Math Models, Spring 2016: HW 4

Note that some problems or parts of problems may not be graded.

1. In class, we derived the governing equation for the motion of a bead sliding on a rotating hoop. We assumed the hoop was circular. Here we derive the governing equation for the motion of a bead sliding on a rotating hoop but assume instead that the hoop is elliptic. Specifically, we assume that the shape of the hoop is described by the equation

$$\frac{x^2}{c^2} + \frac{y^2}{d^2} = 1. \tag{1}$$

(a) As we did in class, we set up our coordinates with $\mathbf{i}$ pointing straight down, so that $\mathbf{j}$ and $\mathbf{k}$ define a horizontal plane. The hoop rotates around the x-axis with the center of the ellipse at the origin. We define

$$\boldsymbol{\lambda}_1(\beta) = \cos(\beta)\mathbf{k} + \sin(\beta)\mathbf{j}, \quad \boldsymbol{\lambda}_2(\beta) = -\sin(\beta)\mathbf{k} + \cos(\beta)\mathbf{j}.$$

We assume that at time t the hoop is contained in the plane defined by $\mathbf{i}$ and $\boldsymbol{\lambda}_1(\omega t)$.

We let $\mathbf{p}(t)$ denote the position vector for the bead at time t. Show that

$$\mathbf{p}(t) = c\cos(\phi(t))\mathbf{i} + d\sin(\phi(t))\boldsymbol{\lambda}_1(\omega t) \tag{2}$$

for some function ϕ. In other words, show that $\mathbf{p}(t)$ is the position vector for a point on the ellipse.

(b) Show that the acceleration of the bead is

$$\begin{aligned}\ddot{\mathbf{p}}(t) &= -c[\cos(\phi(t))(\dot{\phi}(t))^2 + \sin(\phi(t))\ddot{\phi}(t)]\mathbf{i} \\ &\quad + d[-\sin(\phi(t))\dot{\phi}(t)^2 + \cos(\phi(t))\ddot{\phi}(t) - \omega^2\sin(\phi(t))]\boldsymbol{\lambda}_1(\omega t) \\ &\quad + 2d\omega\cos(\phi(t))\dot{\phi}(t)\boldsymbol{\lambda}_2(\omega t).\end{aligned} \tag{3}$$

(c) Assume that the forces that act on the bead are the following. There is gravity, which acts straight down. There is a frictional force, whose magnitude is proportional to $\dot{\phi}$ (with coefficient b) and which acts tangential to the hoop. And there is a reaction force, which always acts perpendicular to the hoop. Write down Newton's Second Law for the bead. This will be a vector equation. (To do this, you need to find a unit vector tangent to the hoop and a unit vector perpendicular to the hoop.)

(d) By taking the dot product of both sides of the vector equation from (c) with an appropriate vector, derive a scalar equation for ϕ that does not involve the reaction force.

(e) Assume if $c = d$. Show that the scalar equation you derived in (d) reduces to the second-order equation for ϕ that we derived in class for a circular hoop. (Note that if $c = d$, then this common value is the radius of the hoop, which we called r in class.)

Additional Instructions

To facilitate the HW grading, please do the following.

- Write your HW solutions double spaced.
- Write your HW solutions in the same order in which the problems are assigned above.
- Clearly label each problem. Draw a dark box or circle around the label—the point is that I should be able to quickly flip through your HW solution and find a given problem.