Quantum mechanics

The Statistical Interpretation

by Hakob Avetisyan, Gate42

CONTENTS

[1] RevModPhys, 42(4), 358 (1970)
[2] AmJPhys, 54(10), 883 (1986)
[3] Nuovo Cimento 38B(1), 75 (1977)
[4] AmJPhys, 55(8), 696 (1987)
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[6] Asher Peres, Quantum Theory: Concepts and Methods
[7] JPhys, A 24, L175 (1991)
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[10] AmJPhys, 61, 443 (1993)

Any successful theory in the physical sciences is expected to make accurate predictions.

Given some well-defined experiment, the theory should correctly specify the outcome or should at least assign the correct probabilities to all the possible out comes.

From this point of view QM is highly successful.

As the fundamental modern theory of atoms, of molecules, of elementary particles, of electromagnetic radiation and of the solid state it supplies methods for calculating the results of experiments in all these realms.

Bernard d'Espagnat

MATH FORMALISM

A theory can be divided into:

- set of primitive concepts & relations between them
- rules to relate the concepts to experiment

1. State

$$\rho = \rho^\dagger \leftarrow \text{self adjoint on Hilbert Space}$$

$$\rho = \sum_n \rho_n |\phi_n\rangle\langle\phi_n| \leftarrow \text{spectral decomposition}$$

$$0 \leq \rho \leq 1 \leftarrow \text{positive semidefinite}$$

$$\sum_n \rho_n = 1 \leftarrow \text{unit trace}$$

Fano, RevModPhys **29**, 74 (1957)

2. Pure State

$$\rho^2 = \rho \leftarrow \text{definition}$$

$$\rho_n = 1, \rho_n' = 0, \text{ for } n \neq n'$$

$$\rho = |\phi_n\rangle\langle\phi_n|$$

if not pure = mixed state

3. Observable

$$R = R^\dagger \leftarrow \text{self adjoint on Hilbert Space}$$

$$R = \sum_n r_n \Pi_n \leftarrow \text{spectral decomposition}$$

$$\Pi_n = \sum_a |a, r_n\rangle\langle a, r_n| \leftarrow \text{orthogonal projectors}$$

4. Average of obs R in state ρ

$$\langle R\rangle = Tr(\rho R), \quad \langle R\rangle = \langle\psi|R|\psi\rangle \leftarrow \text{Born's rule}$$

By introducing the characteristic function $\langle \exp(i\xi R)\rangle$, can obtain the entire statistical distribution of the observable R in the state ρ.

5. The only values which an observable may take on are its eigenvalues, and the probabilities of each of the eigenvalues can be calculated:

$$P(r_n) = Tr(\rho \Pi_n), \quad P(r_n) = \sum_a |\langle \psi | a, r_n \rangle|^2$$

6. Hilbert Space is a direct sum of coherent subspaces

$$\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2 \oplus \cdots \oplus \mathbb{H}_m \quad \leftarrow \text{superposition principle}$$

in each $\mathbb{H}_i$, every vector may represent a pure state

7. Any self-adjoint operator, all of whose eigenvectors lie within coherent subspaces, represents an observable.

8. The dynamical law

$$\rho(t) = U(t, t_0)\rho(t_0)U(t, t_0)^{-1}, \quad U(t, t_0) \text{ is unitary}$$

Correspondence rules

state, observable $\longleftrightarrow$ empirical reality

Observable

dynamical variable whose value can be measured

If $pq - qp \neq 0 \rightarrow$ no general rule to constract operator to represent $f(p, q)$

Fortunately, can determine the Hamiltonian operator for any finite number of particles interacting through velocity-independent potentials in the presence of an arbitrary external electromagnetic field

State

provides a description of certain statistical properties of an ensemble of similarily prepared systems, but need not provide a complete description of an individual system.

Example:

system: single electron

ensemble: electrons output from acceleration machine one by one. They are similar in momentum, not similar in position

(Uncertainty principle: not possible to prepare systems with ALL observable properties similar)

beam of electrons = many-body problem

QM ensemble (prepd experimentally) $\neq$ Thermod ensemble (calc device)

QM: Prep –> meas –> see statistics

Thermodyn: Calc on ensemble –> compare with measurement on single system

EPR Theorem

1. "Every element of physical reality must have a counterpart in the physical theory." -- necessary condition for complete theory.
2. If, without in any way disturbing a system, we can predict with certainty (p=1), the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity. -- sufficient for ∃ element of reality.

$$|\psi\rangle = (|01\rangle - |10\rangle)/\sqrt{(2)} \quad \leftarrow \text{singlet state}$$

measure σ_{z1}, know σ_{z2}(w/ p=1) $\Longrightarrow$ $\sigma_{z2}$is an element of reality

$$|\psi\rangle = (|+-\rangle - |-+\rangle)\sqrt{(2)} \quad \leftarrow \text{singlet state}$$

measure σ_{x1}, know σ_{x2}(w/ p=1) $\Longrightarrow$ σ_{x2} is an element of reality

as $\sigma_z\sigma_x - \sigma_x\sigma_z \neq 0$ no state $|\psi\rangle$ can provide a value for both σ_z and σ_x $\Longrightarrow$ $|\psi\rangle$ doesn't provide complete description of an individual system

Theorem (EPR)

a) $|\psi\rangle$ provides complete description of an individual system (Bohr accepted this)

b) Physical conditions of sepatared objects are independent (Einstein accepted this)

a) & b) are incompatible!

Statistical interpretation: repeat many times, get 50%-50% results

Uncertainty Principle

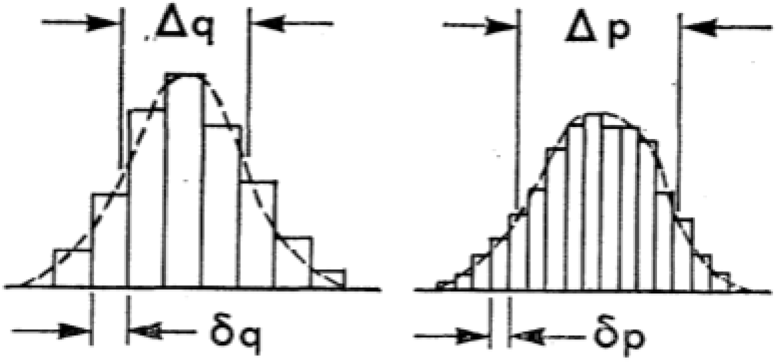
For each $|\psi\rangle$ corresponds statistical distribution of values of observables

$$(\Delta A)^2 = \langle (A - \langle A \rangle)^2 \rangle$$

$$(\Delta A)^2 (\Delta B)^2 \geq [\langle AB + BA \rangle - \langle A \rangle \langle B \rangle]^2/4 + \langle 1/i(AB - BA)\rangle$$
$$/4$$

Experimentally: measure A on similarly prepared systems$\rightarrow$ constract distribution $\rightarrow$ std(A)$\rightarrow$ same for B

$\Longrightarrow std(A)std(B) \geq$ some lower limit for a particular state ρ



$\delta q, \quad \delta p$ are the errors of individual measurements

Uncertainty Principle = statement about state preparation (refers to "future"), not measurement (refers to "past")

Filtering type of state preparation nowadays is refered as measurement (confusingly)

Measurement

Measurement is an interaction between the object and apparatus so that a correspondence is set up between the initial state of the object and the final state of the apparatus.

The apparatus must be treated as quantum while interacts with object, and classical, once measurement is over.

To measure observable R, should set up correspondence

Final state of the object has no significance to the success of the measurement. Equation of motion must lead to

$$U|r,0\rangle = |r,\alpha_r\rangle \leftarrow \text{if } R \text{ is not changed by interaction}$$

$$U|r,0\rangle = |\phi_r,\alpha_r\rangle \leftarrow \text{general case}$$

α_r is the "pointer" position label, $|r\rangle$ is the eigenvector of R

$\{|\phi_r,\alpha_r\rangle\}$ are orthonormal, because $\alpha_r \neq \alpha_{r'}$, $\{|\phi_r\rangle\}$ need not be orthonormal

$$U|\psi,0\rangle = U\sum_r |r\rangle\langle r|\psi,0\rangle = \sum_r \langle r|\psi\rangle|\phi_r,\alpha_r\rangle \equiv |final\rangle \leftarrow$$

for general $|\psi\rangle$, not e-state of R

P("pointer"=α_r)= $|\langle r|\psi\rangle|^2$ $\leftarrow$identical to $P(r)$ for obs R before interacting with apparatus

No need of "collapse"

After the interaction between the object and the apparatus, the final state operator for the combiried system:

$$\rho_{final} = |final\rangle\langle final| = \sum_r \sum_{r'} \langle r|\psi\rangle\langle\psi|r'\rangle|\phi_r, \alpha_r\rangle\langle\phi_{r'}, \alpha_{r'}$$

If "collapse" accepted:

$$\rho_{mixed} = \sum_r |\langle r|\psi\rangle|^2 |\phi_r, \alpha_r\rangle\langle\phi_r, \alpha_r|$$ ← correct answer for P("pointer"= α_r), BUT

$$\langle M\rangle = Tr(\rho_{final}M) = \sum_{rr'} \langle r|\psi\rangle\langle\psi|r'\rangle\langle\phi_{r'}, \alpha_{r'}|M|\phi_r, \alpha_r\rangle$$

$$\langle M\rangle = Tr(\rho_{mix}M) = \sum_r |\langle r|\psi\rangle|^2 \langle\phi_r, \alpha_r|M|\phi_r, \alpha_r\rangle$$

Will not agree if $M \neq I \otimes M_2$ or $M \neq M_1 \otimes I$ and if $|\phi_r\rangle$ are not orthogonal

Joint Probability Distribution

Extention of the formalism

Goal: construct $P(q,p;\psi)$ for any $|\psi\rangle$ s.t.

(1) $\int P(q,p;\psi)dp \equiv P(q;\psi) = |\langle\psi|q\rangle|^2$

(2) $\int P(q,p;\psi)dq \equiv P(p;\psi) = |\langle\psi|p\rangle|^2$

(3) $P(q,p;\psi) \geq 0$

Turns out (3) is not satisfied for all quantum states, but even in those cases (1) and (2) hold

A method for constracting a single observable pdf: characteristic function for arbitrary obs A:

$$M(\xi;\psi) = \langle \exp(i\xi A)\rangle = \sum_{n=0}^{\infty} \frac{(i\xi)^n}{n!} \langle A^n \rangle$$

$$M(\xi;\psi) = \int \exp(i\xi A) P(A;\psi) dA$$

By QM: $\langle A^n \rangle = \langle \psi | \hat{A}^n | \psi \rangle$

$$\implies P(A;\psi) = (1/2\pi) \int \exp(-i\xi A) M(\xi;\psi) d\xi$$

Might try

$$M(\theta,\lambda;\psi) = \langle \exp(i(\theta q + \lambda p))\rangle = \sum_{n,m=0}^{\infty} \frac{(i\theta)^n (i\lambda)^m}{n!m!} \langle q^n p^m \rangle$$

$$\implies P(q,p;\psi) = (1/2\pi)^2 \iint \exp(-i(\theta q + \lambda p)) M(\theta,\lambda;\psi) d\theta d\lambda$$

no unique way to take products of noncommuting operators $q^n p^m$ and none are satisfactory, because $P(q,p;\psi)$ takes negative values for some cases. Different approaches even introduced complex valued pdf-s. All failed. Its OK, because

QM has nothing to say about simultaneous measurements of non-commuting observables. ONLY for single obs, or COMMUTING set of obs

Q: Does QM satisfy the axioms of Probability Theory?

Probability Theory in QM

AmJPhys 54(10), 883 (1986)

The axioms

1. $0 \leq P(A|B) \leq 1$

2. $P(A|A) = 1$

3. $P(\tilde{A}|B) = 1 - P(A|B)$

4. $P(A,B|C) = P(A|C)P(B|A,C)$

Limit frequency interpretation: n repetitions, m times A occurs, $C = \{A, \tilde{A}\}$

$$P(A|C) = \lim_{n\to\infty}(m/n) \longleftarrow \exists ?$$

Propensity interpretation: $P(A|C)$ is a measure of tendency or propensity of physical conditions C to produce result A

A ="particle's position lies in $[q_1, q_2]$ interval"

C_q ="config of position measuring device" $\rightarrow P(A|C_q)$

No occasion to consider $P(A, B|C)$ if $AB - BA \neq 0$

Propensity ← LOLN → Limit Frequency

$$P(|f - p| \geq \epsilon | E^n) \leq \frac{1}{n\epsilon^2} \langle (K_i - p)^2 \rangle$$

where $p = P(A|E), \;\; f = m/n = (1/n) \sum K_i$

Probability in QM

$$P(R = r_n|\psi) = |\langle r_n|\psi\rangle|^2$$

ψ = state prep procedure, $|\psi\rangle$ = math obj to calc pdf of obs

1. $0 \leq P \leq 1 \longleftarrow |\langle r_n|\psi\rangle|^2 \leq \langle r_n|r_n\rangle\langle\psi|\psi\rangle$

2. $P(R = r_n|r_n) = 1$

3. $P(R \neq r_n|\psi) = 1 - \sum_{r_n} |\langle r_n|\psi\rangle|^2$

4. particular case:
$|\psi\rangle = |\psi\rangle_a|\psi\rangle_b \Longrightarrow P(R_a = r_m, R_b = r_n|\psi) = |\langle r_m|\psi\rangle_a|^2|\langle r_n|\psi\rangle_b|^2$

Born rule satisfies all 4 axioms

axiom 4. for general case $P(A, B|\rho) = P(A|\rho)P(B|A, \rho)$

By QM: $P(A, B|\rho) = Tr\left[\rho \Pi_A(\Delta_a)\Pi_B(\Delta_b)\right], \quad [A, B] \neq 0$

where $\Pi_A(\Delta_a) = \sum_{a \in \Delta} |a\rangle\langle a|$

$P(A|\rho) = Tr[\rho \Pi_A(\Delta_a)]$

$P(B|A, \rho)$: system has been subjected to filtering that ensures e-values of $A \in \Delta_a$

$$\Longrightarrow \rho' = \frac{\Pi_A(\Delta_a)\rho\Pi_A(\Delta_a)}{Tr[\Pi_A(\Delta_a)\rho\Pi_A(\Delta_a)]}$$

for RHS of $P(A, B|\rho) = P(A|\rho)P(B|A, \rho)$:

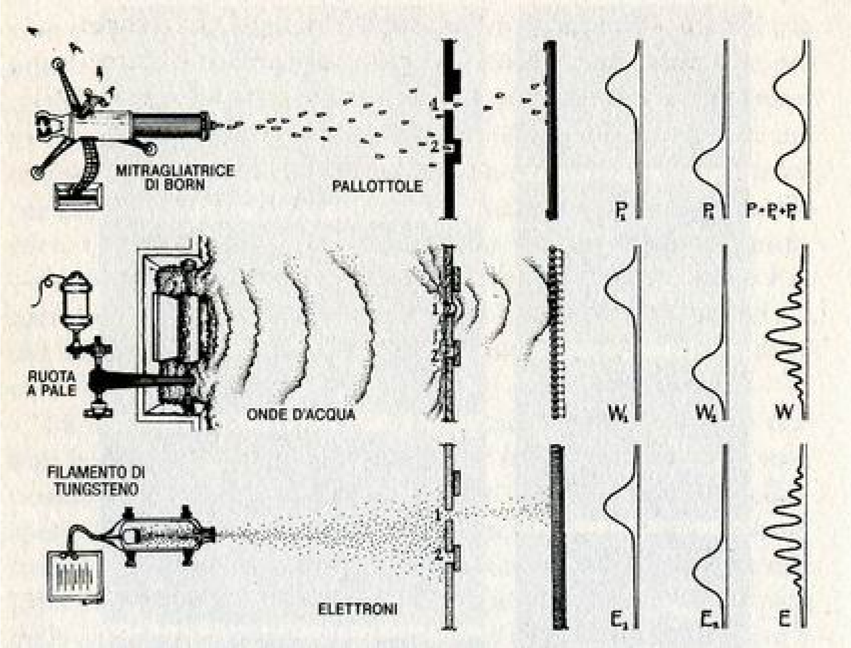
$$P(A|\rho)P(B|A,\rho) = Tr[\rho' \Pi_B(\Delta_b)]Tr[\rho \Pi_A(\Delta_a)]$$

$$P(A|\rho)P(B|A,\rho) = Tr[\Pi_A(\Delta_a)\rho \Pi_A(\Delta_a)\Pi_B(\Delta_b)]$$

$P(A|\rho)P(B|A,\rho) = Tr[\rho \Pi_A(\Delta_a)\Pi_B(\Delta_b)] = P(A,B|\rho)$ only when $[\Pi_A(\Delta_a), \Pi_B(\Delta_b)] = 0$

if $[\Pi_A(\Delta_a), \Pi_B(\Delta_b)] \neq 0$, QM cannot assign meaning to $P(A, B|\rho)$

A. [Double Slit](https://www.youtube.com/watch?v=ToRdROokUhs) Fallacy



As exclusive events, $P_{12}(X) = P_1(X) + P_2(X)$, wrongly concluded Prob Th doesn't hold for QM.

Right way is to condition on state preparation:
$P(X|C_3) \neq P(X|C_1) + P(X|C_2)$. Does not contradict to Prob Th

Superposition Fallacy

As in QM

$$\langle B|A\rangle = \sum_{C'} \langle B|C\rangle\langle C|A\rangle$$

Usually concluded the product-sum rule

$$P(B|A) = \sum_{C'} P(B|C)P(C|A)$$

holds if Q interference terms negligible. Wrong

Correct: start with

$$P(B, C|A) = P(B|C, A)P(C|A)$$

$$\sum_C P(B, C|A) = \sum_C P(B|C, A)P(C|A)$$

$$P(B, C_1|A) + P(B, C_2|A) + \cdots = \sum_C P(B|C, A)P(C|A)$$

$$\sum_C P(B, (C_1 \vee C_2 \vee \cdots)|A) = \sum_C P(B|C, A)P(C|A)$$

$$P(B|A) = \sum_C P(B|C, A)P(C|A)$$

Hidden Variables

If $|\psi\rangle$ not e-vector of obs being measured $\Longrightarrow$ individual event is NOT determined by QM

$\Longrightarrow$ Maybe $\exists$ HVs, not controllable in state prep, s. t. stat distribution of QM are averaged over HVs?

Von Neumann's Thm: No HV Theory is able to reproduce QM predictions (math was OK in the thm, physical conclusion was wrong)

Bohm provides 1st hidden variables theory: PhysRev 85, 166, (1952), PhysRev 85, 180 (1952)

Bell Clarifies Von Neumann's Thm & relation to HV: RevModPhys. 38, 447 (1966)

QM doesn't demand HV, but makes search for them reasonable

Bell's Theorem

No local theory can reproduce QM predictions

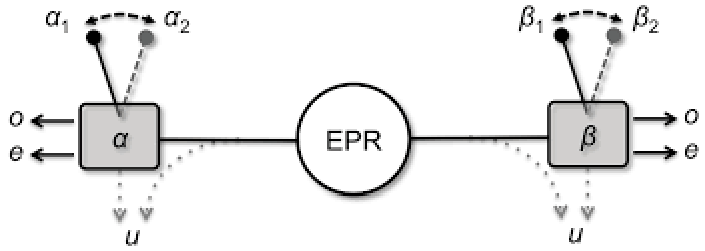
OR: ∃ conflict between QM predictions & Locality

Locality = predictive completeness (wait for it!)

CHSH inequality [Nuovo Cimento 38B(1), 75 (1977)]

Constraint on the statistics of ”coincidences” in a Bell test experiment.

Local theroies must satisfy it. PhysRevLett, 23, 880 (1969).



Natural thought: what happens in α is independent of knob setting of β

j^{th} event = response of α_j and β_j $\in \{\pm 1\}$

Correlation: $C = \frac{1}{N} \sum_j \alpha_j \beta_j \equiv \langle \alpha_j \beta_j \rangle$

Only one setting will be chosen in experiment at a time

Locality assumption:

$\alpha_j = \{\pm 1\}$ whatever knob setting β, $\beta_j = \{\pm 1\}$ whatever knob setting α

def

$$\gamma_j = \alpha_{1j}\beta_{1j} + \alpha_{2j}\beta_{1j} + \alpha_{1j}\beta_{2j} - \alpha_{2j}\beta_{2j}$$

$$\gamma_j = (\alpha_{1j} + \alpha_{2j})\beta_{1j} + (\alpha_{1j} - \alpha_{2j})\beta_{1j} = \text{even integer} \leq 2$$

$$C = \langle \gamma_j \rangle = C^{11} + C^{12} + C^{21} - C^{22} \leq 2$$

QM predition for singlet state

$$|\psi\rangle = \frac{1}{\sqrt(2)} |01\rangle - |10\rangle \Longrightarrow C^{QM}_{singlet} = \cos(2\alpha - a\beta)$$

$$C^{QM}_{singlet} = 2\sqrt(2) \quad \text{for}$$

$$\alpha_1 = \pi/4,\ \alpha_2 = 0,\ \beta_1 = \pi/8,\ \beta_2 = 3\pi/8$$

Bell's Thm: Does QM contradict relativity?

1. Special Relativity demands "no spooky acion at a distance"

2. Locality implies Bell's thm

3. QM violates Bell's inequality

$\Longrightarrow$ QM contradicts relativity! (Does it?)

Simple locality = nonlocal causality (can't transmit info faster than c)

Bell's thm requires stronger locality assumption

stronger locality = simple locality(OK with QM) + predictive completeness(fails for QM)

Back to Spin Correlation Experiment

simple locality:

$$Q_A(x_A|\overrightarrow{n_A}, S_A, S_B^0, \lambda) = \sum_{x_B} P(x_A, x_B|\overrightarrow{n_A}, \overrightarrow{n_B}, S_A, S_B, \lambda)$$

$$Q_B(x_B|\overrightarrow{n_B}, S_B, S_A^0, \lambda) = \sum_{x_A} P(x_A, x_B|\overrightarrow{n_A}, \overrightarrow{n_B}, S_A, S_B, \lambda)$$

λ = state of 2-particle sys ($|\psi\rangle$, or $|\psi\rangle$ "+" HV)

$\overrightarrow{n_A}, \overrightarrow{n_B}$: unit vectors defining the spin-component to be measured

S_A, S_B: premeasurement settings, S^0: device not operating

Proof that QM is OK with simple locality

$$(\overrightarrow{\sigma_A} \cdot \overrightarrow{n_A})|x_A\rangle = x_A|x_A\rangle,\ (\overrightarrow{\sigma_B} \cdot \overrightarrow{n_B})|x_B\rangle = x_B|x_B\rangle$$

$$P(x_A, x_B) = \langle x_A, x_B|\rho_{AB}|x_A, x_B\rangle$$

$$Q_A(x_A) = Tr_B[|x_A\rangle\langle x_A|\rho_{AB}] = Tr[|x_A\rangle\langle x_A|\rho_A] = \langle x_A|\rho_A|x_A\rangle$$

where $\rho_A = Tr_B[\rho_{AB}] = \sum_{x_B}\langle x_B|\rho_{AB}|x_B\rangle$

$$Q_A(x_A) = \sum_{x_B}\langle x_A, x_B|\rho_{AB}|x_A, x_B\rangle = \sum_{x_B} P(x_A, x_B)$$

Strong Locality

$$P(x_A, x_B | \overrightarrow{n_A}, \overrightarrow{n_B}, S_A, S_B, \lambda) = Q_A(x_A | \overrightarrow{n_A}, S_A, S_B^0, \lambda) \cdot Q_B(x_B | \overrightarrow{n_B}, S_B, S_A^0, \lambda)$$

does not hold if $\lambda = |\psi\rangle$ is quantum

For more general λ, observed would be

$$P_{obs}(x_A, x_B) = \int Q_A(x_A | \overrightarrow{n_A}, S_A, S_B^0, \lambda) \cdot Q_B(x_B | \overrightarrow{n_B}, S_B, S_A^0, \lambda) \rho(\lambda) d\lambda$$

No $\rho(\lambda)$ can obey the above & QM predictions (Bell's thm)

Predictive Completeness

$|\psi\rangle$ is predictively complete w.r.t measurement M (post-preparation), if results of non-M (pre-preparation) measurements provide no info relevant to predicting result of M

M = spin-component meas on 1st particle

non-M = meas on 2nd particle at spacelike separation

Predictive completeness to hold $\Rightarrow$

$$P(x_A, x_B | \overrightarrow{n_A}, \overrightarrow{n_B}, S_A, S_B, \lambda) = \sum_{x_B'} P(x_A, x_B' | \overrightarrow{n_A}, \overrightarrow{n_B}, S_A, S_B, \lambda) \cdot$$
$$\sum_{x_A'} P(x_A', x_B | \overrightarrow{n_A}, \overrightarrow{n_B}, S_A, S_B, \lambda)$$

The factoring means absence of predictively relevant info. Each factor MAY depend on BOTH $\overrightarrow{n_A}$ and $\overrightarrow{n_B}$

Strong locality $\equiv$ simple locality + pred completeness

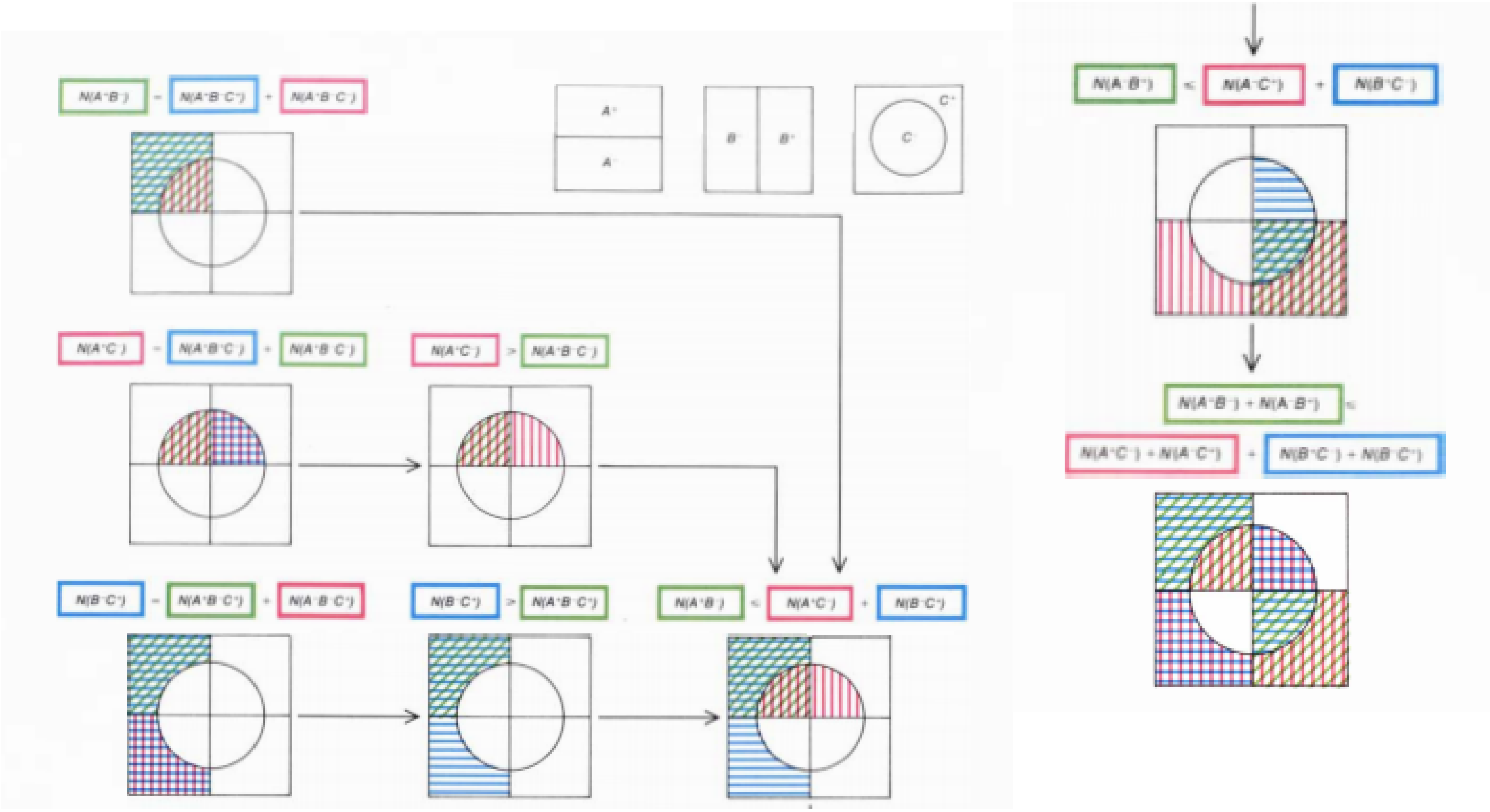
Bell inequality violation (by QM and experiment) is due to failure of the above formula

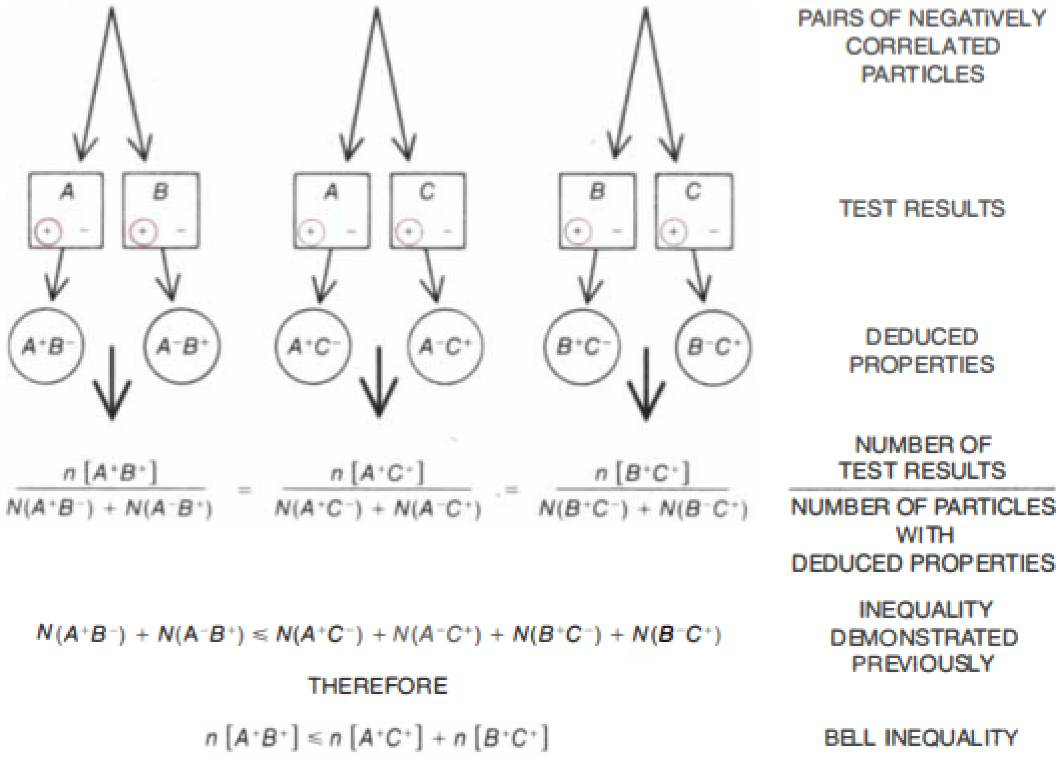
QM & Reality

1. Reality: $\exists$ property even when not measured

2. Inference: can apply freely, draw legit conclusions from obs

3. EPR Locality (predictive completeness)



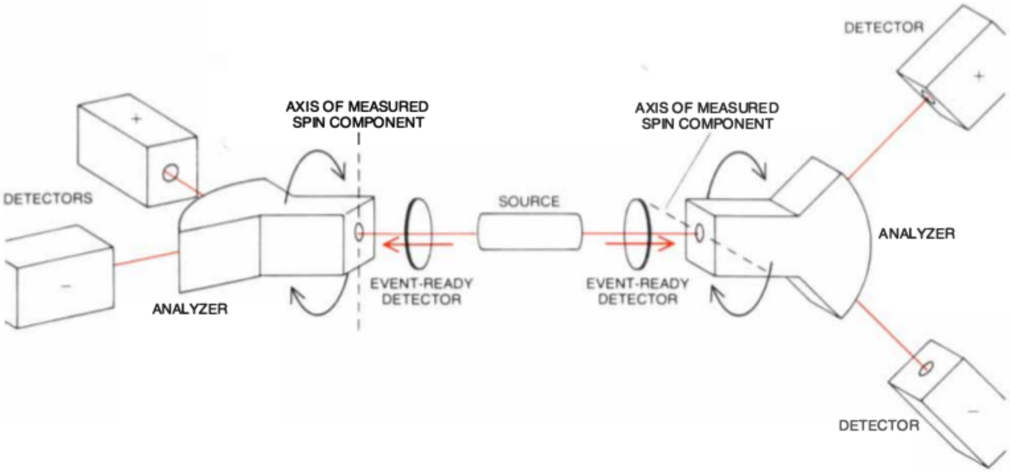


A, B, C represent spin directions (non-commuting)

$N(A^- B^+)$ = number of single particles w. properties A^-, B^+

$N(A^* B^*)$ not possible to measure experimentally .

$n[A^* B^*]$ refers to PAIR of particles: can measure and test


DETECTORS
AXIS OF MEASURED SPIN COMPONENT
ANALYZER
EVENT-READY DETECTOR
SOURCE
AXIS OF MEASURED SPIN COMPONENT
EVENT-READY DETECTOR
DETECTOR
ANALYZER
DETECTOR

Kochen-Specker Theorem

QM states imply statistical restrictions on the results of measurements

QM would be incomplete in the sense that a typical state description of an individual system could be supplemented in terms of HVs

Bell's thm: given locality - an HV model can't match stat predictions of QM

KS thm: given noncontextuality - an HV model can't match stat predictions of QM

these new contradictions between QM and HV don't depend on a particular state, therefore are free from stat inferences. Only operator algebra is needed

KS: given a premise of noncontextuality (NC), certain sets of QM observables cannot consistently be assigned values at all (even before the question of their statistical distributions arises)

NC: If a QM system possesses a property (value of an observable), then it does so independently of any measurement context, i.e. independently of HOW that value is eventually measured.

VD: All observables defined for a QM system have definite values at all times.

O: There is a one-one correspondence between properties of a q system and projection operators on the system's Hilbert space

KS says $\exists$ contradiction between VD + NC + O and QM

Statement of the KS Theorem

Let H be of dimension $x \geq 3$. $\exists\{y_1, \cdots, y_M\}$ s.t. (KS1) & (KS2) are contradictory:

(KS1) All $y_1, \cdots, y_M$ simultaneously have values, $v(y_1), \cdots, v(y_M)$

(KS2) Values of all observables conform to the following constraints:

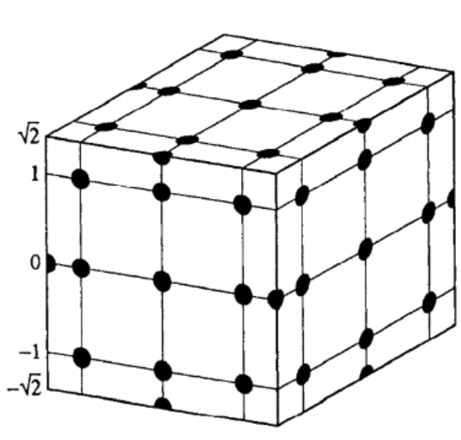
(a) If y_i, y_j, y_k are all compatible and $y_k = y_i + y_j, \rightarrow v(y_k) = v(y_i) + v(y_j)$

(b) if y_i, y_j, y_k are all compatible and $y_k = y_i \cdot y_j, \rightarrow v(y_k) = v(y_i) \cdot v(y_j)$

Proof of KS Thm [Peres, JPhys, A 24, L175 (1991)]

33 vectors (rays), invariant under interchange of x,y,z and reversal of each axis, 16 bases (triads)

for each triad can associate value 1 (GREEN) to one of rays, and 0 (RED) to other two



Ortho Triad	Other Rays	1st green because of
$\mathbf{001}$ 100 010	110 $1\bar{1}0$	choice of z axis
$\mathbf{101}$ $\bar{1}01$ $\mathit{010}$	$\cdots$ $\cdots$	choice of x vs -x
$\mathbf{011}$ $0\bar{1}1$ $\mathit{100}$	$\cdots$ $\cdots$	choice of y vs -y
$\mathbf{1\bar{1}2}$ $\bar{1}12$ $\mathit{110}$	201 021	choice of x vs y
$\mathbf{102}$ $\mathit{\bar{2}01}$ $\mathit{010}$	$\bar{2}11$ $\cdots$	ortho to 2nd and 3rd
$\mathbf{211}$ $\mathit{0\bar{1}1}$ $\mathit{\bar{2}11}$	$\bar{1}02$ $\cdots$	ortho to 2nd and 3rd
$\mathbf{201}$ $\mathit{010}$ $\mathit{\bar{1}02}$	$\bar{1}\bar{1}2$ $\cdots$	ortho to 2nd and 3rd
$\mathbf{112}$ $\mathit{1\bar{1}0}$ $\mathit{\bar{1}\bar{1}2}$	$0\bar{2}1$ $\cdots$	ortho to 2nd and 3rd
$\mathbf{012}$ $\mathit{100}$ $\mathit{0\bar{2}1}$	$1\bar{2}1$ $\cdots$	ortho to 2nd and 3rd
$\mathbf{121}$ $\mathit{\bar{1}01}$ $\mathit{1\bar{2}1}$	$0\bar{1}2$ $\cdots$	ortho to 2nd and 3rd

KS is a geometrical statement. It's independent of QM but profoundly affects the interpretation of it

KS (contrary to Bell) doesn't involve stat correlations in ensemble of systems

KS compares the results of various measurements that can be performed on a single system

no need of taking averages over unspecified HVs

or over fictitious experimental runs (as in CHSH case)

The problem cannot be of pure logic

Empirical premises inderlying KS thm

9 premises

7 phenomenological, can be tested experimentally, can be derived from QM

2 counterfactual: possible to imagine the reulst of unperformed experiments (such a way that results have correlations mimicking experiments)

1. Elemetary tests

∃ "yes-no" tests: projectors A, B, C, ... with outcomes a, b, c ... =1 (yes), =0 (no)

E.g.

$$P_m = I - (\vec{\mathbf{m}} \cdot \vec{\mathbf{J}})^2 \qquad \longleftarrow 3x3, Tr(P_m) = 1$$

$$P_\theta = \frac{1}{2}[I + \sigma_x \sin(\theta) + \sigma_z \cos(\theta)] \qquad \longleftarrow 2x2, Tr(P_\theta) = 1$$

2. State preparation

∀ "yes-no" exp, ∃ ways of preparing systems, s.t. outcome is predictible with probability 1

predictable with prob=1 iff

$$Tr(\rho P) = 0 \ \ or \ \ 1$$

3. Compatibility of ”yes-no” experiments

prepare system in a way that result of A is predictable & repeatable

perform compatible B test ([A,B]=0)

subsequent execution of A shall yield the same result (as if B hasn’d been performed)

two different spin 1/2

$$[1/2(I + \vec{\mathbf{m}} \cdot \overrightarrow{\sigma_1}), 1/2(I + \vec{\mathbf{n}} \cdot \overrightarrow{\sigma_2})] = 0 \qquad \longleftarrow \; \forall \vec{\mathbf{m}}, \vec{\mathbf{n}}$$

single spin 1

$$[I - (\vec{\mathbf{m}} \cdot \vec{\mathbf{J}})^2, I - (\vec{\mathbf{n}} \cdot \vec{\mathbf{J}})^2] = 0 \qquad \longleftarrow \; \forall \vec{\mathbf{m}} \cdot \vec{\mathbf{n}} = 0$$

4. Symmetry and transitivity

Compatibility is a symmetric property but it's not transitive

if $[A, B] = [A, C] = 0$ it doesn't follow that $[B, C] = 0$

E.g.

$$\vec{\mathbf{m}} \cdot \vec{\mathbf{n}} = \vec{\mathbf{m}} \cdot \vec{\mathbf{r}} \neq \vec{\mathbf{n}} \cdot \vec{\mathbf{r}}$$

$$A = (\vec{\mathbf{m}} \cdot \vec{\mathbf{J}})^2, \quad B = (\vec{\mathbf{n}} \cdot \vec{\mathbf{J}})^2, \quad C = (\vec{\mathbf{r}} \cdot \vec{\mathbf{J}})^2$$

$$A = \vec{\mathbf{m}} \cdot \overrightarrow{\sigma_1}, \quad B = \vec{\mathbf{n}} \cdot \overrightarrow{\sigma_2}, \quad C = \vec{\mathbf{r}} \cdot \overrightarrow{\sigma_2}$$

5. Constraints

When state prep is such that compatible tests have predictable results → results may be constrained

E.g.

$$P_m = \frac{1}{2}(I + \vec{\mathbf{m}} \cdot \vec{\sigma})$$

has predictable outcome p_m, then $p_m + p_{-m} = 1$

E.g. $\vec{\mathbf{m}}, \vec{\mathbf{n}}, \vec{\mathbf{r}}$ are orthogonal triad

$$P_m = I - (\vec{\mathbf{m}} \cdot \vec{J})^2, \;\; P_n = I - (\vec{\mathbf{n}} \cdot \vec{J})^2, \;\; P_r = I - (\vec{\mathbf{r}} \cdot \vec{J})^2$$

then

$$P_m + P_n + P_r = I$$

QM predicts (also experimentally) for predictable tests, the outcomes satisfy $p_m + p_n + p_r = 1$, one test is positive, others are negative

6. Further constraints

Even if outcomes of constrained tests are NOT predictable, there will be proper constraints, if the tests are actually performed. Outcomes aren't completely random (microscopic determinism?)

For a complete set of orthogonal projectors:

$$P_i P_j = P_j P_i = \delta_{ij} P_i, \quad \sum P_i = I$$

then for $p_i = \pm 1$

$$\left\langle \left(\sum p_i \right)^2 \right\rangle - \left\langle \left(\sum p_i \right) \right\rangle^2 = \left\langle \left(\sum P_i \right)^2 \right\rangle - \left\langle \left(\sum P_i \right) \right\rangle^2 \equiv 0$$

so that $\sum p_i = 1$

7. Correlations

If outcomes of compatible tests are neither predictable nor constrained, there may still be statistical correlations between these outcomes

E.g.

$$P_{1\alpha} = \frac{1}{2}(I + \vec{\alpha} \cdot \vec{\sigma_1}) \quad P_{2\beta} = \frac{1}{2}(I + \vec{\beta} \cdot \vec{\sigma_2})$$

$$\Longrightarrow \quad \langle p_{1\alpha} p_{1\beta} \rangle = \frac{1}{2}(1 - \vec{\alpha} \cdot \vec{\beta})$$

which already violates Bell's inequality

8. Counterfactual realism

a) Possible to imagine hypothetical results for any unperformed test, and to do coalculations where these unknown results are treated as if they were definite nunbers

⇕ purely intelectual activity

b) Possible to imagine the results of any set of compatible tests, and to treat them in calculations as they had definite values, satisfying the same constraints or correlations that are imposed on the results of real tests

(example in next slide)

E.g. 2 orthogonal triads

$$\vec{\mathbf{m}} \cdot \vec{\mathbf{n}} = \vec{\mathbf{m}} \cdot \vec{\mathbf{r}} = \vec{\mathbf{n}} \cdot \vec{\mathbf{r}} = 0, \quad \vec{\mathbf{m}} \cdot \vec{\mathbf{s}} = \vec{\mathbf{m}} \cdot \vec{\mathbf{t}} = \vec{\mathbf{s}} \cdot \vec{\mathbf{t}} = 0, \quad \vec{\mathbf{n}} \cdot \vec{\mathbf{s}} \neq 0$$

can imagine $P_m = I - (\vec{\mathbf{m}} \cdot \vec{\mathbf{J}})^2$ is measures either with P_n, P_r or P_s, P_t

then results must obey

$$p_m + p_n + p_r = 1, \quad p_m + p_s + p_t = 1$$

experimentally can be realized only one at a time. Are the p_m's the same?

Can't be tested experimentally (mutually exclusive setups) $\Longrightarrow$ need one more postulate

9. Counterfactual compatibility

No-contextuality hypothesis

The hypothetical result of an unperformed test doesn’t depend on the choice of compatible tests tha may be perfomeed together woth it

Summary of KS

Although 9. cannot be tested directly, some of its logical consequences can be shown to conflict with QM and experiment

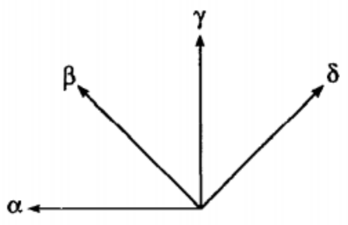
E.g.

$$p_{1\gamma} + p_{2\beta} + p_{1\alpha}p_{2\delta} - p_{1\alpha}p_{2\beta} - p_{1\gamma}p_{2\beta} - p_{1\gamma}p_{2\delta} = 0 \quad or \quad 1$$

$\Longrightarrow$ average of LHS ought to be in range [0,1]

for singlet state

$$\tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{4}\left(1 + \frac{1}{\sqrt{(2)}}\right) - \tfrac{3}{4}\left(1 - \frac{1}{\sqrt{(2)}}\right) = \tfrac{1}{2} + \frac{1}{\sqrt{(2)}} > 1$$



Rationale of contextuality: A test Is $\vec{\alpha} \cdot \vec{\sigma_1} = 1?$ or Is $\vec{m} \cdot \vec{J} = 0?$ has a well defined answer only if state prep satisfies $Tr(\rho P) = 0 \; or \; 1$, so that the required answer is predictable

For any other state prep, these questions, which are represented by degenerate operators , are ambiguous. The answer depends on which other (compatible) tests are performed

The same operator may correspond to different observables. The symbol P_m has different meaning if measured alone or with P_n or P_r