12.1-2 Counting
12.3-4 Probability

SECTION 12.1 The Counting Principle

Counting by Making a List

Combinatorics is the study of counting the different outcomes of some task. For example, if a coin is flipped, the side facing upward will be a head or a tail. The outcomes can be listed as {H, T}. There are two possible outcomes.

If a regular six-sided die is rolled, the possible outcomes are , , , , , . The outcomes can also be listed as {1, 2, 3, 4, 5, 6}. There are six possible outcomes.

EXAMPLE 1 Counting by Forming a List

List and then count the number of different outcomes that are possible when one letter from the word *Tennessee* is chosen.

Solution

The possible outcomes are {T, e, n, s}. There are four possible outcomes.

CHECK YOUR PROGRESS 1 List and then count the number of different outcomes that are possible when one letter is chosen from the word *Mississippi*.

Solution See page S43.

In combinatorics, an **experiment** is an activity with an observable outcome. The set of all possible outcomes of an experiment is called the **sample space** of the experiment. Flipping a coin, rolling a die, and choosing a letter from the word *Tennessee* are experiments. The sample spaces are {H, T}, {1, 2, 3, 4, 5, 6}, and {T, e, n, s}, respectively.

An **event** is one or more of the possible outcomes of an experiment. Flipping a coin and having a head show on the upward face, rolling a 5 when a die is tossed, and choosing a T from one of the letters in the word *Tennessee* are all examples of events. An event is a *subset* of the sample space.

EXAMPLE 2 Listing the Elements of an Event

One number is chosen from the sample space

Sample space

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$$

List the elements in the following events.

an event

a. The number is even. {2, 4, 6, 8, 10, 12, 14, 16, 18, 20}

b. The number is divisible by 5. {5, 10, 15, 20}

c. The number is a prime number. {2, 3, 5, 7, 11, 13, 17, 19}

Solution

a. {2, 4, 6, 8, 10, 12, 14, 16, 18, 20}

b. {5, 10, 15, 20}

c. {2, 3, 5, 7, 11, 13, 17, 19}

CHECK YOUR PROGRESS 2 One digit is chosen from the digits 0 through 9. The sample space *S* is {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}. List the elements in the following events.

a. The number is odd. {1, 3, 5, 7, 9}

b. The number is divisible by 3. {0, 3, 6, 9}

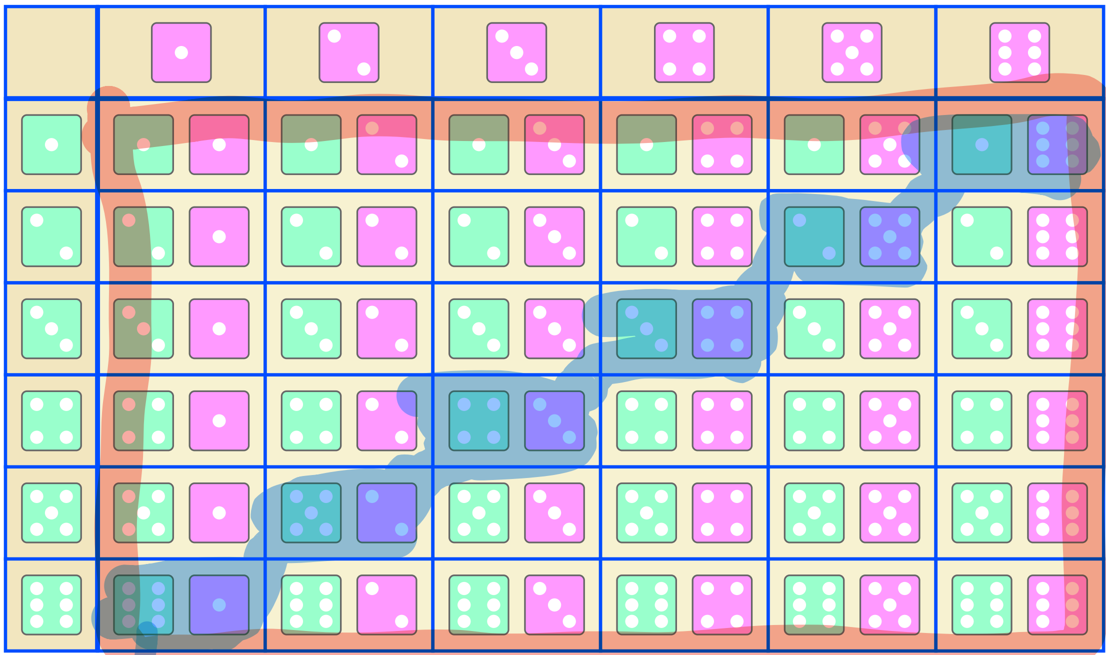
c. The number is greater than 7. {8, 9}

Solution See page S43.

Counting by Making a Table

Each of the experiments given above illustrates a *single-stage experiment.* A **single-stage experiment** is an experiment for which there is a single outcome. Experiments that have two, three, or more stages are called **multi-stage experiments**. To count the number of outcomes of such an experiment, a systematic procedure is helpful. Using a table to record results is one such procedure.

Consider the two-stage experiment of rolling two dice, one red and one green. How many different outcomes are possible? To determine the number of outcomes, make a table with the different outcomes of rolling the red die across the top and the different outcomes of rolling the green die down the side.



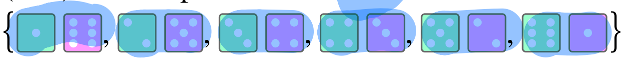
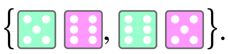
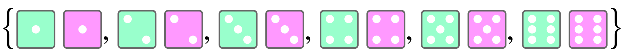
Outcomes of Rolling Two Dice

36 outcomes in sample space

By counting the number of entries in the diagram above, we see that there are 36 different outcomes of the experiment of rolling two dice. The sample space is

S = 

From the table, several different events can be discussed.

- The sum of the pips (dots) on the upward faces is 7. There are six outcomes of this event. They are .
- The sum of the pips on the upward faces is 11. There are two outcomes of this event. They are .
- The numbers of pips on the upward faces are equal. There are six outcomes of this event. They are .

EXAMPLE 3 Counting Using a Table

Two-digit numbers are formed from the digits 1, 3, and 8. Find the sample space and determine the number of elements in the sample space.

Solution

Use a table to list all the different two-digit numbers that can be formed by using the digits 1, 3, and 8.

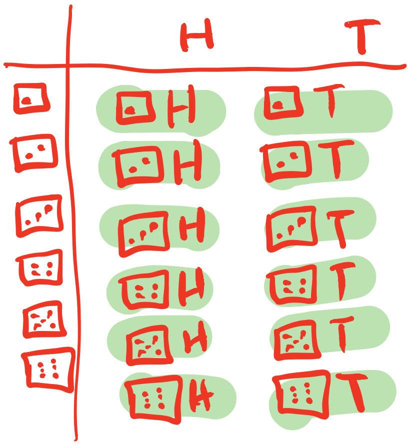
	1	3	8
1	11	13	18
3	31	33	38
8	81	83	88

The sample space is {11, 13, 18, 31, 33, 38, 81, 83, 88}. There are nine two-digit numbers that can be formed from the digits 1, 3, and 8.

CHECK YOUR PROGRESS 3 A die is tossed and then a coin is flipped. Find the sample space and determine the number of elements in the sample space.

Solution See page S43.

12



Counting by Using a Tree Diagram

A **tree diagram** is another way to organize the outcomes of a multi-stage experiment. To illustrate the method, consider a computer store offering special prices on its most popular laptop models. A customer can choose from two sizes of RAM, three screen sizes, and two preloaded application packages. How many different laptops can customers choose?

step1 step2 step

We can organize the information by letting M_1 and M_2 represent the two sizes of RAM; S_1, S_2, and S_3 represent the three screen sizes; and A_1 and A_2 represent the two application packages (see Figure 12.1).

step3 step 2 step1

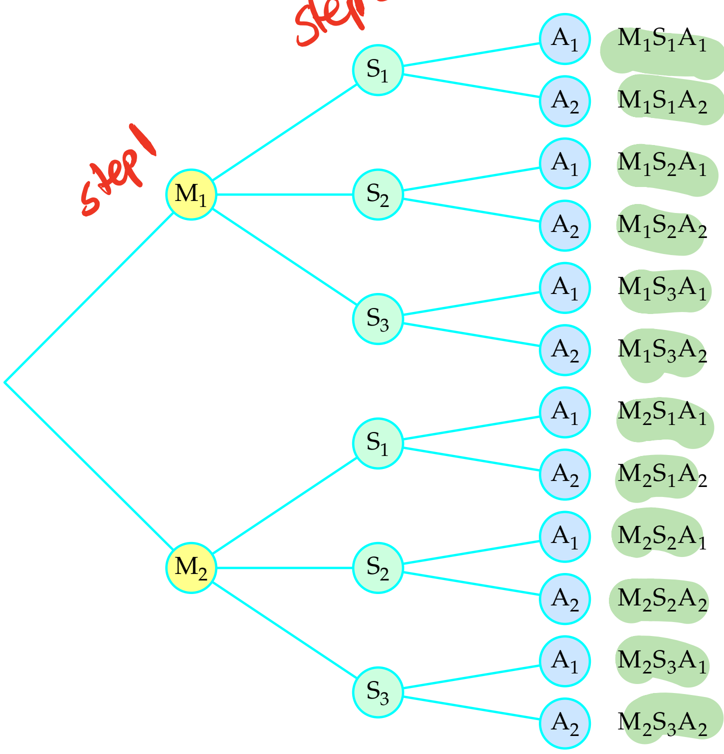


FIGURE 12.1 There are 12 possible laptops.

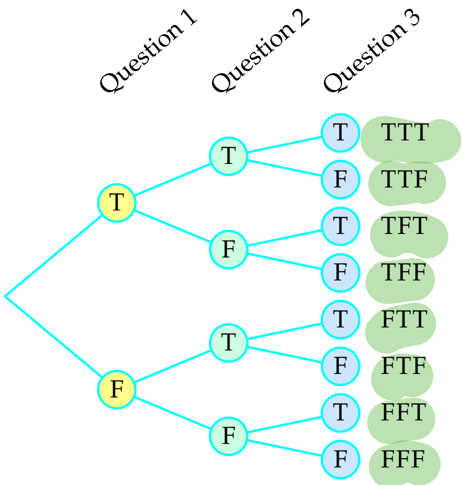


FIGURE 12.2

EXAMPLE 4 Counting Using a Tree Diagram

A true/false test consists of 10 questions. Draw a tree diagram to show the number of ways to answer the first three questions.

Solution

See the tree diagram in Figure 12.2. There are eight possible ways to answer the first three questions.

CHECK YOUR PROGRESS 4 Draw a tree diagram to determine the sample space for Example 3.

Solution See page S43.

The Counting Principle

For each of the previous problems, the possible outcomes were listed and then counted to determine the number of different outcomes. However, it is not always possible or practical to list and count outcomes. For example, the number of different five-card poker hands that can be drawn from a standard deck of 52 playing cards is 2,598,960. Trying to create a list of these hands would be quite time consuming.

Consider again the problem of selecting a laptop. By using a tree diagram, we listed the 12 possible laptops. Another way to arrive at this result is to find the product of the numbers of choices available for RAM sizes, screen sizes, and application packages.

$$\begin{bmatrix}\text{number of}\\\text{RAM sizes}\end{bmatrix} \times \begin{bmatrix}\text{number of}\\\text{screen sizes}\end{bmatrix} \times \begin{bmatrix}\text{number of}\\\text{application packages}\end{bmatrix} = \begin{bmatrix}\text{number of}\\\text{laptops}\end{bmatrix}$$
$$2 \times 3 \times 2 = 12$$

For the example of tossing two dice, there were 36 possible outcomes. We can arrive at this result without listing the outcomes by finding the product of the number of possible outcomes of rolling the red die and the number of possible outcomes of rolling the green die.

with replacement

6 · 6

$$\begin{bmatrix}\text{outcomes}\\\text{of red die}\end{bmatrix} \times \begin{bmatrix}\text{outcomes}\\\text{of green die}\end{bmatrix} = \begin{bmatrix}\text{number of}\\\text{outcomes}\end{bmatrix}$$
$$6 \times 6 = 36$$

This method of determining the number of outcomes of a multi-stage experiment without listing them is called the **counting principle**.

Counting Principle

Let E be a multi-stage experiment. If $n_1, n_2, n_3, \ldots, n_k$ are the number of possible outcomes of each of the k stages of E, then there are $n_1 \cdot n_2 \cdot n_3 \cdot \cdots \cdot n_k$ possible outcomes for E.

EXAMPLE 5 Counting by Using the Counting Principle

In horse racing, betting on a *trifecta* refers to choosing the exact order of the first three horses across the finish line. If there are eight horses in a race, how many trifectas are possible, assuming there are no ties?

Solution

8 · 7 · 6 = 336

Any one of the eight horses can be first, so $n_1 = 8$. Because a horse cannot finish both first and second, there are seven horses that can finish second; thus $n_2 = 7$. Similarly,

without replacement

there are six horses that can finish third; $n_3 = 6$. By the counting principle, there are $8 \cdot 7 \cdot 6 = 336$ possible trifectas.

$\boxed{9 \cdot 8 \cdot 7}$ =

CHECK YOUR PROGRESS 5 Nine runners are entered in a 100-meter dash for which a gold, silver, and bronze medal will be awarded for first, second, and third place finishes, respectively. In how many possible ways can the medals be awarded? (Assume that there are no ties.)

Solution See page S43.

MATH

Coding Characters for Web Pages

So that any web page will look the same on computers all over the world, standards must be adopted and adhered to by web page developers. Developers must be able to code the English alphabet, punctuation marks, special characters such as \$, ¢, and %, and all the numerals. Developers also need to be able to display letters in the Cyrillic alphabet, such as Ж and Й; letters in the French alphabet, like ç; the Japanese symbol あ; and a host of other characters and special symbols.

The standard default coding system endorsed by the World Wide Web Consortium (W3C) is Unicode Transformation Format-8 (UTF-8). This coding system uses one to four bytes to represent each character.

Here are some of the representations of characters used in UTF-8.

Character	UTF-8
¢	11000010 101000010
A	01000001
Ж	11010000 10010110
あ	11100011 10000001 10000010

We can use the counting techniques discussed in this chapter to count the number of unique characters that can be created with UTF-8. There are over one million possible characters.

Browsers use other character sets as well, such as ISO Latin 1. You can see which character set your browser is using by selecting Preferences and viewing the font characteristics.

TAKE NOTE

A *byte* is a sequence of eight digits using just 0 and 1. For instance, 01100101 is a byte.

Counting With and Without Replacement

Consider an experiment in which three balls colored red, blue, and green are placed in a box. A person reaches into the box and repeatedly pulls out a colored ball, keeping note of the color picked. The sequence of colors will depend on whether the balls are returned to the box after each pick. We say that the experiment can be performed *with replacement* or *without replacement*.

Consider the following two situations.

$9 \cdot 8 \cdot 7 \cdot 6 = \boxed{3024}$

1. How many four-digit numbers can be formed from the digits 1 through 9 if no digit can be repeated? no repeating → without replacement

$9 \cdot 9 \cdot 9 \cdot 9 = \boxed{6561}$

2. How many four-digit numbers can be formed from the digits 1 through 9 if a digit can be used repeatedly? with replacement

In the first case, there are nine choices for the first digit ($n_1 = 9$). Because a digit cannot be repeated, the first digit chosen cannot be used again. Thus there are only eight choices for the second digit ($n_2 = 8$). Because neither of the first two digits can be used as the third digit, there are only seven choices for the third digit ($n_3 = 7$). Similarly, there are six choices for the fourth digit ($n_4 = 6$). By the counting principle, there are $9 \cdot 8 \cdot 7 \cdot 6 = 3024$ four-digit numbers in which no digit is repeated.

In the second case, there are nine choices for the first digit ($n_1 = 9$). Because a digit can be used repeatedly, the first digit chosen can be used again. Thus there are nine choices for the second digit ($n_2 = 9$) and, similarly, nine choices for the third and fourth digits ($n_3 = 9, n_4 = 9$). By the counting principle, there are $9 \cdot 9 \cdot 9 \cdot 9 = 6561$ four-digit numbers when digits can be used repeatedly.

The set of four-digit numbers created without replacement includes numbers such as 3867, 7941, and 9128. In these numbers, no digit is repeated. However, numbers such as 6465, 9911, and 2222, each of which contains at least one repeated digit, can be created only with replacement.

QUESTION Does a multi-stage experiment performed with replacement generally have more or fewer possible outcomes than the same experiment performed without replacement?

more outcomes

fewer

EXAMPLE 6 Counting With and Without Replacement

5 letters

From the letters a, b, c, d, and e, how many four-letter groups can be formed if

a. a letter can be used more than once? with replacement

b. each letter can be used exactly once? without replacement

at most

$5 \cdot 5 \cdot 5 \cdot 5 = \boxed{625}$

$5 \cdot 4 \cdot 3 \cdot 2 = \boxed{120}$

Solution

a. Because each letter can be repeated, there are $5 \cdot 5 \cdot 5 \cdot 5 = 625$ possible four-letter groups.

b. Because each letter can be used only once, there are $5 \cdot 4 \cdot 3 \cdot 2 = 120$ four-letter groups in which no letter is repeated.

CHECK YOUR PROGRESS 6 In how many ways can three awards be given to five students if

a. each student may receive more than one award? with replacement

b. each student may receive no more than one award? without

$5 \cdot 5 \cdot 5 = 5^3 = \boxed{125}$

$5 \cdot 4 \cdot 3 = \boxed{60}$

Solution See page S43.

ANSWER More. Each stage of an experiment (after the first) performed without replacement will have fewer possible outcomes than the preceding stage. Performed with replacement, each stage of the experiment has the same number of outcomes.

EXCURSION

Decision Trees

Decision trees are tree diagrams that are used to solve problems that involve many choices. To illustrate, suppose we are given eight coins, one of which is counterfeit and slightly heavier than the other seven. Using a balance scale, we must find the counterfeit coin.

EXTENSIONS

35. Computer Programming A main component of any computer programming language is its method of repeating a series of computations. Each programming language has its own syntax for performing those "loops." In one programming language, BASIC, the structure is similar to the display below.

```
FOR I = 1 TO 10 ——— Start with I = 1.
    FOR J = 1 TO 15 — Start with J = 1.
        SUM = I + J
    NEXT J ← Increase J by 1 until J > 15.
NEXT I ← Increase I by 1 until I > 10.
END
```

The program repeats each loop until the index variables, I and J in this case, exceed a certain value. After this program is executed, how many times will the instruction SUM = I + J have been executed? What is the final value of SUM?

36. Review the rules of the game of checkers, and make a tree diagram that shows all of the first two moves that are possible by one player of a checker game. Assume that no moves are blocked by the opponent's checkers. (*Hint:* It may help to number the squares of the checkerboard.)

12.1-2 Counting 12.3-4 Probability

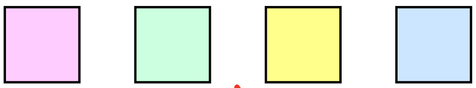
SECTION 12.2 Permutations and Combinations

fancy formulas to save time

1. Factorial

Factorial

Suppose four different colored squares are arranged in a row. One possibility is shown below.



all

How many different ways are there to order the colors? There are four choices for the first square, three choices for the second square, two choices for the third square, and one choice for the fourth square. By the counting principle, there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ different arrangements of the four squares. Note from this example that the number of arrangements equals the product of the natural numbers n through 1, where n is the number of objects. This product is called a *factorial*.

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

$3! = 3 \cdot 2 \cdot 1 = 6$

n Factorial

***n* factorial** is the product of the natural numbers n through 1 and is symbolized by $\boldsymbol{n!}$.

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$

CALCULATOR NOTE

The factorial of a number becomes quite large for even relatively small numbers. For instance, 58! is the approximate number of atoms in the known universe. The number 70! is greater than 10 with 100 zeros after it. This number is larger than most scientific calculators can handle.

Here are some examples:

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$
$$8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40{,}320$$
$$1! = 1$$

On some occasions it will be necessary to use 0! (zero factorial). Because it is impossible to define zero factorial in terms of a product of natural numbers, a standard definition is used.

Zero Factorial

$$0! = 1$$

reason 1

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$
$3! = 3 \cdot 2 \cdot 1 = 6$ $\div 4$
$2! = 2 \cdot 1 = 2$ $\div 3$
$1! = 1 = 1$ $\div 2$
$\boxed{0! = 1}$ $\div 1$

$0! = 1$
completes the pattern.

A factorial can be written in terms of smaller factorials. This is useful when calculating large factorials. For example,

$$10! = 10 \cdot 9!$$
$$10! = 10 \cdot 9 \cdot 8!$$
$$10! = 10 \cdot 9 \cdot 8 \cdot 7!$$

reason 2

$5! = 5 \cdot \underbrace{\boxed{4 \cdot 3 \cdot 2 \cdot 1}}_{4!}$

$5! = 5 \cdot 4!$

$3! = 3 \cdot 2!$

$\boxed{1! = 1 \cdot 0!}$

$(1) = (1)(0!)$

has to be 1

In other words, making $0! = 1$ allows our formula to work without needing special cases (to always work)

$\boxed{\text{reason } 3}$ The 3rd reason that $0! = 1$ is as follows.

$3! = 6 \quad \{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}$

$2! = 2 \quad \{(1,2), (2,1)\}$

$1! = 1 \quad \{(1)\}$

$0! = 1 \quad \{()\}$

A factorial can be written in terms of smaller factorials. This is useful when calculating large factorials. For example,

$$10! = 10 \cdot 9!$$
$$10! = 10 \cdot 9 \cdot 8!$$
$$10! = 10 \cdot 9 \cdot 8 \cdot 7!$$

EXAMPLE 1 **Simplify Factorials**

Evaluate: **a.** $5! - 3!$ **b.** $\dfrac{9!}{6!}$

Solution

a. $5! - 3! = (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) - (3 \cdot 2 \cdot 1) = 120 - 6 = 114$

b. $\dfrac{9!}{6!} = \dfrac{9 \cdot 8 \cdot 7 \cdot \cancel{6!}}{\cancel{6!}} = 9 \cdot 8 \cdot 7 = 504$

– **CHECK YOUR PROGRESS 1** Evaluate: **a.** $7! + 4!$ **b.** $\dfrac{8!}{4!}$

Solution See page S43.

a) $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 + 4 \cdot 3 \cdot 2 \cdot 1 = 5040 + 24 = 5064$

b) $\dfrac{8!}{4!}$

There's a trick here!

$\dfrac{8 \cdot 7 \cdot 6 \cdot 5 \cdot \cancel{4 \cdot 3 \cdot 2 \cdot 1}}{\cancel{4 \cdot 3 \cdot 2 \cdot 1}}$

$= 8 \cdot 7 \cdot 6 \cdot 5 = \boxed{1680}$

2. Permutations

- might not be asked to use all the objects
- objects might not be all distinct

Permutations

Determining the number of possible ordered arrangements of a group of distinct objects, as we did with the squares earlier, is one application of the counting principle. Each arrangement of this type is called a *permutation*.

Permutation

A **permutation** is an arrangement of objects in a definite order.

For example, abc and cba are two different permutations of the letters a, b, and c. As a second example, 122 and 212 are two different permutations of one 1 and two 2s.

The counting principle is used to count the number of different permutations of any set of objects. We will begin our discussion using sets of *distinct* objects; that is, sets in which no two objects are the same. For instance, the objects a, b, c, d are distinct, whereas the objects □, ☆, ○, ☆ are not.

Most music players allow the user to create a playlist, which is a list of songs that can be played on the device. Many of these players have a *shuffle* feature that plays the songs in a playlist in a different order each time.

Suppose a playlist consists of two songs, a rock song and a reggae song. If the shuffle feature is used, the songs could be played in two orders:

rock then reggae or reggae then rock

Thus there are two choices for the first song (rock or reggae) and there is one choice for the second song (whichever song was not played first). By the counting principle, there are $2 \cdot 1 = 2! = 2$ permutations or orders in which to play the songs.

With three songs in a playlist, one rock, one reggae, and one country, there are three choices for the first song, two choices for the second song, and one choice for the third song. By the counting principle, there are $3 \cdot 2 \cdot 1 = 3! = 6$ permutations in which the three songs could be played.

Permutation 1	Permutation 2	Permutation 3	Permutation 4	Permutation 5	Permutation 6
Rock	Rock	Reggae	Reggae	Country	Country
Reggae	Country	Rock	Country	Rock	Reggae
Country	Reggae	Country	Rock	Reggae	Rock

POINT OF INTEREST

If a playlist has 10 songs, there are $10! = 3{,}628{,}800$ possible permutations of the songs in shuffle mode. If each song were 2 min long, it would take over 120,000 h to listen to every permutation.

With four songs in a playlist, there are $4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$ orders in which the songs could be played. In general, if there are n songs in a playlist, then there are $n!$ permutations, or orders, in which the songs could be played.

Suppose now that you have a playlist that consists of eight songs but you have time to listen to only three of the songs. In shuffle mode, any one of the eight songs could play first, then any one of the seven remaining songs could play second, and then any one of the remaining six songs could play third. By the counting principle, there are $8 \cdot 7 \cdot 6 = 336$ permutations in which the songs could be played.

The following formula can be used to determine the number of permutations of n distinct objects (the songs in the example above), of which k are selected.

Permutation Formula for Distinct Objects

The number of permutations of n distinct objects selected k at a time is

number of ways to order k out of n candidates

$$P(n, k) = \frac{n!}{(n-k)!}$$

Applying this formula to the situation above in which there were eight songs ($n = 8$), of which only three songs ($k = 3$) could be played, we have

$$P(\overset{n}{8}, \overset{k}{3}) = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{\boxed{8 \cdot 7 \cdot 6} \cdot 5!}{5!} = \boxed{8 \cdot 7 \cdot 6} = 336$$

There are 336 permutations of playing the songs. This is the same answer we obtained using the counting principle.

EXAMPLE 2 **Counting Permutations**

A university tennis team consists of six players who are ranked from 1 through 6. If a tennis coach has 10 players from which to choose, how many different ranked tennis teams can the coach select?

by the counting principle

$10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5$

Solution

Because the players on the tennis team are ranked from 1 through 6, a team with player A in position 1 is different from a team with player A in position 2. Therefore, the number of different teams is the number of permutations of 10 players selected 6 at a time.

$$P(10, 6) = \frac{10!}{(10-6)!} = \frac{10!}{4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4!}{4!}$$

by formula above

$$= 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 151{,}200$$

There are 151,200 possible tennis teams.

by counting principle

CHECK YOUR PROGRESS 2 A college golf team consists of five players who are ranked from 1 through 5. If a golf coach has eight players from which to choose, how many different ranked golf teams can the coach select?

Solution See page S43.

EXAMPLE 3 Counting Permutations

In 2015, 18 horses were entered in the Kentucky Derby. How many different finishes of first through fourth place were possible?

Solution

Because the order in which the horses finish the race is important, the number of possible finishes of first through fourth place is $P(18, 4)$.

by the P(n,k) formula

$$\boxed{P(18, 4)} = \boxed{\frac{18!}{(18-4)!}} = \frac{18!}{14!} = \frac{18 \cdot 17 \cdot 16 \cdot 15 \cdot \cancel{14!}}{\cancel{14!}}$$

by the counting principle

$$= 18 \cdot 17 \cdot 16 \cdot 15 = \boxed{73{,}440}$$

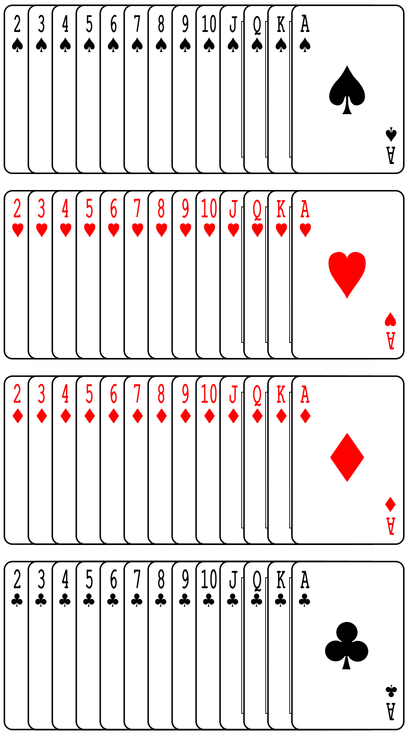
There were 73,440 possible finishes of first through fourth places.

CHECK YOUR PROGRESS 3 There were 43 cars entered in the 2015 Daytona 500 NASCAR race. In how many different ways could the first, second, and third place prizes have been awarded?

Solution See page S43.

MATH

How Many Shuffles?



A standard deck of playing cards

A standard deck of playing cards consists of 52 different cards divided into four suits: spades (♠), hearts (♥), diamonds (♦), and clubs (♣). Each shuffle of the deck results in a new arrangement of the cards. We could say that each shuffle results in a new permutation of the cards. There are $P(52, 52) = 52! \approx 8 \times 10^{67}$ (that's 8 with 67 zeros after it) possible arrangements.

Suppose a deck has each of the four suits arranged in order from 2 through ace. How many shuffles are necessary to achieve a randomly ordered deck in which any card is equally likely to occur in any position in the deck?

Two mathematicians, Dave Bayer of Columbia University and Persi Diaconis of Harvard University, have shown that seven shuffles are enough. Their proof has many applications to complicated counting problems. One problem in particular is that of analyzing speech patterns. Solving this problem is critical to enabling computers to interpret human speech.

Applying Several Counting Techniques

The permutation formula is derived from the counting principle. This formula is a convenient way of expressing the number of ways in which the items in an ordered list can be arranged. Both the permutation formula and the counting principle are needed to solve some counting problems.

EXAMPLE 4 Counting Using Several Methods

Using 12.2 Part 1. Factorial

$9! = 362880$

Five women and four men are to be seated in a row of nine chairs. How many different seating arrangements are possible if

a. there are no restrictions on the seating arrangements? $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

b. the women sit together and the men sit together?

Solution

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \; 4 \cdot 3 \cdot 2 \cdot 1 + 4 \cdot 3 \cdot 2 \cdot 1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = \boxed{2 \cdot 5!\,4!}$

Because seating arrangements have a definite order, they are permutations.

a. If there are no restrictions on the seating arrangements, then the number of seating arrangements is $P(9, 9)$. $\rightarrow P(n,k) = \frac{n!}{(n-k)!}$

Using 12.2 Part 2, Permutations

In general

$$\boxed{P(n,n) = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!}$$

$$P(9, 9) = \frac{9!}{(9-9)!} = \frac{9!}{0!} = 9! = 362{,}880$$

$0! = 1$

There are 362,880 seating arrangements.

b. This is a multi-stage experiment, so both the permutation formula and the counting principle will be used. There are 5! ways to arrange the women and 4! ways to arrange the men. We must also consider that either the women or the men could be seated at the beginning of the row. There are two ways to do this. By the counting principle, there are $2 \cdot 5! \cdot 4!$ ways to seat the women together and the men together.

$$\boxed{2 \cdot 5! \cdot 4! = 5760}$$

There are 5760 arrangements in which women sit together and men sit together.

CHECK YOUR PROGRESS 4 There are seven tutors, three juniors and four seniors, who must be assigned to the 7 h that a math center is open each day. If each tutor works 1 h per day, how many different tutoring schedules are possible if

a. there are no restrictions?

b. the juniors tutor during the first 3 h and the seniors tutor during the last 4 h?

Solution See page S43.

12.2 Topic 3.

Permutations of Indistinguishable Objects

Up to this point we have been counting the number of permutations of *distinct* objects. We now look at the situation of arranging objects when some of them are identical. In the case of identical or indistinguishable objects, a modification of the permutation formula is necessary. The general idea is to count the number of permutations as if all of the objects were distinct and then remove the permutations that look alike.

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \div 3! \div 2!$

Consider the permutations of the letters *bbbcc*. We first assume all the letters are different by labeling them as $b_1b_2b_3c_1c_2$. Using the permutation formula, there are $5! = 120$ permutations. Now we need to remove repeated permutations. Note that

$\boxed{\ldots b \ldots b \ldots b}$

$$b_1b_2b_3c_1c_2 \quad b_1b_3b_2c_1c_2 \quad b_2b_1b_3c_1c_2 \quad b_2b_3b_1c_1c_2 \quad b_3b_2b_1c_1c_2 \quad b_3b_1b_2c_1c_2$$

$\boxed{\ldots c \ldots c \ldots}$

are all distinct permutations that end with c_1c_2. However, if we replace each b_1, b_2, and b_3 with b, all six of these permutations will look the same. Thus there are $3! = 6$ times too many arrangements for each arrangement of c_1 and c_2. A similar argument applies to the c's. There are $2! = 2$ times too many permutations of the c's for each arrangement of b's.

$$\boxed{\frac{5!}{3!\,2!}} = 10$$

over counted by 3! because bbb

bbbcc	bbbcc	bbbcc	bbbcc	bbbcc	bbbcc
bbcbc	bbcbc	bbcbc	bbcbc	bbcbc	bbcbc
bbccb	bbccb	bbccb	bbccb	bbccb	bbccb
bcbbc	bcbbc	bcbbc	bcbbc	bcbbc	bcbbc
bcbcb	bcbcb	bcbcb	bcbcb	bcbcb	bcbcb
bccbb	bccbb	bccbb	bccbb	bccbb	bccbb
cbbbc	cbbbc	cbbbc	cbbbc	cbbbc	cbbbc
cbbcb	cbbcb	cbbcb	cbbcb	cbbcb	cbbcb
cbcbb	cbcbb	cbcbb	cbcbb	cbcbb	cbcbb
ccbbb	ccbbb	ccbbb	ccbbb	ccbbb	ccbbb

bbbcc	bbbcc	bbbcc	bbbcc	bbbcc	bbbcc
bbcbc	bbcbc	bbcbc	bbcbc	bbcbc	bbcbc
bbccb	bbccb	bbccb	bbccb	bbccb	bbccb
bcbbc	bcbbc	bcbbc	bcbbc	bcbbc	bcbbc
bcbcb	bcbcb	bcbcb	bcbcb	bcbcb	bcbcb
bccbb	bccbb	bccbb	bccbb	bccbb	bccbb
cbbbc	cbbbc	cbbbc	cbbbc	cbbbc	cbbbc
cbbcb	cbbcb	cbbcb	cbbcb	cbbcb	cbbcb
cbcbb	cbcbb	cbcbb	cbcbb	cbcbb	cbcbb
ccbbb	ccbbb	ccbbb	ccbbb	ccbbb	ccbbb

over counted by 2! by cc

answer $* \, 3! * 2! = 5!$

That's why

answer $= \dfrac{5!}{3!\,2!}$

$\boxed{\dfrac{5!}{3!2!}} = 10$

Combining the results above, the number of permutations of *bbbcc* is

$$\frac{5!}{3! \cdot 2!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(3 \cdot 2 \cdot 1) \cdot (2 \cdot 1)} = 10$$

There are 10 distinct permutations of *bbbcc*.

formula

Permutations of Objects, Some of Which Are Identical

The number of distinguishable permutations of n objects of r different types, where k_1 identical objects are of one type, k_2 are of another, and so on, is given by

$$\frac{n!}{k_1! \cdot k_2! \cdot \cdots \cdot k_r!}$$

where $k_1 + k_2 + \cdots + k_r = n$.

EXAMPLE 5 Permutations of Identical Objects

A password requires 7 characters. If a person who lives at 155 Nunn Road wants a password to be an arrangement of the characters 155NUNN, how many different passwords are possible?

Solution

We are looking for the number of permutations of the characters 155NUNN. With $n = 7$ (number of characters), $k_1 = 1$ (number of 1's), $k_2 = 2$ (number of 5's), $k_3 = 3$ (number of N's), and $k_4 = 1$ (number of U's) we have

$$\frac{7!}{1! \cdot 2! \cdot 3! \cdot 1!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3!}{2 \cdot 3!} = 420$$

There are 420 possible passwords.

CHECK YOUR PROGRESS 5 Eight coins—3 pennies, 2 nickels, and 3 dimes—are placed in a single stack. How many different stacks are possible if

a. there are no restrictions on the placement of the coins?

b. the dimes must stay together?

Solution See page S43.

$\boxed{\dfrac{8!}{3!2!3!}}$

$= \dfrac{8 \cdot 7 \cdot \overset{2}{\cancel{6}} \cdot 5 \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{3} \cdot \cancel{2} \cdot \cancel{1} \cdot \cancel{2} \cdot \cancel{1} \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}$

$= \boxed{560}$

Combinations

For some arrangements of objects, the order in which the objects are arranged is important. These are permutations. If a telephone extension is 2537, then the digits must be dialed in exactly that order. On the other hand, if you were to receive a \$1 bill, a \$5 bill, and a \$10 bill, you would have \$16 regardless of the order in which you received the bills. A **combination** is a collection of objects in which the order of the objects is not important. The three-letter sequences acb and bca are *different* permutations but the *same* combination.

12.2
Topic 4.
Combinations

QUESTION From a group of 45 applicants, five identical scholarships will be awarded. Is the number of ways in which the scholarships can be awarded determined by permutations or combinations?

ANSWER Combinations. The order in which the scholarship winners are chosen is not important.

The formula for finding the number of combinations is derived in much the same manner as the formula for finding the number of permutations of identical objects. Consider the problem of finding the number of possible combinations when choosing three letters from the letters a, b, c, d, and e, without replacement. For each choice of three letters, there are 3! permutations. For example, choosing the letters a, d, and e gives the following six permutations.

ade aed dea dae ead eda

Because there are six permutations and each permutation represents the *same* combination, the number of permutations is six times the number of combinations. This is true each time three letters are selected. Therefore, to find the number of combinations of five objects chosen three at a time, divide the number of permutations by 3! = 6. The number of combinations of five objects chosen three at a time is

$$\frac{P(5,3)}{3!} = \frac{5!}{3!\cdot(5-3)!} = \frac{5!}{3!\cdot 2!} = \frac{5\cdot 4\cdot 3!}{3!\cdot 2!} = \frac{5\cdot 4}{2\cdot 1} = 10$$

"n choose k"

CALCULATOR NOTE

Some calculators can compute permutations and combinations directly. For instance, on a TI-83/84 calculator, enter 11 nCr 5 for $C(11, 5)$. The nCr operation is accessible in the probability menu after pressing the MATH key.

• $_{11}C_5$ means

• $C(11,5)$ means

• $\binom{11}{5}$

• "11 choose 5"

Combination Formula

"Number of ways to choose k out of n objects"

The number of combinations of n objects chosen k at a time is

$$\boxed{C(n,k)} = \frac{P(n,k)}{k!} = \frac{n!}{k!\cdot(n-k)!}$$

intuition 1 intuition 2

In Section 12.1, we stated that there were 2,598,960 possible 5-card poker hands. This number was calculated using the combination formula. Because the 5-card hand ace of hearts, king of diamonds, queen of clubs, jack of spades, 10 of hearts is exactly the same as the 5-card hand king of diamonds, jack of spades, queen of clubs, 10 of hearts, ace of hearts, the order of the cards is not important, and therefore the number of hands is a combination. The number of different 5-card poker hands is the combination of 52 cards chosen 5 at a time, which is given by $C(52, 5)$.

$$C(52,5) = \frac{52!}{5!\cdot(52-5)!} = \frac{52!}{5!\cdot 47!} = \frac{52\cdot 51\cdot 50\cdot 49\cdot 48\cdot 47!}{5!\cdot 47!}$$

$$= \frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5\cdot 4\cdot 3\cdot 2\cdot 1} = 2{,}598{,}960$$

using 12.2 Part 2

Intuition 1

$$\frac{11\cdot 10\cdot 9\cdot 8\cdot 7}{5\cdot 4\cdot 3\cdot 2\cdot 1}$$

Intuition 2 – using 12.2 Part 3

1 2 3 4 5 6 7 8 9 10 11

Y Y Y Y Y N N N N N N

$$\frac{11!}{5!\,6!}$$

EXAMPLE 6 Counting Using the Combination Formula

An emergency room at a hospital has 11 nurses on staff. Each night a team of 5 nurses is on duty. In how many different ways can the team of 5 nurses be chosen?

Solution

This is a combination problem, because the order in which the nurses are chosen is not important. The 5 nurses N_1, N_2, N_3, N_4, N_5 are the same as the 5 nurses N_3, N_5, N_1, N_2, N_4.

$$C(11,5) = \frac{11!}{5!\cdot(11-5)!} = \frac{11!}{5!\cdot 6!} = \frac{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot \cancel{6!}}{5!\cdot\cancel{6!}}$$

$$= \frac{11\cdot 10\cdot 9\cdot 8\cdot 7}{5\cdot 4\cdot 3\cdot 2\cdot 1} = \boxed{462}$$

There are 462 possible teams of 5 nurses.

CHECK YOUR PROGRESS 6 A restaurant employs 16 waiters and waitresses. In how many ways can a group of 9 waiters and waitresses be chosen for the lunch shift?

Solution See page S44.

"Committee" → combinations

EXAMPLE 7 Counting Using the Combination Formula and the Counting Principle

A committee of 5 is chosen from 5 mathematicians and 6 economists. How many different committees are possible if the committee must include 2 mathematicians and 3 economists?

$\frac{5\cdot 4}{2\cdot 1}\cdot\frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1}$

$10 \cdot 20$

$= \boxed{200}$

Solution

Because a committee of professors A, B, C, D, and E is exactly the same as a committee of professors B, D, E, A, and C, choosing a committee is an example of choosing a combination. There are 5 mathematicians from whom 2 are chosen, which is equivalent to $C(5, 2)$ combinations. There are 6 economists from whom 3 are chosen, which is equivalent to $C(6, 3)$ combinations. Therefore, by the counting principle, there are $C(5, 2) \cdot C(6, 3)$ ways to choose 2 mathematicians and 3 economists.

$$C(5, 2) \cdot C(6, 3) = \frac{5!}{2! \cdot 3!} \cdot \frac{6!}{3! \cdot 3!} = 10 \cdot 20 = 200$$

There are 200 possible committees consisting of 2 mathematicians and 3 economists.

CHECK YOUR PROGRESS 7 An IRS auditor randomly chooses 5 tax returns to audit from a stack of 10 tax returns, 4 of which are from corporations and 6 of which are from individuals. In how many different ways can the auditor choose the tax returns if the auditor wants to include 3 corporate and 2 individual returns?

Solution See page S44.

MATH MATTERS

Buying Every Possible Lottery Ticket

A lottery prize in Pennsylvania reached \$65 million. A resident of the state suggested that it might be worth buying a ticket for every possible combination of numbers. To win the \$65 million, a player has to correctly select 6 of 50 numbers. Each ticket costs \$1. Because the order in which the numbers are drawn is not important, the number of different possible tickets is $C(50, 6) = 15{,}890{,}700$. Thus it would cost the resident \$15,890,700 to purchase tickets for every possible combination of numbers.

It might seem that an approximately \$16 million investment to win \$65 million is reasonable. Unfortunately, when prize levels reach such lofty heights, many more people play the lottery. This increases the chances that more than one person will select the winning combination of numbers. In fact, eight people chose the winning numbers, and each received approximately \$8 million. Now the \$16 million investment does not look very appealing.

-Cards

A standard deck of playing cards contains 4 suits: spades, hearts, diamonds, and clubs. Each suit has 13 cards: 2 through 10, jack, queen, king, and ace. See the Math Matters on page 699.

EXAMPLE 8 Counting Problems with Cards

From a standard deck of playing cards, 5 cards are chosen. How many 5-card combinations contain

a. 2 kings and 3 queens?

b. 5 hearts?

c. 5 cards of the same suit?

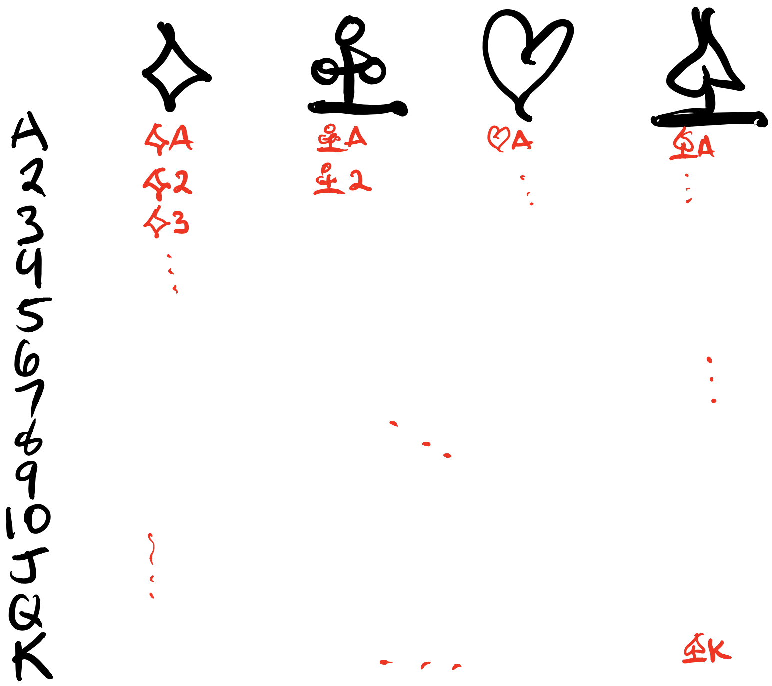
$C(4,2)\cdot C(4,3) = \frac{4\cdot 3}{2\cdot 1}\cdot\frac{4\cdot 3\cdot 2}{3\cdot 2\cdot 1} = 6\cdot 4 = \boxed{24}$

$C(13,5) = \frac{13\cdot 12\cdot 11\cdot 10\cdot 9}{5\cdot 4\cdot 3\cdot 2\cdot 1} = \boxed{1287}$

$1287 + 1287 + 1287 + 1287 = 4\cdot 1287 = \boxed{5148}$

A standard deck of playing cards contains 4 suits: spades, hearts, diamonds, and clubs. Each suit has 13 cards: 2 through 10, jack, queen, king, and ace. See the Math Matters on page 699.

deck has $4 * 13 = 52$ cards



52 cards