Multivariable Calculus - Exercise 1

Mittwoch, 2. Januar 2019 09:01

1 Contourlines I

Draw the contour lines of the function $f(x,y) = 2x + 3y + 1$ in the rectangular domain $[0,6] \times [-3,3]$. You should be able to solve this problem without the help of the computer (or pocket calculator). Describe the contour lines.

$f(x,y) = c$ $\quad 2x + 3y + 1 = c$ $\quad y = -\frac{2x-1}{3}$ $\quad x=1 \rightarrow y = -\frac{1}{3}$

$2x + 3y + (1-c) = 0$

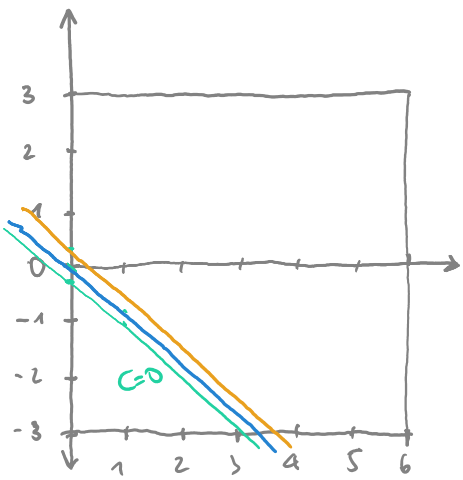
$c=0: \quad 0 = 2x + 3y + 1$

$y = 0 = -\frac{2x+1}{3}$

$x = 0: \quad y = -\frac{1}{3} \rightarrow \begin{bmatrix} 0 \\ -\frac{1}{3} \end{bmatrix}$

$x = 1: \quad y = -1 \rightarrow \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

rectangular domain $[0,6] \times [-3,3]$



$C=1: \quad 1 = 2x + 3y + 1$

$y = 0 = -\frac{2x}{3}$

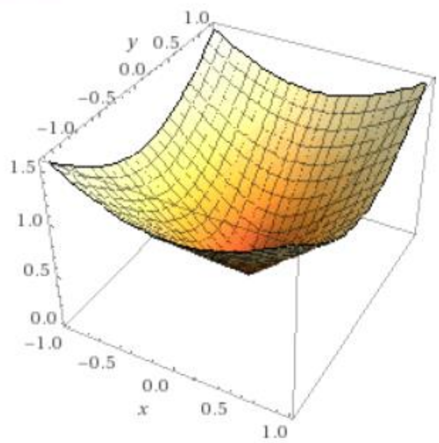
$x=0: \quad y = 0$

$x=1: \quad y = -\frac{2}{3}$

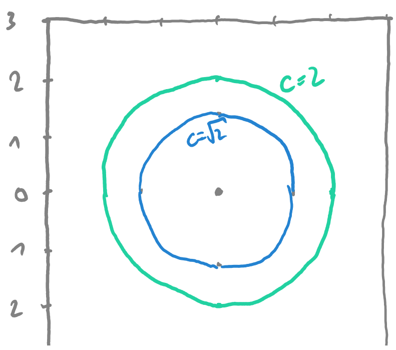
2 Contourlines II

Draw the contour lines of the function $f(x,y) = \sqrt{x^2+y^2}$ in the square domain $[-3,3]^2$. Describe the contour lines.

3D plot:



$f(x,y) = \sqrt{x^2+y^2} = (x^2+y^2)^{1/2} = c$



$c = \sqrt{2}, \quad \sqrt{x^2+y^2} = \sqrt{2}$

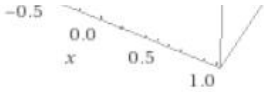
$x^2 + y^2 = 2$

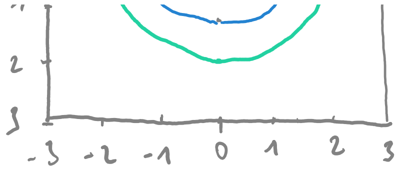
$y = 0 = \sqrt{2-x^2}$

$x = 0 = \sqrt{2-y^2}$

$x_1 = \sqrt{2} \qquad x_2 = -\sqrt{2}$

$y_1 = \sqrt{2} \qquad y_2 = -\sqrt{2}$





$y_1 = \sqrt{2}$ $\quad y_2 = -\sqrt{2}$

$c = 2, \quad x^2 + y^2 = 4$

$y = 0 = \sqrt{4 - x^2}$

$x = 0 = \sqrt{4 - y^2}$

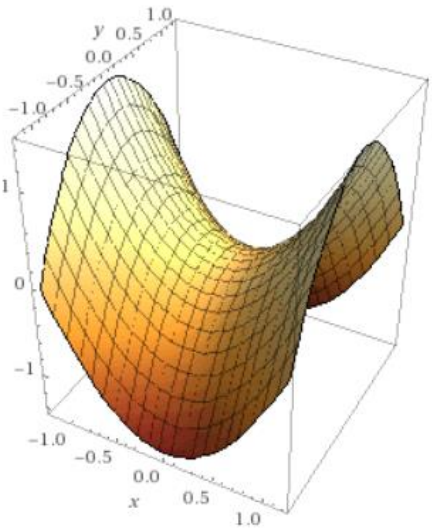
$x_1 = 2 \quad x_2 = -2$

$y_1 = 2 \quad y_2 = -2$

3 Contourlines III

Sketch the contourlines of the function $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto z = f(x, y) = x^2 - y^2$. What kind of curves do You see? Highlight the countourline with niveau $z = 1$. Describe the contour lines.

3D plot:



for level curves we consider

$$\boxed{x^2 - y^2 = c, \quad c = \text{const.}}$$

$c = 0 \rightarrow x^2 - y^2 = (x - y)(x + y) = 0$

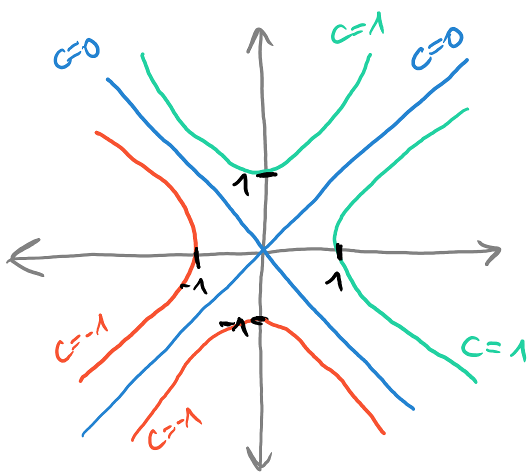
$y = \pm x$

$c = 1 \rightarrow x^2 - y^2 = 1$ Hyperbolia

$y = \sqrt{x^2 - 1}$

$c = -1 \rightarrow x^2 - y^2 = -1$ Hyperbolia

$y = \sqrt{x^2 + 1}$



4 Partial derivatives I

Find V_r if $V = \frac{1}{3}\pi r^2 h$.

$$V_r = \frac{\partial V}{\partial r} = 2 \cdot \frac{1}{3}\pi r h = \frac{2}{3}\pi r h$$

$$V_r = \frac{\partial V}{\partial r} = 2 \cdot \frac{1}{3}\pi r h = \frac{2}{3}\pi r h$$

5 Partial derivatives II

Find all partial derivatives of $f(x,y,z) = \frac{x^2 y^3}{z}$.

$$f_x(x,y,z) = \frac{2xy^3}{z}$$

$$f_y(x,y,z) = \frac{3x^2 y}{z}$$
3

$$f_z(x,y,z) = \frac{x^2 y}{z} = (x^2 y) z^{-1} = -1 (x^2 y) z^{-2}$$

$$= -\frac{x^2 y}{z^2}$$

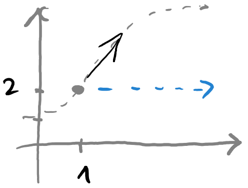
6 Partial derivatives III

(a) The surface S is given, for some constant a, by

$$z = 3x^2 + 4y^2 - axy.$$

Find the values of a which ensure that S is sloping upward when we move in the positive x-direction from the point $(1,2)$.

(b) With the values of a from part (a), if you move in the positive y-direction from the point $(1,2)$, does the surface slope up or down? Explain.



surface S is $z = 3x^2 + 4y^2 - axy$, a: constant

we derive function depending on x

$$\frac{\partial z}{\partial x} = 6x - ay \qquad (a^1 \to 1a^0 = 1)$$

now we place Point $\begin{bmatrix}1\\2\end{bmatrix}$ into function $\frac{\partial z}{\partial x}$

$$6 \cdot 1 - a2 = 6 - 2a$$

to ensure that S is sloping upwards $\rightarrow_x$

$$6 - 2a > 0$$

$$a < \frac{-6}{-2} \rightarrow \underline{a < 3}$$

a) S is sloping up if $a < 3$ is choosen.

b) in y-direction $\frac{\delta z}{\delta y} = 8y - ax$

b) in y-direction $\frac{\delta z}{\delta y} = 8y - ax$

$P(1,2) \rightarrow 16 - a$

from a) we know $a < 3 \rightarrow 16 - (0..1..2..) > 13$

so $16 - a > 13$

7 Gradient I

Find the gradient of the following functions:

(a) $f(x,y) = \frac{3}{2}x^5 - \frac{4}{7}y^6$, (b) $z = xe^y$, (c) $z = \sin(x/y)$, (c) $f(a,b) = \frac{2a+3b}{2a-3b}$.

Assume f is a scalar function depending on two variables x and y.

The **Gradient** of f is a vector, whose compoents are the partial derivatives of f, i.e.

$$\nabla f(\mathbf{x}) = \begin{bmatrix} f_x(\mathbf{x}) \\ f_y(\mathbf{x}) \end{bmatrix} \quad \text{or} \quad \nabla f(x,y) = \begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix}$$

a) $f(x,y) = \frac{3}{2}x^5 - \frac{4}{7}y^6$

$\nabla f(x) = \begin{bmatrix} \frac{15}{2}x^4 \\ -\frac{24}{7}y^6 \end{bmatrix}$

b) $z(x,y) = xe^y$

$\nabla z(x,y) = \begin{bmatrix} e^y \\ xe^y \end{bmatrix}$

c) $z(x,y) = \sin\left(\frac{x}{y}\right) \overset{!}{=} \sin\left(x \cdot \frac{1}{y}\right) \overset{!}{=} \sin(x^1 \cdot y^{-1})$

$x' = \sin(x \cdot y^{-1})' = 1 \cdot y^{-1} \cdot \cos\left(\frac{x}{y}\right)$

$y' = \sin(x \cdot y^{-1}) = -1xy^{-2} \cdot \cos\left(\frac{x}{y}\right)$

$\nabla z(x,y) = \begin{bmatrix} y^{-1}\cos\left(\frac{x}{y}\right) \\ -xy^{-2}\cos\left(\frac{x}{y}\right) \end{bmatrix} = \frac{1}{y^2}\begin{bmatrix} y\cos\left(\frac{x}{y}\right) \\ -x\cos\left(\frac{x}{y}\right) \end{bmatrix}$

d) $f(a,b) = \frac{2a+3b}{2a-3b} \overset{!}{=} (2a+3b)(2a-3b)^{-1}$

$f_a = \frac{\partial f}{\partial a} = 2(2a-3b)^{-1} + (2a+3b) \cdot (-1)(2a-3b)^{-2} \cdot 2$

8 Gradient II

Find the gradient of the following functions at the point:

(a) $f(x,y) = x^2y + 7xy^3$, at $(1,2)$ (b) $f(r,h) = 2\pi rh + \pi r^2$, at $(2,3)$ (c) $f(m,n) = 5m^2 + 3n^4$, at $(5,2)$.

a) $f(x,y) = x^2y + 7xy^3 \rightarrow P(1,2)$

$$\nabla f(x,y) = \begin{bmatrix} 2xy + 7y^3 \\ x^2 + 21xy^2 \end{bmatrix}, \quad \nabla f(1,2) = \begin{bmatrix} 60 \\ 85 \end{bmatrix}$$

b) $f(r,h) = 2\pi rh + \pi r^2 \rightarrow P(2,3)$

$$\nabla f(r,h) = \begin{bmatrix} 2\pi h + 2\pi r \\ 2\pi r \end{bmatrix}, \quad \nabla f(2,3) = \begin{bmatrix} 10\pi \\ 4\pi \end{bmatrix}$$

c) $f(m,n) = 5m^2 + 3n^4 \rightarrow P(5,2)$

$$\nabla f(m,n) = \begin{bmatrix} 10m \\ 12n^3 \end{bmatrix}, \quad \nabla f(5,2) = \begin{bmatrix} 50 \\ 96 \end{bmatrix}$$

9 Directional derivative I

Calculate the directional derivative of the function $f(x,y) = x^2 + y^2$ at $\mathbf{x}_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ in the direction of $\mathbf{e} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. Draw the contourlines around that point and the gradient of f at that point. Check with the formula which uses the gradient of f to compute the directional derivative.

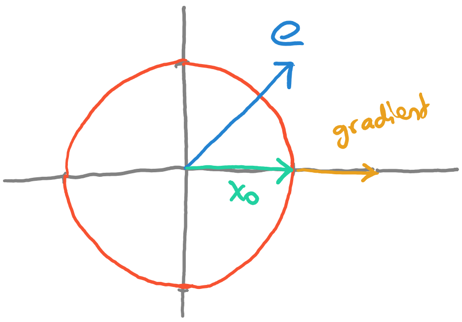
$f(x,y) = x^2 + y^2$ $\quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, in direction $e = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\nabla f(x,y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$ $\quad \nabla f_{x_0}(1,0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$$e = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xrightarrow{\text{unit vector}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Def(x_0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \circ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{2}{\sqrt{2}} = \underline{\underline{\sqrt{2}}}$$

<u>Contour lines</u>



10 Directional derivative II

Calculate the directional derivative of the real valued function in 3D, $f(x,y,z) = z\sin x + \ln(x^2 - y^2)$ at $\mathbf{x}_0 = \begin{bmatrix}1 & 0 & 1\end{bmatrix}^T$ in the direction of $\mathbf{e} = \begin{bmatrix}1 & 1 & 0\end{bmatrix}^T$. Check with the formula which uses the gradient of f to compute the directional derivative.

$$f(x,y,z) = z \cdot \sin x + \ln(x^2 - y^2) \qquad x_0 = [1 \; 0 \; 1]^T \xrightarrow{dir} e\,[1 \; 1 \; 0]^T$$

$$\nabla f(x,y,z) = \begin{bmatrix} z \cdot \cos x + \dfrac{2x}{x^2 - y^2} \\ -\dfrac{2y}{x^2 - y^2} \\ \sin x \end{bmatrix}, \qquad \nabla f(1,0,1) = \begin{bmatrix} \cos(1) + 2 \\ 0 \\ \sin(1) \end{bmatrix}$$

$$e = [1 \; 1 \; 0]^T \xrightarrow{\text{unit vector}} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$D_e f(x_0) = \begin{bmatrix} \cos 1 + 2 \\ 0 \\ \sin 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{2}} \\ 0 \\ 0 \end{bmatrix} = \frac{\cos 1 + 2}{\sqrt{2}}$$