Linear Algebra - Exercise 4

Montag, 31. Dezember 2018 12:22

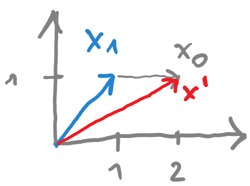
1 Translation is not linear

Show that the Translation by the vector $\mathbf{x}_0$

$$\mathbf{y} = T_{\mathbf{x}_0}(\mathbf{x}) \; = \; \mathbf{x} + \mathbf{x}_0$$

is not linear, i.e. show that the following does not hold:

$$T_{\mathbf{x}_0}(\mathbf{x}_1 + \mathbf{x}_2) = T_{\mathbf{x}_0}(\mathbf{x}_1) + T_{\mathbf{x}_0}(\mathbf{x}_2) \quad \forall \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3,$$
$$T_{\mathbf{x}_0}(\alpha \mathbf{x}) = \alpha T_{\mathbf{x}_0}(\mathbf{x}) \qquad \forall \mathbf{x} \in \mathbb{R}^3 \text{ and } \forall \alpha \in \mathbb{R}.$$



$$x_0 = \begin{array}{c} \begin{array}{ccc} x & y & \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T \end{array}$$ relative from x_1

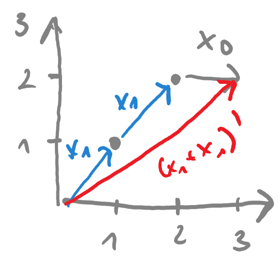
Translationsvector

$$T = T_{x_0} = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = x'$$

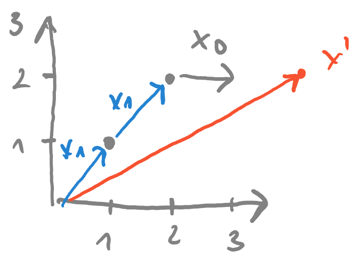
Question: is $T_{x_0}(x_1 + x_1) = T_{x_0}(x_1) + T_{x_0}(x_1)$?

$$x_1 + x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$T_{x_0} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \xrightarrow{\text{depicted}}$$



$$\overline{T_{x_0}(x_1) + T_{x_0}(x_1)} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$



$\neq$

Answer: No, translation by vector x_0 is not linear

$\rightarrow$ you can't sum up translations.

2 Composing translations

Write down the individual translation matrices

1. $\boldsymbol{T}_1$ translate by vector $\mathbf{x}_1 = \begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix}^T$
2. $\boldsymbol{T}_2$ translate by vector $\mathbf{x}_2 = \begin{bmatrix} x_2 & y_2 & z_2 \end{bmatrix}^T$
3. $\boldsymbol{T}_3$ translate by vector $\mathbf{x}_1 + \mathbf{x}_2$

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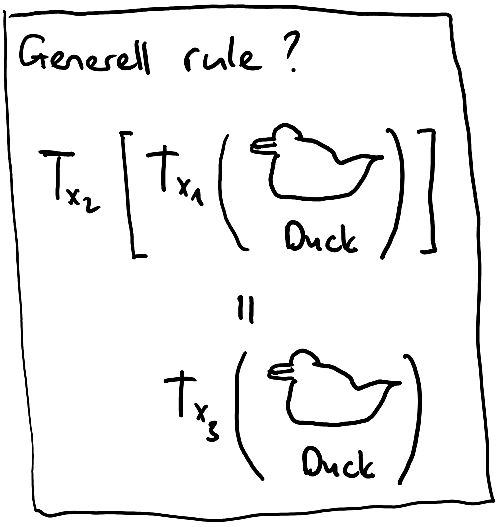
How are these three matrices $\boldsymbol{T}_1$, $\boldsymbol{T}_2$ and $\boldsymbol{T}_3$ related? Is there a general rule for the composition of translation matrices?

1. $T_1 = T_{x_1} = \begin{bmatrix} 1 & 0 & 0 & x_1 \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & z_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2. $T_2 = T_{x_2} = \begin{bmatrix} 1 & 0 & 0 & x_2 \\ 0 & 1 & 0 & y_2 \\ 0 & 0 & 1 & z_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3. $T_3 \neq T_{x_1} + T_{x_2}$!

$$T_3 = T_{x_1+x_2} = \begin{bmatrix} 1 & 0 & 0 & x_1 + x_2 \\ 0 & 1 & 0 & y_1 + y_2 \\ 0 & 0 & 1 & z_1 + z_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



3 Inverse of the translation matrix

A translation in 3D by the vector $\mathbf{x}_0 = \begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix}^T$ can be described in homogeneous coordinates by the 4×4 matrix

$$\boldsymbol{T} = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find it's inverse $\boldsymbol{T}^{-1}$ and show that $\boldsymbol{T}\boldsymbol{T}^{-1} = \boldsymbol{T}^{-1}\boldsymbol{T} = \boldsymbol{I}$. Note: You can find the inverse by common sense!

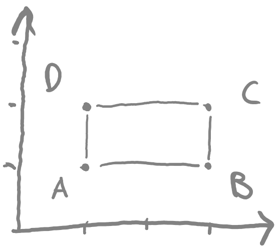
if $TT^{-1} = I$ then lets set T^{-1} using common sense.

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & -x_0 \\ 0 & 1 & 0 & -y_0 \\ 0 & 0 & 1 & -z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4 Scaling 2D

The rectangle with the vertices $A(1,1)$, $B(3,1)$, $C(3,2)$, and $D(1,2)$ should be rescaled by a factor of 2 in the x- and a factor $1/2$ in the y-direction. The vertex A has to be held fixed. Calculate the scaling matrix and apply it the the rectangle. Draw the original and scaled rectangle.



$$S = \begin{bmatrix} 2 & 0 & x \\ 0 & .5 & y \\ 0 & 0 & 1 \end{bmatrix}$$

Punktematrix

$$P = \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

We need to move the square to 0-Point with T and move it back to its original position with T^{-1}

$$T = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

we can apply T, S, T^{-1} seperately ore all together

$$TST^{-1} = \quad \overset{S}{\begin{bmatrix} 2 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \overset{T^{-1}}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}$$

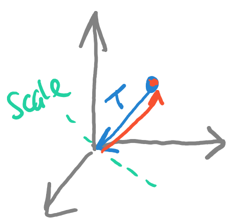
$$T \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ 0 & .5 & -1 \\ 0 & 0 & 1 \end{bmatrix} \underline{\begin{bmatrix} 2 & 0 & 1 \\ 0 & .5 & -.5 \\ 0 & 0 & 1 \end{bmatrix}} \; TST^{-1}$$

now we apply TST^{-1} to P.

$$P' = \begin{bmatrix} 1 & 5 & 5 & 1 \\ 1 & 1 & 3/2 & 3/2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

5 Scaling 3D

Suppose You want to scale a 3D-object with the scaling center in $\begin{bmatrix} t_x & t_y & t_z & 1 \end{bmatrix}^T$ (homogeneous coord.) and scaling factors s_x, s_y and s_z in x-, y- and z-direction. Calculate the transformation matrix $\boldsymbol{S}_t$ and verify, that the center stays fixed under the transformation.



1. GoTo ∅
2. Do Scaling
3. GoTo origin

$$T = \begin{bmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \end{bmatrix} \quad T^{-1} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \end{bmatrix} \quad S = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T^{-1} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad S = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T^{-1} S T = \begin{bmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_x & 0 & 0 & -S_x t_x \\ 0 & S_y & 0 & -S_y t_y \\ 0 & 0 & S_z & -S_z t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_x & 0 & 0 & t_x - S_x t_x \\ 0 & S_y & 0 & t_y - S_y t_y \\ 0 & 0 & S_z & t_z - S_z t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6 Rotation around (a_x, a_y) in 2D

You want to rotate a 2D-object around (a_x, a_y) by an angle θ. Try to find a general transformation matrix to accomplish this. Remember: You need a translation to the origin, the rotation and the translation back to (a_x, a_y) and the matrices are multiplied where the order of the matrices is reversed.

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

rotates the plane by angle θ (counterclockwise) around origin $(0, 0)$.

We need to move $(a_x, a_y) \to (0, 0)$ and back.

$$T = \begin{bmatrix} 1 & 0 & -a_x \\ 0 & 1 & -a_y \\ 0 & 0 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & 0 & a_x \\ 0 & 1 & a_y \\ 0 & 0 & 1 \end{bmatrix}$$

shift $\to$ rotation $\to$ shift.

$$T^{-1} R T = \begin{bmatrix} 1 & 0 & a_x \\ 0 & 1 & a_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -a_x \\ 0 & 1 & -a_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta & -a_x \cos\theta + a_y \cos\theta \\ \sin\theta & \cos\theta & -a_x \sin\theta - a_y \cos\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & a_x - a_x \cos\theta + a_y \cos\theta \\ \sin\theta & \cos\theta & a_y - a_x \sin\theta - a_y \cos\theta \\ 0 & 0 & 1 \end{bmatrix}$$

7 Rotation around the origin in 3D

On the slides we have presented to 3×3 rotation matrix $\boldsymbol{Q}$ for a rotation around the axis $\mathbf{a} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^T$ by an angle θ. Check that $\boldsymbol{Q}\mathbf{a} = \mathbf{a}$, i.e. the direction of the rotation axis is invariant under this rotation.

$$Q = (\cos\theta) I + (1 - \cos\theta) = \begin{bmatrix} a_1^2 & a_1 a_2 & a_1 a_3 \\ a_1 a_2 & a_2 & a_2 a_3 \\ a_1 a_3 & a_2 a_3 & a_3^3 \end{bmatrix} - \sin\theta \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix}$$

8 Projection onto plane through the origin

What is the 3×3 projection matrix $\boldsymbol{I} - \mathbf{n}\mathbf{n}^T$ onto the plane $2x/3 + 2y/3 + z/3 = 0$? In homogeneous coordinates add 0, 0, 0, 1 as an extra row and column in $\boldsymbol{P}$. Project $(3, 3, 3)$ onto the plane.

plane $\frac{2x}{3} + \frac{2y}{3} + \frac{z}{3} = 0$

$$n = \frac{1}{3}\begin{bmatrix} 2 & 2 & 1 \end{bmatrix}^T \xrightarrow{\text{normalize}} \frac{1}{3} \cdot \sqrt{2^2 + 2^2 + 1^2} = \frac{1}{3} \cdot \frac{1}{\sqrt{9}} = \frac{1}{3} \cdot \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} = \frac{1}{9}$$

$$\overset{!}{=} \frac{1}{9}\begin{bmatrix} 2 & 2 & 1 \end{bmatrix}^T$$

$$I - nn^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \cdot \frac{1}{9}\begin{bmatrix} 2 & 2 & 1 \end{bmatrix}^T$$

$$= \frac{1}{9}\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$$

in homogeneous coord.

$$P_h(3,3,3) = \frac{1}{9}\begin{bmatrix} 5 & -4 & -2 & 0 \\ -4 & 5 & -2 & 0 \\ -2 & -2 & 8 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 3 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1 \\ -1 \\ 4 \\ 1 \end{bmatrix}$$

if we insert $\frac{1}{3}[-1 \ -1 \ -1 \ 4 \ 1]^T$

$$\underbrace{\frac{2}{3}\left(-\frac{1}{3}\right)}_{x} + \underbrace{\frac{2}{3}\left(-\frac{1}{3}\right)}_{y} + \underbrace{\frac{1}{3}\left(\frac{4}{3}\right)}_{z} = 0$$

$= 0$

9 Projection onto an arbitrary plane

With the same 4×4 matrix $\boldsymbol{P}$ from Exercise 8, multiply $\boldsymbol{T}_+\boldsymbol{P}\boldsymbol{T}_-$ to find the projection matrix onto the plane $2x/3 + 2y/3 + z/3 = 1$. The Translation $\boldsymbol{T}_-$ moves a point on that plane (choose a simple one) to $(0, 0, 0, 1)$. The inverse matrix $\boldsymbol{T}_+$ moves it back. Project $(3, 3, 3)$ onto the plane.

plane $\frac{2x}{3} + \frac{2y}{3} + \frac{z}{3} = 1$

$P(3,3,3) \longrightarrow (\overset{x}{3}, \overset{y}{3}, \overset{z}{0})$

to move P down the z-axis by -3

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$n = \frac{1}{3} \cdot \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 2 & 1 \end{bmatrix}^T \qquad \sqrt{2^2 + 2^2 + 1^2}$$

$$= \frac{1}{9}\begin{bmatrix} 2 & 2 & 1 \end{bmatrix}^T$$

$$I - nn^T = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \frac{1}{3}\begin{bmatrix}2\\2\\1\end{bmatrix}\cdot\frac{1}{3}[2\ 2\ 1]^T = \frac{1}{9}\begin{bmatrix}5&-4&-2\\-4&5&-2\\-2&-2&8\end{bmatrix}$$

homogeneous $\rightarrow \frac{1}{9}\begin{bmatrix}5&-4&-2&0\\-4&5&-2&0\\-2&-2&8&0\\0&0&0&9\end{bmatrix} = P_n$

combined transformation matrix

$$T^{-1}P_nT = \frac{1}{9}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&3\\0&0&0&1\end{bmatrix}\begin{bmatrix}5&-4&-2&0\\-4&5&-2&0\\-2&-2&8&0\\0&0&0&9\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&-3\\0&0&0&1\end{bmatrix}$$

$$\frac{1}{9}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&3\\0&0&0&1\end{bmatrix}\begin{bmatrix}5&-4&-2&6\\-4&5&-2&6\\-2&-2&8&-24\\0&0&0&9\end{bmatrix}$$

$$T^{-1}P_nT = \frac{1}{9}\begin{bmatrix}5&-4&-2&6\\-4&5&-2&6\\-2&-2&8&3\\0&0&0&9\end{bmatrix}$$

let's check with point $(0,0,3)$. It lies on the plane and shouldn't move

$$P_{n,E}\begin{bmatrix}0\\0\\3\\1\end{bmatrix} = \frac{1}{9}\begin{bmatrix}5&-4&-2&6\\-4&5&-2&6\\-2&-2&8&3\\0&0&0&9\end{bmatrix}\begin{bmatrix}0\\0\\3\\1\end{bmatrix} = \frac{1}{9}\begin{bmatrix}0\\0\\27\\9\end{bmatrix} = \begin{bmatrix}0\\0\\3\\1\end{bmatrix}$$

It did not move $\rightarrow$ great