Multivariable Calculus - Exercise 2

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1 Total differential I: The *Cow-culus* Exercise

A cow's udder is in the shape of a hemisphere.

1. If its diameter is measured to be 26 cm with a possible error of 0.5 cm, then use differentials to approximate the
   (a) propagated error
   (b) relative error
   (c) percent error

   in computing its volume.
2. Estimate the maximum allowable percent error in measuring the diameter if the error in computing the volume must not exceed 3%.

Note: In this exercise the diameter is the quantity being measured, while the volume is being computed. You should therefore express the volume as a function of the diameter.

1.

diameter $= 26\text{cm} \pm \underbrace{0.5 \text{ cm}}_{\Delta d}$

$\boxed{V = \frac{1}{12}\pi d^3}$ formula for Volume with diameter

a) what is the propagated error?

Change in the Volume V if diameter d is changed by a small quantity ∇d.

$$\nabla V = \frac{V}{\partial d} = \frac{1}{4}\pi d^2 \cdot \Delta d = \frac{1}{4}\pi (26 \text{ cm})^2 \cdot 0.5\text{cm} = \underline{265.5\text{cm}^3}$$

the propagated error is ∇V: 265.5cm^3, at a possible error of 0.5cm

b) what is the relative error?

$$\frac{\nabla V}{V} = \frac{\frac{1}{4}\pi d^2 \cdot \Delta d}{\frac{1}{12}\pi d^3} = \frac{3 \cdot \Delta d}{d}$$

insert values: $\frac{3 \cdot 0.5\text{cm}}{26\text{cm}} = \underline{0.058\text{cm}}$

c) what is the percent error?

$$\boxed{\text{rel. error} \cdot 100} = \underline{5.769\,\%}$$

2. What is the max. allowed possible error in measuring Δd if the percent error may not exceed 3%?

reverse: 3% | $\frac{3 \cdot x}{26cm} = 0.03cm$

$x = \frac{26 \cdot 0.03}{3} = 0.26cm$

the diameter has to be measured with a max. possible error of 0.26cm.

2 Total differential II

(a) Find the (total) differential of $g(u,v) = u^2 + uv$.

(b) Use your answer to part (a) to estimate the change in g as you move from $(1,2)$ to $(1.2, 2.1)$.

a) total differential

$$dg = g_u(u,v)\,du + g_v(u,v)\,dv = \underline{du(2u+v) + dv(u)}$$

b) Δg in moving from $(1,2)$ to $(1.2, 2.1)$

$u = 1,\ v = 2$

$\Delta u = 0.2,\ \Delta v = 0.1$ | $0.2(2+2) + 0.1(1) = \underline{0.9}$

test: let's compare this to the real change of function g

$$g(1.2, 2.1) - g(1,2) = (1.2^2 + 1.2 \cdot 2.1) - (1^2 + 1 \cdot 2)$$
$$= (1.44 + 2.52) - 3$$
$$= 3.96 - 3 = \underline{0.96}$$

pretty close: $0.9 \leftrightarrow 0.96$

So the total differential helps you to approximate the change in a function, when a variable (input) is changed.

3 Total differential III

In a room, the temperature is given by $T = f(x,t)$ degrees Celsius, where x is the distance from a heater (in meters) and t is the elapsed time (in minutes) since the heater has been turned on.

A person standing 3 m from the heater 5 min after it has been turned on observes the following:

1. T is increasing $1.2\,^\circ\mathrm{C\,min^{-1}}$, and
2. by walking away from the heater, T decreases by $2\,^\circ\mathrm{C\,m^{-1}}$ as time is held constant.

Estimate (using the total differential) how much cooler or warmer it would be 2.5 m from the heater after 6 min.

$T = f(x,t)$, 1. Person at $P(3,5)$ notices $\Delta T = 1.2°C/min$

2. $\Delta T = -2°C/m$ at fixed time

The partial differentials can be found in the text above.

$$\frac{\partial T}{\partial x} = -2°C \cdot \Delta x \quad , \quad \frac{\partial T}{\partial t} = 1.2°C \cdot \Delta t$$

The total differential can be found by adding them together.
at 3m, 5min:

$$dT = f_x(x,t) + f_t(x,t) = 1.2°C \cdot dt - 2°C \cdot dx$$

How much will the temperature change if we move from 3m to 2.5m and stay 1min.

$$dT = 1.2°C \cdot 1min - 2°C \cdot (-0.5m) = 1.2°C + 1°C = 2.2°C$$

by moving closer to the heater the Temp. will inc. by 2.2°C in 1 min.

4 Linearization I

From a differentiable function $f(x,y)$ we know $f(1,3) = 7$ and $\nabla f(1,3) = [2,\text{-}5]^T$.

(a) Find the equation of the tangent line to the level curve of f through the point $(1,3)$.

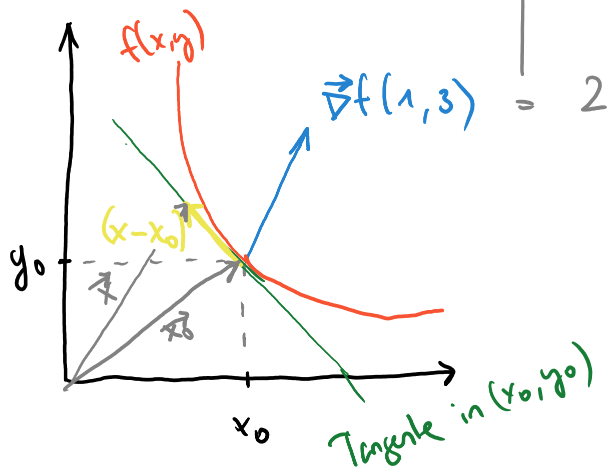
(b) Find the equation of the tangent plan to the surface $z = f(x,y)$ at the point $(1,3)$.

a) the tangent is perpendicular to the gradient

$$0 = \nabla f(x_0)(x - x_0) = \begin{bmatrix} 2 \\ -5 \end{bmatrix} \left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right)$$

$\wedge$ (x,y)

$$0 = \nabla f(x_0)(x-x_0) = \begin{bmatrix} \ \\ -5 \end{bmatrix}\left(\begin{bmatrix} \ \\ y_0 \end{bmatrix} - \begin{bmatrix} \ \\ 3 \end{bmatrix}\right)$$



$$= 2(x_0-1) - 5(y_0-3) = 2x_0 - 2 - 5y_0 + 15$$

$$= 2x_0 - 5y_0 + 13$$

$$y = \frac{2x_0 + 13}{5}$$

b) we need to add a new varable surface $s = f(x,y)$

$g(x,y,s) = f(x,y) - s$ the surface $s = f(x,y)$ is the contour surface of $g(x,y,s) = 0$

we know that the contour surface is perpendicular to the gradient of g.

$$0 = \nabla g(x_0)(x-x_0) = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}\right) = 2(x-1) - 5(y-3) - (z-7)$$

$$= 2x - 2 - 5y + 15 - z + 7$$

$$= 2x - 5y - z + 20$$

5 Linearization II

Suppose the gradient of f at $\mathbf{x}_0 = [x_0, y_0]^T$ is nonzero. Using linearization and the definition of the total derivative the change in f when we move from $\mathbf{x}_0$ to $\mathbf{x}$ is

$$\mathrm{d}f = \nabla f(\mathbf{x}_0) \bullet (\mathbf{x} - \mathbf{x}_0)$$

Therefore, the tangent line to the contourline satisfies

$$\mathrm{d}f = 0 \quad \Leftrightarrow \quad \nabla f(\mathbf{x}_0) \bullet (\mathbf{x} - \mathbf{x}_0) = 0$$

Therefore, the contourline through $\mathbf{x}_0$ is perpendicular to the gradient $\nabla f(\mathbf{x}_0)$ at this point.

Verify this for

(a) $f(x,y) = y - x^2$ and $\mathbf{x}_0 = [1,1]^T$.

(b) $g(x,y) = x^2 - y^2$ and $\mathbf{x}_0 = [2, \sqrt{3}]^T$.

Check by drawing the contour and the tangent line!

a) $f(x,y) = y - x^2$ $\quad x_0 = [1, 1]^T$

find the gradient $\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]^T$

$$\nabla f = \begin{bmatrix} -2x \\ 1 \end{bmatrix} \longrightarrow \vec{\nabla} f(1,1) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

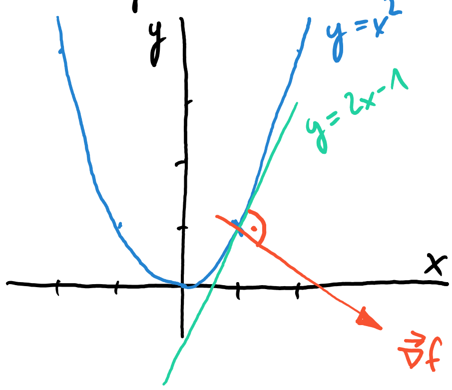
$$\vec{\nabla} f(1,1) \cdot (x - x_0) = 0$$

$$= \begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \end{bmatrix} = -2(x-1) + (y-1)$$

$$= -2x + 2 + y - 1 = \underline{-2x + y + 1 = 0}$$

function	gradient
$y = x^2$	$-2x + y + 1 = 0$

for point (1,1)



$$\vec{\nabla} f = -2 \cdot 1 + 1 + 1 = 0$$

7 Chain-Rule I

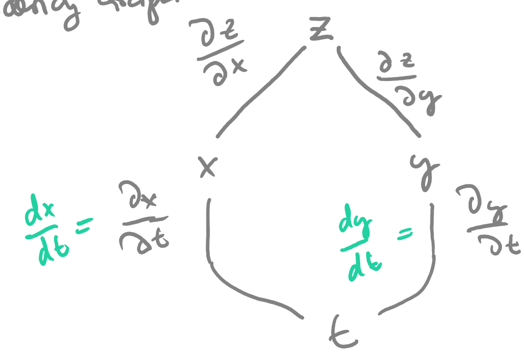
Find dz/dt using the chain rule if

(a) $z = xy^2, x = e^{-t}, y = \sin t$

(b) $z = \ln(x^2 + y^2), x = 1/t, y = \sqrt{t}$

(c) $z = (x+y)e^y, x = 2t, y = 1 - t^2$.

a) Dependency Graph



$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{dz}{dt} = y^2 \cdot (-e^{-t}) + 2xy \cdot \cos t$$

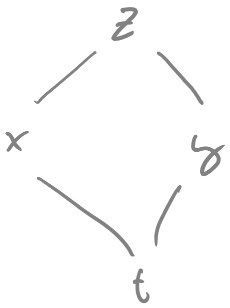
we know $y = \sin t$ and $x = e^{-t}$.

$$\frac{dz}{dt} = y^2 \cdot (-e^{-t}) + 2xy \cdot \cos t$$

we know $y = \sin t$ and $x = e^{-t}$:

$$\underline{\frac{dz}{dt} = -e^{-t} \cdot \sin(t)^2 + 2e^{-t} \sin t \cdot \cos t}$$

b)



same dependency graph

$z = \ln(x^2+y^2)$, $x = \frac{1}{t}$, $y = \sqrt{t}$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= 2x\left(\frac{1}{x^2+y^2}\right) \cdot (-2t^{-2}) + 2y\left(\frac{1}{x^2+y^2}\right) \cdot \left(0.5\, y^{-0.5}\right)$$

$$= \frac{2}{t}\left(\frac{1}{\frac{1}{t^2}+t}\right) \cdot (-2t^{-2}) + 2\sqrt{t}\left(\frac{1}{\frac{1}{t^2}+t}\right) \cdot \left(0.5 \sqrt{t}^{-0.5}\right)$$

$$= \frac{2}{t(t^{-1}+1)} \cdot (-t^{-3}) + \frac{1}{2} = \underline{\frac{2 \cdot (-t^{-3}) + 1}{t(t^{-1}+1)}}$$

8 Chain-Rule II

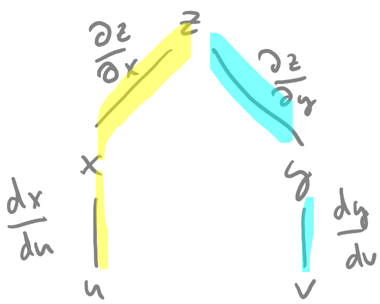
Find $\partial z/\partial u = z_u$ and $\partial z/\partial v = z_v$ using the chain rule if

(a) $z = xe^y$, $x = \ln u$, $y = v$

(b) $z = xe^y$, $x = u^2+v^2$, $y = u^2-v^2$

(c) $z = \ln(xy)$, $x = (u^2+v^2)^2$, $y = (u^3+v^3)^2$.

a) Dep. Graph



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{dx}{du} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dv}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{dx}{du} + \frac{\partial z}{\partial y} \cdot \frac{dy}{du}$$

$$= e^y \cdot \frac{1}{u} + xe^y \cdot 1 = e^y \left(\frac{1}{u} + x\right)$$

$$= e^u \left(\frac{1}{u} + \ln u\right)$$

$$\frac{\partial z}{\partial v} = e^y \cdot 0 + (xe^y) \cdot 0 = 0$$