# Linear Algebra - Exercise 3

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## 1 Determinant

Compute the determinant of the following matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 8-\lambda \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 6 \\ 0 & 0 & -1 \end{bmatrix}.$$

Which of the matrices have an inverse, i.e. are not singular (do not compute the inverse, though)?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \qquad \det(A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$\stackrel{!}{=} 1 \cdot 8 - 2 \cdot 3 = \underline{2}$$

$$B = \begin{bmatrix} 1-\lambda & 2 \\ 3 & 8-\lambda \end{bmatrix} \qquad \det(B) = (1-\lambda)(8-\lambda) - 6$$

$$\stackrel{!}{=} 8 - 9\lambda + \lambda^2 - 6$$

$$\stackrel{!}{=} \underline{\lambda(\lambda-9)+2}$$

$$C = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 6 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{\text{extend}} \begin{bmatrix} 1 & 2 & 2 & 1 & 2 \\ 0 & 3 & 6 & 0 & 3 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

$$m(\bullet) - m(\bullet)$$

$$\det(C) = (1 \cdot 3 \cdot -1) + (2 \cdot 6 \cdot 0) + (2 \cdot 0 \cdot 0)$$

$$- ((2 \cdot 3 \cdot 0) + (1 \cdot 6 \cdot 0) + (2 \cdot 0 \cdot 1))$$

$$\stackrel{!}{=} \underline{-3}$$

## 2 Again: Determinant

First compute the determinant using Sarrus' formula. Then derive the reduced row echelon form of the matrices and again compute the determinant by multiplying the pivot elements.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & 8 \\ 3 & 3 & 10 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & 4 \\ 3 & 1 & 6 \end{bmatrix}$$

Is the matrix invertible (do not compute the inverse, though)?

$$\det(A) = 20 - 24 + 12 - 12 - 24 + 20$$

$$\stackrel{!}{=} \underline{-8}$$

$$\text{rref}(A) = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & 8 \\ 3 & 3 & 10 \end{bmatrix} \xrightarrow{R3-(R2+R1)} \begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & 8 \\ 0 & 2 & 0 \end{bmatrix} \xrightarrow{R2-2R1} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 4 & 4 \\ 0 & 2 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{R2}{4},\ \frac{R3}{2}} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R1+R2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[R3\cdot(-1)]{R3-R2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad 1 \cdot 1 \cdot 1 = \underline{1}$$

det(A) != 0, so there is an inverse of A.

## 3 Again: Determinant

Compute the determinant by Laplace expansion (along an appropriate column or row).

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Compare with the determinant You obtain, if using the reduced row echelon form of the matrix. Is the matrix invertible?

to compute the determinant using Laplace expansion imagine another matrix above A, this chessboard matrix contains only 1, -1

$$A^{+-} = \begin{vmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{vmatrix}$$

$$A = \begin{vmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{vmatrix}$$

note this row has 2 zeros, I only need 2 3x3 matrices.

$1 \cdot 1$

$$1 \cdot \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{vmatrix}$$

$2 \cdot 1$ $\quad (-1)(-1)$

$$= 2 \cdot \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + 1 \begin{vmatrix} -1 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= 2 \cdot (2-1) + 1(-1) = \underline{1}$$

$(-1) \cdot (-1)$

$$+1 \begin{vmatrix} -1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{vmatrix}$$

$(-1)\,1$ $\quad (-1)(-1)$

$$= -1 \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 0 & -1 \\ 0 & 1 \end{vmatrix}$$

$$= -1(2-1) + 1(0) = \underline{-1}$$

$$\det(A) = 1 \cdot 1 + 1 \cdot (-1) = \underline{\underline{0}}$$

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is the above matrix invertible? No, $\det(A) = 0$

## 4 Again: Determinant

If a $4 \times 4$ matrix has $\det(\boldsymbol{A}) = 1/2$, find $\det(2\boldsymbol{A})$ and $\det(-\boldsymbol{A})$ and $\det(\boldsymbol{A}^2)$ and $\det(\boldsymbol{A}^{-1})$.

You have to apply the rules on slide 6!

rules:

$\det(A^T) = \det(A)$ $\qquad \det(AB) = \det(A) \cdot \det(B)$

$\det(I) = 1$ $\qquad \det(nA) = n^{\dim(A)} \cdot \det(A)$

$\det(A^{-1}) = \frac{1}{\det(A)}$ $\qquad \det(-A) = (-1)^{\dim(A)} \cdot \det(A)$

so if $\det(A) = \frac{1}{2}$

4x4 matrix

- $\det(2A) = 2^4 \cdot \det(A) = 16 \cdot \frac{1}{2} = \underline{8}$
- $\det(-A) = (-1)^4 \cdot \det(A) = \underline{\frac{1}{2}}$
- $\det(A^2) = \det(A) \cdot \det(A) = \left(\frac{1}{2}\right)^2 = \underline{\frac{1}{4}}$

- $\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{\frac{1}{2}} = 2$

## 5 Eigenvalues and -vectors

Find the eigenvalues and eigenvectors of

$$A_1 = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Check the trace and determinant.

Usually a vector x changes direction when multiplied by A. Certain exceptional vectors are in the same direction as Ax.

$$Ax = \lambda x$$

Such a vector x is called eigenvector and the corresponding $\lambda$ is called eigenvalue.

$A_1 = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ for each eigenvalue $\lambda$, we have to find

$$(A - \lambda I)x = 0$$

1) $A - \lambda I$

$\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

deterministic polynomial

$$\det(A - \lambda I) = \begin{bmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{bmatrix}$$

$$= (3-\lambda)(-3-\lambda) - 16$$

$$= -9 + 3\lambda + \lambda^2 - 3\lambda - 16$$

$$= \lambda^2 - 25 = (\lambda+5)(\lambda-5)$$

the roots of the characteristic polynomial is $\lambda_1 = -5$ and $\lambda_2 = 5$. $(-5+5=0)$ $(5-5=0)$

**Eigenvector for $\lambda_1 = -5$**

$$A - (-5I) = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e. $x_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

**Eigenvectors for $\lambda_2 = 5$**

$(A - 5I)x = \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, i.e. $x_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Note in each case $A - \lambda I$ has dependent columns,

Note in each case $A - \lambda I$ has dependent columns,
i.e. $\dim(N(A-\lambda I)) \geq 1$.

$$S = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}, \quad S^{-1} = S^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$

## 6 Again: Eigenvalues and -vectors

Find the eigenvalues and eigenvectors of

$$A = \begin{bmatrix} 2 & 5 \\ 3 & -0 \end{bmatrix}$$

Verify, that $A = S\Lambda S^{-1}$ where $S$ contains the eigenvectors as columns. Make sure, they are in the same order as the eigenvalues in $\Lambda$.

Compute $\exp(A)$ using this factorization.

Check Your results with Octave.

```
octave:1> format bank
octave:2> A = [2 5; 3 0]
A =

  2.00  5.00
  3.00  0.00

octave:3> [S, l] = eig(A)
S =

  0.86  -0.71
  0.51  0.71

l =

Diagonal Matrix

  5.00     0
     0  -3.00

octave:4> S * l * inv(S)
ans =

  2.00  5.00
  3.00  0.00
```

## 7 Again: Eigenvalues and -vectors

Find the eigenvalues of $A$ and $B$ (easy for triangular matrices) and $A+B$:

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A+B.$$

Are the eigenvalues of $A+B$ equal (or not equal) to the eigenvalues of $A$ plus eigenvalues of $B$? Explain why!

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix} \xrightarrow{\det} (3-\lambda)(1-\lambda)$$

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$

$$B - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} \xrightarrow{\det} (1-\lambda)(3-\lambda)$$

$$\lambda_1 = 1$$
$$\lambda_2 = 3$$

$(A+B) - \lambda I = \begin{bmatrix} 4-\lambda & 1 \\ \end{bmatrix} \xrightarrow{\det} (4-\lambda)(4-\lambda) - 1$

$\lambda_2 = 3$

$$(A+B)-\lambda I = \begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix} \xrightarrow{\det} (4-\lambda)(4-\lambda) - 1$$

$$= 16 - 8\lambda + \lambda^2 - 1$$

$$= \lambda^2 - 8\lambda + 15$$

$$\overset{!}{=} (\lambda-5)(\lambda-3)$$

$\lambda_1 = 5$

$\lambda_2 = 3$

Sum of eigenvalues $A = \lambda_1 + \lambda_2 = 4$

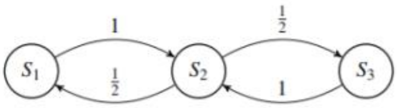
$B = \lambda_1 + \lambda_2 = 4$

$A + B = 8$

$(A+B) = 8$

## 8 Again: Eigenvalues and -vectors

Write down the $3 \times 3$ transition matrix $\boldsymbol{P}$ for the Markov chain shown on the right. Make sure, the columns add to 1. The first column contains the probabilities for moving from state $S_1$ to one of the states $S_1$, $S_2$ and $S_3$.




What is the long term probability for each state? You have find the eigenvector to the eigenvalue 1.

Note: it can be shown, that every stochastic matrix, i.e. one whos columns add to 1 has an eigenvalue 1.

read

$$P = \begin{array}{c|ccc} & S_1 & S_2 & S_3 \\ \hline S_1 & 0 & 1 & 0 \\ S_2 & 0.5 & 0 & 0.5 \\ S_3 & 0 & 1 & 0 \\ \hline & 0.5 & 2 & 0.5 \end{array}$$

Ok transpose Matrix

$$= \begin{array}{c|ccc} & S_1 & S_2 & S_3 \\ \hline S_1 & 0 & 0.5 & 0 \\ S_2 & 1 & 0 & 1 \\ S_3 & 0 & 0.5 & 0 \end{array}$$

read

to furfil constraint of column sums = 1,
E transposed matrix and changed reading direction.

find eigenvalue $P - \lambda I$

$$P - \lambda I = \begin{bmatrix} 0-\lambda & 0.5 & 0 \\ 1 & 0-\lambda & 1 \\ 0 & 0.5 & 0-\lambda \end{bmatrix} \xrightarrow{\det} -\left((0-\lambda)\cdot 1 \cdot 0.5\right) - \left((0-\lambda)\cdot 1 \cdot 0.5\right)$$

$$\overset{!}{=} 0.5\lambda + 0.5\lambda = \lambda$$

$$\begin{bmatrix} 0 & 0.5 & 0-\lambda \end{bmatrix} \overset{!}{=} 0.5\lambda + 0.5\lambda = \underline{\lambda}$$

I stop at this point, because I have no idea what I am doing.

## 9 Singular Value Decomposition (SVD)

Find the SVD of the singular matrix $A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$. The rank is $r = 1$.

1) Eigenvalues of $A^T A$

$$A^T A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$$

$$(A^T A) - \lambda I = \begin{bmatrix} 5-\lambda & 5 \\ 5 & 5-\lambda \end{bmatrix}$$

$\downarrow$ det

$$\det(\cdot\cdot) = (5-\lambda)(5-\lambda) - 25 \overset{|}{=} \lambda^2 - 10\lambda + 25 - 25 \overset{|}{=} \lambda^2 - 10\lambda = (\lambda - 10)(\lambda - 0)$$

$$\sigma_2^2 = \lambda_1 = 10$$
$$\sigma_1^2 = \lambda_2 = 0$$

Eigenvector for $\lambda_1 = 10$

$$(A^T A) - 10 I = \begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Eigenvector for $\lambda_1 = 0$

$$(A^T A) - 0 I = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A^T A = V \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} V^T$$

1) first column of $U$:

$$u_1 = A \frac{v_1}{\sigma_1} = \frac{1}{\sqrt{10}} \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 2\sqrt{2} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$

II) Second column of U:

$$u_2 = A\frac{v_2}{\sigma_2} = \frac{1}{\sqrt{0}}\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$u_2$ is useless!

But $U^T U = I \rightarrow \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix} = U$

conclution:
I invert col. 2.

$$\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}}_{U}\underbrace{\begin{bmatrix} \overset{\sigma_1}{\sqrt{10}} & 0 \\ 0 & \underset{\sigma_2}{0} \end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}}_{V^T}$$