Recursion

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Recursion Definition

Recursion is a programming process where a method calls itself repeatedly until it reaches a defined point of termination.

Recursive Example

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- As a Java method:

```java
public static int factorial(int n) {
    if (n < 0) {
        throw new IllegalArgumentException("arg must be nonnegative");
    } else if (n == 0) {
        return 1;                          // base case
    } else {
        return n * factorial(n - 1);       // recursive case
    }
}
```

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  - Explicit calls to the current method.
  - Each recursive call **must** be defined so that it advances towards a base case.

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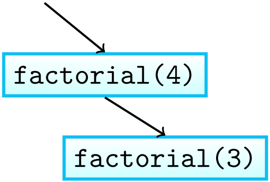
- A box for each recursive call
- An arrow from each caller to callee
- An arrow from each callee to caller showing the return value.

Visualizing Recursion

`factorial(4)`

Visualizing Recursion



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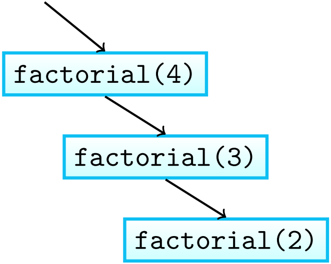

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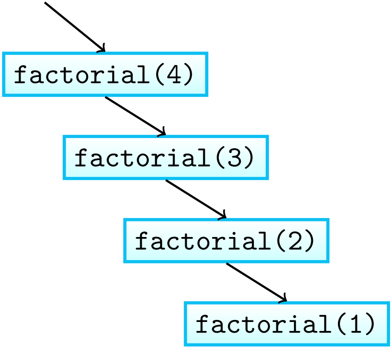

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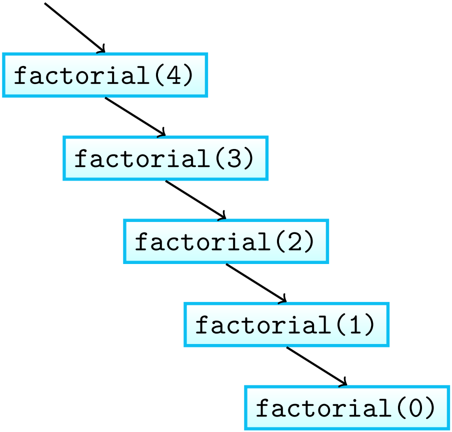

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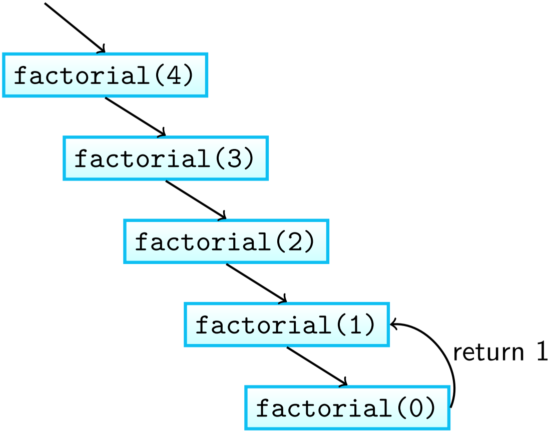

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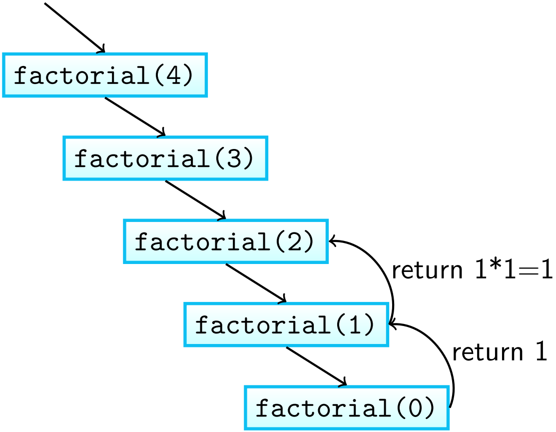

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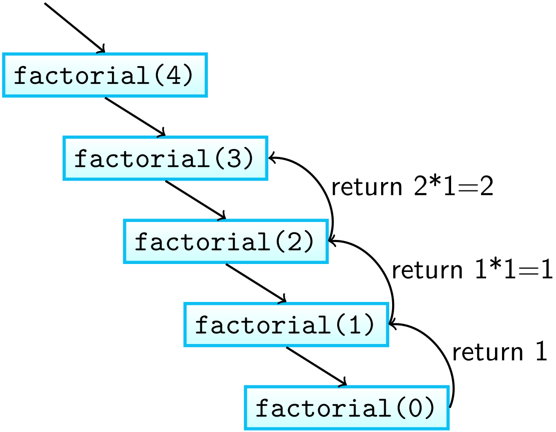

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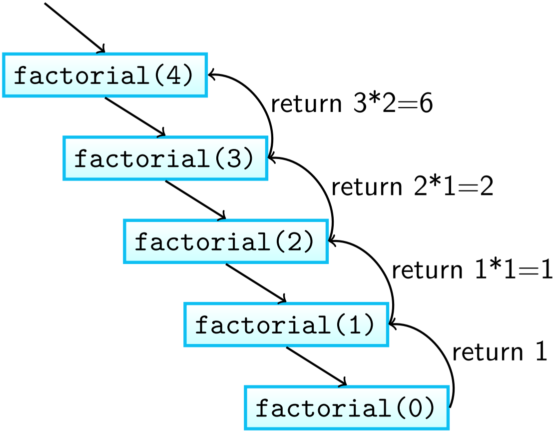

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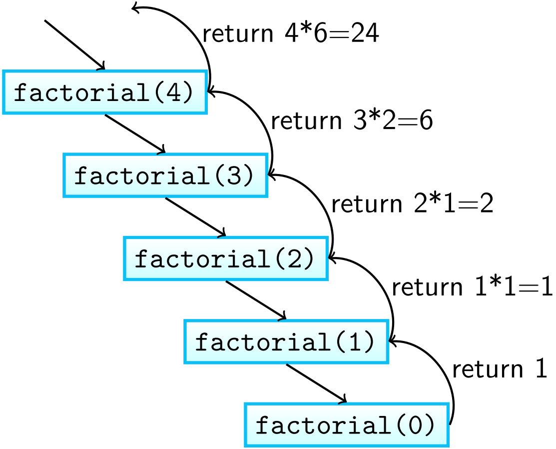

Binary Search Method

Search for an integer in an ordered list

procedure $\textsc{BinarySearch}(data, target, low, high)$
    if $low > high$ **then**
        return `False`
    else
        $mid \leftarrow (low + high)/2$
        if $target = data[mid]$ **then**
            return `True`
        else if $target < data[mid]$ **then**
            return $\textsc{BinarySearch}(data, target, low, mid - 1)$
        else
            return $\textsc{BinarySearch}(data, target, mid + 1, high)$
        end if
    end if
end procedure

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  - If `target > data[mid]`, then we recursively call the method on the second half of the sequence.

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(Blue represents the area being considered, while green represents the element that is checked.)

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  - Thus, each recursive call divides the search region in half; hence, there can be at most $\log n$ levels.

Computing Powers Example

- The power function, $p(a, n) = a^n$, can be defined recursively:

$$p(a, n) = \begin{cases} 1 & \text{if } n = 0 \\ a \times p(a, n-1) & \text{else} \end{cases}$$

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- This leads to an power function that runs in $O(n)$ time (because we make n recursive calls).
- We can do better than this, however.

Recursive Squaring Example

- We can derive a more efficient recursive algorithm by using repeated squaring:

$$p(a, n) = \begin{cases} 1 & \text{if } n = 0 \\ a \times p(a, (n-1)/2)^2 & \text{if} a > 0 \text{ is odd} \\ p(a, (n-1)/2)^2 & \text{if} a > 0 \text{ is even} \end{cases}$$

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- For example (knowing beforehand that $2^2 = 4$),
$2^4 = 2^{(4/2)^2} = (2^{4/2})^2 = (2^2)^2 = 4^2 = 16$
$2^5 = 2^{1+(4/2)^2} = 2 \times (2^{4/2})^2 = 2 \times (2^2)^2 = 2 \times 4^2 = 32$
$2^6 = 2^{(6/2)^2} = (2^{6/2})^2 = (2^3)^2 = 8^2 = 64$
$2^7 = 2^{1+(6/2)^2} = 2 \times (2^{6/2})^2 = 2 \times (2^3)^2 = 2 \times 8^2 = 128$

Recursive Squaring Method

```
procedure Power(a, n)
    if n = 0 then
        return 1
    end if
    if n is odd then
        y ← Power(a, (n − 1)/2)
        return a × y × y
    else
        y ← Power(a, n/2)
        return y × y
    end if
end procedure
```

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- Such methods might receive some optimization benefits at runtime.
  - Note that Java doesn't perform any optimizations for tail recursion.