Appendices

Appendix 1

I. Determining the theoretical Front/Rear axle slip balance

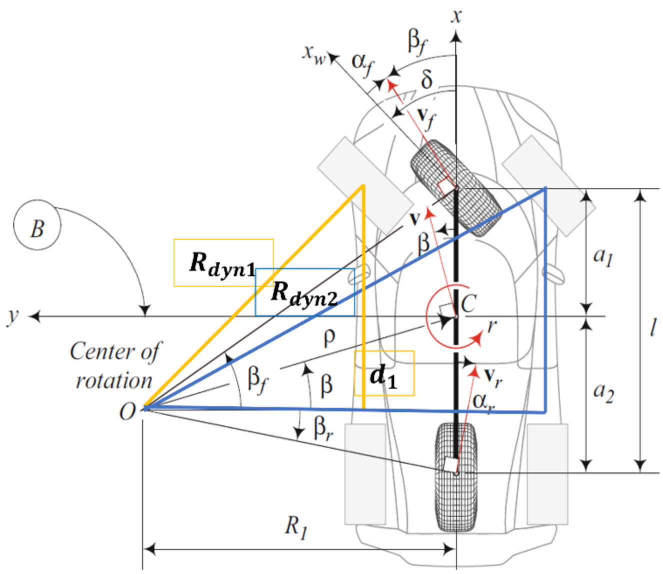

Figure A.1. A schematic of a two-wheel vehicle with instantaneous center of rotation.

There are several steps to consider for the calculation of the theoretical Front/Rear slip balance. First, we derive the dynamic radiuses of the wheels connected to the front axle.

The distance between the center of rotation and the middle of the rear axle track as shown in Figure **A.1**, is computed as:

$$R_{stat} = R_1 = \frac{l}{tan(\delta)} \approx \frac{l * t_s}{\delta}, \tag{A.1}$$

a. Front axle average slip balance :

The importance of the steering angle amplitude outweighs that of the wheel sideslip angle when considering the front axle of the vehicle. The steering wheel angle primarily influences the front axle wheels deflection, as the impact of the the wheel sideslip angle is nearly insignificant. As a result, the front wheel sideslip angle α_f is not considered in the formula for the dynamic radius at the front axle. However, it does play a role in determining the dynamic radius at the rear α_r.

To calculate the two front axle wheel radii, we use the Pythagoras theorem for a rectangular triangle. The front axle inner wheel dynamic radius of turn is denoted as R_{dyn1}, while the front axle outer wheel dynamic radius of turn is represented by R_{dyn2}. Both radii are expressed as follow,

$$R_{dyn1} = \sqrt{d_1^2 + \left(R_{stat} - \frac{t_f}{2}\right)^2} \leftarrow front\ inner\ wheel \tag{A.2}$$

$$R_{dyn2} = \sqrt{d_1^2 + \left(R_{stat} + \frac{t_f}{2}\right)^2} \leftarrow rear\ outer\ wheel \tag{A.3}$$

$$\boldsymbol{R_{FAdyn}} = \sqrt{\boldsymbol{d_1^2} + (\boldsymbol{R_{stat}})^2} \leftarrow front\ axle\ average\ dynamic\ radius \tag{A.4}$$

R_{FAdyn} : the front axle average dynamic radius of turn.

d_1 : length of the opposite side of the rectangular triangle formed by $\left(R_{stat} - t_f/2\right)$ (Adjacent side) and R_{dyn1} (hypotenuse).

d_2 : the difference between the wheelbase $\boldsymbol{l}$ and $\boldsymbol{d_1}$,

$$d_2 = l - d_1 \tag{A.5}$$

Now, using the two dynamic radii of turn, we'll deduce the speed of each wheel on the front axle. This will allow us to calculate the average speed of the whole axle as follow,

$$V_1 = R_{dyn1} * \dot{\psi}_{meas} = \boldsymbol{r_{dyn}} * \omega_1 \tag{A.6}$$

$$V_2 = R_{dyn2} * \dot{\psi}_{meas} = \boldsymbol{r_{dyn}} * \omega_2 \tag{A.7}$$

The average speed is,

$$\frac{V_1 + V_2}{2} = \frac{R_{dyn1} + R_{dyn2}}{2} * \dot{\psi}_{meas} = \boldsymbol{R_{FAdyn}} * \dot{\psi}_{meas} \tag{A.8}$$

where V_1 and V_2 are the front axle inner and outer wheels speed respectively. The vehicle yaw rate measure $\dot{\psi}_{meas}$, the dynamic wheel radius $\boldsymbol{r_{dyn}}$, and the wheels angular velocities ω_1 and ω_2 appear in the expression of the average speed of the front axle. So that,

$$\boldsymbol{R_{FAdyn}} = \frac{R_{dyn1} + R_{dyn2}}{2} = \sqrt{d_1^2 + (R_{stat})^2}. \tag{A.10}$$

Equation A.8 is rewritten as,

$$\frac{V_1 + V_2}{2} = \left(\frac{R_{dyn1} + R_{dyn2}}{2}\right) * \dot{\boldsymbol{\psi}}_{\boldsymbol{meas}} = \left(\frac{\omega_1 + \omega_2}{2}\right) * \boldsymbol{r_{dyn}} \tag{A.11}$$

such that the average front axle angular velocity reads as,

$$\boldsymbol{\omega_{FAmoy}} = \frac{\omega_1 + \omega_2}{2} \tag{A.12}$$

Finally, rearranging equation A.11 yields,

$$\left(R_{dyn1} + R_{dyn2}\right) * \dot{\psi}_{meas} = (\omega_1 + \omega_2) * r_{dyn} = V_1 + V_2 \tag{A.13}$$

Knowing the vehicle velocity V_v is

$$V_v = R_{stat} * \dot{\psi}_{meas} = \frac{l}{\delta} * \dot{\psi}_{meas} = \frac{l}{\frac{\delta}{t_s}} \dot{\psi}_{meas} \quad \Leftrightarrow \quad \dot{\boldsymbol{\psi}}_{\boldsymbol{meas}} = \frac{\frac{\boldsymbol{\delta}}{\boldsymbol{t_s}}}{\boldsymbol{l}} * \boldsymbol{V_v} \tag{A.14}$$

we can establish a relationship between the average front axle slip S_{FAmoy} and the front axle dynamic radii of turn $\left(R_{dyn1} + R_{dyn2}\right)$ through the following formulation,

$$S_{FAmoy} = (\omega_1 + \omega_2) * \frac{r_{dyn}}{V_v} = \left(R_{dyn1} + R_{dyn2}\right) * K_u(\delta) \tag{A.15}$$

with,

$$K_u(\delta) = \frac{\delta}{l * t_s} \tag{A.16}$$

Based on equation A.15, we can infer that if we possess the radii, we can determine the slip. Now, recall,

$$\boldsymbol{R_{FAdyn}} = \sqrt{d_1^2 + (R_{stat})^2} = \sqrt{(l - d_2)^2 + \left(\frac{l}{tan\,(\delta)}\right)^2} \tag{A.17}$$

The rear axle average dynamic radius R_{RAdyn}, using Pythagoras theorem gives,

$$d_2^2 + {R_{stat}}^2 = R_{RAdyn}^2 \quad \Longrightarrow \quad d_2 = \sqrt{R_{RAdyn}^2 - {R_{stat}}^2} \tag{A.18}$$

It can be also expressed as,

$$R_{RAdyn} = \frac{R_{stat}}{\cos\,(\alpha_r)} \quad \Longrightarrow \quad d_2 = \sqrt{\left(\frac{R_{stat}}{\cos\,(\alpha_r)}\right)^2 - (R_{stat})^2} \tag{A.19}$$

Finally, the length d is equal to

$$d_2 = R_{stat}\sqrt{\frac{1}{(\cos{(\alpha_r)})^2} - 1} = \frac{l}{tan\ (\delta)}\sqrt{\frac{1}{(\cos{(\alpha_r)})^2} - 1} \tag{A.20}$$

By plugging equation A.20 into A.17, we obtain,

$$R_{FAdyn} = \sqrt{\left(l - \sqrt{\frac{l^2}{tan\ (\delta)^2(\cos{(\alpha_r)})^2} - \left(\frac{l}{tan\ (\delta)}\right)^2}\right)^2 + \left(\frac{l}{tan\ (\delta)}\right)^2} \tag{A.21}$$

b. <u>Rear axle average slip balance:</u>

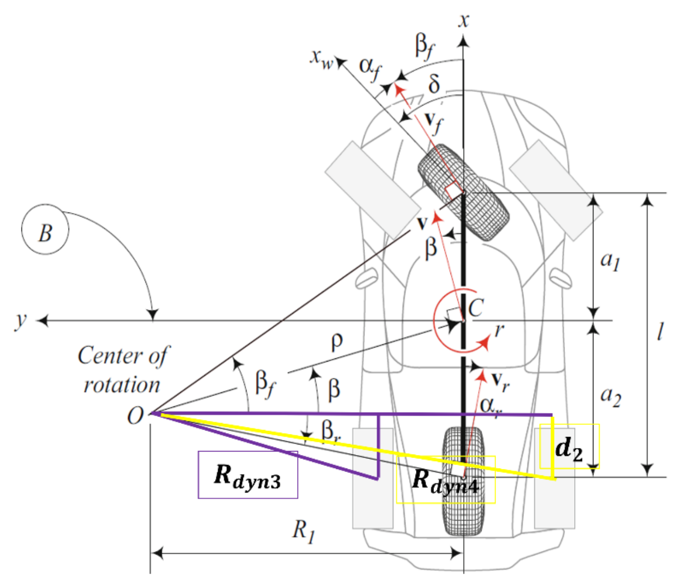

Figure A.2. A schematic of a two-wheel vehicle with instantaneous center of rotation.

Similarly, the rear axle inner wheel dynamic radius of turn is denoted as R_{dyn3}, while the rear axle outer wheel dynamic radius of turn is represented by R_{dyn4}. Both radii are expressed as follow,

$$R_{dyn3} = \sqrt{d_2^2 + \left(R_{stat} - \frac{t_f}{2}\right)^2} \tag{A.22}$$

$$\boldsymbol{R_{dyn4}} = \sqrt{\boldsymbol{d_2^2} + \left(\boldsymbol{R_{stat}} + \frac{\boldsymbol{t_f}}{\boldsymbol{2}}\right)^2} \tag{A.23}$$

With the rear axle average dynamic radius of turn

$$R_{RAdyn} = \frac{R_{stat}}{cos\ (\alpha_r)} \tag{A.24}$$

d_2 is now,

$$d_2 = \sqrt{R_{RAdyn}^2 - R_{stat}{}^2} = \sqrt{\frac{l^2}{tan\ (\delta)^2 (\cos\ (\alpha_r))^2} - \left(\frac{l}{tan\ (\delta)}\right)^2} \tag{A.25}$$

Replacing d_2 into equation A.23 yields,

$$R_{dyn4} = \sqrt{\frac{l^2}{tan\ (\delta)^2 (\cos\ (\alpha_r))^2} - \left(\frac{l}{tan\ (\delta)}\right)^2 + \left(\frac{l}{tan\ (\delta)} + \frac{t_f}{2}\right)^2} \tag{A.26}$$

Finally, the difference in slip between the slip of the outer rear wheel and the average slip of the front wheels is expressed as follows:

$$\boldsymbol{\Delta S_{theo}} = \boldsymbol{100} * \left(\frac{\boldsymbol{R_{dyn4}} - \boldsymbol{R_{FAdyn}}}{\boldsymbol{R_{FAdyn}}}\right). \tag{A.27}$$

Replacing each radius by their corresponding expression yields,

$$\Delta S_{theo} = \frac{\left(\sqrt{\left(\frac{\boldsymbol{l^2}}{\mathbf{tan}(\boldsymbol{\delta})^2 \left(\mathbf{cos}\left(\frac{\boldsymbol{V_v^2} * \boldsymbol{\delta}}{\boldsymbol{l}} \boldsymbol{\delta_{sr}}\right)\right)^2} - \left(\frac{\boldsymbol{l}}{\mathbf{tan}(\boldsymbol{\delta})}\right)^2\right) + \left(\frac{\boldsymbol{l}}{\mathbf{tan}(\boldsymbol{\delta})} + \frac{\boldsymbol{t_f}}{\boldsymbol{2}}\right)^2} - \sqrt{\left(\boldsymbol{l} - \sqrt{\frac{\boldsymbol{l^2}}{\boldsymbol{tan}\ (\boldsymbol{\delta})^2 \left(\mathbf{cos}\ (\frac{\boldsymbol{V_v^2} * \boldsymbol{\delta}}{\boldsymbol{l}} \boldsymbol{\delta_{sr}})\right)^2} - \left(\frac{\boldsymbol{l}}{\boldsymbol{tan}\ (\boldsymbol{\delta})}\right)^2}\right)^2 + \left(\frac{\boldsymbol{l}}{\boldsymbol{tan}\ (\boldsymbol{\delta})}\right)^2}\right)}{\left(\sqrt{\left(\boldsymbol{l} - \sqrt{\frac{\boldsymbol{l^2}}{\boldsymbol{tan}\ (\boldsymbol{\delta})^2 \left(\mathbf{cos}\ (\frac{\boldsymbol{V_v^2} * \boldsymbol{\delta}}{\boldsymbol{l}} \boldsymbol{\delta_{sr}})\right)^2} - \left(\frac{\boldsymbol{l}}{\boldsymbol{tan}\ (\boldsymbol{\delta})}\right)^2}\right)^2 + \left(\frac{\boldsymbol{l}}{\boldsymbol{tan}\ (\boldsymbol{\delta})}\right)^2}\right)} \tag{A.28}$$

This natural slip is used to determine which control to use in the case of vehicle' understeer or oversteer behavior. It is important to note that, we consider only the outer rear wheel radius instead of the average radius because the lateral guidance performance of a vehicle depend on the ability of the rear axle to follow the lead of the front to change direction. For this reason, only the wheel with the greater speed is considered in the theoretical slip calculation.

Appendix 2

II. Determining the Ackermann Front/Rear axle slip balance

For Ackermann slip balance computation, the center of rotation is taken such that it formed a perpendicular line with respect to the vehicle wheelbase l as shown in Figure **A.3.**

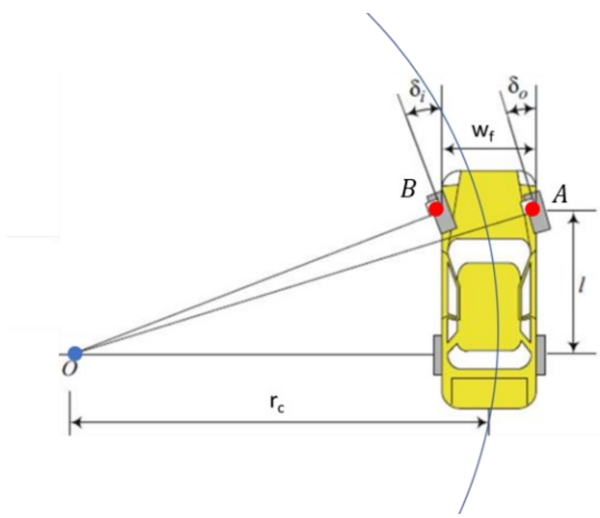

Figure A.3. A schematic of a two-wheel vehicle with fixed center of rotation.

Given a circle with a radius of $r_c = R_{stat}$ and $t_f = w_f$ representing the track width of the vehicle, when we have knowledge of the longitudinal speed V_v of the vehicle, we can calculate the respective distances (circumferences) covered by the inside wheel and the outside wheel.

The front axle inner wheel velocity V_1 and the rear axle outer wheel velocity V_2 is expressed in the equations below,

$$\begin{cases} V_2 = \sqrt{l^2 + \left(R_{stat} + \frac{t_f}{2}\right)^2} * \dot{\psi}_{meas} = r_{dyn} * \omega_2 \\ V_1 = \sqrt{l^2 + \left(R_{stat} - \frac{t_f}{2}\right)^2} * \dot{\psi}_{meas} = r_{dyn} * \omega_1 \\ r_{dyn} * \Delta\omega = \left(\sqrt{l^2 + \left(\frac{l}{\frac{\delta}{t_S}} + \frac{t_f}{2}\right)^2} - \sqrt{l^2 + \left(\frac{l}{\frac{\delta}{t_S}} - \frac{t_f}{2}\right)^2}\right) * \dot{\psi}_{meas} \\ \dot{\psi}_{meas} = \frac{\frac{\delta}{t_S}}{l} V_v \end{cases} \tag{A.29}$$

with,

$$t_f = w_f, \quad \text{and} \quad r_c = R_{stat}. \tag{A.30}$$

Using the four equations above we derive the front axle natural Ackermann slip balance as follows,

$$r_{dyn} * \Delta\boldsymbol{\omega} = r_{dyn} * (\omega_1 - \omega_2) = \left(\sqrt{l^2 + \left(\frac{l}{\frac{\delta}{t_S}} + \frac{t_f}{2}\right)^2} - \sqrt{l^2 + \left(\frac{l}{\frac{\delta}{t_S}} - \frac{t_f}{2}\right)^2}\right) * \frac{\frac{\delta}{t_S}}{l} V_v \tag{A.31}$$

equivalaent to,

$$\Delta S_{nf}(\delta) = \frac{\Delta\omega * r_{dyn}}{V_v} = \left(\sqrt{l^2 + \left(\frac{l}{\frac{\delta}{t_S}} + \frac{t_f}{2}\right)^2} - \sqrt{l^2 + \left(\frac{l}{\frac{\delta}{t_S}} - \frac{t_f}{2}\right)^2}\right) * \frac{\frac{\delta}{t_S}}{l}. \tag{A.32}$$

Equation A.32 can be rewritten as,

$$\Delta \boldsymbol{S_{nf}(\delta)} = \left(\sqrt{\boldsymbol{l^2 + \left(\frac{l}{\alpha} + \frac{1}{2} t_f\right)^2}} - \sqrt{\boldsymbol{l^2 + \left(\frac{l}{\alpha} - \frac{1}{2} t_f\right)^2}}\right) * \frac{\boldsymbol{|\alpha|}}{\boldsymbol{l}} \tag{A.33}$$

with,

$$\alpha = \frac{\delta}{t_S}. \tag{A.34}$$

In the similar vein, the rear axle natural Ackermann slip balance reads as,

$$\Delta \boldsymbol{S}_{\boldsymbol{nr}}(\boldsymbol{\delta}) = \boldsymbol{t}_{\boldsymbol{f}} * \frac{|\boldsymbol{\alpha}|}{\boldsymbol{l}}. \tag{A.35}$$

Appendix 3

III. Modal analysis of an over-actuated vehicle under lateral acceleration

Modal analysis or modal synthesis is a well-known technique for designing transfer function models, which involves the construction of a source-filter synthesis model. The filter transfer function consists of several first- and/or second-order filter sections.

This approach represents the physical system as a weighted sum of individual modes, each driven by an external excitation. The basic concept of modal analysis is to model any vibratory structure as roughly the summation of the individual contribution of each natural mode. This also applies to rigid body systems with finite degrees of freedom. generally, we can write their equation of motions.

$$m_s \ddot{x} + c\dot{x} + kx = f_x(t) \tag{A.36}$$

with m_s the mass of the system, $\boldsymbol{c}$ the damping coefficient, $\boldsymbol{k}$ the stiffness coefficient and $f_x(t)$ the excitation force.

The fundamental concept of modal analysis is known as the expansion theorem. This theory proposes that we can express any given set of motions as a combination of individual contributing modes. In other words, the expansion theorem suggests that the overall motion of a system can be decomposed into simpler individual modes that are combined to form the complete motion of the system.

$$\{x(t)\} = \{u\}^1 q_1(t) + \{u\}^2 q_2(t) + \cdots + \{u\}^n q_n(t) \tag{A.37}$$

It can be written in a compact form as

$$\{x(t)\} = [\{u\}^1 \quad \{u\}^2 \quad \dots \quad \{u\}^n] \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{bmatrix} \tag{A.38}$$

$q_1(t)$ is a time-dependent amplitude that represents the importance of the contribution of mode 1 ($\{u\}^1$) to the motion of $\{x(t)\}$. The vector,

$$q(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{bmatrix} \tag{A.39}$$

is called modal coordinate or natural coordinate. In the Laplace domain it is define as,

$$\left(\frac{X(s)}{U(s)}\right) = q(s) \tag{A.40}$$

It is possible to express equation 1.44., in this different way,

$$\begin{gathered} \dot{\psi} * \left(\left(\frac{1}{\omega_{\dot{\psi}}^2}\right) * s^2 + \left(\frac{2\zeta_{\dot{\psi}}}{\omega_{\dot{\psi}}}\right) * s + 1\right) = g_0 * \left(1 + \tau_{\dot{\psi}\delta_f} * s\right)\delta_f \\ + g_0 * \left(1 + \tau_{\dot{\psi}\delta_r} * s\right)\delta_r \\ + g_0 * \frac{\left(\alpha\left(\delta_{s_f} - \delta_{s_r}\right) - \delta_{s_f}\right)}{m\alpha l(\alpha - 1)} * \left(1 + \tau_{\dot{\psi}M_{yawC}} * s\right) M_{yawC}. \end{gathered} \tag{A.41}$$

Recall,

$$m\ddot{x} + c\dot{x} + kx = f_x(t) \quad \Longrightarrow \quad \left(\frac{1}{\omega_{\dot{\psi}}^2}\right) * \ddot{x} + \left(\frac{2\zeta_{\dot{\psi}}}{\omega_{\dot{\psi}}}\right) * \dot{x} + x = u(t) \tag{A.42}$$

In the Laplace domain,

$$X(s)\left(\left(\frac{1}{\omega_{\dot{\psi}}^2}\right) * s^2 + \left(\frac{2\zeta_{\dot{\psi}}}{\omega_{\dot{\psi}}}\right) * s + 1\right) = U(s). \tag{A.43}$$

By replacing equation 1.59., into equation 1.57., and solving for $\dot{\boldsymbol{\psi}}$, yields,

$$\dot{\psi} = \begin{bmatrix} g_0 * \left(1 + \tau_{\dot{\psi}\delta_f} * s\right) \\ g_0 * \left(1 + \tau_{\dot{\psi}\delta_r} * s\right) \\ g_0 * \frac{\left(\alpha\left(\delta_{s_f} - \delta_{s_r}\right) - \delta_{s_f}\right)}{m\alpha l(\alpha - 1)} * \left(1 + \tau_{\dot{\psi}M_{yawC}} * s\right) \end{bmatrix}^T * \left(\frac{X(s)}{U(s)}\right) * \begin{bmatrix} \delta_f \\ \delta_r \\ M_{yawC} \end{bmatrix} \tag{A.44}$$

which is equivalent to,

$$\boldsymbol{\dot{\psi}} = \boldsymbol{g_0} \begin{bmatrix} \left(\mathbf{1} + \boldsymbol{\tau_{\dot{\psi}\delta_f}} * \boldsymbol{s}\right) \\ \left(\mathbf{1} + \boldsymbol{\tau_{\dot{\psi}\delta_r}} * \boldsymbol{s}\right) \\ \frac{\left(\boldsymbol{\alpha}\left(\boldsymbol{\delta_{s_f}} - \boldsymbol{\delta_{s_r}}\right) - \boldsymbol{\delta_{s_f}}\right)}{\boldsymbol{m\alpha l}(\boldsymbol{\alpha} - \mathbf{1})} * \left(\mathbf{1} + \boldsymbol{\tau_{\dot{\psi}M_{yawC}}} * \boldsymbol{s}\right) \end{bmatrix}^{\boldsymbol{T}} * \begin{bmatrix} \boldsymbol{q_{m1}(t)} \\ \boldsymbol{q_{m2}(t)} \\ \boldsymbol{q_{m3}(t)} \end{bmatrix} \tag{A.45}$$

with further simplifications we can express the yaw rate as a linear combination of the three individual mode shapes present in the yaw dynamics of a car.

$$\begin{aligned} \boldsymbol{\dot{\psi}}(t) = g_0 \Bigg(& \left(1 + \tau_{\dot{\psi}\delta_f} * s\right) * \boldsymbol{q_{m1}}(t) + \left(1 + \tau_{\dot{\psi}\delta_r} * s\right) * \boldsymbol{q_{m2}}(t) \\ & + \frac{\left(\alpha\left(\delta_{s_f} - \delta_{s_r}\right) - \delta_{s_f}\right)}{m\alpha l(\alpha - 1)} * \left(1 + \tau_{\dot{\psi}M_{yawC}} * s\right) * \boldsymbol{q_{m3}}(t) \Bigg) \end{aligned} \tag{A.46}$$

These modes represent the different ways in which a vehicle respond to an external force, moment, or steering input. With $\boldsymbol{q_{m1}}(t)$, $\boldsymbol{q_{m2}}(t)$, and $\boldsymbol{q_{m3}}(t)$ the modal coordinates corresponding to the three natural modes.

The sideslip angle of the vehicle at the center of gravity is expressed by,

$$\boldsymbol{\beta}(t) = g_0 \begin{pmatrix} \frac{\left(\alpha l - \delta_{s_r} {v_x}^2\right)}{v_x} * \left(1 + \tau_{\beta\delta_f} * s\right) * \boldsymbol{q_{m1}}(t) \\ + \frac{\left((\alpha - 1)l - \delta_{s_f} {v_x}^2\right)}{v_x} * \left(1 + \tau_{\beta\delta_r} * s\right) * \boldsymbol{q_{m2}}(t) \end{pmatrix}. \tag{A.47}$$

The lateral acceleration is expressed as a linear combination of the two previous variables.

$$\boldsymbol{\gamma} = \left(\dot{v}_y + \dot{\psi}.v_x\right) = v_x * \left(\boldsymbol{\dot{\psi}} + s.\boldsymbol{\beta}\right) \tag{A.48}$$

with,

$$\dot{v}_y = \dot{\beta} v_x \tag{A.49}$$

Finally, the interest of expressing the physical quantities affecting the yaw dynamics of a vehicle in the modal base of the yaw-sideslip mode shape allows differentiating the dynamics of the natural modes of the vehicle in a cornering phase, from the transmission zeros with respect to a wheel steering control input of front and/or rear axles or yaw moment control action.