

Skip Lists

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(edits by M. Hudachek-Buswell)

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- However, searching for a particular item will still be $O(n)$.
- This could be reduced by having some items (where these items are randomly chosen) be in another list, and have these items act like a “marker”. That way, you can reduce the search time by searching fewer items to see where in the list it might be.

Skip Lists

- A skip list is similar to a linked list with the items in ascending order, but a skip list has multiple **levels**.

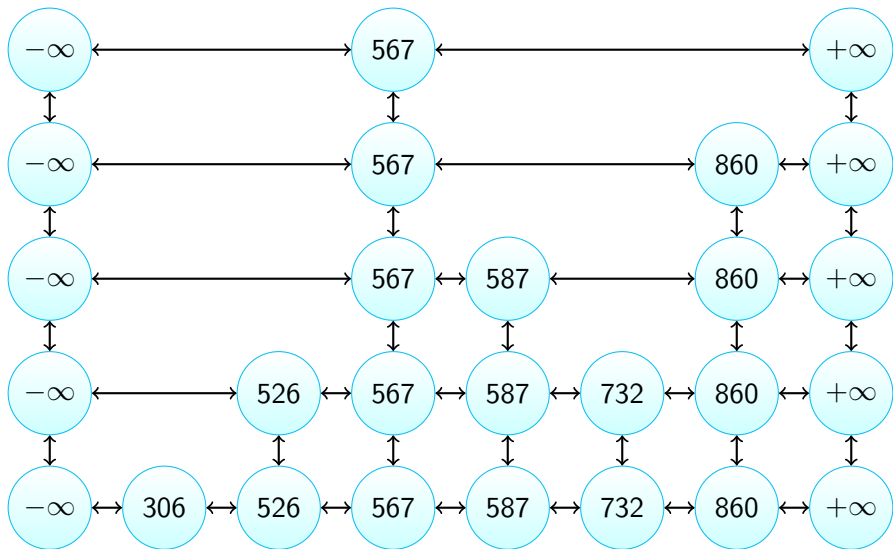
Skip Lists

- A skip list is similar to a linked list with the items in ascending order, but a skip list has multiple **levels**.
- Each level has (ideally) half of the items on the level below it. This means that the first level (usually called level 1) would have all of the items, the second level would have half of all of the items, the third level would have quarter of all of the items, etc.

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- When adding a new item, a **coin flipper** is used to determine if the item gets promoted to the next level and, if so, how many times.

Example Skip List



Coin Flipper

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- If the coin comes up heads, then the item is promoted to the next level, and another flip is done. If the coin comes up tails, then the item is not promoted, and no more flips are done.

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- A coin flipper can either be someone flipping an actual coin or some method that randomly gives heads or tails (or some representation of it). The idea is that this “coin” is random.
- If the coin comes up heads, then the item is promoted to the next level, and another flip is done. If the coin comes up tails, then the item is not promoted, and no more flips are done.
- Because of this randomness, the resulting skip list might not be perfectly arranged (with only half of the items on a level being present on the level above it).

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- Each node also holds a reference to the previous and next nodes in that level, and the nodes directly above and below that node. The nodes directly above and below a node should hold a reference to the same item.
- Finally, each node also contains the level the node is on; all nodes on the bottom level typically have a level of 1.
- Because each node has four references (up, down, left, right), this type of node might also be known as a **quad-node**.

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- For skip lists, at the starting of each level, there needs to be a phantom node that represents negative infinity. At the ending of each level, there can *optionally* be a phantom node that represents positive infinity.
- The negative infinity phantom nodes are required; the positive infinity phantom nodes are not.
- Because negative infinity and positive infinity might not be representable in all programming languages, some other special value can be used to represent negative infinity and positive infinity.

Searching

- Start at the top-left phantom node.

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Searching

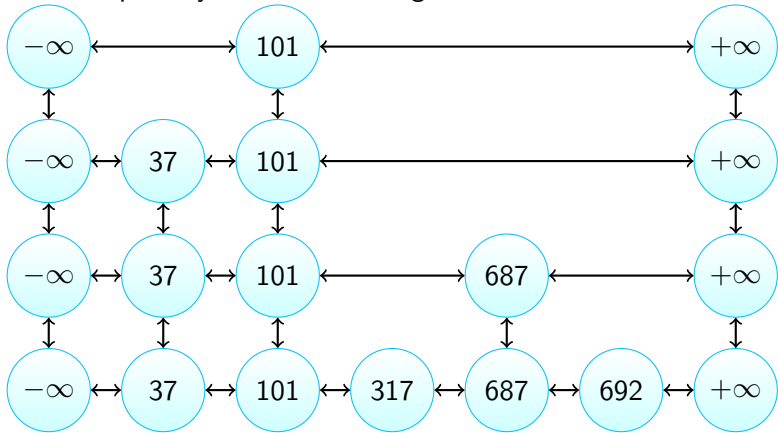
- Start at the top-left phantom node.
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Searching

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- Check the data in the node to the right.
 - If the data you're looking for is less than the data in the node, or there is no node there, go down one level.
 - If the data you're looking for is greater than the data in the node, go to that node.
 - If the data you're looking for is equal to the data in the node, then you've found the data.
- Repeat the previous step, until you find the node or go off of the list, in which case the data you're looking for isn't in the skip list.

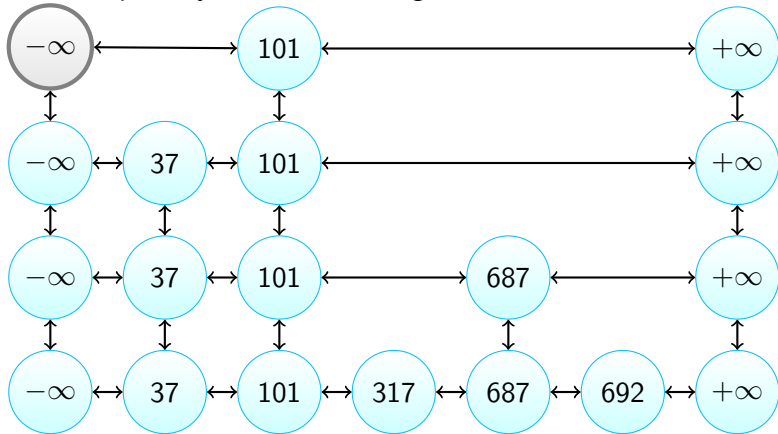
Searching

For example, if you were searching for 687:



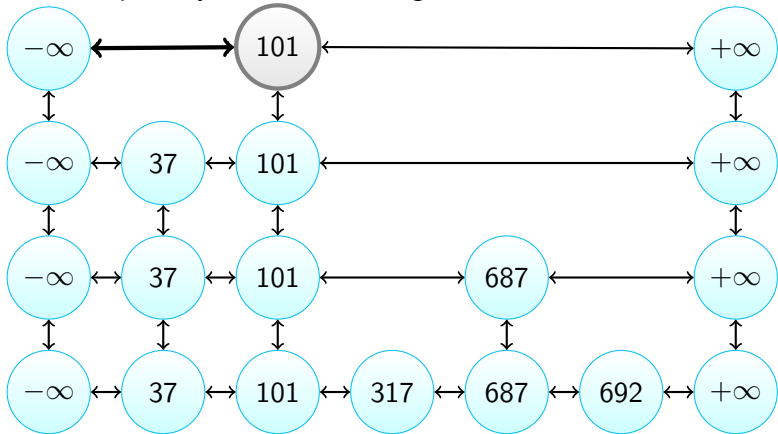
Searching

For example, if you were searching for 687:



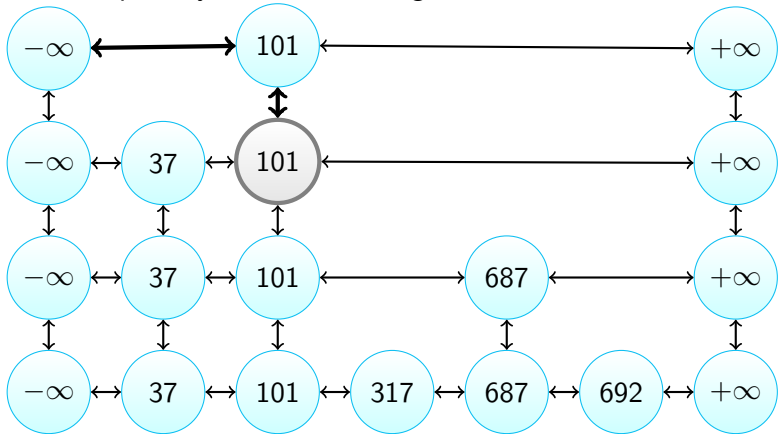
Searching

For example, if you were searching for 687:



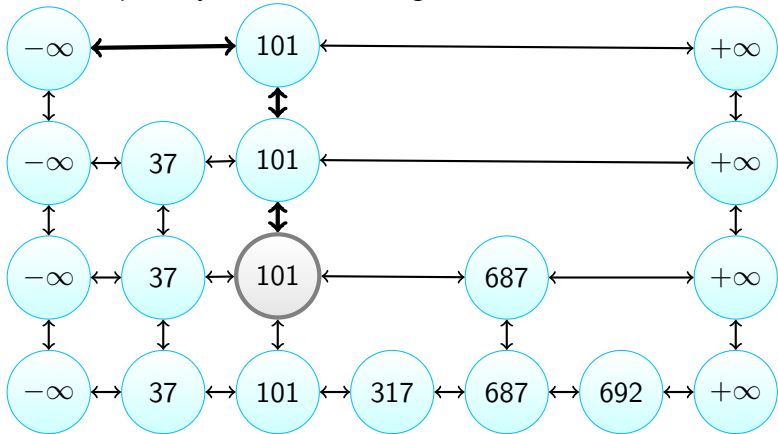
Searching

For example, if you were searching for 687:



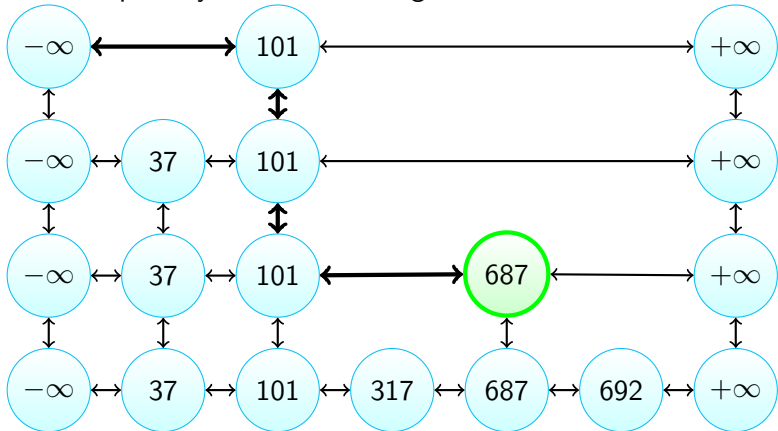
Searching

For example, if you were searching for 687:



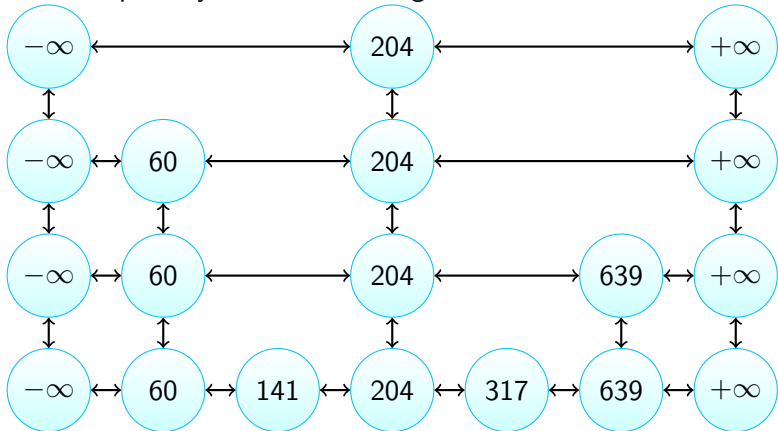
Searching

For example, if you were searching for 687:



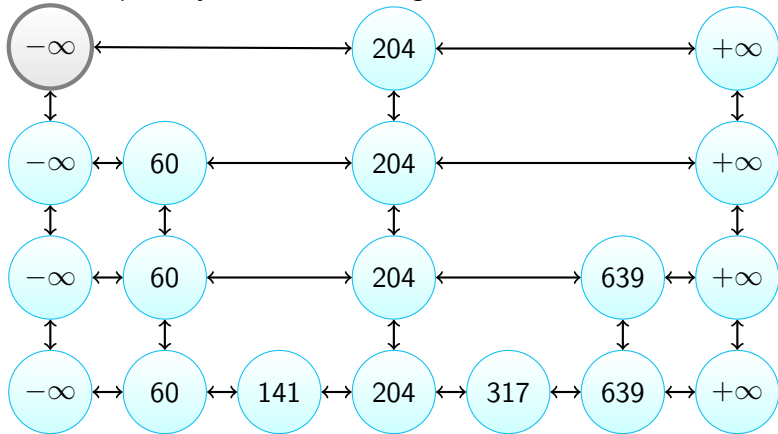
Searching

For example, if you were searching for 141:



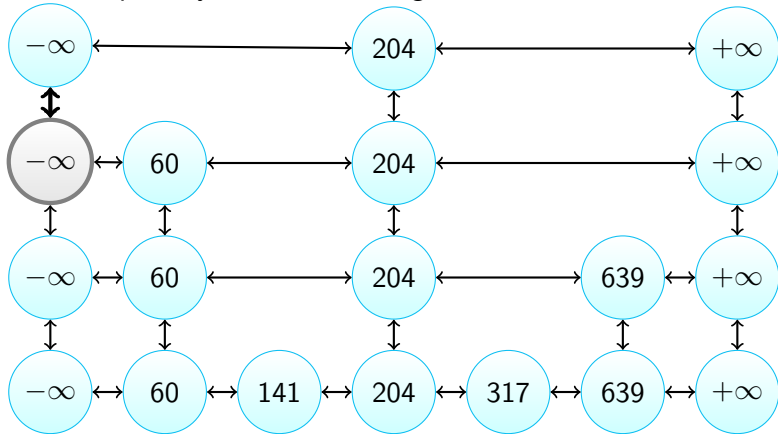
Searching

For example, if you were searching for 141:



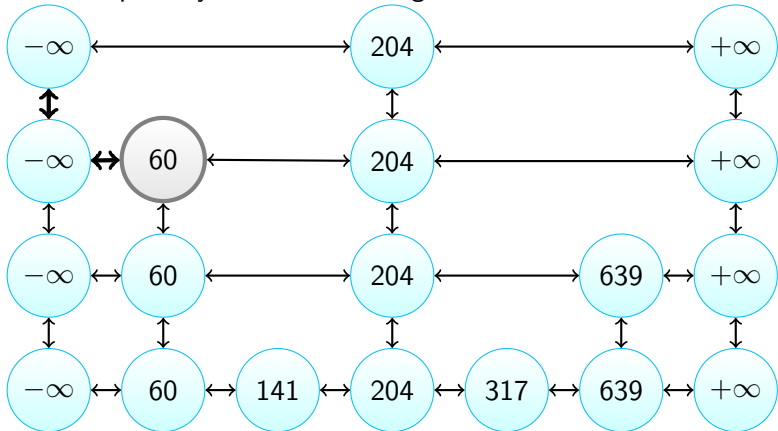
Searching

For example, if you were searching for 141:



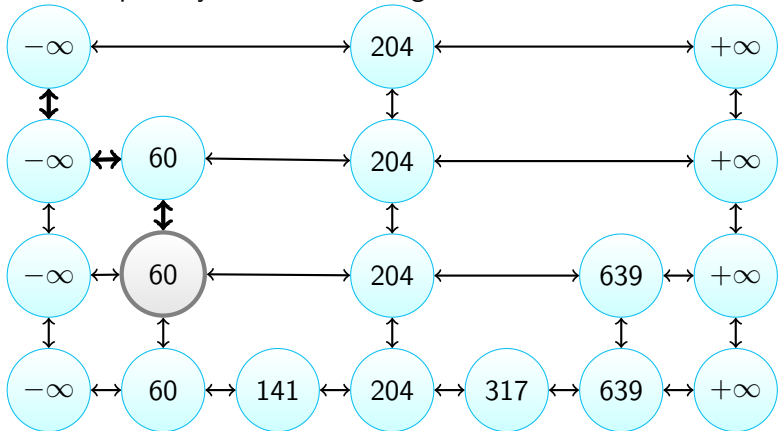
Searching

For example, if you were searching for 141:



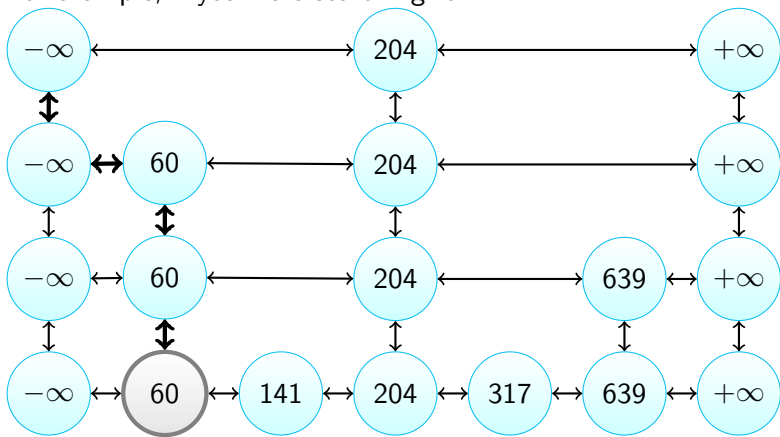
Searching

For example, if you were searching for 141:



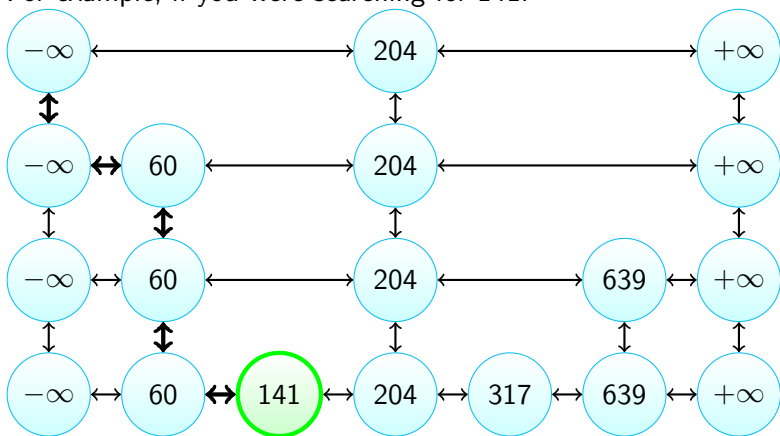
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For example, if you were searching for 141:



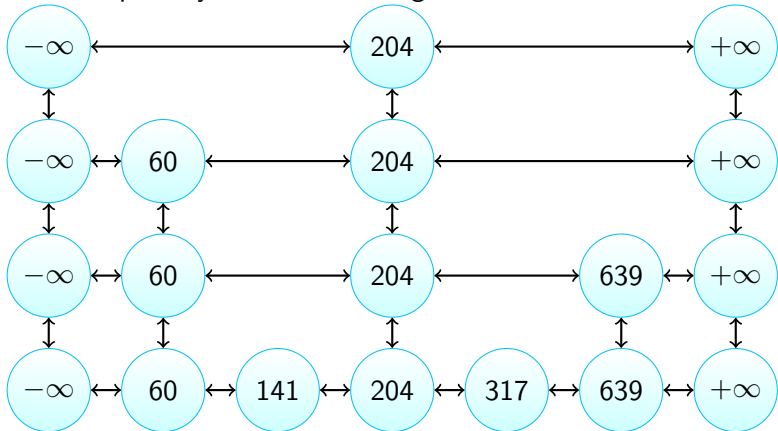
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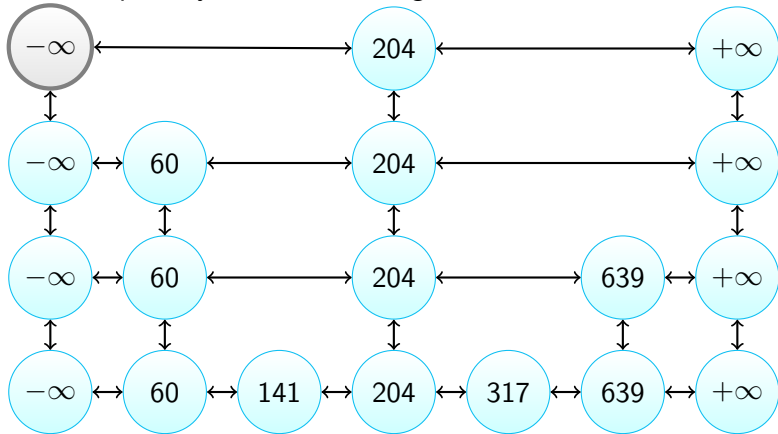
Searching

For example, if you were searching for 405:



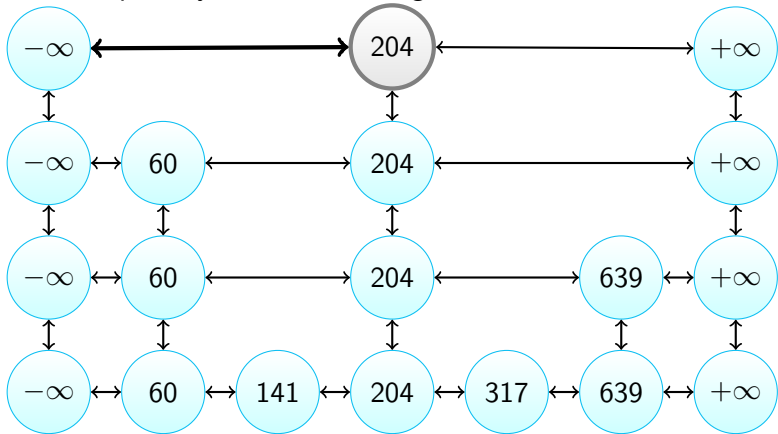
Searching

For example, if you were searching for 405:



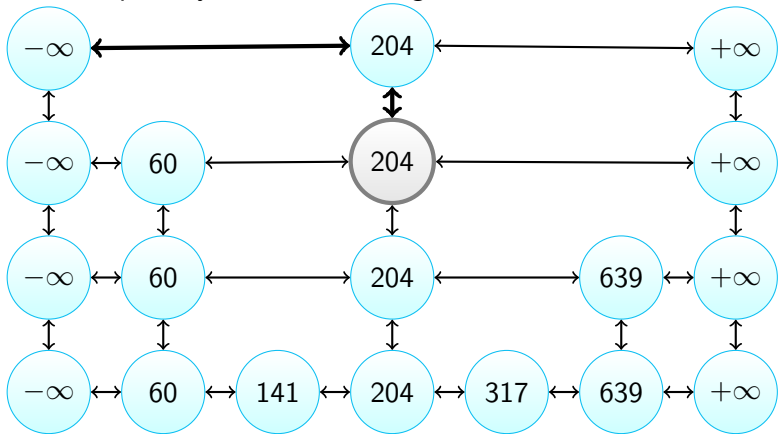
Searching

For example, if you were searching for 405:



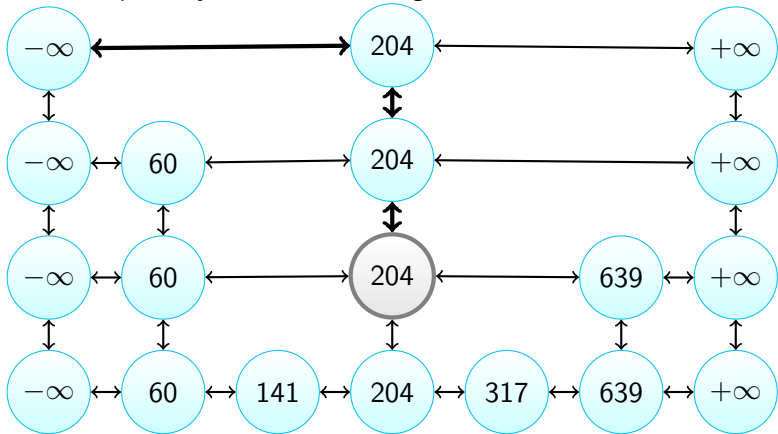
Searching

For example, if you were searching for 405:



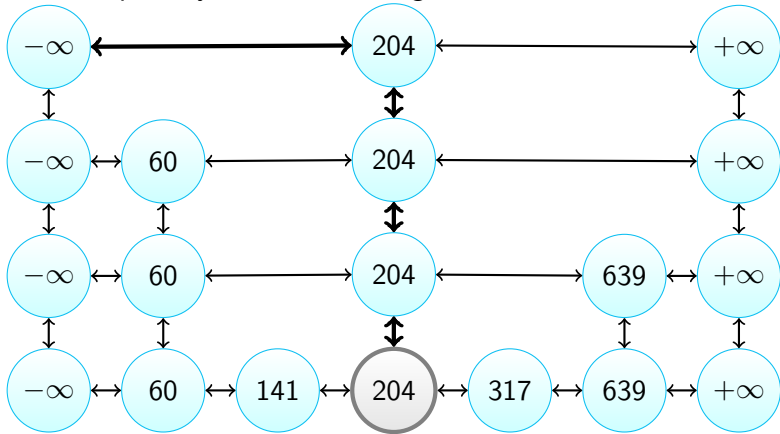
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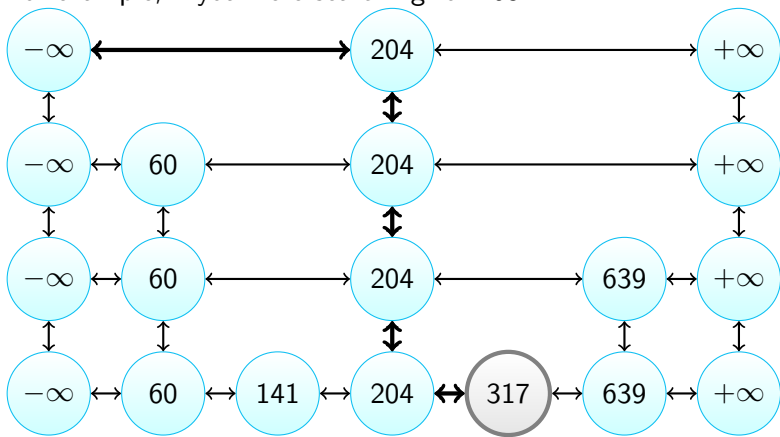
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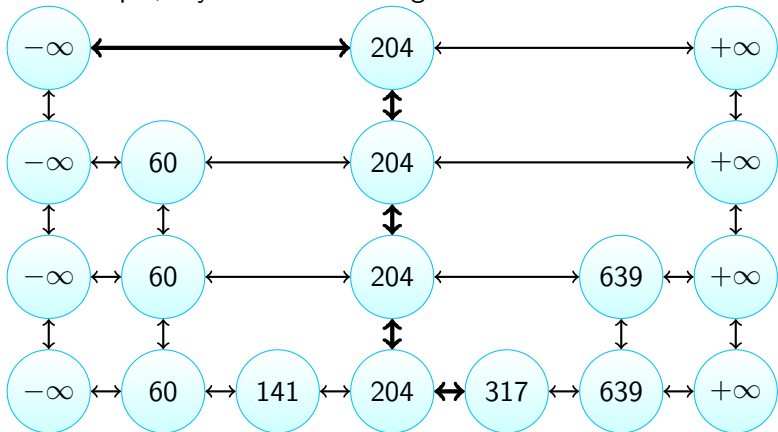
Searching

For example, if you were searching for 405:



Searching

For example, if you were searching for 405:



405 is not in the skip list.

Searching

```
procedure SEARCH(data, node)  
  if node is not valid then  
    return FALSE  
  else  
    while data > node.next.data do  
      node  $\leftarrow$  node.next  
    end while  
    if data = node.next.data then  
      return TRUE  
    else  
      return SEARCH(data, node.down)  
    end if  
  end if  
end procedure
```

Adding

- Use the coin flipper to see how many times the item you are going to add should be promoted.

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- If there aren't enough levels in the skip list for the new item, then create the new levels.

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- Use the coin flipper to see how many times the item you are going to add should be promoted.
- If there aren't enough levels in the skip list for the new item, then create the new levels.
- Traverse the skip list as if you're searching for the item. However, before going down to the level below, if the item needs to be added to this level, add it, then go down.

Adding

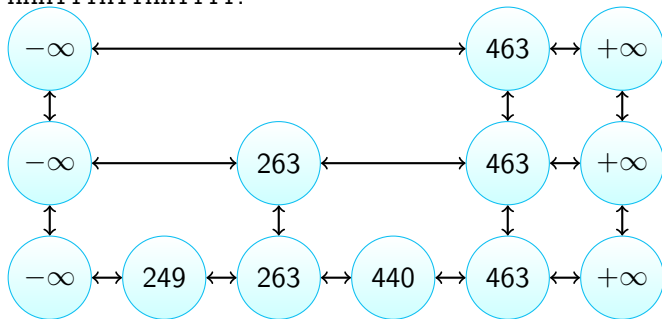
- Use the coin flipper to see how many times the item you are going to add should be promoted.
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- Traverse the skip list as if you're searching for the item. However, before going down to the level below, if the item needs to be added to this level, add it, then go down.
- Repeat the previous step until you reach the bottom level.

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- Use the coin flipper to see how many times the item you are going to add should be promoted.
- If there aren't enough levels in the skip list for the new item, then create the new levels.
- Traverse the skip list as if you're searching for the item. However, before going down to the level below, if the item needs to be added to this level, add it, then go down.
- Repeat the previous step until you reach the bottom level.
- Handling of duplicate items is implementation-defined.

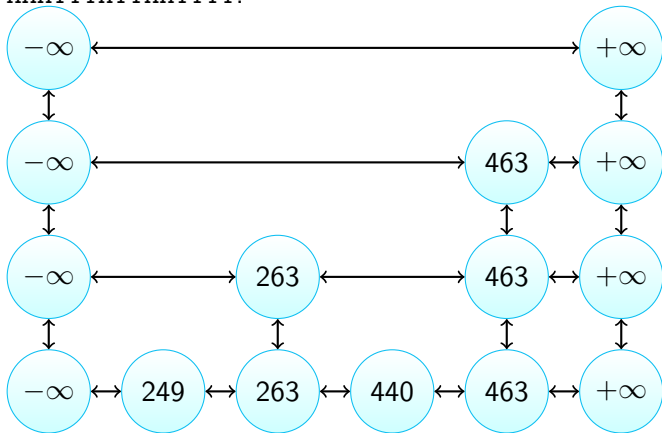
Adding

For example, if you were adding 310, and the coin flipper gave HHHTTTHTTTHHTTTT:



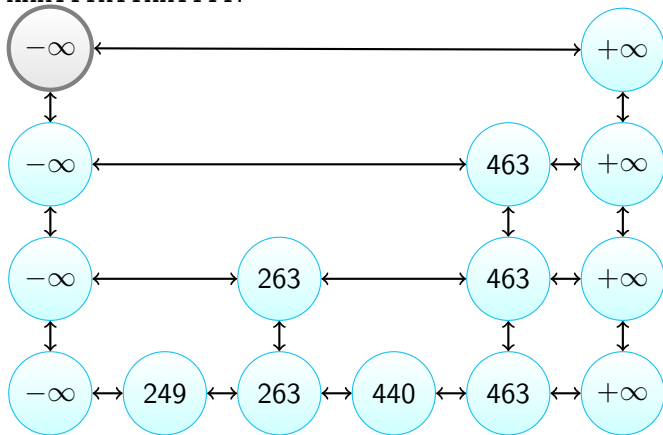
Adding

For example, if you were adding 310, and the coin flipper gave HHHTTTHTTTHHTTTT:



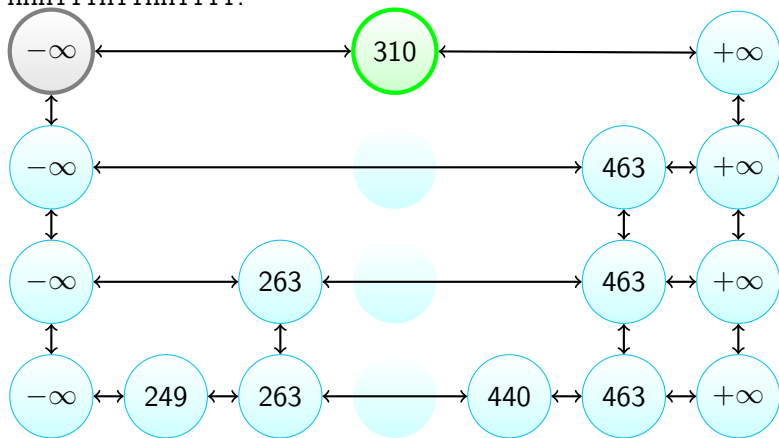
Adding

For example, if you were adding 310, and the coin flipper gave HHHHTTHTTTHHTTTT:



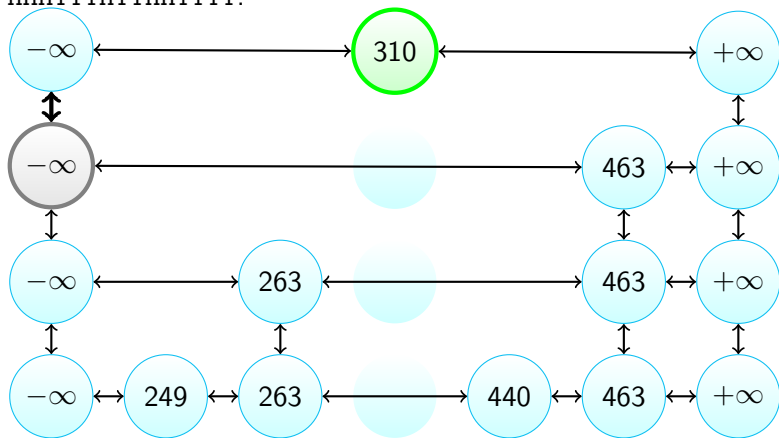
Adding

For example, if you were adding 310, and the coin flipper gave HHHHTTHTTHHTTTT:



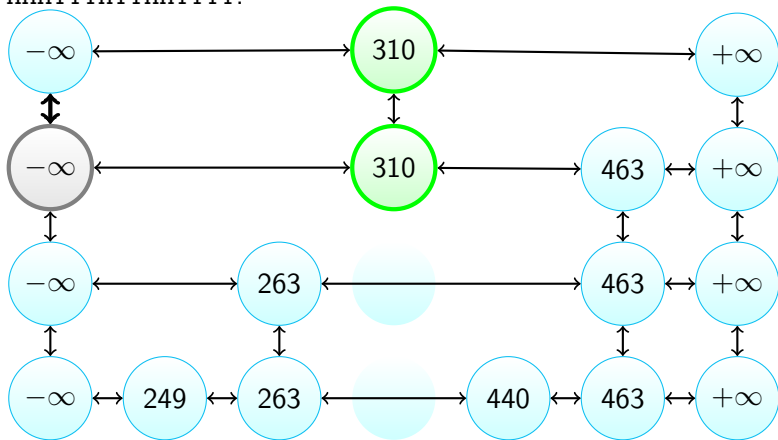
Adding

For example, if you were adding 310, and the coin flipper gave HHHHTTHTTHHTTTT:



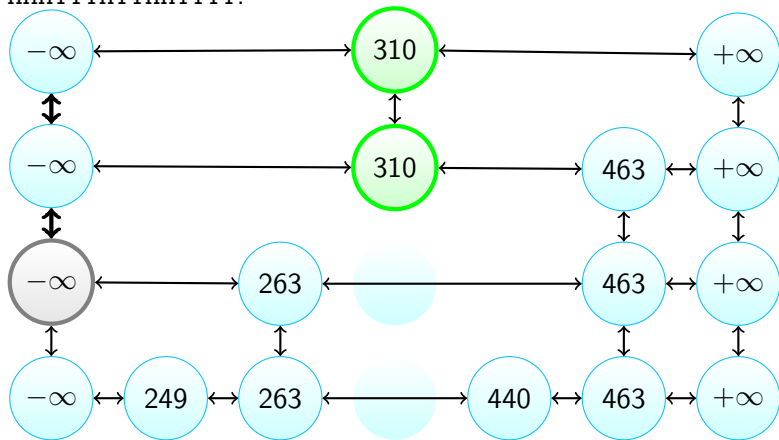
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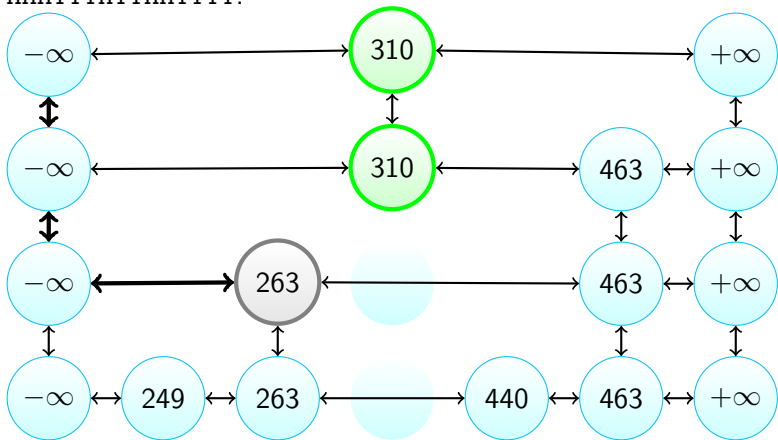
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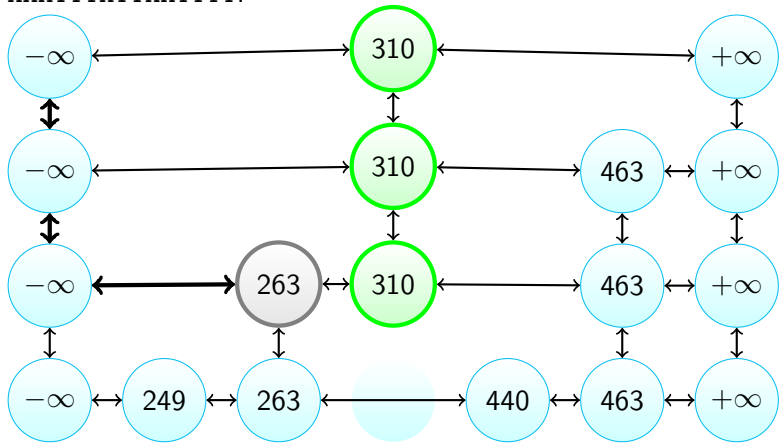
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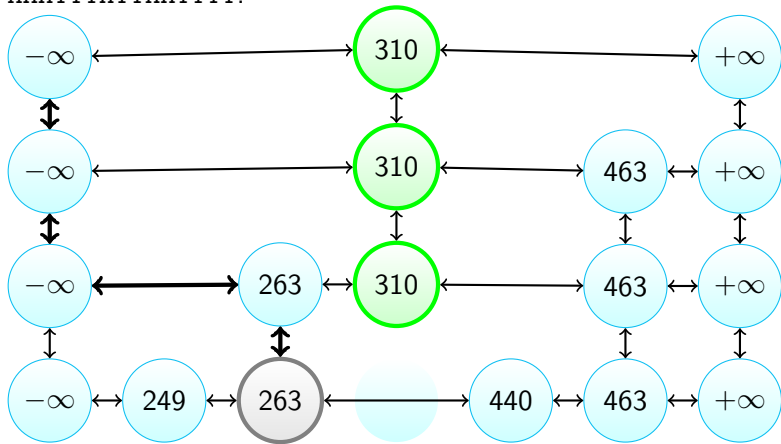
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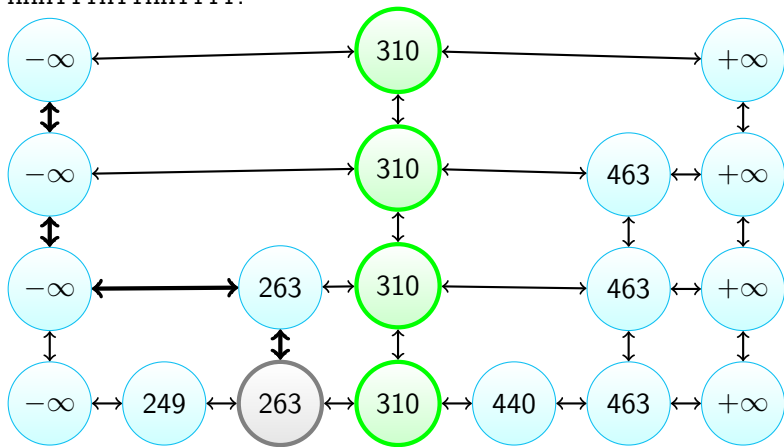
Adding

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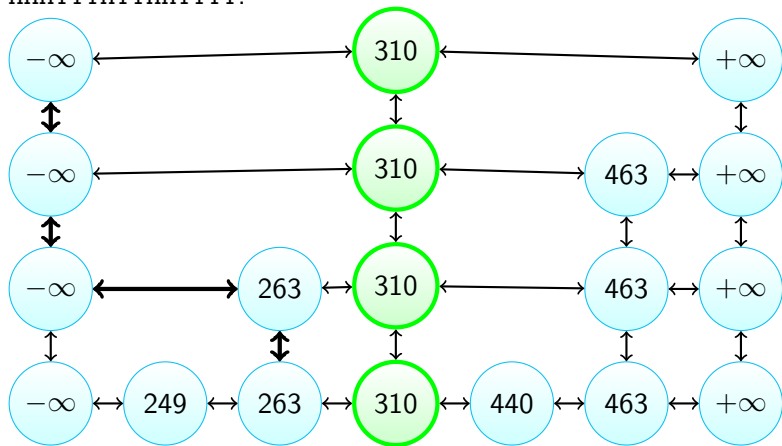
Adding

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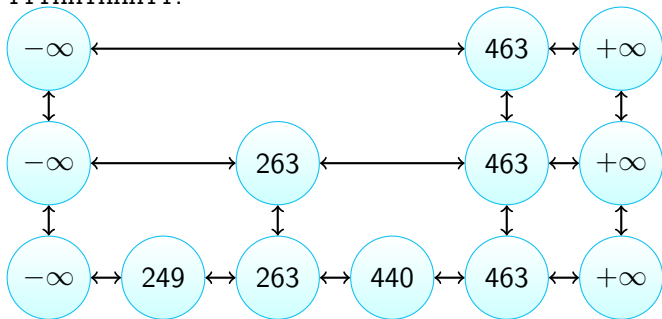
Adding

For example, if you were adding 310, and the coin flipper gave HHHTTTHTTHHTTTT:



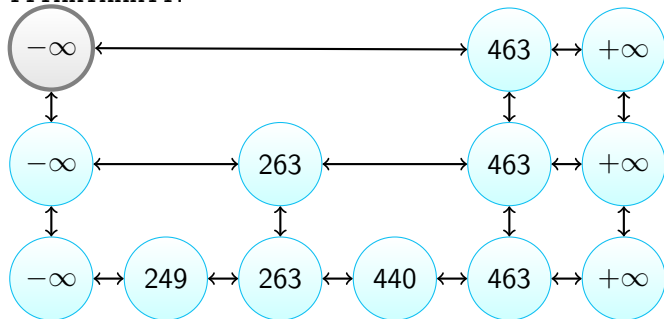
Adding

For example, if you were adding 118, and the coin flipper gave TTTHTTHHHTT:



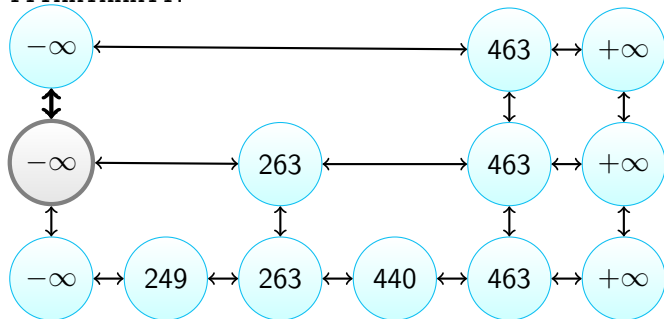
Adding

For example, if you were adding 118, and the coin flipper gave TTTHTHHTT:



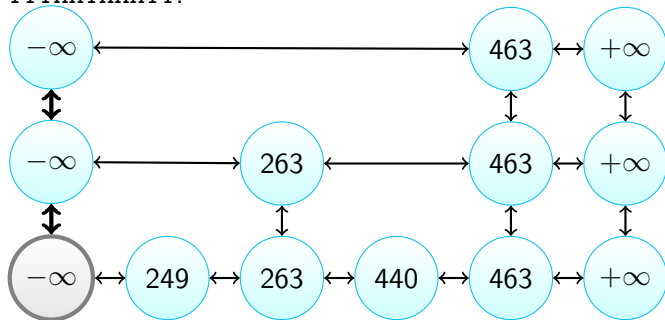
Adding

For example, if you were adding 118, and the coin flipper gave TTTHTHHTT:



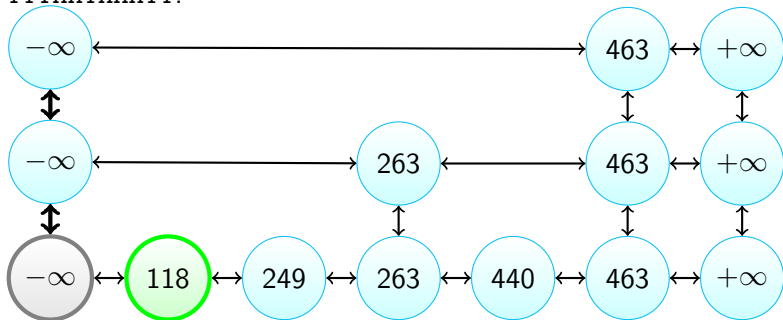
Adding

For example, if you were adding 118, and the coin flipper gave TTTHTHHTT:



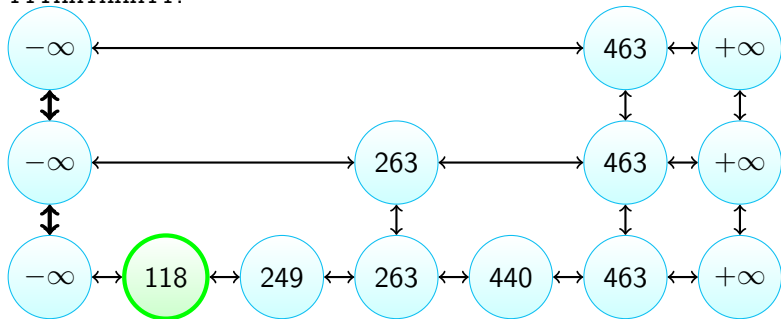
Adding

For example, if you were adding 118, and the coin flipper gave TTTTHHTHHHTT:



Adding

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Adding

```
procedure ADD(data)  
    levels  $\leftarrow$  highest level the data will be on (based on coin  
flipper)  
    while highest level  $<$  levels do  
        create another level  
    end while  
    ADD(data, levels, top-left phantom node, NULL)  
end procedure
```

Adding

```
procedure ADD(data, levels, node, upperNode)  
  if node is not valid then  
    return  
  else  
    while data > node.next.data do  
      node  $\leftarrow$  node.next  
    end while  
    if node.level  $\leq$  levels then  
      Add new node containing data after node  
      node.up  $\leftarrow$  upperNode  
      upperNode  $\leftarrow$  newly-added node  
    end if  
    ADD(data, levels, node.down, upperNode)  
  end if  
end procedure
```

Removing

- Follow the same steps as searching until you find the item in the skip list or go off of the skip list (in which case the data is not there).

Removing

- Follow the same steps as searching until you find the item in the skip list or go off of the skip list (in which case the data is not there).
- Disconnect the node from the rest of the nodes on that level, and move down to the level below.

Removing

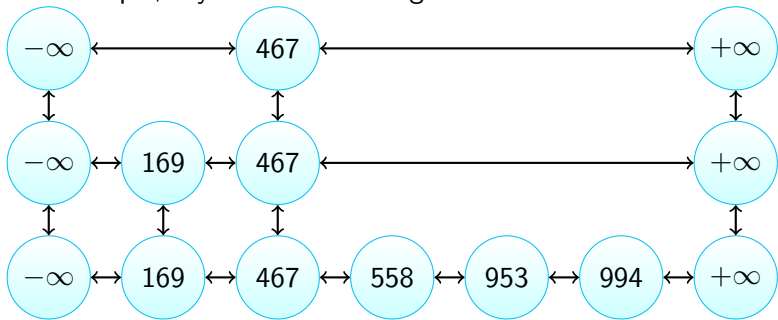
- Follow the same steps as searching until you find the item in the skip list or go off of the skip list (in which case the data is not there).
- Disconnect the node from the rest of the nodes on that level, and move down to the level below.
- Repeat the previous step until you go off of the skip list.

Removing

- Follow the same steps as searching until you find the item in the skip list or go off of the skip list (in which case the data is not there).
- Disconnect the node from the rest of the nodes on that level, and move down to the level below.
- Repeat the previous step until you go off of the skip list.
- Remove any empty levels in the skip list.

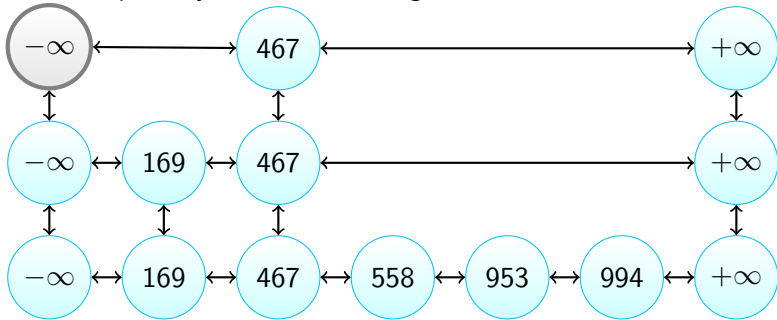
Removing

For example, if you were removing 467:



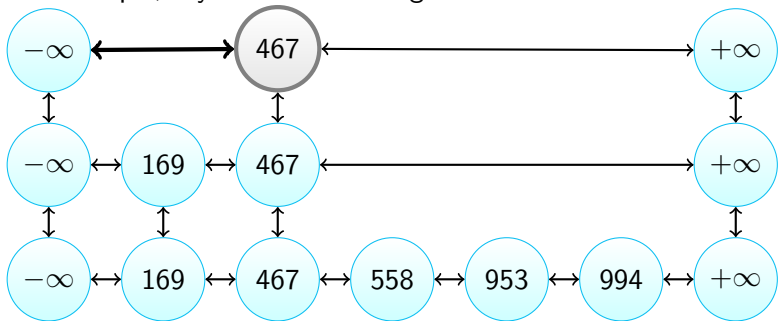
Removing

For example, if you were removing 467:



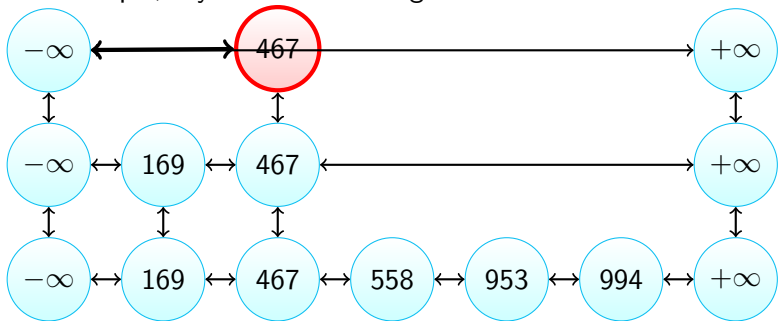
Removing

For example, if you were removing 467:



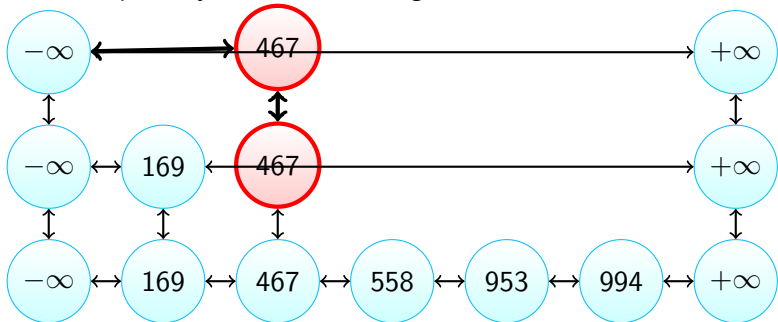
Removing

For example, if you were removing 467:



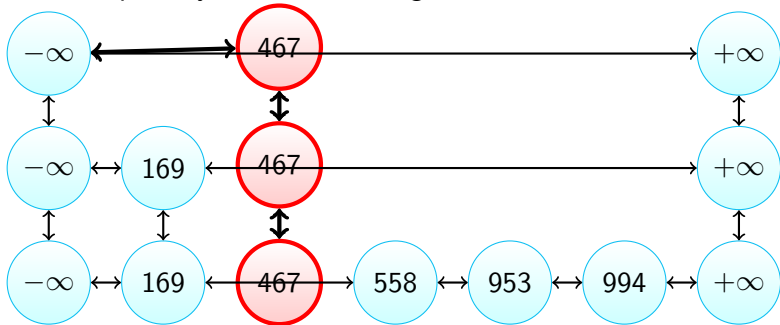
Removing

For example, if you were removing 467:



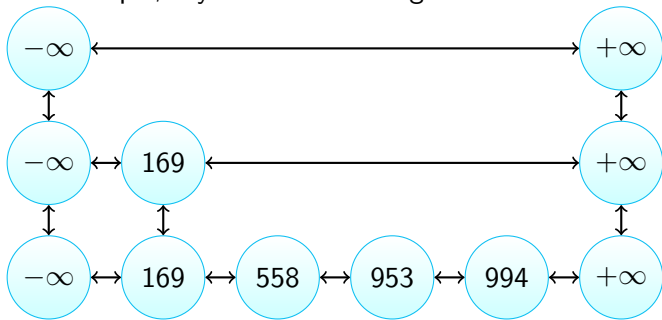
Removing

For example, if you were removing 467:



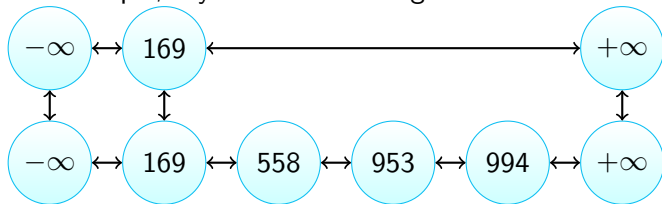
Removing

For example, if you were removing 467:



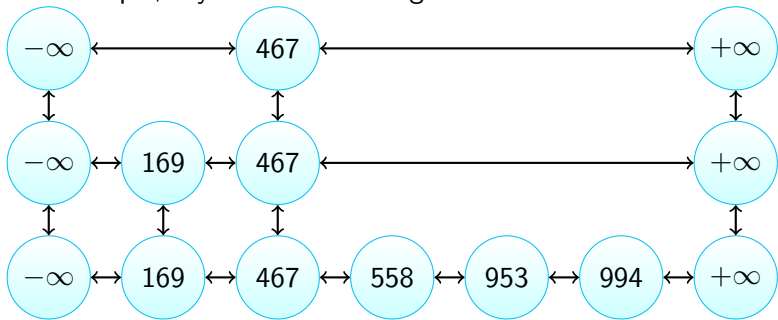
Removing

For example, if you were removing 467:



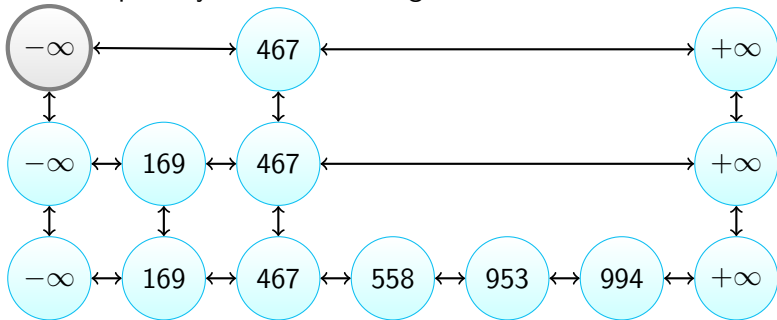
Removing

For example, if you were removing 169:



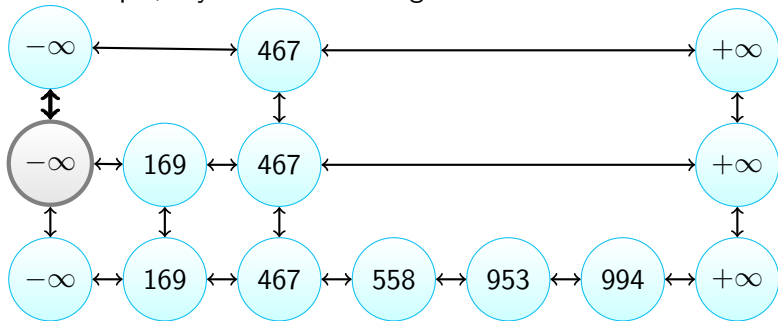
Removing

For example, if you were removing 169:



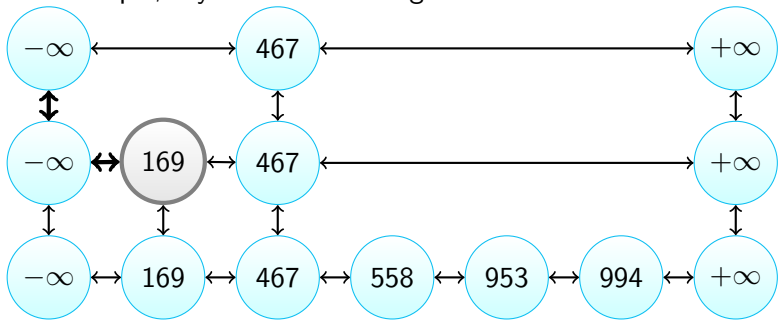
Removing

For example, if you were removing 169:



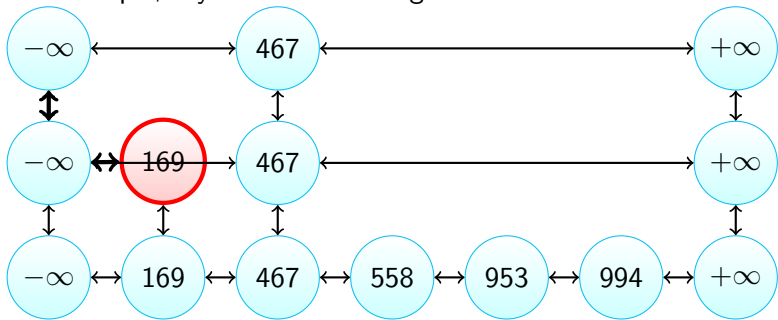
Removing

For example, if you were removing 169:



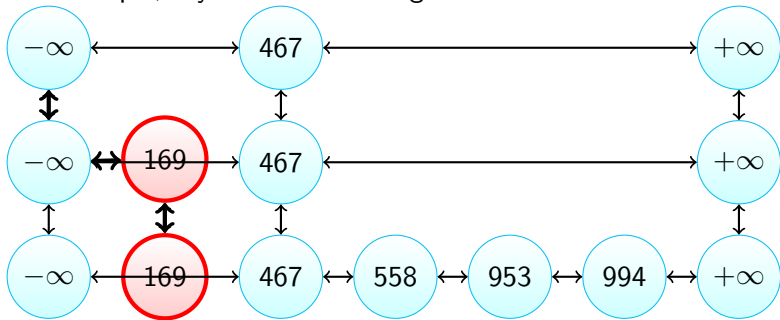
Removing

For example, if you were removing 169:



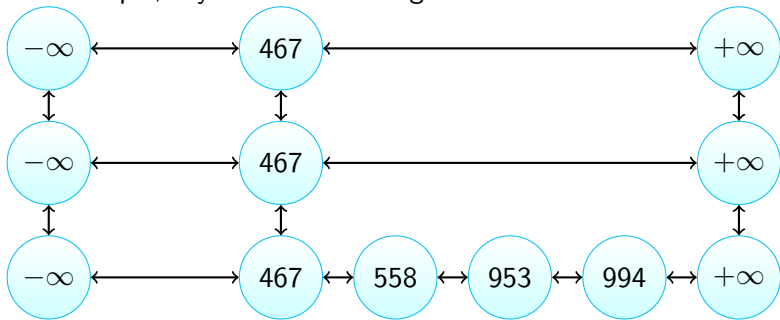
Removing

For example, if you were removing 169:



Removing

For example, if you were removing 169:



Removing

```
procedure REMOVE(data, node)  
  if node is not valid then  
    return  
  else  
    while data > node.next.data do  
      node  $\leftarrow$  node.next  
    end while  
    if data = node.next.data then  
      REMOVE(node.next)  
      remove any empty levels  
    else  
      REMOVE(data, node.down)  
    end if  
  end if  
end procedure
```

Removing

```
procedure REMOVE(node)  
  while node is valid do  
    disconnect/remove node from this level  
    node  $\leftarrow$  node.down  
  end while  
end procedure
```


Performance

- In the best case, a truly random coin flipper is used, the skip list has $\log n$ levels (where n is the number of data items), and each level has half of the items in the level below. In this case, adding, searching, and removing are $O(\log n)$.

Performance

- In the best case, a truly random coin flipper is used, the skip list has $\log n$ levels (where n is the number of data items), and each level has half of the items in the level below. In this case, adding, searching, and removing are $O(\log n)$.
- The average case for all three operations are $O(\log n)$ as well, because each level will contain roughly half of the items on the level below it.

Performance

- In the worst case, the coin flipper used is not truly random, or it just so happens that all of the items added into the skip list are at the same level. In this case, adding, searching, and removing are $O(n)$.

Performance

- In the worst case, the coin flipper used is not truly random, or it just so happens that all of the items added into the skip list are at the same level. In this case, adding, searching, and removing are $O(n)$.
- Note that unlike all of the data structures we've seen, because each item is stored multiple times in a skip list, the space complexity (the rate at which the space used increases) is $O(n \log n)$ in the worst case (on average, it is roughly $O(n)$). All of the other data structures we've seen have a space complexity of $O(n)$.