Lucerne University of
Applied Sciences and Arts
HOCHSCHULE
LUZERN
Technik & Architektur

Module TA.BA_IMATH
Final Exam HS 2015

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Student:					
# marks:					
Total marks:			Grade:		

Thursday, January, 21th 2016 (08:30-10:30), E204

Important Notes (README):

- **Allowed auxiliaries:** Pocket calculator (z.B. TI nSpire CX, CAS), Your lecture notes and exercises, Your own formulary and two of Your joice.
- Write down all intermediate results of the solution process (including CAS Commands if You use CAS). Numerical results should be accurate to four (4) digits. Sketches have to be correct qualitatively.
- **Write Your solution in the free space after the statement of the problem. If You need more space, add another sheet of paper and make sure You write Your name and the problem number on that page**.
- **Grading:** A correct and complete solution of one problem statement yields five marks. For the final grade the four (4) problems with the highest number of marks are used.

1 Linear Algebra

1. (2 marks) Find the dimension and construct a basis for the four subspaces associated with the matrix

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix}$$

2. (3 marks) Find the eigenvalues $\lambda_1 = \sigma_1^2$, $\lambda_2 = \sigma_2^2$ and the corresponding unit eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$ of $\boldsymbol{A}^T\boldsymbol{A}$. Then find $\mathbf{u}_1 = \boldsymbol{A}\mathbf{v}_1/\sigma_1$ and $\mathbf{u}_2 = \boldsymbol{A}\mathbf{v}_2/\sigma_2$.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

Verify that $\mathbf{u}_1$ is a unit eigenvector of $\boldsymbol{A}\boldsymbol{A}^T$.

1. ?

2.

I) $A^TA = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}$

II) $(A^TA) - \lambda I = \begin{bmatrix} 10-\lambda & 20 \\ 20 & 40-\lambda \end{bmatrix}$

$\det(\dots) = (10-\lambda)(40-\lambda) - 400$

$\overset{!}{=} 400 - 50\lambda + \lambda^2 - 400$

$\overset{!}{=} \lambda^2 - 50\lambda = (\lambda - 50)(\lambda - 0)$

$\lambda_1 = 50 \quad \sigma_1 = \sqrt{50}$

$\lambda_2 = 0 \quad \sigma_2 = \sqrt{0}$

IIIa) Eigenvector zu $\lambda_1 = \sigma_1^2 = 50$

$(A^TA) - \lambda_1 I = \begin{bmatrix} -40 & 20 \\ 20 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$v_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \xrightarrow{\text{norm}} \frac{1}{\|\vec{x}\|} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \Big| \quad \|\vec{x}\| = \sqrt{1^2 + 2^2} \overset{!}{=} \sqrt{5}$

IIIb) Eigenvector zu $\lambda_2 = \sigma_1^2 = 0$

$(A^TA) - \lambda_2 I = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$v_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\qquad \begin{bmatrix} 2 \\ -1 \end{bmatrix} \xrightarrow{\text{norm}} \frac{1}{\|\vec{x}\|} \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad \Big| \quad \|\vec{x}\| = \sqrt{2^2 + (-1)^2} \overset{!}{=} \sqrt{5}$

IVa) Find $u_1 = A \frac{v_1}{\sigma_1}$ (1. Spalte U)

$u_1 = \frac{v_1}{\sigma_1} = \frac{1}{\sqrt{50}} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\begin{bmatrix} \frac{\sqrt{2}\sqrt{5}}{\sqrt{50}} \\ \frac{2\sqrt{2}\sqrt{5}}{\sqrt{50}} \end{bmatrix}$ $\boxed{\frac{1}{\sqrt{5}} = \frac{\sqrt{2}\sqrt{5}}{\sqrt{50}} \quad \Big| \quad \sqrt{50} = \sqrt{2}\cdot\sqrt{5}\cdot\sqrt{5}}$

$\begin{bmatrix} \frac{1}{\sqrt{50}} & \frac{2}{\sqrt{50}} \\ \frac{3}{\sqrt{50}} & \frac{6}{\sqrt{50}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}\sqrt{5} + 4\sqrt{2}\sqrt{5}}{50} \\ \frac{3\sqrt{2}\sqrt{5} + 12\sqrt{2}\sqrt{5}}{50} \end{bmatrix} = \begin{bmatrix} \frac{5\sqrt{10}}{50} \\ \frac{15\sqrt{10}}{50} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{10}}{10} \\ \frac{3\sqrt{10}}{10} \end{bmatrix} = \begin{bmatrix} 0.316 \\ 0.948 \end{bmatrix}$

V) Find U

$U = \begin{bmatrix} 0.316 & 0.948 \\ 0.948 & -0.316 \end{bmatrix}$

VI) Verify that u_1 is a unit eigenvector of A^TA.

$A^TA = A^T \underbrace{\begin{bmatrix} 0.316 & 0.948 \\ 0.948 & -0.316 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{10} & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix}}_{V^T}$

2 Linear Algebra and Multivariate Calculus

1. (3 marks) You want to describe the (orthogonal) projection onto the plane $x + 2y + 2z - 3 = 0$ using a 4×4-matrix $\boldsymbol{P}$ in homogeneous coordinates. Assume the translation matrix

$$\begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad T^{-1} \quad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

1. (3 marks) You want to describe the (orthogonal) projection onto the plane $x + 2y + 2z - 3 = 0$ using a 4×4-matrix $\boldsymbol{P}$ in homogeneous coordinates. Assume the translation matrix

$$\boldsymbol{T} = \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

moves the point $(3, 0, 0)$ into the origin. You write the projection matrix in the form $\boldsymbol{P} = \boldsymbol{T}^T \boldsymbol{P}_0 \boldsymbol{T}$. Compute $\boldsymbol{P}_0$.

2. (2 marks) Apply the chain rule to compute the partial derivative $\partial z / \partial u = z_u$ if $z = \ln(xy)$ where $x = (u^2 + v^2)^2$, $y = (u^3 + v^3)^2$.

1. plane $x + 2y + 2z = 3$, $n = \frac{1}{\sqrt{5}}[1\ 2\ 2]^T = \frac{1}{3}[1\ 2\ 2]^T$

T moves $(3,0,0) \to (0,0,0)$

$$I - nn^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}[1\ 2\ 2]^T$$

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{bmatrix} \frac{1}{9}$$

$$= I - \begin{bmatrix} 1/9 & 2/9 & 2/9 \\ 2/9 & 4/9 & 4/9 \\ 2/9 & 4/9 & 4/9 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 8 & -2 & -2 \\ -2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix}$$

$$P_0 = \begin{bmatrix} I - nn^T & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ 0\ 0\ 0 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 8 & -2 & -2 & 0 \\ -2 & 5 & -4 & 0 \\ -2 & -4 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

$$T^{-1} P_0 T = \frac{1}{3}\begin{bmatrix} 1 & & & 3 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 8 & -2 & -2 & 0 \\ -2 & 5 & -4 & 0 \\ -2 & -4 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 1 & & & -3 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 8 & -2 & -2 & -24 \\ -2 & 5 & -4 & 6 \\ -2 & -4 & 5 & 6 \\ 0 & 0 & \overset{0}{4} & 9 \end{bmatrix}$$

$$P = \frac{1}{3}\begin{bmatrix} 8 & -2 & -2 & 3 \\ -2 & 5 & -4 & 6 \\ -2 & -4 & 5 & 6 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

$$P(3,0,0) = \frac{1}{3}\begin{bmatrix} 8 & -2 & -2 & 3 \\ -2 & 5 & -4 & 6 \\ -2 & -4 & 5 & 6 \\ 0 & 0 & 0 & 9 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 36 \\ 0 \\ 0 \\ 9 \end{bmatrix} = \begin{bmatrix} 12 \\ 0 \\ 0 \\ 3 \end{bmatrix} \to \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

einsetzen gleichung:

$x + 2y + 2z = -3$

$12 = -3$

$4 - 3 = 0$

Something went wrong...

3 Multivariate Calculus and Numerical Errors

1. (2.5 marks) The temperature in space ($\mathbb{R}^3$) is given by the function

$$T(x, y, z) = z \sin x + \frac{1}{2} \ln\left(x^2 - y^2\right)$$

   You are in $(\pi,\ 0,\ 1)$. In which direction would You move, if You want to move to a cooler place as quickly as possible. Approximate the temperature change using linearization if we move from $(\pi,\ 0,\ 1)$ to $(\pi + 0.1, -0.1, 1.1)$. Does the temperature increase or decrease?

2. (2.5 marks) Using Your pocket calculator compute

$$A = \sqrt{x+1} - \sqrt{x} \qquad \text{with} \quad x = 3'412'789'678'105'619$$

   as accurately as possible and with as few operations as possible.

4 Numerical Integration

1. (3 marks) Apply Simpson's rule to compute

$$I = \int_0^{\pi} \sin(x)\,\mathrm{d}x$$

   using 2, and 4 subintervals, respectively.

2. (2 marks) What is the lowest number of subintervals (an even number) to achieve a result accurate to six digits after the comma.

5 Ordinary Differential Equations

Consider the second-order differential equation

$$y'' = -3y' - y, \quad y(0) = 0 \quad \text{and} \quad y'(0) = 3,$$

1. (1.5 marks) Express this second-order ODE as an equivalent system of first-order ODEs, including the initial conditions for the system.
2. (1.5 marks) Perform one step of the forward Euler method **by hand** for this ODE system using a step size of $h = 0.1$.
3. (2 marks) Perform one step of the backward Euler method **by hand** for this ODE system using a step size of $h = 0.1$.