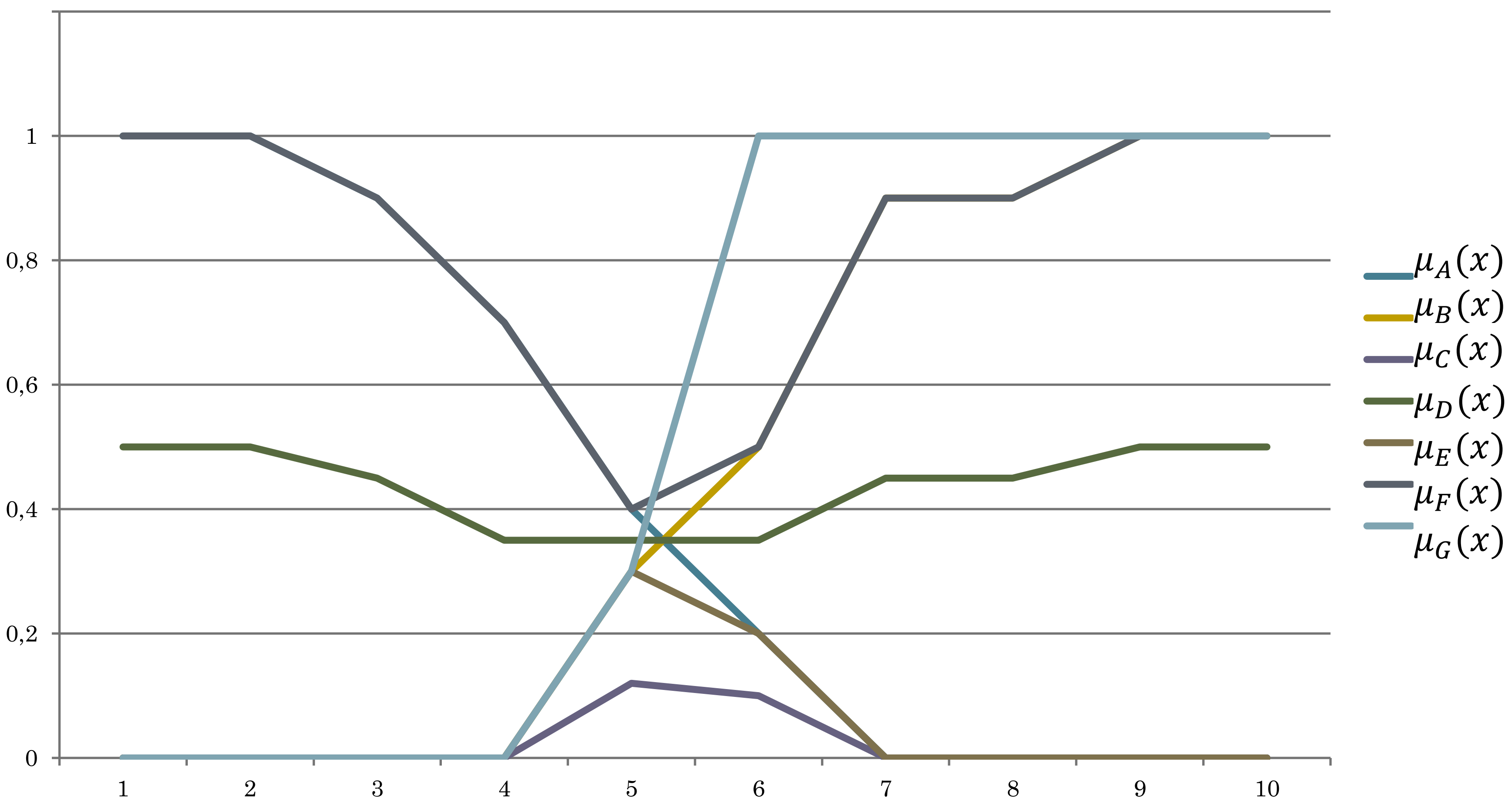
or shortly

$$F = A \cup B = \{(1,1), (2,1), (3,0.9), (4,0.7), (5,0.4), (6,0.5), (7,0.9), (8,0.9), (9,1), (10,1)\}.$$

- The α operation on the fuzzy sets A and B is

$$G = A\alpha B = \{(1,0), (2,0), (3,0), (4,0), (5,0.3), (6,1), (7,1), (8,1), (9,1), (10,1)\}.$$

Example – Membership Functions



Properties of Binary Operations

- The intersection and union of fuzzy sets A and B in X has the properties as follows.
  - Associativity

$$(A \cap B) \cap C \equiv A \cap (B \cap C)$$
$$(A \cup B) \cup C \equiv A \cup (B \cup C)$$

  - Distributivity

$$(A \cap B) \cup C \equiv (A \cup B) \cap (B \cap C)$$
$$(A \cup B) \cap C \equiv (A \cap B) \cup (B \cap C)$$

  - Absorption

$$A \cup (A \cap B) \equiv A$$
$$A \cap (A \cup B) \equiv A$$

  - Idempotency

$$A \cap A = A$$
$$A \cup A = A$$

Properties of Binary Operations

- Commutativity

$$A \cap B = B \cap A$$
$$A \cup B = B \cup A$$

- De Morgan's law

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

- A union of a fuzzy set A with its complement $\overline{A}$ is not necessary a universal set X, i.e.,

$$A \cup \overline{A} \neq X$$

- An intersection of a fuzzy set A with its complement $\overline{A}$ is not necessary an empty set X, i.e.,

$$A \cap \overline{A} \neq 0$$

Similarities Between Fuzzy Sets

- Similarity between fuzzy sets A and B can be measured using Minkowski's distances

$$D_M(A,B) = \left[\sum_{i=1}^{\mathrm{card}(X)} |\mu_A(x_i) - \mu_B(x_i)|^s \right]^{\frac{1}{s}}, \qquad \text{where} \quad s \geq 1$$

- Minkowski's distances for $s = 1, s = 2$ and $s \rightarrow \infty$ are most commonly used.

Similarities Between Fuzzy Sets

- $s = 1$; Hamming's distance:

$$D_H(A,B) = \sum_{i=1}^{\mathrm{card}(X)} |\mu_A(x_i) - \mu_B(x_i)|$$

- $s = 2$; Euclidean distance:

$$D_E(A,B) = \sqrt{\sum_{i=1}^{\mathrm{card}(X)} |\mu_A(x_i) - \mu_B(x_i)|^2}$$

- $s \to \infty$; Chebyshev distance:

$$D_E(A,B) = \max_{\forall x \in X}\{|\mu_A(x) - \mu_B(x)|\}$$

FUZZY RELATIONS

- Let A and B be two fuzzy sets in the universal sets X and Y.

- The Cartesian product of A and B, denoted by $A \times B$, is defined by their membership function $\mu_A(x)$ and $\mu_B(y)$ as

$$\mu_{A\times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}, \qquad \forall x \in X, \forall y \in Y$$

- The Cartesian product $A \times B$ is thus a fuzzy set of ordered pairs (x, y) for all $x \in X$ and $y \in Y$, with the membership $\mu_{A\times B}(x, y)$.

- In a sense, the Cartesian product of two fuzzy sets can be seen as a fuzzy relation.

- Fuzzy relations offer the capability to capture the uncertainty and vagueness in relations between sets and elements of a set.

Fuzzy Relations

- In general, a fuzzy relation is a fuzzy set defined on the Cartesian product of crisp sets $A_1, A_2, \ldots, A_n$ where tuples $(x_1, x_2, \ldots, x_n)$ may have varying degrees of membership within the relation.

- The membership grade indicates the strength of the relation present between the elements of the tuple.

- In the case of two fuzzy sets, the fuzzy relation is defined as

$$R = \left\{ \left((x, y), \mu_R(x, y) \right) \right\}, \qquad \forall (x, y) \in X \times Y.$$

or

$$R = \bigcup_{\forall (x,y) \in X \times Y} \frac{(x, y)}{\mu_R(x, y)}.$$

Domain and Range of Fuzzy Relation

- Let A be a domain and B be a range of a fuzzy relation

$$R = \{((x,y),\mu_R(x,y))\}, \qquad \forall(x,y) \in X \times Y.$$

- Then the membership function of the fuzzy set A in X is

$$\mu_A(x) = \max_{y\in Y}\{\mu_R(x,y\}, \qquad x \in X,$$

- and the membership function of the fuzzy set B in Y is

$$\mu_B(y) = \max_{x\in X}\{\mu_R(x,y\}, \qquad y \in Y.$$

- If the domain A in X and range B in Y is given then the membership function of the fuzzy relation is defined as

$$\mu_R(x,y) = \min_{\forall(x,y)\in X\times Y}\{\mu_A(x),\mu_B(y)\}.$$

EXAMPLE

- Let the universal sets X and Y be

$$X = \{0,1,2,3,4,5,6,7,8,9\} \text{ and}$$
$$Y = \{0,1,2,3,4,5,6,7,8,9\}.$$

- Let the membership function $\mu_R(x,y)$ of the fuzzy relation R for $x \in X$ and $y \in Y$ be

$$\mu_R(x,y) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0 \\ 0 & 0 & 0.3 & 0.6 & 0.6 & 0.6 & 0.6 & 0.6 & 0.3 & 0 \\ 0 & 0 & 0.3 & 0.6 & 1 & 1 & 1 & 0.6 & 0.3 & 0 \\ 0 & 0 & 0.3 & 0.6 & 1 & 1 & 1 & 0.6 & 0.3 & 0 \\ 0 & 0 & 0.3 & 0.6 & 1 & 1 & 1 & 0.6 & 0.3 & 0 \\ 0 & 0 & 0.3 & 0.6 & 0.6 & 0.6 & 0.6 & 0.6 & 0.3 & 0 \\ 0 & 0 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

- The membership function of the fuzzy domain A in X is then

$$\mu_A(x) = \max_{y \in Y}\{\mu_R(x, y\} = \{0, 0.3, 0.6, 1, 1, 1, 0.6, 0.3, 0, 0\}.$$

- The membership function of the fuzzy range B in Y is then

$$\mu_B(y) = \max_{x \in X}\{\mu_R(x, y\} = \{0, 0, 0.3, 0.6, 1, 1, 1, 0.6, 0.3, 0\}.$$

- The fuzzy sets A and B are thus

$$A = \{(0,0), (1,0.3), (2,0.6), (3,1), (4,1), (5,1), (6,0.6), (7,0.3), (8,0), (9,0)\}$$

and

$$B = \{(0,0), (1,0), (2,0.3), (3,0.6), (4,1), (5,1), (6,1), (7,0.6), (8,0.3), (9,0)\}.$$

EXAMPLE

- The membership function $\mu_R(x, y)$ of the fuzzy relation R can now be re-calculated from the membership functions $\mu_A(x)$ and $\mu_B(x)$

$$\mu_R(x, y) = \min_{\forall (x,y) \in X \times Y} \{\mu_A(x), \mu_B(y)\}.$$

$$\mu_R(x, y) = \begin{array}{cc} & Y \longrightarrow \\ X \big\downarrow & \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0 \\ 0 & 0 & 0.3 & 0.6 & 0.6 & 0.6 & 0.6 & 0.6 & 0.3 & 0 \\ 0 & 0 & 0.3 & 0.6 & 1 & 1 & 1 & 0.6 & 0.3 & 0 \\ 0 & 0 & 0.3 & 0.6 & 1 & 1 & 1 & 0.6 & 0.3 & 0 \\ 0 & 0 & 0.3 & 0.6 & 1 & 1 & 1 & 0.6 & 0.3 & 0 \\ 0 & 0 & 0.3 & 0.6 & 0.6 & 0.6 & 0.6 & 0.6 & 0.3 & 0 \\ 0 & 0 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Composition of Fuzzy Relations

- The operation composition combines the fuzzy relations between different sets, say (x, y) and (y, z) ; $x \in X, y \in Y, z \in Z$.
- Let us consider the relations

$$
\begin{aligned}
R_1 &= \left\{ \left((x, z), \mu_{R_1}(x, z) \right) \right\}, \qquad \forall (x, z) \in X \times Z. \\
R_2 &= \left\{ \left((z, y), \mu_{R_2}(z, y) \right) \right\}, \qquad \forall (z, y) \in Z \times Y.
\end{aligned}
$$

- The max-min composition, denoted by $R_1 \circ R_2$, defines the composite fuzzy relation with the membership function

$$
\mu_{R_1 \circ R_2}(x, y) = \max_{z \in Z} \left\{ \min_{\forall x \in X, \forall y \in Y} \left\{ \mu_{R_1}(x, z), \mu_{R_2}(z, y) \right\} \right\}, \forall (x, y) \in X \times Y.
$$

Composition of Fuzzy Relations

- The min-max composition, denoted by $R_1 \diamond R_2$, defines the composite fuzzy relation with the membership function

$$\mu_{R_1 \diamond R_2}(x, y) = \min_{z \in Z}\left\{\max_{\forall x \in X, \forall y \in Y}\{\mu_{R_1}(x, z), \mu_{R_2}(z, y)\}\right\}, \forall(x, y) \in X \times Y.$$

- The max-min and min-max compositions are related

$$\overline{R_1} \circ \overline{R_2} = \overline{R_1 \diamond R_2}$$

Example

- Let the universal sets X, Y and Z be

$$X = \{1, 2\}, Y = \{0, 1, 2\} \text{ and } Z = \{0, 1, 2\}.$$

- Let the membership function $\mu_{R_1}(x, z)$ of the fuzzy relation R_1 for $x \in X$ and $z \in Z$, and the membership function $\mu_{R_2}(z, y)$ of the fuzzy relation R_2 for $z \in Z$ and $y \in Y$ be

$$\mu_{R_1}(x, z) = \begin{bmatrix} 0.1 & 0.3 & 0 \\ 0.8 & 1 & 0.3 \end{bmatrix} \quad \mu_{R_2}(z, y) = \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.2 & 1 & 0.6 \\ 0.5 & 0 & 0.4 \end{bmatrix}$$

- Compute the max-min composition $R_1 \circ R_2$!

Example

- Let us calculate the value of the membership function $\mu_{R_1 \circ R_2}(x, y)$ for $x = x_1 = 1$ and $y = y_1 = 0$.

$$\mu_{R_1 \circ R_2}(x_1, y_1) = \max \begin{Bmatrix} \min\{\mu_R(x_1, z_1), \mu_R(z_1, y_1)\}, \\ \min\{\mu_R(x_1, z_2), \mu_R(z_2, y_1)\}, \\ \min\{\mu_R(x_1, z_3), \mu_R(z_3, y_1)\}, \end{Bmatrix}$$

$$= \max\{\min\{0.1, 0.8\}, \min\{0.3, 0.2\}, \min\{0, 0.5\}\} = 0.2.$$

- In the same way, the values of the membership function $\mu_{R_1 \circ R_2}(x, y)$ for all the other values of x in y is calculate and the result is

$$\mu_{R_1 \circ R_2}(x, y) = \begin{bmatrix} 0.2 & 0.3 & 0.3 \\ 0.8 & 1 & 0.6 \end{bmatrix}$$

Basic Properties of Fuzzy Relations

- The basic properties of a fuzzy relation R that are often considered are the following.
  - Reflectivity, if

$$\mu_R(x,x) = 1, \qquad \forall (x,x) \in X \times Y.$$

  - Symmetricity, if

$$\mu_R(x,y) = \mu_R(y,x), \qquad \forall x \in X, \forall y \in Y.$$

  - Min-max transitivity, if

$$\mu_R(x,y) \geq \max_{z \in Z} \left\{ \min_{\forall x \in X, \forall y \in Y} \{\mu_R(x,z), \mu_R(z,y)\} \right\}.$$

- Similarity fuzzy relation is always reflective, symmetrical, and transitive.

Fuzzy Linguistic Terms and Variables

- Linguistic variables are used every day to express what is important and its context.
- The proposition '*This soup is hot*" represents an opinion independent of measuring system, and it has information that most listeners will understand.
- Linguistic variables are used in ordinary daily activities, including procedural instructions, such as cooking recipes etc.

Linguistic Terms

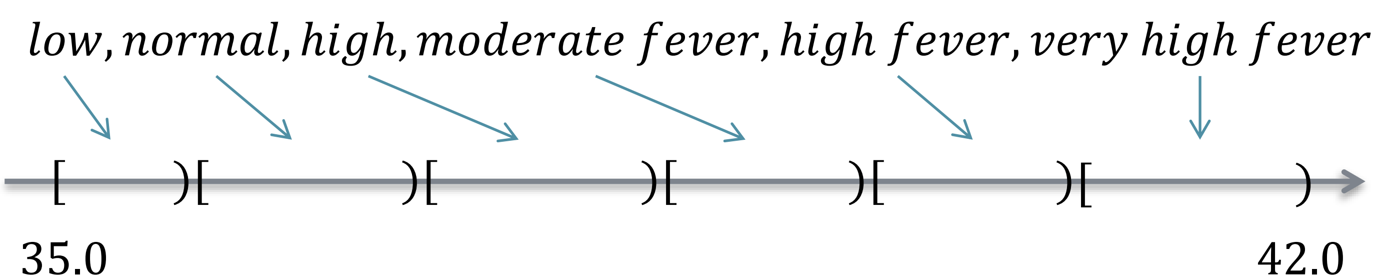
- A linguistic term T_i is a term that is considered to be true for some interval in the ordered universal set X (of basic physical measurements etc) and false outside the interval.
- The number of linguistic terms defined in the given universal X set can be between 1 and card(X).
- The number of linguistic terms in X defines its granularity.
- If the numbers of linguistic terms is close to 1 then its granularity is considered to be "coarse" and if the number is close to card(X) then it is considered to be "fine".

EXAMPLE

- Let the universal set X be the body temperatures between 35.0 and 42.0 °C.

- A doctor may consider 5 linguistic terms that relate to the body temperature ranging from $T_i \in \{low, normal,\ high, moderate\ fever, high\ fever, very\ high\ fever\}$.

- Each of the terms is associated with one of the temperature intevals, e.g., the term *normal* could be associated with the interval [36.1,37.0], and similarly of the other terms.



Fuzzy Linguist Term

- A fuzzy linguistic term F_i is a fuzzy set in the universal set X

$$F_i = \{(x, \mu_{T_i}(x))\}, \qquad \forall x \in X,$$

where T_i denote the i-th common linguistic term in X.

- A fuzzy linguistic variable H is then a variable that takes it value from the set of fuzzy linguist terms $\{F_i : i = 1, 2, \dots, N_T\}$.
- $T(H)$ then denotes a set of linguistic terms $\{T_i : i = 1, 2, \dots, N_T\}$ and $F(H)$ a set of fuzzy linguist terms $\{F_i : i = 1, 2, \dots, N_T\}$.

EXAMPLE

- Let H denote a linguistic variable of body temperature.
- A universal set X of the basic measurements of body temperatures could then be

$$X = \{35.0, 35.1, 35.2, \ldots, 41.9, 42.0\},$$

a set of linguistic terms

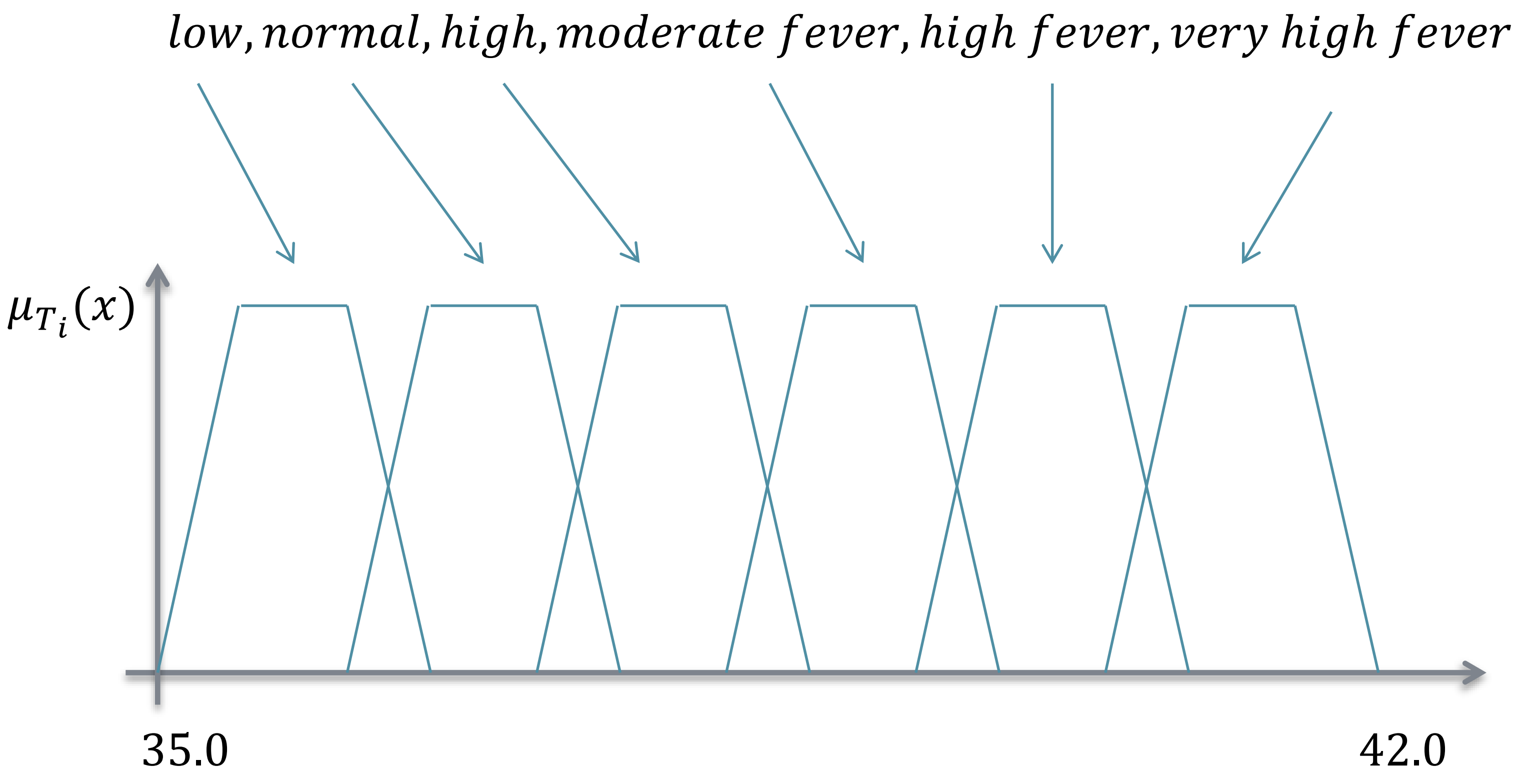
$$T(H) = \{low, normal, \ldots, very\ high\ fever\}$$

and a set of fuzzy linguistic terms $\mathrm{F(H)} = \{F_i : i = 1, 2, \ldots, N_T\}$, where, e.g.,

$$\begin{aligned}\text{"}normal\text{"} = F_1 = \{&(35.0, 0), \ldots, (35.8, 0),\\ &(35.9, 0.1), (36.0, 0.3), (36.1, 0.5), (36.2, 0.8), (36.3, 0.9),\\ &(36.4, 1), \ldots, (36.8, 1),\\ &(36.9, 0.9), (37.0, 0.7), (37.1, 0.4), (37.2, 0.3), (37.3, 0.1),\\ &(37.4, 0), \ldots, (42.0, 0)\}\}\end{aligned}$$

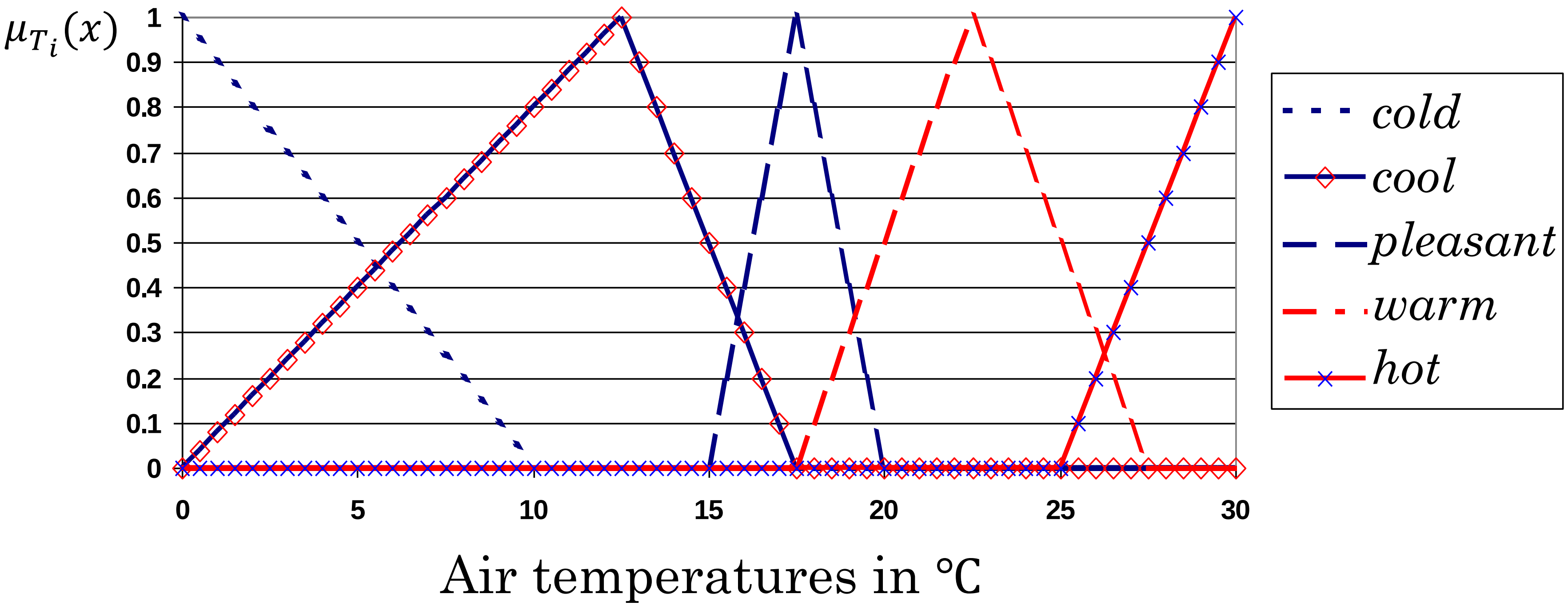
EXAMPLE

- The membership functions of all the fuzzy linguistic terms can be illustrated as shown below.



EXAMPLE

- The membership functions of fuzzy linguistic terms are not necessary uniform as it can be seen for the example below



Linguistic Terms

- A set of linguistic terms of a fuzzy linguistic variable T(H) can be "enriched" using adverbs, such as: more, less, very little, quite like etc.
- These "enriched" versions of the fuzzy linguist terms $F_i = \{(x, \mu_{T_i}(x))\}$ can be obtained using the operators of
  - Concentration

$$\mu_{adv(T_i)} = \mu_{con(F_i)} = \left(\mu_{T_i}(x)\right)^2$$

  - Dilatation

$$\mu_{adv(T_i)} = \mu_{dil(F_i)} = \sqrt{\mu_{T_i}(x)}$$

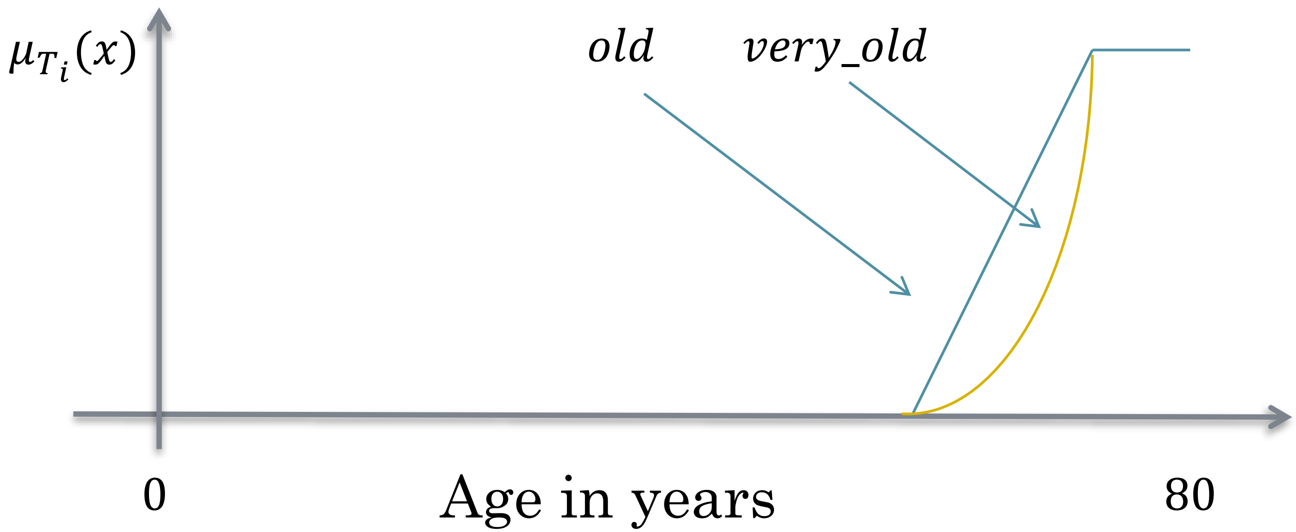
EXAMPLE

- Let us consider the following fuzzy linguistic term.

$$"old" = \{(60, 0.2), (65, 0.4), (70, 0.6), (75, 0.8), (80, 1)\}.$$

- Using the concertation operator the fuzzy linguistic term for *"very_old"* can be defined.

$$"very\_old" = con("old") = \{(60, 0.04), (65, 0.16), (70, 0.36), (75, 0.64), (80, 1)\}.$$



Fuzzy Rules

- A fuzzy rule is defined as a conditional statement in the form

$$\text{IF } x = A \text{ THEN } y = B$$

where x and y are fuzzy linguistic variables, and A and B are fuzzy linguistic terms determined by fuzzy sets on the universal sets X and Y, respectively.

- Examples:
  - IF pressure is high, THEN volume is small.
  - IF temperature is normal THEN maintain fan.
  - IF the speed is high, THEN apply the brake a little.

Fuzzy Linguistic Variables

- At the root of fuzzy set theory lies the idea of linguistic variables that are considered as fuzzy variables.
- The range of possible values of a linguistic variable represents the universe of discourse of that variable.
- For example, the universe of discourse of the linguistic variable *speed* might have the range between 0 and 200 km/h and may include such fuzzy subsets as *very slow*, *slow*, *medium*, *fast*, and *very fast*.

Fuzzy Rules vs. Classical Rules

- Fuzzy **IF-THEN** rules differ from the classical two-valued **IF-THEN** rules.
- For example, the following classical **IF-THEN** rules are based on binary logic (the implication is either true or false).

Rule: 1	
IF	speed is > 100
THEN	min_stopping_distance = 300

Rule: 2	
IF	speed is < 40
THEN	max_stopping_distance = 40

- The variable *speed* can have any numerical value between 0 and 200 km/h, and stopping distance can take either value 300 or 40.
- As is can be seen, the classical rules are expressed in the language of binary/Boolean logic.

FUZZY RULES VS. CLASSICAL RULES

- We can also represent the two stopping distance rules in a fuzzy form

Rule: 1	
IF	speed is fast
THEN	stopping_distance is long

Rule: 2	
IF	speed is slow
THEN	stopping_distance is short

- In fuzzy rules, the linguistic variable *speed* also has the range (the universe of discourse) between 0 and 200 km/h, but this range includes fuzzy sets, such as *slow*, *medium* and *fast etc*.
- The universe of discourse of the linguistic variable *stopping_distance* can be between 0 and 300 m and may include such fuzzy sets as *short*, *medium* and *long*.

Fuzzy Logic

- Fuzzy rules play an essential role in fuzzy logic.
- The extension of crisp logic to fuzzy logic is made in the same way as the extension of crisp set theory to fuzzy set theory.
- In fuzzy logic, the truth values are multi-valued, as absolute true, partially true, absolute false etc. represented numerically as s real value between 0 to 1.
- The truth value of the fuzzy proposition $x = A$ is thus given by the membership function $\mu_A(x)$ of the fuzzy set A.
- The fuzzy logic is similar to the crisp logic supported by the basic logical connectives, i.e., negation, conjunction, disjunction and implication.

Fuzzy Logical Connectives

- The fuzzy negation of the proposition $x = A$ is denoted as $\neg\,(x = A)$.

- The fuzzy truth value of the proposition $\neg\,(x = A)$, denoted as $\mu_{\neg A}(x)$ is defined as the value of the fuzzy membership function of the complement $\bar{A}$ of the fuzzy set A, i.e.,

$$\mu_{\neg A}(x) = \mu_{\bar{A}}(x) = 1 - \mu_A(x), \qquad \forall x \in X.$$

- The fuzzy truth value of the fuzzy logical conjunction $(\mathrm{x} = A) \wedge (y = B)$, denoted as $\mu_{A\wedge B}(x, y)$, is defined as the value of the fuzzy membership function of the fuzzy intersection (Cartesian product) relation between A and B, i.e.,

$$\mu_{A\wedge B}(x, y) = \mu_{A\cap B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$$

Fuzzy Logical Connectives

- In the same way, the fuzzy truth value of the fuzzy logical disjunction $(\mathrm{x} = A) \vee (y = B)$, denoted as $\mu_{A \vee B}(x, y)$, is defined as the value the fuzzy membership function of the fuzzy union relation between A and B, i.e.,

$$\mu_{A \vee B}(x, y) = \mu_{A \cup B}(x, y) = \max\{\mu_A(x), \mu_B(y)\}$$

- The truth value of the fuzzy implication `IF` $x = A$ `THEN` $y = B$ or $(\mathrm{x} = \mathrm{A}) \rightarrow (y = B)$, denoted as $\mu_{A \rightarrow B}(x, y)$ is defined as the value of any valid fuzzy relation $\mu_R(x, y)$ between the two fuzzy sets A and B.

- The truth value of the fuzzy implication $\mu_{A \rightarrow B}(x, y) = \mu_R(x, y)$ can be derived from the three basic logical connectives based on the logical equivalences defined in prepositional calculus.

Fuzzy Logical Implication

- For instance, if we assume the tautologies of the following two logical equivalences derived from the propositional calculus:

$$(A \to B) \equiv (\neg A) \vee B$$
$$(A \to B) \equiv (\neg A) \vee (A \wedge B)$$
$$(A \to B) \equiv (\neg A \wedge \neg B) \vee B$$

$$\overline{A} \cap \overline{B} = \overline{A \cup B}$$

then the following truth functions can be derived from the truth functions of the basic fuzzy logical connectives.

$$\mu_{A\to B}(x,y) = \max\{1 - \mu_A(x), \mu_B(y)\}$$
$$\mu_{A\to B}(x,y) = \max\{1 - \mu_A(x), \min\{\mu_A(x), \mu_B(y)\}\}$$
$$\mu_{A\to B}(x,y) = \max\{1 - \max\{\mu_A(x), \mu_B(y)\}, \mu_B(y)\}$$

- The first truth function was proposed by Kleen and Dienes, and the second one by Zadeh.

Fuzzy Logical Implication

- The proposed two truth functions for the fuzzy implication are not equivalent, even though the logical equivalences derived from the propositional calculus are tautologies.

- Several other fuzzy relations were proposes for the truth function of the fuzzy implication, among them are:

Mamdani	$\mu_{A\rightarrow B}(x,y) = \min\{\mu_A(x), \mu_B(y)\}$
Larsen	$\mu_{A\rightarrow B}(x,y) = \mu_A(x) \cdot \mu_B(y)$
Lukasiewicz	$\mu_{A\rightarrow B}(x,y) = \min\{1, \left(1 - \mu_A(x) + \mu_B(y)\right)\}$
Bounded product	$\mu_{A\rightarrow B}(x,y) = \max\{0, (\mu_A(x) + \mu_B(y) - 1)\}$

etc.

- The choice of the truth function depends on the application of the fuzzy rules.

Example

- Let us consider the fuzzy implication IF $x = A$ THEN $y = B$, where the two fuzzy sets A in X and B in Y are given as follows.

$$A = \{(0,0),(1,0.3),(2,0.6),(3,1),(4,1),(5,1),(6,0.6),(7,0.3),(8,0),(9,0)\}$$
$$B = \{(0,0),(1,0),(2,0.3),(3,0.6),(4,1),(5,1),(6,1),(7,0.6),(8,0.3),(9,0)\}.$$

- The truth function for the given fuzzy implication, which is defined according to Mamdani, would then be

$$\mu_{A\to B}(x,y) = \begin{array}{c} \xrightarrow{\quad Y \quad} \\ X \downarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0 \\ 0 & 0 & 0.3 & 0.6 & 0.6 & 0.6 & 0.6 & 0.6 & 0.3 & 0 \\ 0 & 0 & 0.3 & 0.6 & 1 & 1 & 1 & 0.6 & 0.3 & 0 \\ 0 & 0 & 0.3 & 0.6 & 1 & 1 & 1 & 0.6 & 0.3 & 0 \\ 0 & 0 & 0.3 & 0.6 & 1 & 1 & 1 & 0.6 & 0.3 & 0 \\ 0 & 0 & 0.3 & 0.6 & 0.6 & 0.6 & 0.6 & 0.6 & 0.3 & 0 \\ 0 & 0 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Compound Logical Implication

- Fuzzy **IF-THEN** rules are often defined in non-canonical forms with a compound antecedent, and the truth functions for such compound rules can be derived from the truth function of the basic logical connectives.

- For example, the fuzzy rule

$$\text{IF } x_1 = A \quad \text{AND} \quad x_2 = B \quad \text{THEN } y = C$$

represent the fuzzy implication

$$((x_1 = A) \wedge (x_2 = B)) \rightarrow (y = C)$$

and its truth function $\mu_R((x_1, x_2), y) = \mu_{(A \wedge B) \rightarrow C}((x_1, x_2), y)$ can be then defined, e.g., according to Mamdani as

$$\mu_R((x_1, x_2), y) = \min\{\min\{\mu_A(x_1), \mu_B(x_2)\}, \mu_C(y)\}$$

Compound Logical Implication

- The fuzzy rule

$$\text{IF } x = A \quad \text{THEN } y = B \quad \text{ELSE} \quad y = C$$

represent the compound fuzzy implication

$$((x = A) \to (y = B)) \wedge ((\neg(x = A)) \to (y = C))$$

- In accordance with the logical equivalence

$$(A \to B) \wedge ((\neg A) \to C) \equiv (A \wedge B) \vee ((\neg A) \wedge C)$$

its truth function is defined as

$$\mu_R(x, y) = \max\{\min\{\mu_A(x), \mu_B(y)\}, \min\{1 - \mu_A(x), \mu_C(y)\}\}$$

Fuzzy Reasoning

- Fuzzy rules can be seen as the rules that can "fire" only to some extent, or in other words, they can "fire" partially.
- If the antecedent is true to some degree of truth, then the consequent is also true only to some degree of truth.
- When a knowledge in an expert domain is represented using fuzzy rules then the most common inference rule of fuzzy reasoning is the generalised Modus Ponens rule.

Generalized Modus Ponens

- Fuzzy reasoning is based on the use of the generalized Modus Ponens rule of inference.

$$\begin{array}{c} \text{Premise 1 (Rule): IF } x = A \text{ THEN } y = B \\ \text{Premise 2 (Fact): } x = A' \\ \hline \text{Conclusion: } y = B' \end{array}$$

- The truth function $\mu_{B'}(y)$ of the conclusion $(y = B')$ is derived from the truth functions $\mu_{A\rightarrow B}(x,y) = \mu_R(x,y)$ and $\mu_{A'}(x)$ as follows.

$$\mu_{B'}(y) = \mu_{A'}(x) \circ \mu_R(x,y) = \max_x \{\min\{\mu_{A'}(x), \mu_R(x,y)\}\}$$

where $\mu_R(x,y)$ is any valid truth function $\mu_{A\rightarrow B}(x,y)$ of the implication $(x = A) \rightarrow (y = B)$.

Example

- Let the truth function $\mu_{A\to B}(x,y)$ of the fuzzy rule IF $x = A$ THEN $y = B$ be

$$\mu_{A\to B}(x,y) = \mu_R(x,y) = \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 \\ 0.6 & 0.7 & 0.6 \end{bmatrix}$$

- Let then the truth function $\mu'_A(x)$ of the fact $x = A'$ for all the values of x be

$$\mu_{A'}(x) = [0.05, 0.1, 0.2, 0.4].$$

EXAMPLE

- The truth function of the conclusion $y = B'$ is then

$$\mu_{B'}(y) = \mu_{A'}(x) \circ \mu_R(x,y) =$$

$$= \max_{x \in X} \left\{ \min_{\forall y \in Y} \{\mu_{A'}(x), \mu_R(x,y)\} \right\} =$$

$$= [0.05, 0.1, 0.2, 0.4] \circ \begin{bmatrix} 0.1 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0.2 \\ 0.4 & 0.4 & 0.4 \\ 0.6 & 0.3 & 0.6 \end{bmatrix} =$$

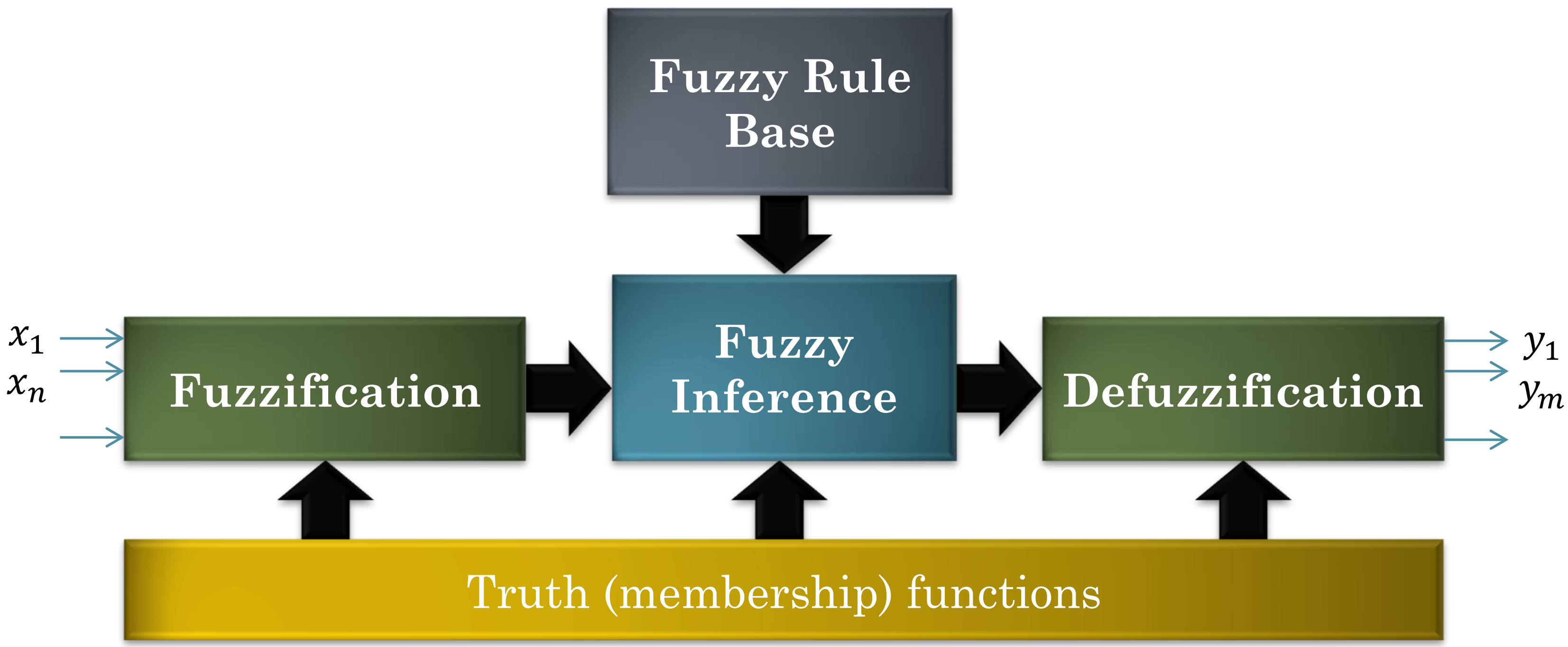
$$= \max_{x \in X} \left\{ \begin{bmatrix} 0.05 & 0.05 & 0.05 \\ 0.1 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0.2 \\ 0.4 & 0.3 & 0.4 \end{bmatrix} \right\} = [0.4, 0.3, 0.4]$$

Fuzzy Systems

- Any system that is based on fuzzy logic may be viewed as a fuzzy system.
- The design of fuzzy systems and fuzzy controllers working in a closed–loop are based on the following three main phases.
  - Fuzzification of input information
  - Fuzzy inferencing using fuzzy rules
  - De-fuzzification of the results from the reasoning process

Fuzzy Systems

- Fuzzy inferencing is the core component of a fuzzy system.
- Fuzzy inferencing combines the facts obtained from the fuzzification process with the fuzzy rule base and conducts the fuzzy reasoning process.



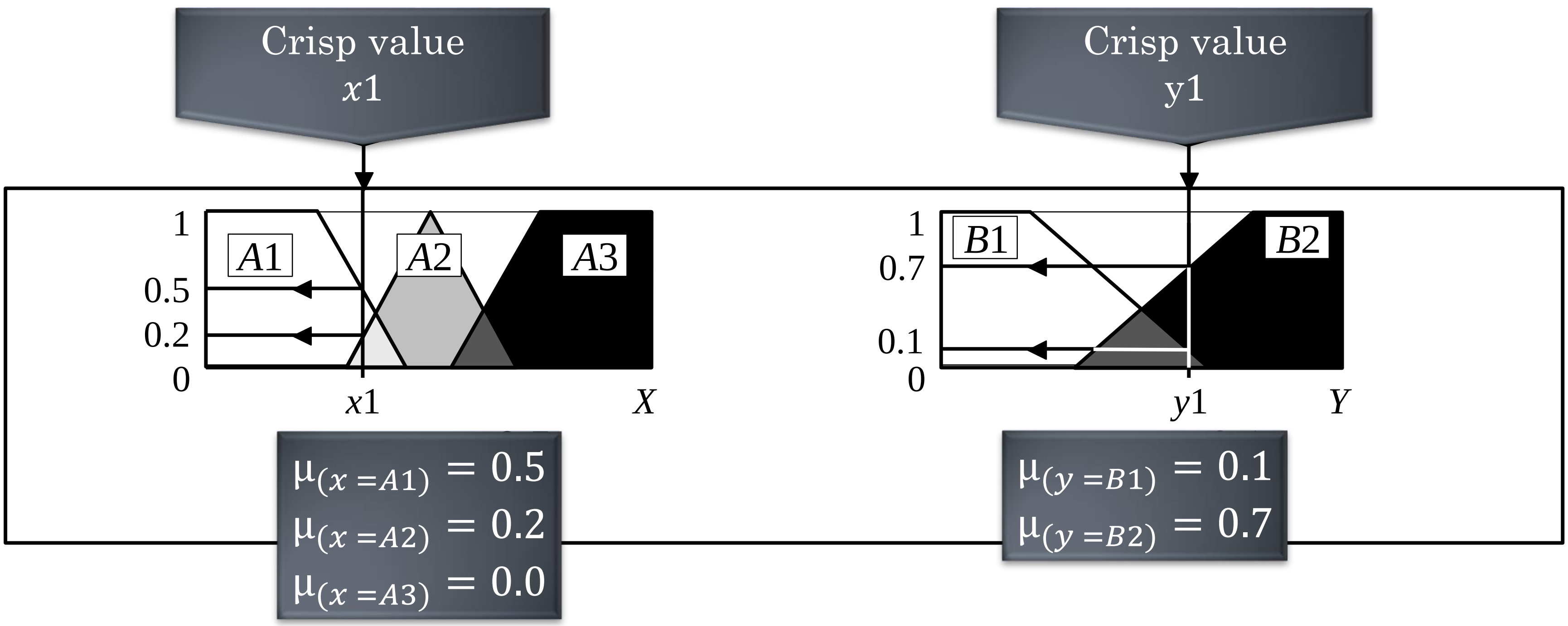
EXAMPLE

- Let us consider an example of three IF-THEN fuzzy rules that may have two fuzzy variables in their antecedents:

Rule: 1	
IF x is A3	IF project_funding is adequate
OR y is B1	OR project_staffing is small
THEN z is C1	THEN risk is low
Rule: 2	
IF x is A2	IF project_funding is marginal
AND y is B2	AND project_staffing is large
THEN z is C2	THEN risk is normal
Rule: 3	
IF x is A1	IF project_funding is inadequate
THEN z is C3	THEN risk is high

STEP 1

- First, the crisp inputs, x1 and y1, are taken and the degree to which these inputs belong to each of the appropriate fuzzy sets is determined.

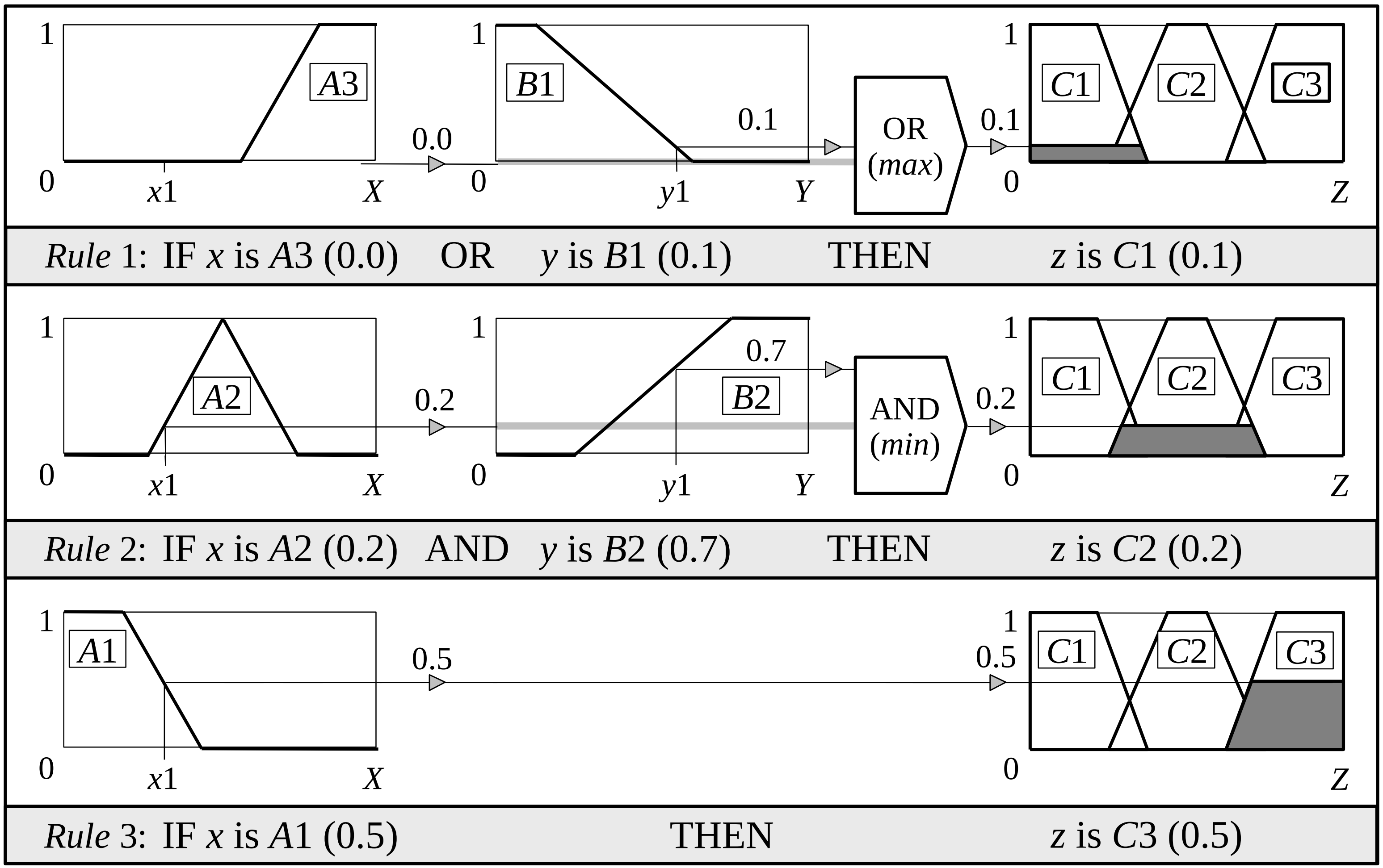


STEP 2

- The second step is to take the fuzzified inputs, $\mu_{(x=A1)} = 0.5, \mu_{(x=A2)} = 0.2, \mu_{(y=B1)} = 0.1$ and $\mu_{(y=B2)} = 0.7$, and apply them to the antecedents of the fuzzy rules.
- If a given fuzzy rule has multiple antecedents, the fuzzy operator (AND or OR) is used to obtain a single number that represents the result of the antecedent evaluation.
- This number (the truth value) is then applied to the consequent membership function using the Mamdani truth function for the implication.

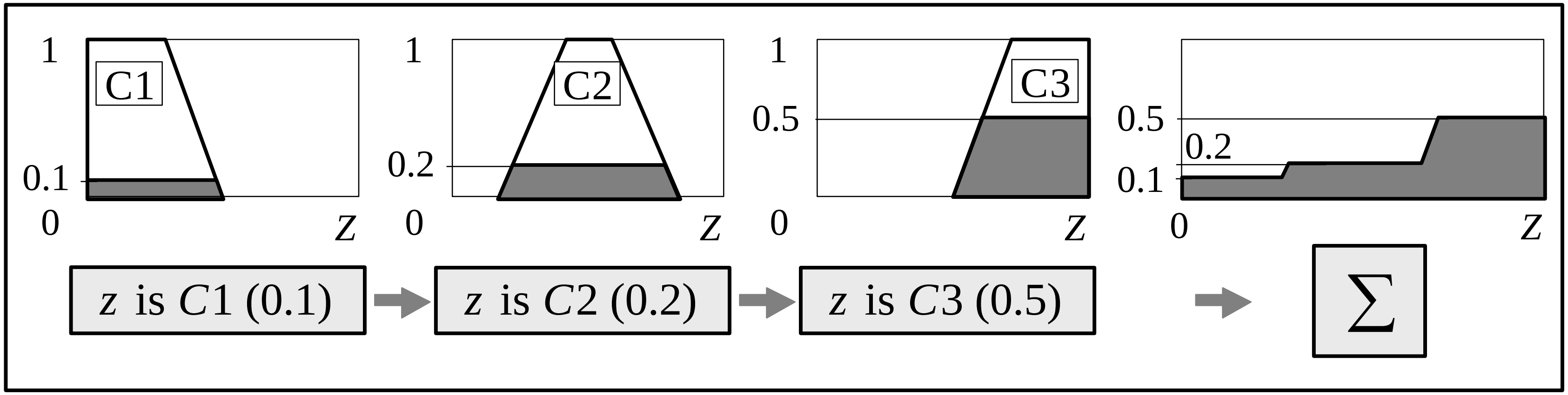
STEP 2

- Evaluation of the fuzzy rules



STEP 3

- Aggregation of the rule outputs is the process of unification of the outputs of all rules.
- The input of the aggregation process is the list of consequent membership functions, and the output is one fuzzy set for each output variable.



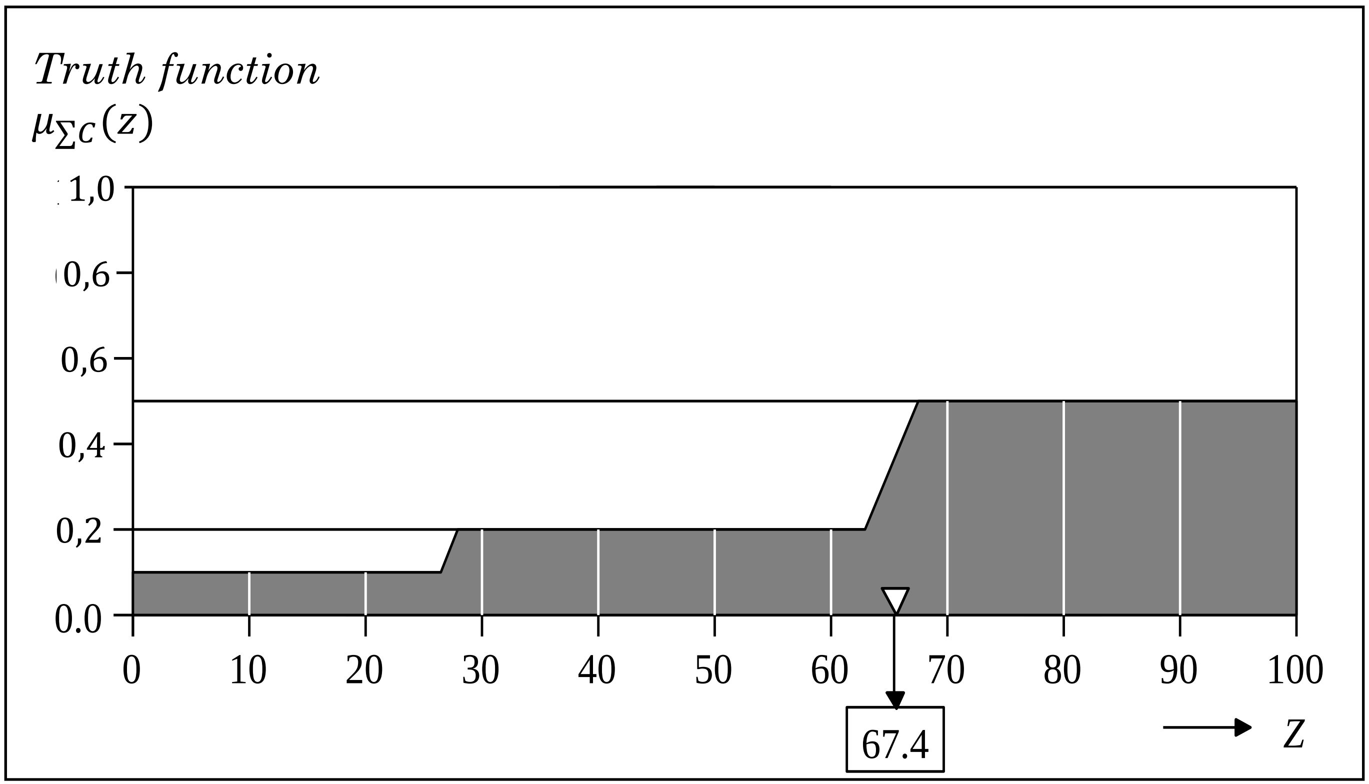
Step 4

- The last step in the fuzzy inference process is defuzzification.
- Fuzziness helps us to evaluate the rules, but the final output of a fuzzy system has to be a crisp number.
- The input for the defuzzification process is the aggregate output fuzzy set and the output is a single number.
- There are several defuzzification methods, but probably the most popular one is the center of gravity method.

Step 4

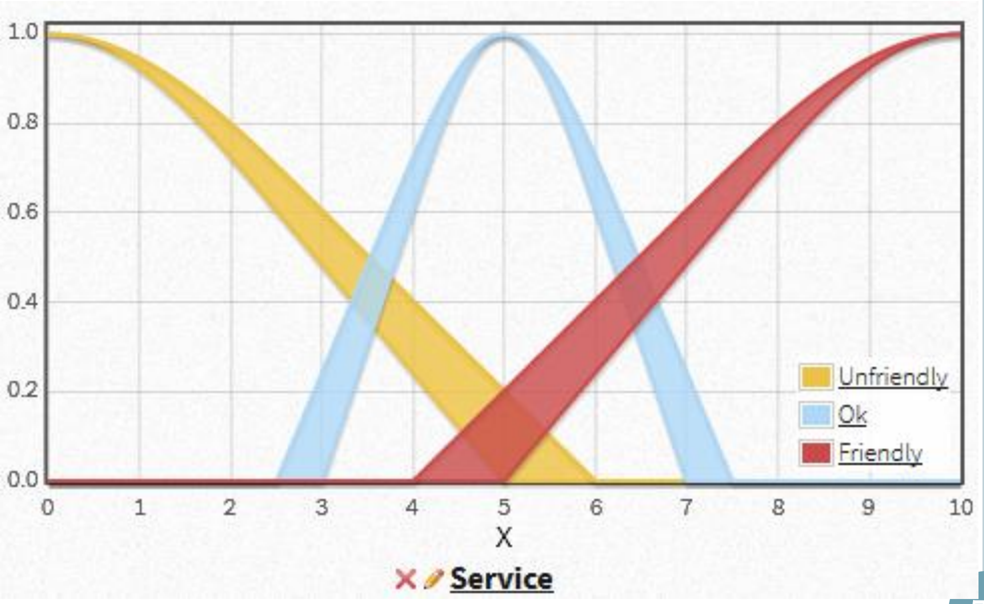
- Determining the crisp value of the aggregated conclusion using the center of gravity (COG) method

$$COG = \frac{(0+10+20)\times 0.1 + (30+40+50+60)\times 0.2 + (70+80+90+100)\times 0.5}{0.1+0.1+0.1+0.2+0.2+0.2+0.2+0.5+0.5+0.5+0.5} = 67.4$$



Final Remarks

- Fuzzy logic and fuzzy systems are widely and successfully used in process modelling and control as tools for handling with uncertainties of information in complex systems.
- The presented conventional fuzzy logic was generalized to the so-called interval type-2 and general type-2 fuzzy sets and systems.
- A type-2 fuzzy set incorporates uncertainty about the membership function into fuzzy set theory.



Fuzzy Logic Programming Tools

- Many different fuzzy logic tools have been developed and are available to researchers, among them are:
  - MathWorks Fuzzy Logic Toolbox
    http://www.mathworks.com/products/fuzzy-logic
  - jFuzzyLogic - The fuzzy logic library in Java
    http://jfuzzylogic.sourceforge.net
  - Fuzzylite for C++
    http://www.fuzzylite.com
  - A Java based toolkit for type-1 and ype-2 fuzzy logic and fuzzy logic systems
    http://juzzy.wagnerweb.net
  - …

Questions

- What is the main difference between crisp and fuzzy sets?
- Give an example of a fuzzy set, its core, complement, support, and the crisp value according to the center of gravity method.
- What are the main binary operations on two fuzzy sets and what are their properties?
- What is a fuzzy linguistic variable and give its example?
- Give several examples of fuzzy rules.
- Describe fuzzy implication and how it truth function is defined?
- Describe the most common fuzzy reasoning rule.
- What are the main processes of a fuzzy system?