Heaps and Priority Queues

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(edits by M. Hudachek-Buswell)

February 5, 2017

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- This can be trivially done with a list, but insertion will be $O(n)$ in the worst case.
- Instead of using a list, a data structure known as a *binary heap*, can be used.

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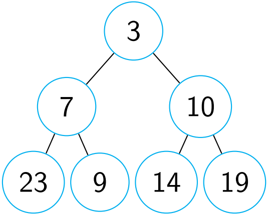
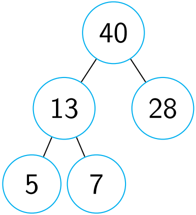
- A heap where the root is the largest item is called a maxheap.
- A heap where the root is the smallest item is called a minheap.
- Heaps can also be represented as an array.

Example Heaps



minheap



maxheap

Properties of a Heap

Heaps have two properties:

Order Property In a heap, the parent node is always larger (in the case of a maxheap) or smaller (in the case of a minheap) than its children. There is no relationship between teh children.

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In addition, only the root (the largest or the smallest) item of a heap can be accessed; you cannot search through a heap.

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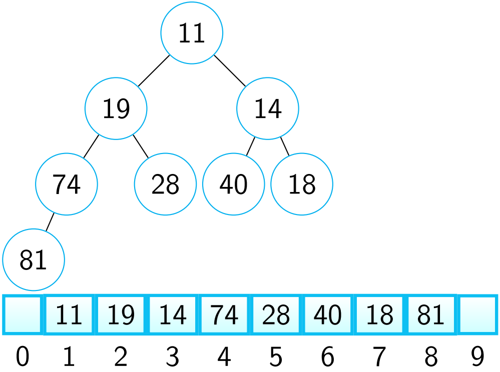
- The left child of an item in index i would then be at index $2i$, and the right child would be at index $2i + 1$.
- The parent of an item in index i would be at $\frac{i}{2}$ (this is assuming you are doing integer division).

Heap as an Array



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- Compare the newly added item to its the parent. If the order property is broken (i.e. the parent is larger than the child, but it is supposed to be a minheap, or the parent is smaller than the child, but it is supposed to be a maxheap), swap the parent and child. This is referred to as *heapify*, and the algorithm is recursive.

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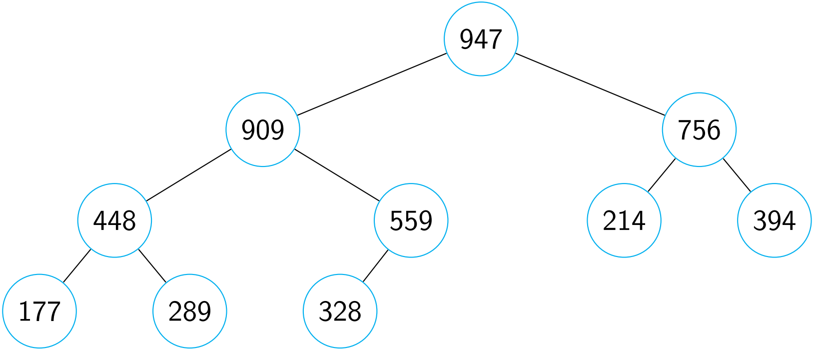
- Repeat the previous step until: 1) you don't make a swap, or 2) you reach the root of the heap.

Adding

Example: Adding 810 into the heap below. Which heap is it? What is the size of the heap? Where do we add the new node? What index?



(Maxheap of size 10. Add to the right of 559, index 11)

Adding

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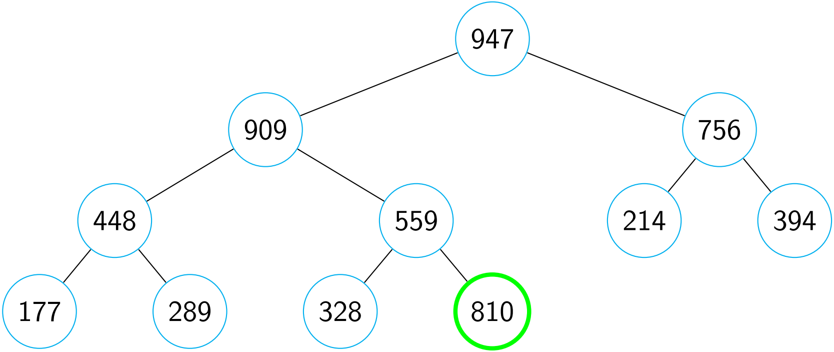

Size of the heap is 11. The order property is violated for parent node of 810.

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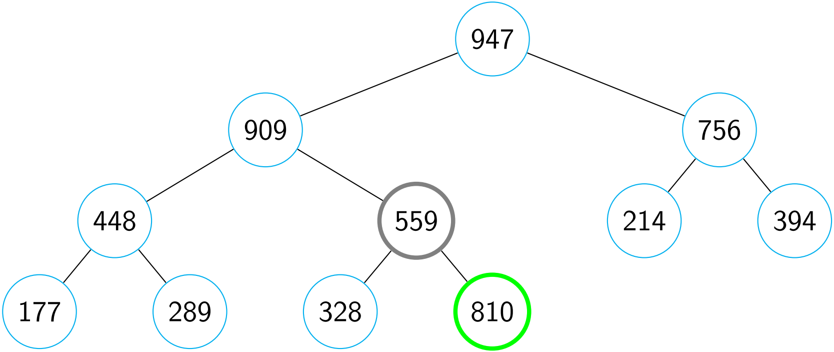

Need to restore order by heapifying parent and child.

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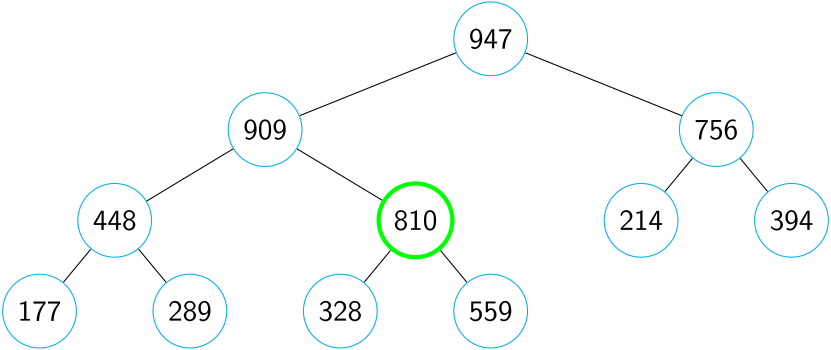

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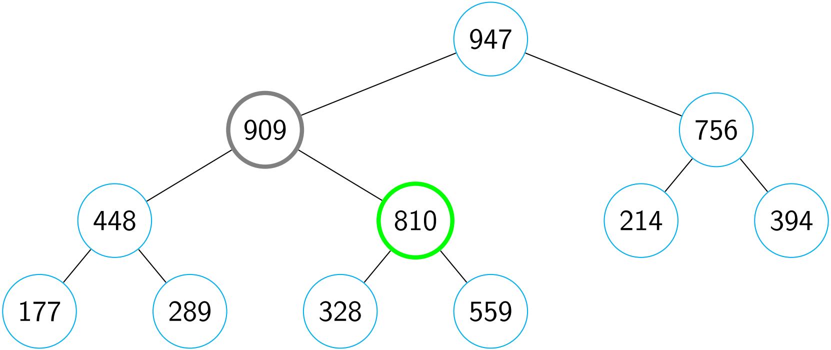

Now check order property again.

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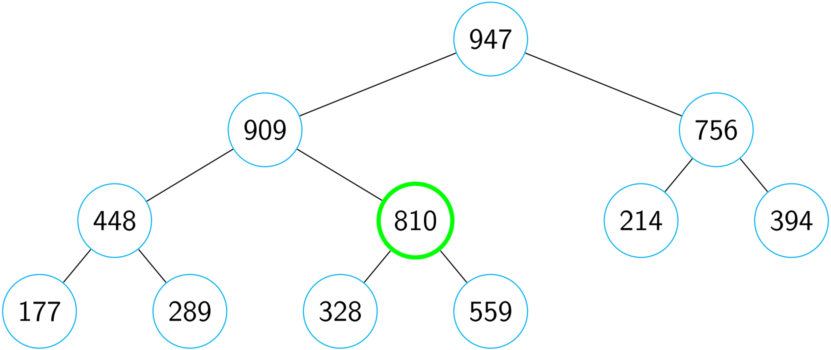
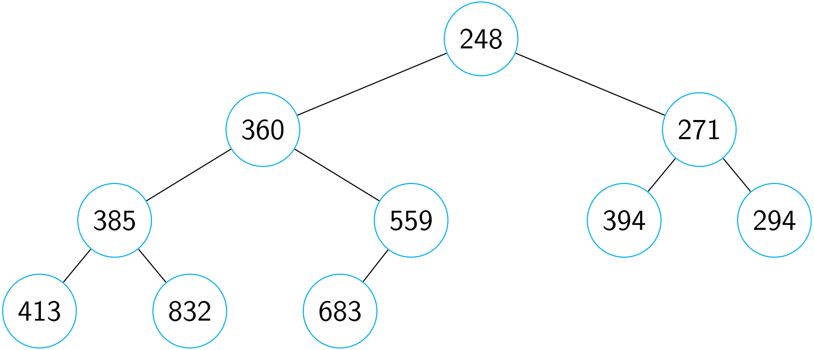


Order property is intact so the add is complete.

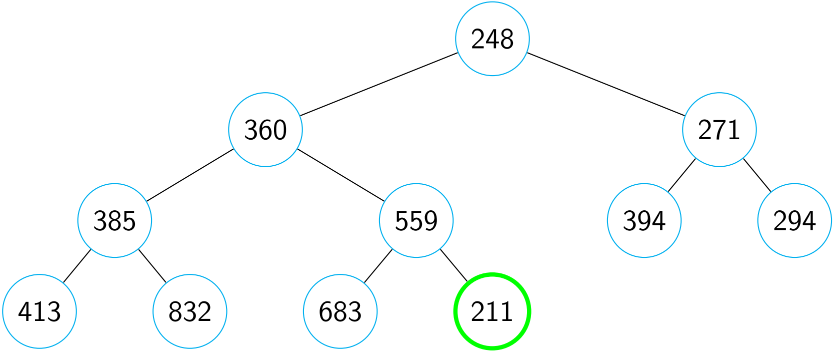
Adding

Example: Adding 211 into the minheap



Adding

Example: Adding 211 into the minheap



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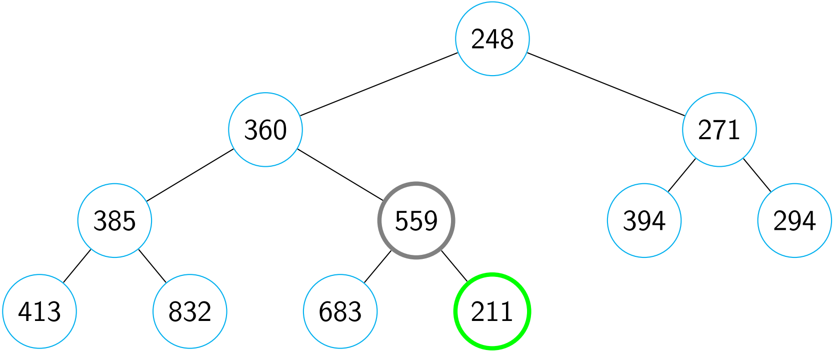

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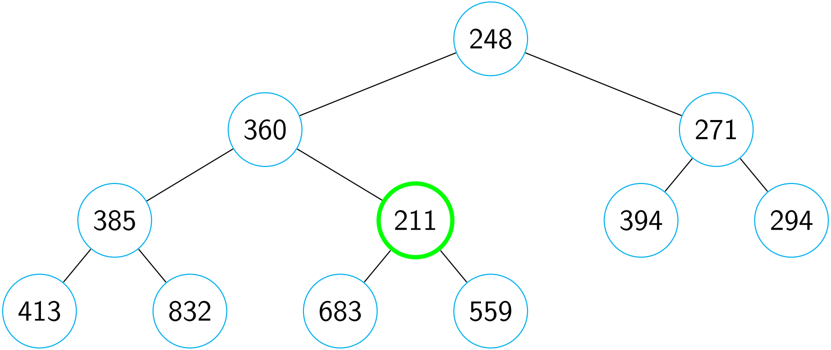

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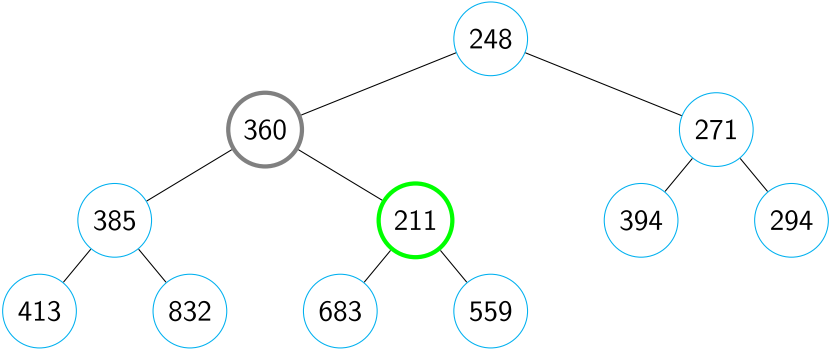

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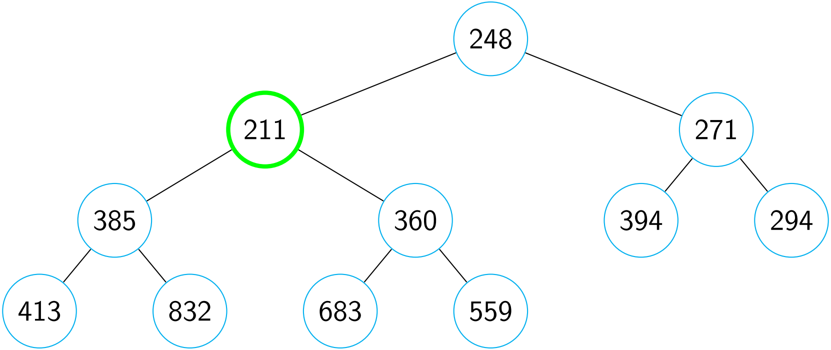

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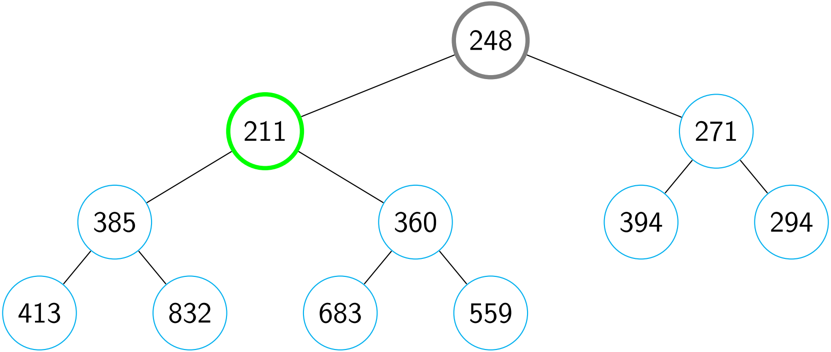

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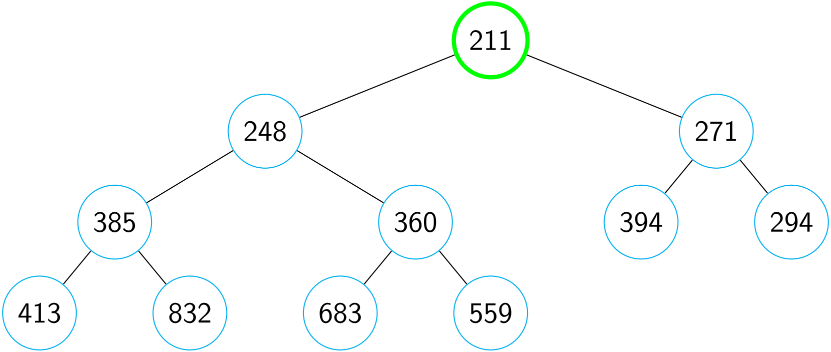

Adding

procedure ADD(*data*)
    index $\leftarrow$ next empty slot in the heap
    heap[*index*] $\leftarrow$ *data*
    parentIndex $\leftarrow$ *index*/2
    while *parentIndex* > 1, and *heap*[*parentIndex*] and *heap*[*index*] are not in the correct order **do**
        Swap *heap*[*parentIndex*] and *heap*[*index*]
        index $\leftarrow$ *parentIndex*
        parentIndex $\leftarrow$ *parentIndex*/2
    end while
end procedure

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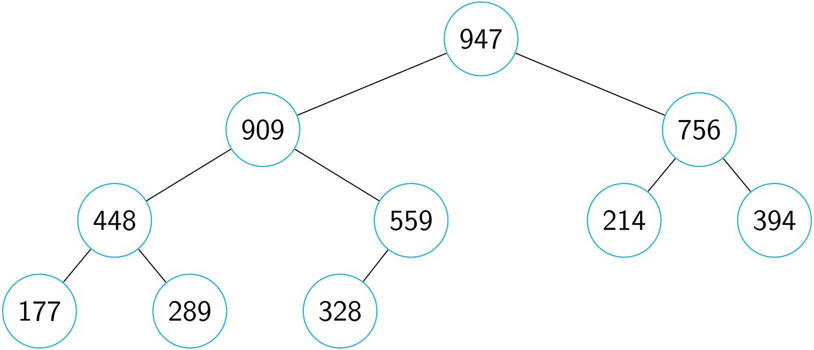
- Repeat the previous step until you don't make a swap or you reach the bottom of the heap.

Removing

Example: Removing 947 from the *max*heap



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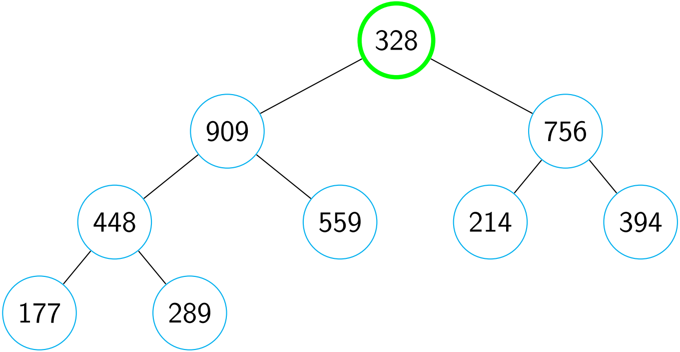

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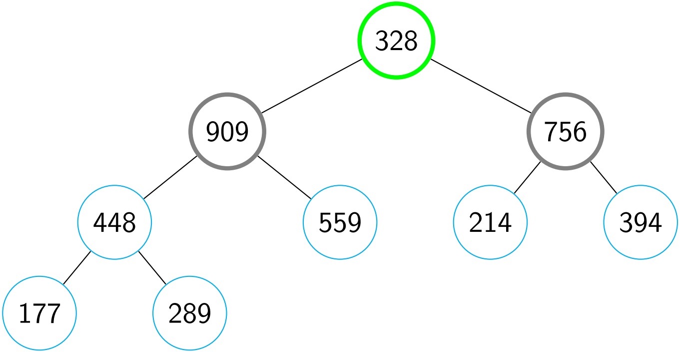

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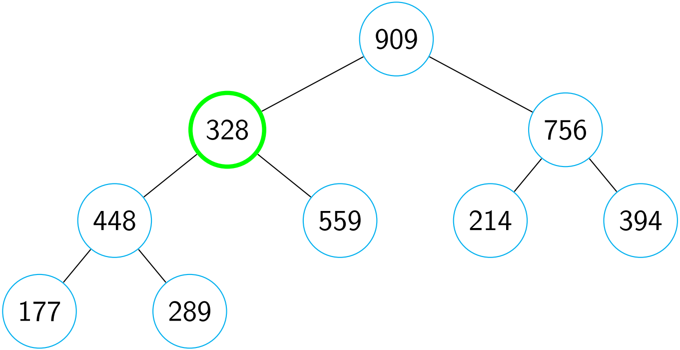

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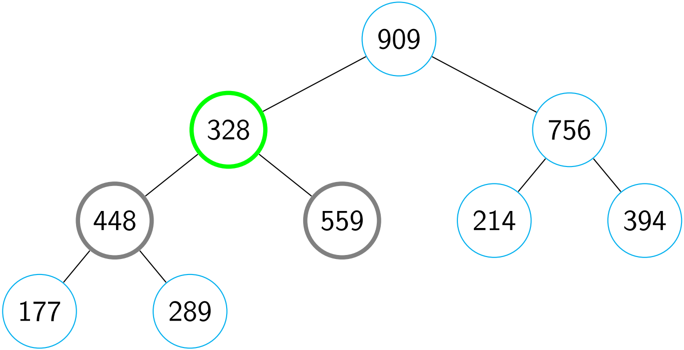

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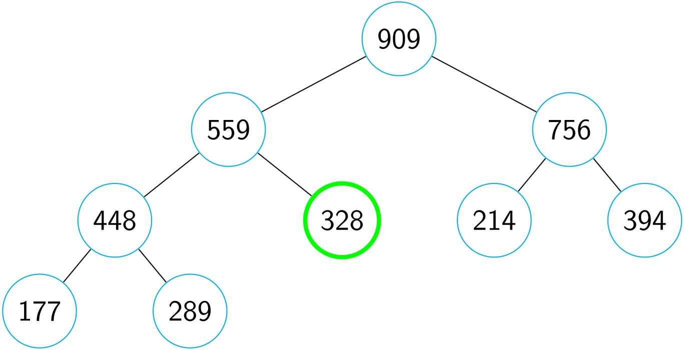

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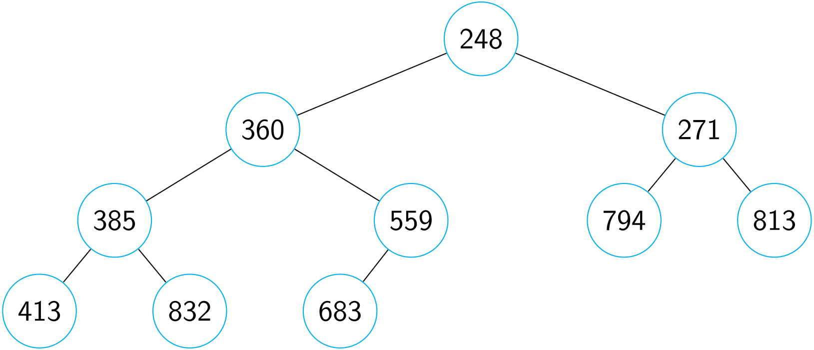
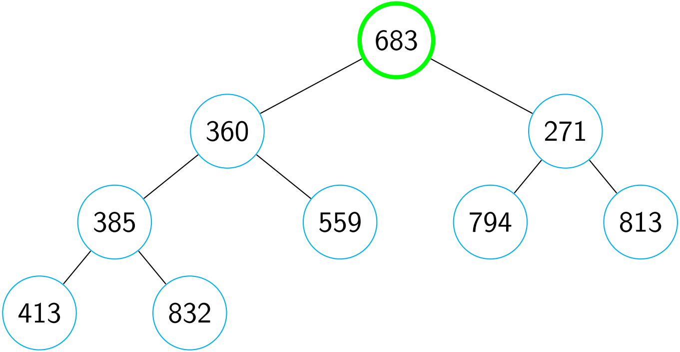
Example: Remove 248 from the minheap



Removing

Example: Remove 248 from the minheap



Removing

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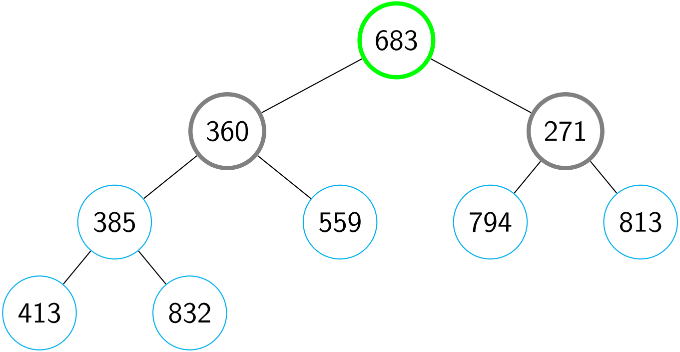

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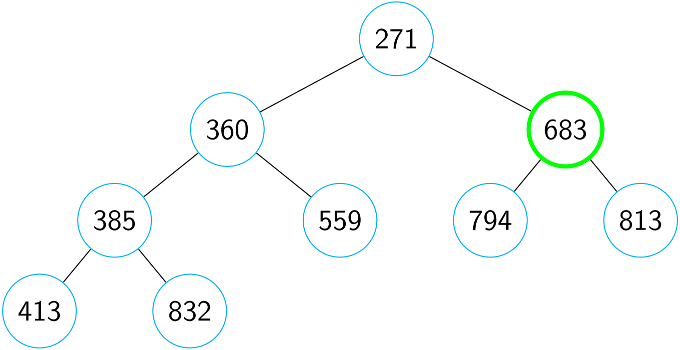

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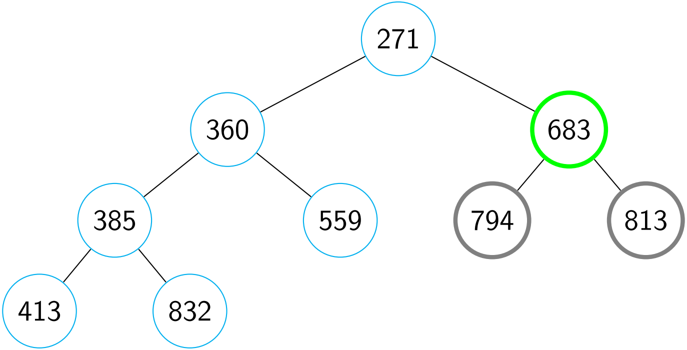

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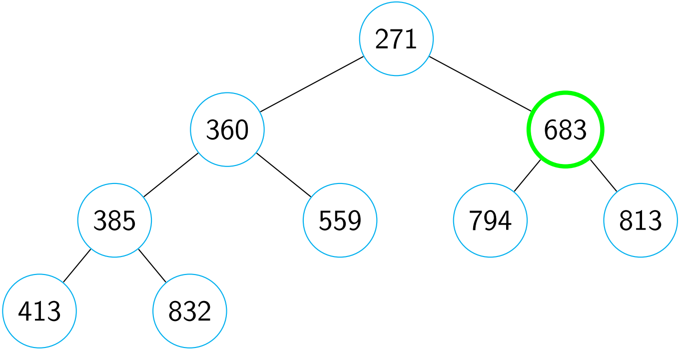

Removing

procedure REMOVING
    index $\leftarrow$ last filled slot in the heap
    data $\leftarrow$ *heap*[1]
    heap[1] $\leftarrow$ *heap*[*index*]
    heap[*index*] $\leftarrow$ `NULL`
    index $\leftarrow$ 1
    while *heap*[*index*] has a child, and *heap*[*index*] and its left and right children are not in the correct order **do**
        Swap the largest/smallest child with *heap*[*index*]
        index $\leftarrow$ index of the largest/smallest child
    end while
    return *data*
end procedure

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- In the best case, however, adding to a heap could be $O(1)$ if you add the items in ascending order (for a minheap) or descending order (for a maxheap), because the item you just added will never get promoted.
- Removing from a heap will be $O(\log n)$ because the item that gets moved to the root may have to go down $\log n$ levels.

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- Priority queues may be used in printer jobs, CPU schedulers, flight boarding, bandwidth management in routers, and as an auxillary data structure in other algorithms.
- Priority queues can be efficiently implemented using a heap.