CS 1332

DATA STRUCTURES AND ALGORITHMS

Programming in Java

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DYNAMIC PROGRAMMING

Chapter 13

Dynamic Programming (DP)

A powerful algorithm technique used for solving a complex problem by breaking it down into a collection of simpler recurrent subproblems, solving each of those subproblems just once, and storing their solutions.

(This does not mean we will have to use recursion.)

Dynamic Programming (DP)

This is considered a Bottom-Up approach:

- Identify smaller subproblems
- Order of solving subproblems
- Make sure subproblems are solved before the larger problem
- The subproblems overlap or are NOT interdependent

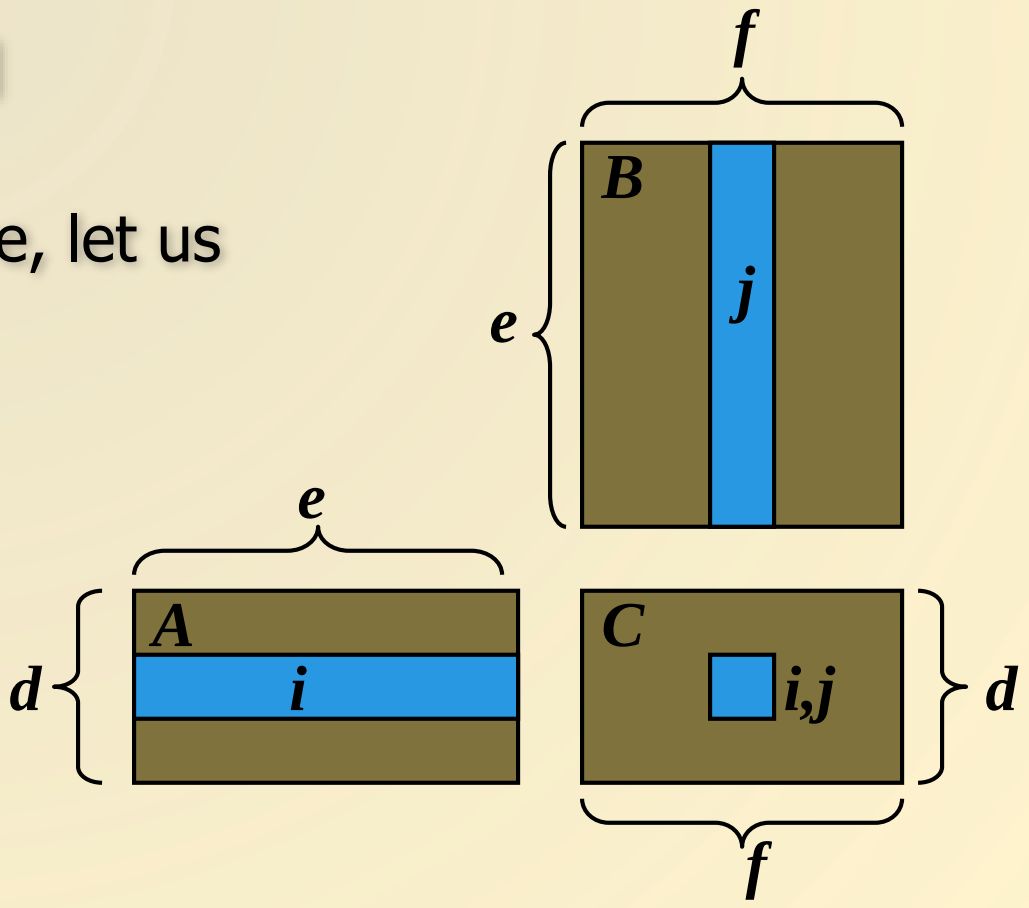
Matrix multiplication is prime example of where we can apply DP

DP for Matrix Chain-Products

- Dynamic Programming is a general algorithm design paradigm.
  - Rather than give the general structure, let us first give a motivating example:
  - **Matrix Chain-Products**
- Review: Matrix Multiplication.
  - $C = A*B$
  - A is $d \times e$ and B is $e \times f$

$$C[i, j] = \sum_{k=0}^{e-1} A[i,k] * B[k, j]$$

  - $O(def)$ time



Matrix Chain-Products

- **Matrix Chain-Product:**
  - Compute $A=A_0*A_1*\ldots*A_{n-1}$
  - A_i is $d_i \times d_{i+1}$
  - Problem: How to parenthesize?
- Example
  - B is 3×100
  - C is 100×5
  - D is 5×5
  - (B*C)*D takes 1500 + 75 = 1575 ops
  - B*(C*D) takes 1500 + 2500 = 4000 ops

A Characterizing Equation

- The global optimal has to be defined in terms of optimal subproblems, depending on where the final multiply is at.
- Let us consider all possible places for that final multiply:
  - Recall that A_i is a $d_i \times d_{i+1}$ dimensional matrix.
  - So, a characterizing equation for $N_{i,j}$ is the following:

$$N_{i,j} = \min_{i \le k < j} \{ N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1} \}$$

A Dynamic Programming Algorithm

- Since subproblems overlap, we don't use recursion.
- Instead, we construct optimal subproblems "bottom-up."
- $N_{i,i}$'s are easy, so start with them
- Then do length 2,3,… subproblems, and so on.
- The running time is $O(n^3)$

Algorithm *matrixChain*(*S*):

Input: sequence ***S*** of ***n*** matrices to be multiplied

Output: number of operations in an optimal parenthesization of ***S***

for $i \leftarrow 1$ **to** $n-1$ **do**

  $N_{i,i} \leftarrow 0$

for $b \leftarrow 1$ **to** $n-1$ **do**

  for $i \leftarrow 0$ **to** $n-b-1$ **do**

    $j \leftarrow i+b$

    $N_{i,j} \leftarrow +\infty$

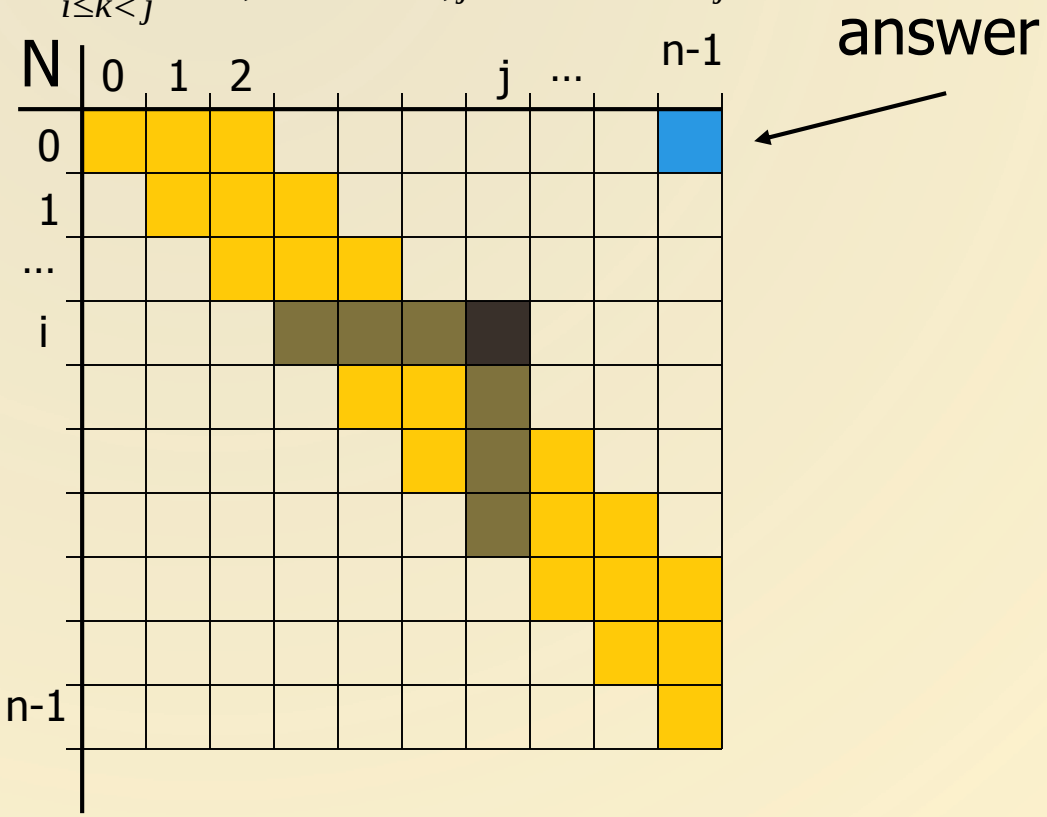
    for $k \leftarrow i$ **to** $j-1$ **do**

      $N_{i,j} \leftarrow \min\{N_{i,j}\ ,\ N_{i,k} + N_{k+1,j} + d_i\, d_{k+1}\, d_{j+1}\}$

DP Algorithm Visualization

$$N_{i,j} = \min_{i \le k < j} \{ N_{i,k} + N_{k+1,j} + d_i d_{k+1} d_{j+1} \}$$

- The bottom-up construction fills in the N array by diagonals
- $N_{i,j}$ gets values from pervious entries in i-th row and j-th column
- Filling in each entry in the N table takes O(n) time.
- Total run time: $O(n^3)$
- Getting actual parenthesization can be done by remembering “k” for each N entry



The DP Technique

- Applies to a problem that at first seems to require a lot of time (possibly exponential), provided we have:
  - **Simple subproblems:** the subproblems can be defined in terms of a few variables, such as j, k, l, m, and so on.
  - **Subproblem optimality:** the global optimum value can be defined in terms of optimal subproblems
  - **Subproblem overlap:** the subproblems are not independent, but instead they overlap (hence, should be constructed bottom-up).

Real DP Example, Subsequences

- A ***subsequence*** of a character string $x_0x_1x_2...x_{n-1}$ is a string of the form $x_{i_1}x_{i_2}...x_{i_k}$, where $i_j < i_{j+1}$.
- Not the same as substring!
- Example String: ABCDEFGHIJK
  - Subsequence: ACEGIJK
  - Subsequence: DFGHK
  - Not subsequence: DAGH

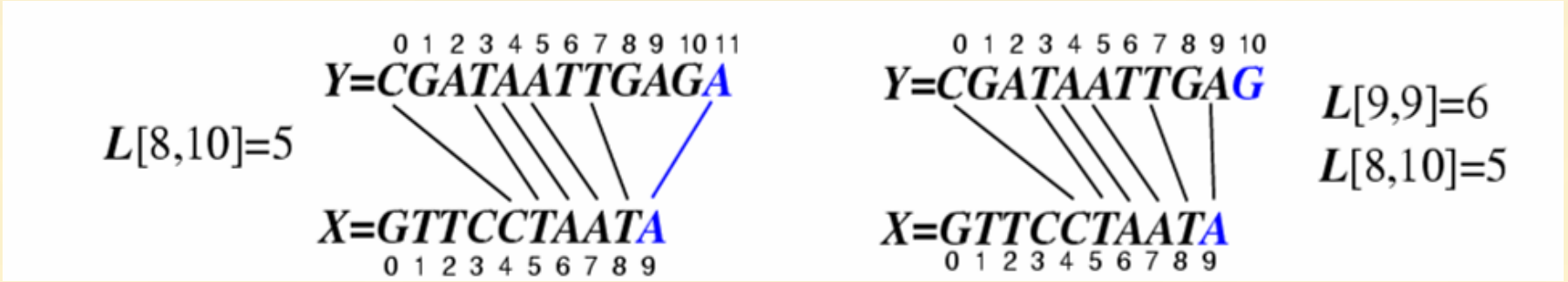
The Longest Common Subsequence (LCS) Problem

- Given two strings X and Y, the longest common subsequence (LCS) problem is to find a longest subsequence common to both X and Y
- Has applications to DNA similarity testing (alphabet is {A,C,G,T})
- Example: ABCDEFG and XZACKDFWGH have ACDFG as a longest common subsequence

DP APPROACH TO THE LCS PROBLEM

- Define L[i,j] to be the length of the longest common subsequence of X[0..i] and Y[0..j].
- Allow for -1 as an index, so L[-1,k] = 0 and L[k,-1]=0, to indicate that the null part of X or Y has no match with the other.
- Then we can define L[i,j] in the general case as follows:
  1. If xi=yj, then L[i,j] = L[i-1,j-1] + 1 (we can add this match)
  2. If xi≠yj, then L[i,j] = max{L[i-1,j], L[i,j-1]} (we have no match here)

Case 1: Case 2:



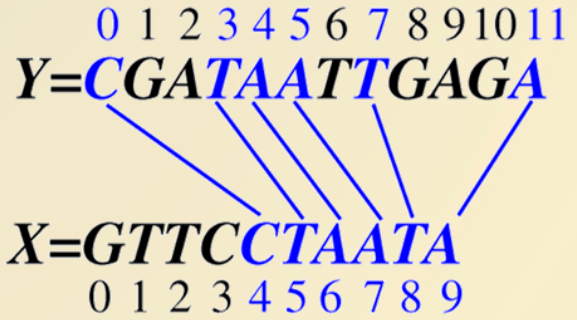
An LCS Algorithm

Algorithm LCS(X,Y):
Input: Strings X and Y with n and m elements, respectively
Output: For i = 0,…,n-1, j = 0,...,m-1, the length L[i, j] of a longest string that is a subsequence of both the string X[0..i] = $x_0x_1x_2…x_i$ and the string Y [0.. j] = $y_0y_1y_2…y_j$

```
for i =1 to n-1 do
  L[i,-1] = 0
for j =0 to m-1 do
  L[-1,j] = 0
for i =0 to n-1 do
  for j =0 to m-1 do
        if x_i = y_j  then
              L[i, j] = L[i-1, j-1] + 1
        else
              L[i, j] = max{L[i-1, j] , L[i, j-1]}
return array L
```

Visualizing the LCS Algorithm

L	-1	0	1	2	3	4	5	6	7	8	9	10	11
-1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	1	1	1	1	1	1	1	1	1	1
1	0	0	1	1	2	2	2	2	2	2	2	2	2
2	0	0	1	1	2	2	2	3	3	3	3	3	3
3	0	1	1	1	2	2	2	3	3	3	3	3	3
4	0	**1**	1	1	2	2	2	3	3	3	3	3	3
5	0	1	1	1	**2**	2	2	3	4	4	4	4	4
6	0	1	1	2	2	**3**	3	3	4	4	5	5	5
7	0	1	1	2	2	3	**4**	4	4	4	5	5	6
8	0	1	1	2	3	3	4	5	**5**	5	5	5	6
9	0	1	1	2	3	4	4	5	5	5	6	6	**6**



Analysis of LCS Algorithm

We have two nested loops

- The outer one iterates *n* times
- The inner one iterates *m* times
- A constant amount of work is done inside each iteration of the inner loop
- Thus, the total running time is O(*nm*)

Answer is contained in L[n,m] (and the subsequence can be recovered from the L table).