Linear Algebra - Exercise 1

24 December 2018 11:15

1 Schwarz inequality and dot product

If $\|\mathbf{v}\| = 5$ and $\|\mathbf{w}\| = 3$, what are the smallest and largest values of $\|\mathbf{v} - \mathbf{w}\|$? What are the smallest and largest values of $\mathbf{v} \bullet \mathbf{w}$? Hint: (i) use geometrical arguments; (ii) for an analytical proof use the Schwarz inequality and the definition of the dot product!

(i) geometrical consideration

the differenz between 2 vectors is smallest if they point in the same directions

therefore:

$$5-3 = 2 \leq \|v-w\| \leq 8 = 5+3$$

(ii) analytical proof

using Cauchy-Schwarz inequality

$$-\|v\|\,\|v\| \leq v \bullet w \leq \|v\|\,\|v\|$$

we find

$$\|v-w\|^2 \geq \|v\|^2 + \|w\|^2 - 2\|v\|\,\|w\| = (\|v\| - \|w\|)^2$$

and similarly

$$\|v-w\|^2 \leq \|v\|^2 + \|w\|^2 + 2\|v\|\,\|w\| = (\|v\| + \|w\|)^2$$

Hence after taking the root on both sides

$$\big|\,\|v\| - \|w\|\,\big| \leq \|v-w\| \leq \big|\,\|v\| + \|w\|\,\big|$$

In our example

$$\big|\,\|v\| - \|w\|\,\big| = |5-3| = 2 \leq \|v-w\| \leq \|v\| + \|w\| = |5+3| = 8$$

2 Unit vectors

Find four perpendicular unit vectors with all components equal to $1/2$ or $-1/2$. Hint: 4-dimensional space!

Hint: perpendicular $\hat{=}$ orthogonal $\hat{=}$ $\perp$

we know $v \bullet w = 0 \Leftrightarrow v \perp w$

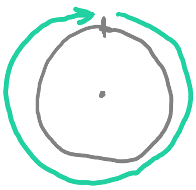
$$v_1 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad v_2 = \frac{1}{2}\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}$$

$$v_3 = \frac{1}{2}\begin{bmatrix}-1\\1\\-1\\1\end{bmatrix}, \quad v_3 = \frac{1}{2}\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}$$

3 Unit vectors in a clock

Find unit vectors $\mathbf{h}(t)$ and $\mathbf{m}(t)$ in the direction of the hour and minute hands of a clock, where t denotes the elapsed time in hours. If $t = 0$ represents noon then $\mathbf{m}(0) = \mathbf{h}(0) = \begin{bmatrix}0 & 1\end{bmatrix}^T$.

At what time will the hands of the clock first be perpendicular? At what time after noon will the hands first forma straight line? Remember, that in the dot product $\mathbf{m}(t) \bullet \mathbf{h}(t)$ we can use $\sin x \sin y + \cos x \cos y = \cos(x - y)$.



m full circle at $t = 1$ $\rightarrow$ h $\frac{1}{12}$th circle

$$m(t) = \begin{bmatrix}\sin x\\ \cos x\end{bmatrix}, \quad h(t) = \begin{bmatrix}\sin y\\ \cos y\end{bmatrix}$$

full circle = $360° = 2\pi$

$$m(t) = \begin{bmatrix}\sin 2\pi t\\ \cos 2\pi t\end{bmatrix}, \quad h(t) = \begin{bmatrix}\sin \frac{\pi t}{6}\\ \cos \frac{\pi t}{6}\end{bmatrix}$$

$$m(t) \bullet h(t) \overset{!}{=} \sin 2\pi t \cdot \sin\frac{\pi t}{6} + \cos 2\pi t \cdot \cos\frac{\pi t}{6}$$

$$\boxed{\sin\alpha \cdot \sin\beta - \cos\alpha \cdot \cos\beta = \cos(\alpha - \beta)}$$

$$= \cos\left(2\pi t - \frac{\pi t}{6}\right) = \cos\left(\frac{11\pi t}{6}\right)$$

the hands are perpendicular when the cosine eq. 0.

$$\frac{11\pi t}{6} = \ldots, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$$\frac{11\pi t}{6} = -\frac{\pi}{2} \quad \Big| \quad 11\pi t = -3\pi \quad \Big| \quad t = -\frac{3}{11}$$

$$\frac{11\pi t}{6} = \frac{\pi}{2} \quad \Big| \quad 11\pi t = 3\pi \quad \Big| \quad t = \frac{3}{11}$$

$$\frac{11\pi t}{6} = \frac{3\pi}{2} \quad \Big| \quad 11\pi t = 9\pi \quad \Big| \quad t = \frac{9}{11}$$

In min and sec $\quad -\frac{3}{11}$

$60 \cdot \left(\frac{3}{11}\right) = 16.\overline{36} = 16$ min

$60 \cdot 0.\overline{36} = 21.\overline{81} = 21.81$ sec

Rest analogue

16 min 21.81 sec , 49 min 5.4$\bar{5}$ sec

the hands are pointing in oposite direction if cosine = -1

$\frac{11\pi t}{6} = \dots, -\pi, \pi, 3\pi$

32 min 43.64 sec

the hands are pointin in the same direction if cosine = 1

$\frac{11\pi t}{6} = \dots, -2\pi, 0, 2\pi, \dots$

1 hour 5 min 27.$\bar{27}$ sec

4 Row and column picture of linear equations

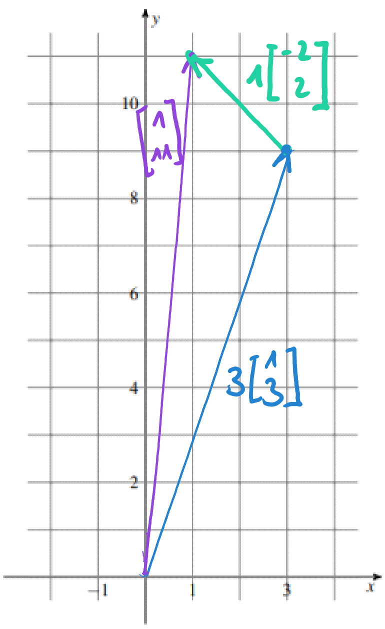
Try to solve the system of two equations

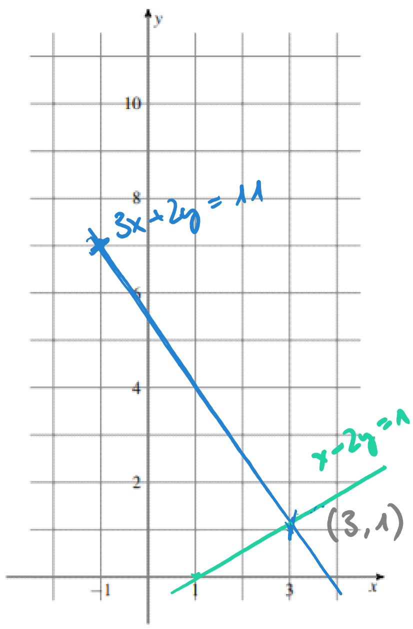
$$\begin{aligned} x - 2y &= \ \ 1 \\ 3x + 2y &= 11 \end{aligned}$$

in the two unknowns x and y using the (i) the row picture and (ii) the column picture. Depict both methods in the (x, y)-plane.

$$\begin{bmatrix} 1 \\ 11 \end{bmatrix} = x \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$x = 3 \qquad y = 1$





$y = \frac{11 - 3x}{2}$, $x = 3$ $\quad \frac{11 - 9}{2} = \underline{1}$

$x = -1 \quad \frac{11 + 3}{2} = \underline{7}$

5 Elimination

What multiple l_{21} of equation 1 should be subtracted from equation 2 such that x disappears from this equation?

$$\begin{aligned} 2x + 3y &= 1 \\ 10x + 9y &= 11 \end{aligned}$$

After this elimination step, write down the upper triangular system and circle the two pivots. Note: the number on the right hand side have no influence on those pivots.

$$\begin{aligned} 2x + 3y &= 1 \\ 10x + 9y &= 11 \end{aligned} \xrightarrow{\text{Augmented M.}} \begin{array}{c} \mathrm{I} \\ \mathrm{II} \end{array} \begin{bmatrix} 2 & 3 & 1 \\ 10 & 9 & 11 \end{bmatrix}$$

dissapear

$$\mathrm{II} - 5\mathrm{I} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -6 & 6 \end{bmatrix} \xrightarrow{\text{equation}} \begin{aligned} 2x + 3y &= 1 \\ -6y &= 6 \end{aligned}$$

pivot elements are 2, -6.

augmented coefficient matrix and elimination matrix

$$\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 10 & 9 & 11 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -6 & 6 \end{bmatrix}$$

6 Back substitution

Solve the triangular system of the previous problem by back substitution, y before x. Verify that x times the first column of the original coefficient matrix plus y times the second column equals the right hand side. If the right hand side changes to $[4, 44]$, what is the new solution?

$$\begin{aligned} 2x + 3y &= 1 \\ -6y &= 6 \end{aligned}$$

1) $-6y = 6 \rightarrow y = -1$

2) $2x - 3 = 1 \rightarrow 2x = 4 \rightarrow x = 2$

$L = \{2, -1\}$

do we get the right hand side if we insert solution?

$$\begin{bmatrix} 2 \\ 10 \end{bmatrix} 2 + \begin{bmatrix} 3 \\ 9 \end{bmatrix} -1 = \begin{bmatrix} 1 \\ 11 \end{bmatrix}$$

7 No or infinitely many substitution

In the following system choose a right side which gives (i) no solution and another right side which gives (ii) infinitely many solutions

$$\begin{aligned} 2x + 2y &= 10 \\ 6x + 4y &= \end{aligned}$$

Draw the corresponding geometrical interpretation in (i) the row picture and in (ii) the column picture.

Augmented coefficient matrix

$$\begin{bmatrix} 2 & 2 & 10 \\ 6 & 4 & a \end{bmatrix} \xrightarrow{\text{II} - 3\text{I}} \begin{bmatrix} 2 & 2 & 10 \\ 0 & -2 & a-30 \end{bmatrix}$$

eliminate

from last row we conclude

$y = 15 - \frac{a}{2} \rightarrow x = 5 - 15 + \frac{a}{2} = \frac{a}{2} - 10$

Always one solution, they are linear independent.

8 Elimination (3 equations)

Reduce this system to upper triangular form by two row operations

$$\begin{aligned} 2x - 3y \quad &= 8 \\ 4x - 7y + \ z &= 20 \\ -2y + 2z &= 0 \end{aligned}$$

Circle the pivots. Solve by back substitution for z, y and x.

$$\begin{aligned} 2x - 3y + 0 &= 8 \\ 4x - 7y + 1z &= 20 \\ -2y + 2z &= 0 \end{aligned} \xrightarrow{\text{Augment. coeff. Matrix}} \begin{bmatrix} 2 & -3 & 0 & 8 \\ 4 & -7 & 1 & 20 \\ 0 & -2 & 2 & 0 \end{bmatrix}$$

eliminate

$$\xrightarrow{\text{I} + (-2\text{I})} \begin{bmatrix} 2 & -3 & 0 & 8 \\ 0 & -1 & 1 & 4 \\ 0 & -2 & 2 & 0 \end{bmatrix} \xrightarrow{\text{III} - 2\text{II}} \begin{bmatrix} 2 & -3 & 0 & 8 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

pivot elements

This system has no solution because $0 = -8$ is a contradiction.

9 Elimination matrices (3D))

Solve the system of equations of the previous problem using the matrix notation. What are the corresponding elimination matrices? Multiply the elimination matrices (from the left) with the coefficient matrix $\boldsymbol{A}$. Do You get an upper triangular matrix?

Elim. Matrix

$$\begin{bmatrix}2&-3&0&8\\4&-7&1&20\\0&-2&2&0\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\-2&1&0\\0&1&0\end{bmatrix}\begin{bmatrix}2&-3&0&8\\0&-1&1&4\\0&-2&2&0\end{bmatrix}$$

Elim. Matrix

$$\begin{bmatrix}2&-3&0&8\\0&-1&1&4\\0&-2&2&0\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}\begin{bmatrix}2&-3&0&8\\0&-1&1&4\\0&0&0&-8\end{bmatrix}$$

Multiplication to verify:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}\begin{bmatrix}1&0&0\\-2&1&0\\0&1&0\end{bmatrix}\begin{bmatrix}2&-3&0&8\\4&-7&1&20\\0&-2&2&0\end{bmatrix}=\begin{bmatrix}2&-3&0&8\\0&-1&1&4\\0&0&0&-8\end{bmatrix}$$

10 Matrix for Row Exchange

Considering the coefficient matrix of the last problem; what matrix $\boldsymbol{P}_{ij}$ exchanges row 2 with row 3? Hint: take the identity matrix and exchange appropriate rows!

$$P_{ij}=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$

11 Gauss-Jordan

Compute the inverse $\boldsymbol{A}^{-1}$ of $\boldsymbol{A}=\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}$. Check the result by computing $\boldsymbol{A}^{-1}\boldsymbol{A}$.

Addon (difficult): When does the inverse of a matrix not exist?

$$A=\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}$$

the inverse of a Matrix does not exist, if the det(A) is 0.

$$I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$[A\,I]=\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\-1&1&0&0&1&0\\0&-1&1&0&0&1\end{array}\right]\xrightarrow{\text{II}+\text{I}}\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&1&1&0\\0&-1&1&0&0&1\end{array}\right]$$

eliminate

$$\xrightarrow{\text{III}+\text{II}}\left[\begin{array}{ccc|ccc}1&0&0&1&0&0\\0&1&0&1&1&0\\0&0&1&1&1&1\end{array}\right]=[I\;A^{-1}]$$

therefore

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

12 The augmented matrix

Write down the augmented matrix $[\boldsymbol{A}\ \mathbf{b}]$ with an extra column:

$$\begin{aligned} x+2y+2z &= 1 \\ 4x+8y+9z &= 3 \\ 3y+2z &= 1 \end{aligned}$$

Apply $\boldsymbol{E}_{21}$ and $\boldsymbol{P}_{32}$ to reach a triangular system. Solve by back substitution. What combined matrix $\boldsymbol{P}_{32}\boldsymbol{E}_{21}$ will do both steps at once?

$$[P_{32}]\,[E_{21}]\,[Ab] = [A']$$

$$[Ab] = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 4 & 8 & 9 & 3 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

eliminate

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 4 & 8 & 9 & 3 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

$$[E_{21}] \quad \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

$$[P_{32}] \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

combined elimination matrix

$$[E'] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -4 & 1 & 0 \end{bmatrix}$$

13 LU-Factorization

Write down the LU-factorization of the coefficient matrix of the previous problem.

L = Lower triangular matrix, U = Upper triangular Matrix

$$Ax = (LU)x = L(Ux) = b$$

we know [Ab] and need to find x

$$Ax = b$$

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 4 & 8 & 9 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

we know $A = (LU)$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 4 & 8 & 9 \\ 0 & 3 & 2 \end{bmatrix} = A'$$

$$A' = \begin{bmatrix} 4 & 8 & 9 \\ 0 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.25 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 4 & 8 & 9 \\ 0 & 3 & 2 \\ 0 & 0 & -0.25 \end{bmatrix}}_{U}$$