Linear Algebra - Exercise 2

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1 Nullspace

Describe the nullspaces of these three matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} A \\ 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix} \quad \text{and} \quad C = [A \ \ 2A] = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}.$$

Also in this case, the nullspace only consists of the zero-vector.

Furthermore we have

$$C = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} = R$$

The free variables x_3 and x_4 are not in the pivot columns. We choose them $(x_3, x_4) = (0,1)$ and $(x_3, x_4) = (1,0)$. In the first case we find $x_2 = -2$ and $x_1 = 0$ and the second case we find $x_2 = 0$ and $x_1 = -2$. Therefore the nullspace is spanned by the two vectors $[0, -2, 0, 1]^T$ and $[-2, 0, 1, 0]^T$.

$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$ find row reduced echelon form: — elim. matrix

1) $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

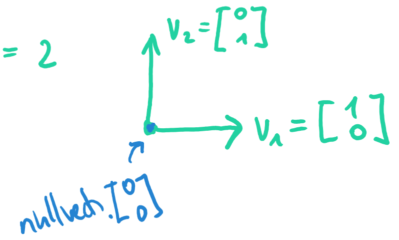
2) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

3) $\begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \underline{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} = \text{rre}$

the vectors $v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $v_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are <u>linearly</u> <u>independet</u>, because v_1 is not a multiple of v_2.

! we know that a combination of 2 independent vectors only add up to the zero vector, if each vector mult. by 0.

$n = 2$



$\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 = \lambda_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$, $\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\lambda_1 = 0$, $\lambda_2 = 0$

$$B = \begin{bmatrix} A \\ 2A \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix} \qquad \underline{\text{rre}}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0{,}5 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \underline{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

$v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ are linearly independent.

$$\lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ 0 \\ 0 \end{bmatrix}$$, only if $\lambda_1, \lambda_2 = 0$ then $\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$C = [A \;\; 2A] = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}$$

$$\text{rre} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0{,}5 \end{bmatrix} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \end{array} = \text{rre}$$

$$\begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 + 2x_4 &= 0 \end{aligned} \quad \Bigg| \quad \begin{aligned} x_1 &= -2x_3 \\ x_2 &= -2x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

$\left.\begin{aligned} x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}\right\}$ freie variablen

We will set x_3, x_4 to $0, 1$ and $1, 0$:

1) $x_3 = 0$, $x_4 = 1$

$$N(C) = [0, -2, 0, 1]^T$$

2) $x_3 = 1$, $x_4 = 0$

$N(C) = [-2, 0, 1, 0]^T$

2 Reduced echelon form of a matrix

What's the solution of the linear system

$$\begin{aligned} x \quad &= 1 \\ 4x+3y &= 1 \\ 2x+3y &= -1 \end{aligned}$$

Use the results from the slides where we computed the reduced row echelon form of the augmented matrix $[\boldsymbol{A} \quad \mathbf{b}]$.

Find the row reduced echelon form of the augmented matrix adjoined with b.

$$[A\ b] = \begin{bmatrix} 1 & 0 & 1 \\ 4 & 3 & 1 \\ 2 & 3 & -1 \end{bmatrix} \xrightarrow{R3-2R1} \begin{bmatrix} 1 & 0 & 1 \\ 4 & 3 & 1 \\ 0 & 3 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -3 \\ 0 & 3 & -3 \end{bmatrix}$$

$$\xrightarrow{R3-R2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R2/3} \begin{array}{c} \begin{array}{ccc} x & y & b \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

back substitution:

$$x = 1, \quad y = -1 \quad \rightarrow \quad \begin{aligned} 2x + 3y &= -1 \\ 2(1) + 3(-1) &= -1 \\ 2 - 3 &= -1 \end{aligned}$$

3 Again: reduced echelon form of a matrix

Find the reduced row echelon form of

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix}$$

What is the rank of $\boldsymbol{A}$ and what is the special solution to $\boldsymbol{A}\mathbf{x} = \mathbf{0}$.

$$Ax = 0 \quad \rightarrow \quad \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

1) reduced row echelon

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 3 & 3 & 10 & 13 \end{bmatrix} \xrightarrow{R3-(R1+R2)} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 2 & 8 & 10 \\ 0 & 2 & 0 & 0 \end{bmatrix} \xrightarrow{R2-2R1} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 4 & 4 & 4 \\ 0 & 2 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{R2}{4},\ \frac{R3}{2}} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \xrightarrow[R_1+R_2]{R_3-R_2} \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix} \xrightarrow[R_1+3R_3]{R_2+R_3} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$$\xrightarrow{R3\cdot(-1)} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = rre \qquad \text{Rank} = 3$$

$$\begin{aligned} x_1 + x_4 &= 0 \qquad & x_1 &= -x_4 \\ x_2 &= 0 & x_2 &= 0 \\ x_3 + x_4 &= 0 & x_3 &= -x_4 \\ & & x_4 &= \text{free variable} \end{aligned}$$

$x_4 = 1$, to solve $Ax=0$, we change the free variable to

$$N(A) = [-1, 0, -1, 1]^T = \vec{x}_1$$

$x_4 = 0$

$$N(A) = [0, 0, 0, 0]^T = \vec{x}_2$$

4 Again: reduced row echelon form of a matrix

Find the reduced row echelon form of

$$\boldsymbol{A} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

What is the rank of $\boldsymbol{A}$ and what is the special solution to $\boldsymbol{A}\mathbf{x} = \mathbf{0}$.

rre

octave → rref(A) $= \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$Ax = 0$

$x_1 = x_4$

$x_2 = x_4$

$x_3 = x_4$

$x_4 =$ free variable

I choose $x_4 = 1$ and $x_4 = 0$

$\vec{x}_1 = [1 \ 1 \ 1 \ 1]^T$

$\vec{x}_2 = [0 \ 0 \ 0 \ 0]^T$

5 Column space

Describe the column spaces (they are subspaces of $\mathbb{R}^2$) for

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}.$$

$C(I)$

$$I = \begin{array}{cc} a_1 & a_2 \end{array} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

the column space is spanned by the two pivot columns $a_1 + a_2$

$a_1 = [1 \ 0]^T$, $a_2 = [0 \ 1]^T$

$C(A)$

$$A = \begin{array}{cc} a_1 & a_2 \end{array} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

the colum space is spanned by the first column $a_1 = [1 \ 2]^T$

$a_2 = 2a_1$ (is a multiplicative of a_1)

$C(B)$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

the column space is spanned by the first and third column

$a_1 = [1 \ 0]^T$ and $a_3 = [3 \ 4]^T$

$a_2 = 2a_1$ (is a multiplicative of a_1)

6 Again: column space

We are given three different vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$. Construct a matrix so that the equations $\boldsymbol{A}\mathbf{x} = \mathbf{b}_1$ and $\boldsymbol{A}\mathbf{x} = \mathbf{b}_2$ are solvable, but $\boldsymbol{A}\mathbf{x} = \mathbf{b}_3$ is not solvable. How can You decide if this is possible? How could You construct $\boldsymbol{A}$?

$A = [b_1 \ b_2]$ if $\vec{b}_1, \vec{b}_2$ are linearly independent.

7 Complete Solution of linear system

Find the condition on $\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^T$ for $\boldsymbol{A}\mathbf{x} = \mathbf{b}$ to be solvable, if

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} b_1 \end{bmatrix}$$

7 Complete Solution of linear system

Find the condition on $\begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}^T$ for $\boldsymbol{A}\mathbf{x} = \mathbf{b}$ to be solvable, if

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

This condition puts $\mathbf{b}$ in the column space of $\boldsymbol{A}$. Find the complete solution $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$ to $\boldsymbol{A}\mathbf{x} = \mathbf{b}$.

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

use augmented matrix A adjoined with b

$$\begin{bmatrix} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ -2 & -3 & b_3 \end{bmatrix} \xrightarrow{R2-R1} \begin{bmatrix} 1 & 1 & b_1 \\ 1 & 2 & b_2 \\ 0 & 0 & b_1+b_2+b_3 \end{bmatrix} \xrightarrow{R2-R1} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & \end{array} \\ \begin{bmatrix} 1 & 1 & b_1 \\ 0 & 1 & b_2-b_1 \\ 0 & 0 & b_1+b_2+b_3 \end{bmatrix} \end{array}$$

Rank $= 2$

only if $b_1+b_2+b_3 = 0$, do we have a solution.
let's assume $b_3 = -(b_1+b_2)$.

$$x_1 + (b_2-b_1) = b_1$$
$$x_1 = b_1-(b_2-b_1) = 2b_1-b_2$$
$$x_2 = b_2-b_1$$

Because rank $= 2$, the column space therefore has dimension 2.

a partial solution is

$$x_p = [2b_1-b_2 \quad b_2-b_1]^T$$

A is a 3×2 matrix, therefore we have $m=3, n=2$
A^T, $m=2, n=3$

$$\boxed{\begin{aligned} \dim(C(A)) + \dim(N(A^T)) &= r + (m-r) = m \\ \dim(C(A^T)) + \dim(N(A)) &= r + (n-r) = n \end{aligned}}$$

we know $\dim(N(A^T)) = m - r = 2 - 2 = 0 = x_n$

$$x = x_p + x_n = x_p = \begin{bmatrix} 2b_1 - b_2 \\ b_2 - b_1 \end{bmatrix}$$

8 Again: Complete Solution of linear system

There are $n = 3$ unknowns but only $m = 2$ equations:

$$\begin{aligned} x + \ \ y + z &= 3 \\ x + 2y - z &= 4 \end{aligned}$$

These are two planes in the xyz space. The planes are not parallel so they intersect in a line. This line of solutions is exactly what elimination will find. The **particular solution** will be **a point on the line**. Adding the nullspace vector $\mathbf{x}_n$ will move us along the line. Then $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$ gives the whole line of solutions.

Answer the following questions:

- Compute the reduced row echelon form $\begin{bmatrix} \boldsymbol{R} & \mathbf{d} \end{bmatrix}$ of the augmented matrix $\begin{bmatrix} \boldsymbol{A} & \mathbf{b} \end{bmatrix}$.

$$[A \ \ b] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & -1 & 4 \end{bmatrix} \xrightarrow{R2 - R1} \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -2 & 1 \end{bmatrix} \xrightarrow{R1 - R2} \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

Rank $r = 2$

$$[R \ \ d] = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

- Looking at $\begin{bmatrix} \boldsymbol{R} & \mathbf{d} \end{bmatrix}$ can You see, whether the linear system has a solution?

$$\boxed{\begin{aligned} \dim(C(A)) + \dim(N(A^T)) &= r + (m - r) = m \\ \dim(C(A^T)) + \dim(N(A)) &= r + (n - r) = n \end{aligned}}$$

$\dim(N(A^T)) = m - r = 4 - 2 = 2$, there is another solution.

- What is the rank of the $\boldsymbol{A}$?

Rank $r = 2$

- Compute the special solution $\mathbf{x}_s$ of $\boldsymbol{A}\mathbf{x} = \mathbf{0}$.

$x_s = Ax = 0$, remove b from $[R \ \ d]$ $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{bmatrix}$

$x_1 = -3x_3$

$x_2 = 2x_3$

for $x_3 = 1$

$x_s = \begin{bmatrix} -3 & 2 & 1 \end{bmatrix}^T$

- Compute the complete solution of $\boldsymbol{A}\mathbf{x} = \mathbf{b}$. Draw this solution in a 3D-plot.

$x_p = Ax = b$

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$\underline{0}$

$x_p = [2 \ 1 \ 0]^T$

- Why is the particular solution not multiplied by a constant, whereas the the special solution is?

therefore the complete solution is $[2 \ 1 \ 0]^T + t[-3 \ 2 \ 1]^T$ where $t \in \mathbb{R}$.

9 Again: Complete Solution of linear system

The complete solution to $\boldsymbol{A}\mathbf{x} = \begin{bmatrix} 1 & 3 \end{bmatrix}^T$ is $\mathbf{x} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T + c\underline{\begin{bmatrix} 0 & 1 \end{bmatrix}}^T$. Find $\boldsymbol{A}$, it's rank and the reduced row echelon form of the augmented matrix..

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

10 Trick to find the nullspace

Suppose that the first r columns are the pivot columns. Then the reduced row echelon form looks like

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{F} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

where $\boldsymbol{I}$ is the $r \times r$ identity matrix with the r pivot columns (and r pivot rows), $\boldsymbol{F}$ is an $r \times (n-r)$ matrix and the matrices $\mathbf{0}$ are filled with zeros.

Then the columns of the **nullspace matrix**

$$\boldsymbol{N} = \begin{bmatrix} -\boldsymbol{F} \\ \boldsymbol{I} \end{bmatrix}$$

solve $\boldsymbol{R}\mathbf{x} = \mathbf{0}$.

Lets try this in a numerical example. The special solution of $\boldsymbol{R}\mathbf{x} = x_1 + 2x_3 + 3x_3 = 0$ are the columns of $\boldsymbol{N}$:

$$\boldsymbol{R} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \qquad \boldsymbol{N} = \begin{bmatrix} -\boldsymbol{F} \\ \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

It's easy to see, that the columns of $\boldsymbol{N}$ satisfy the equation!

11 The fundamental theorem of linear algebra

The matrix $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ has $m = 2$ and $n = 3$ and rank $r = 1$.

The row space is the line along $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$. The nullspace is the plane $x_1 + 2x_2 + 3x_3 = 0$. Their dimensions add to $1 + 2 = 3$.

All columns are multiples of the first column $\begin{bmatrix} 1 & 2 \end{bmatrix}^T$. Therefore $\boldsymbol{A}^T\mathbf{y} = \mathbf{0}$ has the solution $\mathbf{y} = \begin{bmatrix} 2 & -1 \end{bmatrix}^T$. The column space and left nullspace are **orthogonal lines** in $\mathbb{R}^2$. Dimensions $1 + 1 = 2$.

Question: if $\boldsymbol{A}$ has three equal rows, what is its rank? And what are two of the $\mathbf{y}$'s in its left nullspace, i.e. the nullspace of $\boldsymbol{A}^T$?

if A has 3 equal rows, the rank = 1

what are the 2 y's? the nullspace is a plane, hence 2 dimensional.
the dimensions add up to 1+2 = 3.