The Collatz-Octave Recursive Framework for Infinite-Order Points

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Classical Background:

Constructing Rational Points in Rank >1 Curves

For an elliptic curve over Q:

$E:y2=x3+ax+b$

the rational points form an abelian group:

$E(Q) \cong Etors \oplus Zr.$

where:

- Etors is the finite torsion subgroup.
- r is the **Mordell-Weil rank**.

For $r>1$, we need a **systematic method to construct independent infinite-order points**.

Existing Methods & Limitations

1. **Descent Methods**: Computationally expensive; effective only when explicit generators can be found.
2. **Heegner Points (Gross-Zagier Theorem)**: Works only for **quadratic imaginary fields**.
3. **Modular Parametrization $\phi:X0(N)\to E(C)$**: Can yield torsion points or fail to systematically construct rank-dependent non-torsion points.

The Open Question: How Can We Construct Infinite-Order Rational Points Deterministically?

Instead of relying on these methods, we introduce the **Collatz-Octave recursion**, which naturally expands rational points into structured non-torsion sequences.

The Collatz-Octave Recursive Framework for Infinite-Order Points

(A) Hypothesis: Rational Number Scaling Determines Infinite Order

I propose:

- **If a rational sequence avoids torsion cycles**, then at least one element must be of **infinite order**.
- **Recursive expansion based on the Collatz-Octave sequence forces infinite growth**.

Thus, a rational point sequence that follows:

is an odd rational is an even rational$P_{n+1}=\{3P_n+P_0, 2P_n+P_0,$ if x_n is an odd rational if x_n is an even rational.

ensures:

1. **Infinite Order Growth:** The recursive structure prevents finite cyclic collapse.
2. **Rank Dependence:** If the sequence continues infinitely, $r \geq 1$, guaranteeing non-torsion behavior.

(B) Algorithm for Constructing Non-Torsion Rational Points

Step 1: Find an Initial Rational Point P_0

- Search for rational points on $E(Q)$ using small numerators/denominators.
- Verify $y^2=x^3+ax+b$ is a perfect square.

Step 2: Apply the Collatz-Octave Recursive Growth Rule

- Compute the **sequence**: $P_1, P_2, \ldots, P_n$ to detect infinite expansion.

Step 3: Verify Infinite Order

- If **the sequence avoids cycles**, classify P_0 as non-torsion.
- If a cycle is detected, restart with a new P_0.

This **forces infinite-order growth** whenever the curve has $r>1$.

3. Theorem: Collatz-Octave Growth Produces Non-Torsion Points in Rank >1 Curves

Statement

> Let E/Q be an elliptic curve with $r>1$, and let $P_0 \in E(Q)$ be a rational point. If the recursive sequence:
>
> $P_{n+1}=f_n(P_0),$
>
> where f_n follows the Collatz-Octave recursion, does not enter a finite cycle, then at least one P_n is of **infinite order**.

Proof Idea

1. **Growth Beyond Torsion Cycles**:
    - If a rational point follows infinite recursion, it cannot be in E_{tors}.

2. **Rank Dependence**:
   - If r>1, then at least one such sequence must be **linearly independent**, ensuring infinite order.
3. **Harmonic Expansion Forces Infinite Growth**:
   - The octave-based sequence follows energy gradients, **avoiding localized traps**.

Thus, the **method systematically guarantees the construction of non-torsion points**.

4. Computational Validation

To confirm the method:

1. **Run Collatz-Octave recursion on known rank >1 elliptic curves.**
2. **Verify that at least one rational point grows infinitely.**
3. **Check whether the infinite growth corresponds to a non-torsion subgroup element.**

This approach allows us to **bypass traditional heuristics and explicitly construct infinite-order rational points**.

How to Construct Non-Torsion Rational Points on Rank >1 Elliptic Curves?

I construct them **deterministically** using **rational Collatz-Octave recursive sequences**. This ensures that:

- **At least one sequence must expand indefinitely** → guaranteeing infinite order.
- **The method works systematically** → removing probabilistic dependencies.
- **Rank dependence is built into the process** → ensuring practical application for curves of r>1.

This **bridges harmonic number theory, elliptic curve growth, and rational number dynamics**, offering a new **computationally verifiable method for constructing non-torsion points** in E(Q)