int main(int argc, char\* argv []) {

//FUNDAMENTAL CONSTANTS

//CGS UNITS (Centimetre, Gram, Second)

const double gravitational\_constant = 6.67 \* pow(10, -8);

const double speed\_of\_light = 3.0 \* pow(10, 10);

const double radiation\_density\_constant = 7.56 \* pow(10, -15);

const double hydrogen\_fraction = 0.34;

const double helium\_fraction = 0.64;

const double heavy\_fraction = 0.02;

const double boltzmann\_constant = 1.38 \* pow(10, -16);

const double proton\_mass = 1.67 \* pow(10, -24);

const double pi = 3.14159;

//DERIVED CONSTANTS

double mu\_reciprocal = 2.0 \* hydrogen\_fraction + 0.75 \* helium\_fraction + 0.5 \* heavy\_fraction;

double mu = 1.0 / mu\_reciprocal; //Schwarszchild equation 8.2

const double Central\_Pressure = 6.0 \* pow(10, 15);

const double Central\_Temperature = 1.56 \* pow(10, 7);

const double central\_density = 4;

const double Total\_Mass = 2.0 \* pow(10, 33); //Wikipedia (grams)

const double Radius = 7.0 \* pow(10, 10); //Wikipedia (cms)

const double Total\_Luminosity = 6.0 \* pow(10, 33); //Schwarzschild page 42

const double t\_over\_tentosix = 17.0; //Schwarzschild page 83, Table 10.1

const double epsilon1 = 0.0001; //Schwarzschild page 83, Table 10.1

const double nu = 4.0; //Schwarzschild page 83, Table 10.1

**C++ CODE FOR STELLAR STRUCTURE CALCULATIONS**

**For Main Sequence Stars**

double r = 0.0, from\_surface;

double rfactor, temperature\_constants\_factor;

double pressure\_fundamental\_factors, pressure\_stellar\_factors, pressure\_all\_factors;

double pressure, mass, luminosity, temperature;

double zero\_pressure\_distance = 0.0;

bool zero\_pressure\_reached = false;

**PREFACE**

This paper is the third in a series of three papers that look, respectively, at the physics of main sequence stars, techniques of numerical integration and, this part III, the application of numerical integration techniques to the calculation of the internal structure of stars. Internal structure means the profile of pressure, temperature, density and other physical quantities from the centre of the star to the surface. This paper does not describe the evolution of stars.

NOTE TO ME :

This is the structure of the argument and what I want to do :

1. With a constant density approximation the main 4 equations can be analytically integrated but it must be remembered that an approximation is involved so the equations should be used for only a small amount when integrating out from the centre.
2. The 4 integrated equations can be converted into Schwarszchild’s second transformation variables but why not just stay with the physical r unit ?
3. The integrated equations vary with r (or x\*) only which means that their right-hand sides are decoupled.
4. An AM algorithm can take over from the initial outward solution. This will require values for the derivatives at points x\*. At this point the p\*, q\*, f\*, t\* values are distributed across to the other equations (to get the derivatives).
5. There is also what one might call a linear density approximation. This can also give analytic solutions. These can also presumably be converted to p\*, q\*, f\*, t\* form.
6. Is it possible to stay with the untransformed (physical) variables and use numerical methods directly on the initial 4 equations ? At first sight we could :

* Use a Taylor series expansion about the centre
* Use a Heun algorithm
* Use an RK algorithm

1. Don’t worry too much about which integration scheme to adopt. Getting started is better than trying for perfection.

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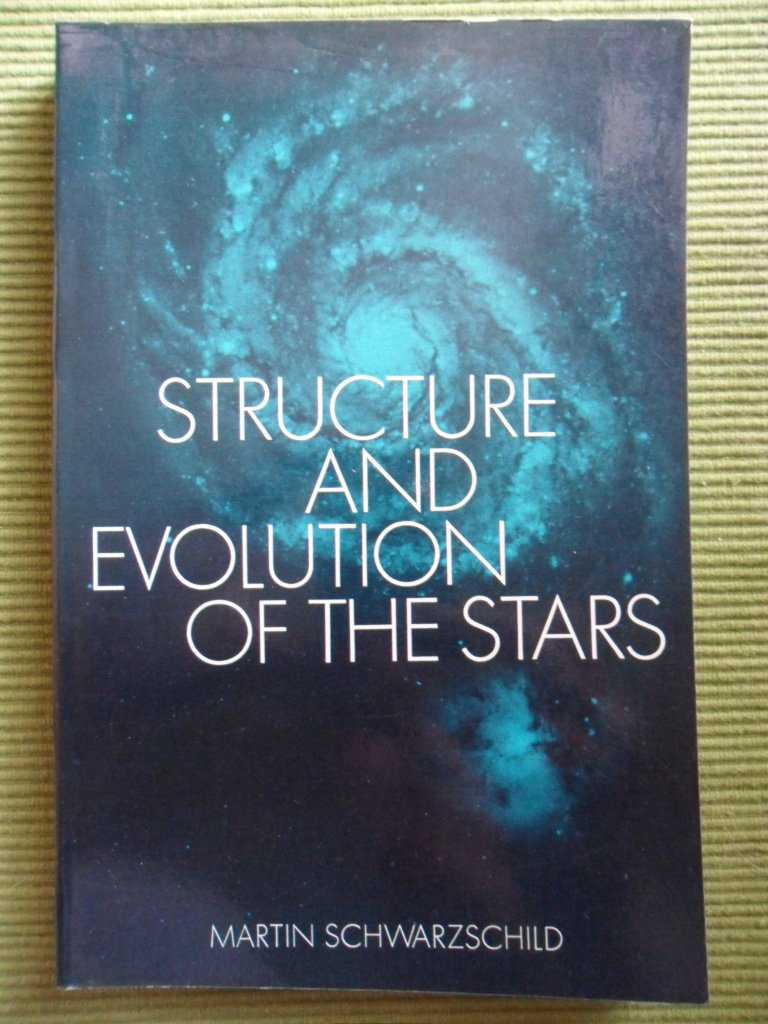
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# **INTRODUCTION**

The problem to be solved in stellar structure calculations is finding the *next* value of a variable given previous values of the variable. *Next* will often mean further out from the centre of the star than the previous values, with the distance between the values constant. Here, distance from the centre of the star is the *independent* variable, the variable that does not depend on anything else. The independent variable could also be a fraction of the mass of the star - the fraction of the mass of the star within a distance from the centre of the star (this will depend on the density).

As mentioned in Part I of this series, most of the physics and a lot of the mathematics is taken from Schwarszchild’s 1965 book “Structure and Evolution of The Stars”.



# **SCHWARSZCHILD CALCULATIONS**

In his 1965 book Schwarszchild defines a transformation of the physical variables (i.e pressure, temperature etc) into a set of dimensionless variables in order to reduce the amount of calculations required for a stellar model. Computational speed is now no longer relevant so the calculations can be done with real physical variables anyway. However, the details of the calculations with transformed variables are looked at below, as well as with the physical variables, just to fill in some of the details that Schwarszchild did not include in his book.

## Calculations With Untransformed Variables

This section examines the details of the mathematics using actual physical values – metres, degrees, Pascals etc.

### Calculations at the Centre

The basic physical equations governing the stellar structure of main sequence stars are, as derived in Part I :

**Figure 1 : The Basic Equations of Stellar Structure**

**Source :** [The Structure and Evolution of the Stars: Amazon.co.uks](https://www.amazon.co.uk/Structure-Evolution-Stars-M-Schwarzschild/dp/0486614794)

The first temperature equation is for radiative heat transfer. The second temperature equation is for convective heat transfer. Only one of these two equations is used at any one time but which one may vary. So only four equations are active at any one time. The two modes are described in Part I of this series.

: pressure at radius r

: mass fraction within radius r

: luminosity at radius r

: temperature at radius r

: density at radius r

: energy production at radius r

: opacity at radius r

In the rest of this section the solution of these equations, in more or less accurate and elegant ways, is set out. Looking at the equations shows that the density is a major complicating factor. The first method assumes a constant density near the centre of the star. The second method allows the density to vary linearly. The last part of this section looks at attempts to solve the equations with no approximations for the density by using the techniques described in Part II.

#### Constant Density Approximation

If the density, energy generation and opacity are considered to be constant for small values of then approximations for the physical variables can be derived. For example, the and / equations above can be integrated and solved for and . These can then be used in an algorithm as an expansion in at the centre. Thus, if the density, energy generation and opacity near the centre are constants ,, then :

For and , there is a complication because appears on the right-hand side of the equation whereas, for and , does not appear on the right-hand side :

so the above cannot be integrated easily because, more exactly :

However, if the approximation is made that , then :

In fact, Schwarszchild gives the equations below in figure 2 as approximate solutions but with the caveat “if one develops the solution at the center in powers of , if one ignores terms of fourth or higher order, and if one designates all quantities at the centre with the subscript c”. It can be seen that they are the same as the equations developed above. This is probably because if one were to use approximate expressions such as (in particular) :

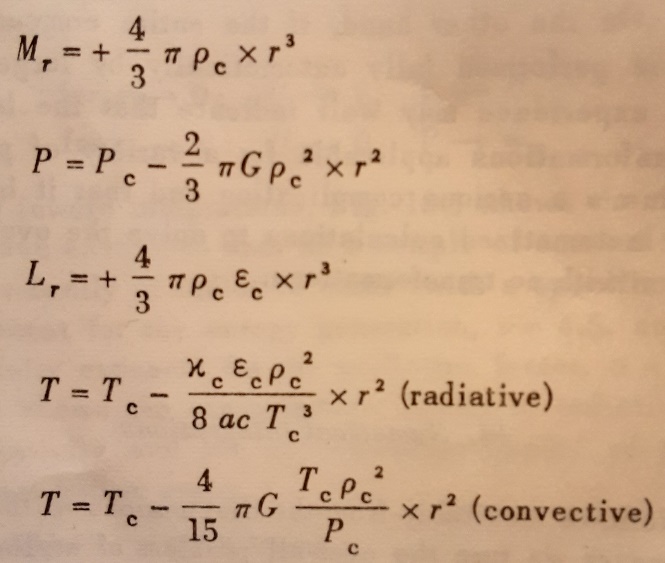
these would produce the higher order terms Schwarszchild refers to. For example, the equation would become :

which could be modified by using the power series expression :

which would give, for (with a constant) :

]

Ignoring “terms of fourth or higher order” leads to the expression in figure 2. This dependence of on also causes complications in the use of Heun’s method and the Runge-Kutta method in section 2.1.1.3 below.



**Figure 2 : Schwarszchild approximate solutions (he does *not* call them constant density solutions)**

**Source :** [The Structure and Evolution of the Stars: Amazon.co.uks](https://www.amazon.co.uk/Structure-Evolution-Stars-M-Schwarzschild/dp/0486614794)

Putting numbers into the equations above with a step size of cms (one hundredth of the stellar radius) yields :

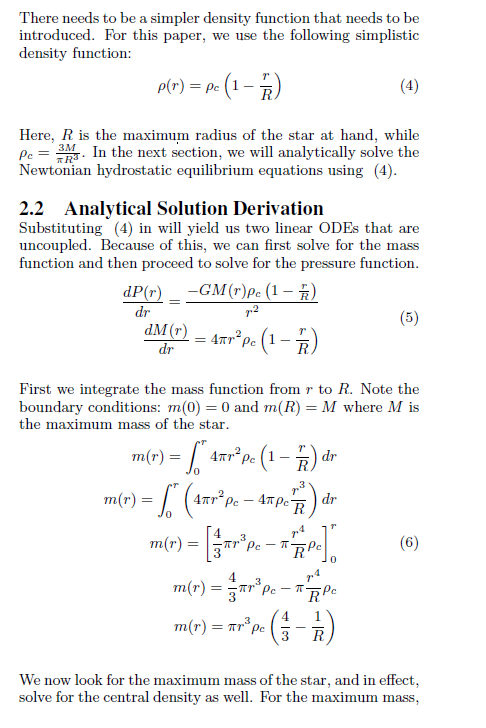
where

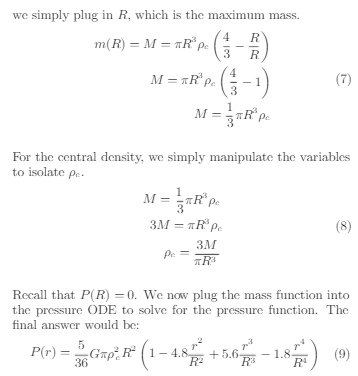
where

These results qualitatively suggest the possibility that an initial set of, say, 5 values can be successfully extended with an Adams-Moulton predictor-corrector algorithm to a realistic solution. For the temperature equation :

#### Linear Density Approximation

There is a small refinement that intuitively one might expect to lead to a significant improvement, approximating the density near the centre as a linear function.





[(PDF) A Comparative Analysis of Numerical Solutions for Solving Stellar Structure Equations (researchgate.net)](https://www.researchgate.net/publication/316007527_A_Comparative_Analysis_of_Numerical_Solutions_for_Solving_Stellar_Structure_Equations)

#### Application of Numerical Techniques

In Part II a number of numerical integration techniques were described, namely, in order of presentation in Part II :

* Taylor series expansion
* Euler’s method
* Heun’s method
* Runge-Kutta methods
* Adams-Bashforth and Adams-Moulton methods

The next few sections explore where each of these techniques leads.

**Taylor Series Expansion**

A Maclaurin series (a Taylor series centred on ) for and can be employed, starting at the centre of the star, with the pressure at the centre being or .

The term  initially looks to be a problem because it has a dependency. However, it also has  in the numerator. From physical reality must be zero. Mathematically, this just means that the  term “wins” against the dependency. After all, the term has an dependency. Hence :

The Maclaurin expansion for is done in much the same way :

In fact, since can be integrated directly, it is not surprising that the Maclaurin series terminates and gives the same result as the constant density approximation (see section 5.1).

The Maclaurin expansion for is similar to that for :

and for :

These lead to (see section 5.1) :

Again, the Maclaurin series gives the same result as the constant density approximation.

**Euler’s Method**

The forward Euler method seems very unpromising since the first derivatives at the centre are all (assumed to be) zero. Section 5.2 shows that the backward Euler method gives the following values for and with the density assumed to be a constant.

and for and :

**Heun’s Method**

In this method the average of the gradient at the centre (zero) and the gradient at is used. This gives the following values for and with the density assumed to be a constant.

and for and :

**Comparison of Taylor, Euler, and Heun Results**

The following table summarises the expressions found for the three methods.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
| Taylor**1** |  |  |  |  |
| Euler**2** |  |  |  |  |
| Heun**2** |  |  |  |  |

1. The Taylor series uses (by definition) the exact values (with constant density assumed) for and i.e the spherical volume term . Hence all values in green.
2. The Euler and Heun methods, if taken literally, use inexact values for and . Hence values in red. Also, the values given involve an additional approximation that is approximately

The naïve values for and (the values in red) for the Euler and Heun methods mean the table is not really very helpful. Instead, if the “correct” values (that is, where is assumed) are used the following table results, where the Taylor values are unchanged.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
| Taylor |  |  |  |  |
| Euler |  |  |  |  |
| Heun |  |  |  |  |

**Heun’s Method With Varying Density**

It is possible to derive an inelegant semi-solution using Heun’s method and assuming the ideal gas equation for the density :

Section 5.3 looks at the derivation of the equation which appears to be a 12th order polynomial in T with and in among the coefficients. Its solution for and would require some guessing of the latter and the numeric solution of the polynomial. However, it seems intuitively unlikely that any great benefit would arise from such a complex equation.

**Runge-Kutta Methods**

Using the algorithm described in section 6.2.2.1 of part II :

= + + 2 + 2 + )  
  
and :

= , )

= +

= +

= + )

In this case, for the pressure, and with :

=

=

=

=

Using :

gives :

which is the same value as for the Taylor series method.

The Runge-Kutta method now gets more complex for the and pair because of the value.

=

**Adams-Bashforth and Adams-Moulton Methods**

These are basically sophisticated extrapolation methods and might be considered as the method(s) to take over once a core set of values has been developed by the methods mentioned above so far. They are therefore used as described in section 2.1.3 where the initial centre and surface values are then integrated towards each other.

#### Choice of Algorithm for Initial Outward Integration

For the initial outward integration the choice out of the methods described in section 2.1.1.3 must be between Taylor series values or Runge-Kutta values. The Euler and Heun methods seem too crude though there is no clear proof of this in the text above. However, the constant density approximation is probably better anyway. Generally speaking, the algorithms in 2.1.1.3 are (by definition) approximations in addition to the constant density approximation. There is also the question of whether to always use the centre as the reference point from which to generate values or whether to step to new points when they are generated. Also, the approximation is not necessary for the Taylor and Runge-Kutta methods as long as they use the centre point. Stepping to new points would require the approximation.

After the accumulation of, say, five or six points around the centre using the constant density approximation one of the Adams-Bashforth (or Adams-Moulton) methods can take over, using derivatives at the successively generated points. After each iteration the newly generated (integrated) values are swapped to the other equations to generate the new derivatives. But how long can the constant density approximation be used ? Why not just continue those equations ? That isn’t feasible because there will inevitably be a density gradient just from the basic physics. However, the density itself is calculable from the gas equation (as long as the gas equation applies). Maybe the algorithm could switch from an initial constant density assumption to one in which the density is also calculated from and and fed back into the formulae ?

### Calculations at the Surface

In the same way as an approximation is used to get a small number of central values, a different approximation is used to get small number of surface values. The surface values are then used to start an Adams-Bashforth integration using the standard equations as given in figure 1 for the calculation of the derivatives at each point.

#### Analytic Solutions

On the basis that, at the surface :

and

Schwarzschild derives the following for the calculation of and near the surface, for the radiative case :

.

where :

: pressure

: temperature

: total mass

: total luminosity

: radius of the star

: gravitational constant

: mass of the proton

: Boltzmann’s constant

: radiation density constant

: speed of light

: distance from centre

with the fractional abundances of hydrogen, helium and heavier elements by weight:

and, for the opacity, as given in Part 1 of this series :

### Outward and Inward Integration

### The Radiative Convective Transition

## Calculations with Transformed Variables

Schwarszchild explained in his book that the number of calculations needed to solve the differential equations in section 2.1.1 is very large due to the necessity of fitting together an outward integration (from the centre) and an inward integration (from the surface) coupled with the fact that the values of total luminosity , and radius , must be varied to get the fit. He therefore introduced a transformation of the variables to variables. He thought that that might not be necessary if the calculations could be done by a computer :

“When all the numerical work is to be performed by a large electronic computing machine, the most efficient way to attack our problem may be by following the general formulation discussed above, in spite of the large number of integrations necessary. When, however, the numerical work has to be performed by more modest means – as has been the case for most of the investigations which we will review in the following chapters – it appears necessary for efficiency’s sake to apply all methods which reduce the number of necessary integrations, even though the miscellany of these labor-saving methods spoils the simplicity of the general formulation”.

However, given the prospect of searching a four-dimensional space of and , it might be necessary to use the transformations anyway, unless perhaps some optimising search algorithm is available.

### Initial Transformation

Schwarszchild introduced, first, an initial transformation and then a supplementary transformation on the initial set of transformations. The initial transformation is :

Why these particular transformations ?

### Supplementary Transformation

#### Derivation of the Constants

#### Estimates of the Range of the Variables

### The Integration Algorithm

#### Calculation of the Derivatives

#### The Outgoing Solution

#### The Inward Solution

## Calculations with Mass as Independent Variable

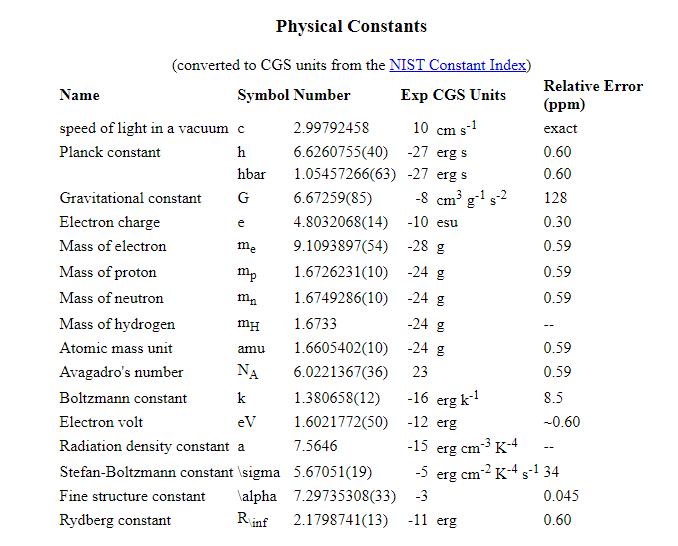
# **HENYEY CALCULATIONS**

## The Linear Algebra of the Solution

# **PHYSICAL CONSTANTS**

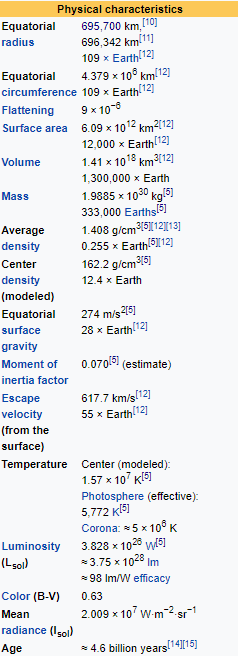
## Universal Physical Constants

Schwarszchild’s book used CGS cvalues throughout, so this paper also uses CGS values.



**Figure (X) : Universal Physical Constants**

## Solar Characteristics



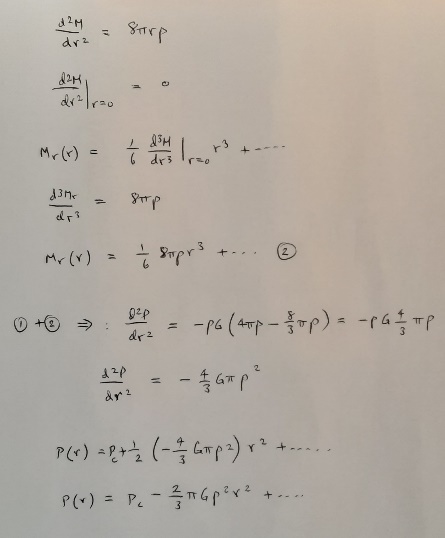
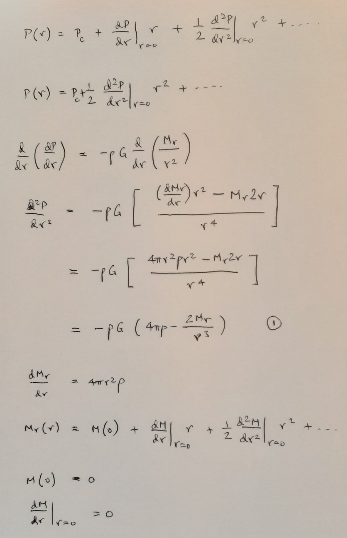
**Figure (X) : Solar Values**

**Source :** [**Sun - Wikipedia**](https://en.wikipedia.org/wiki/Sun)

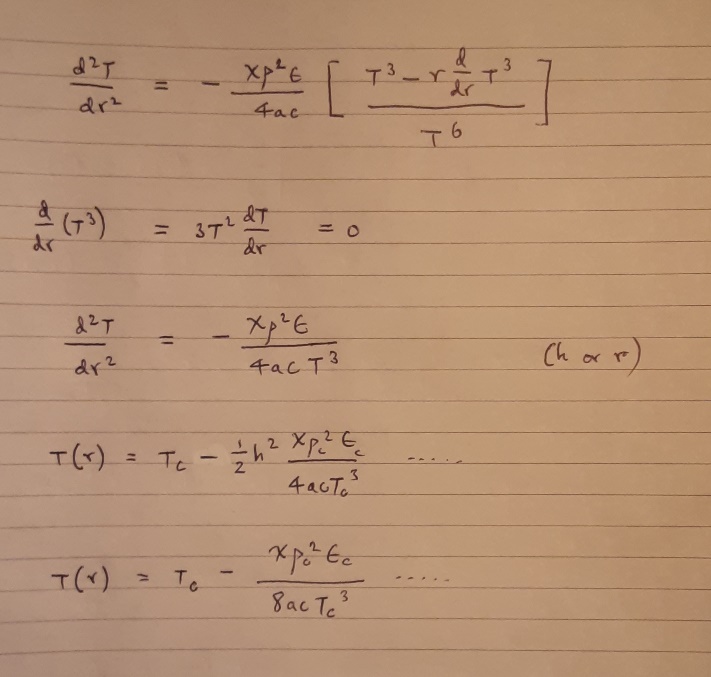
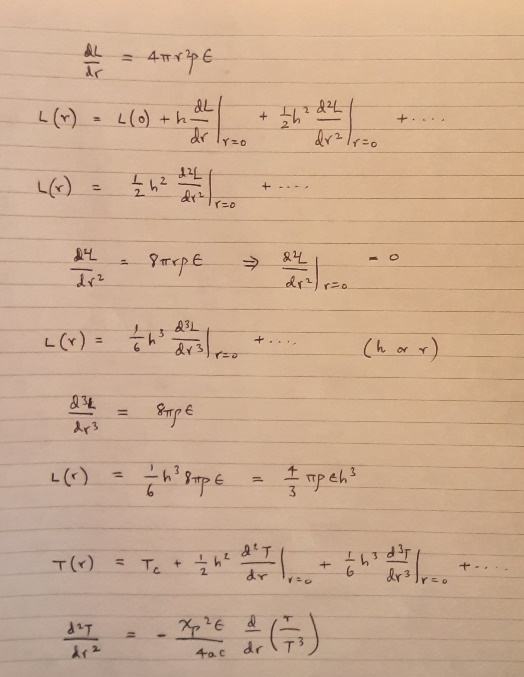
# **WORKINGS**

## Taylor Series Expansion

The density in the derivatives is the central density since the derivatives are evaluated at .

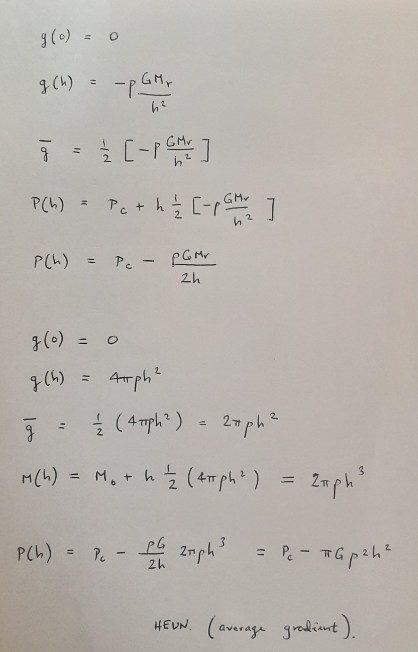
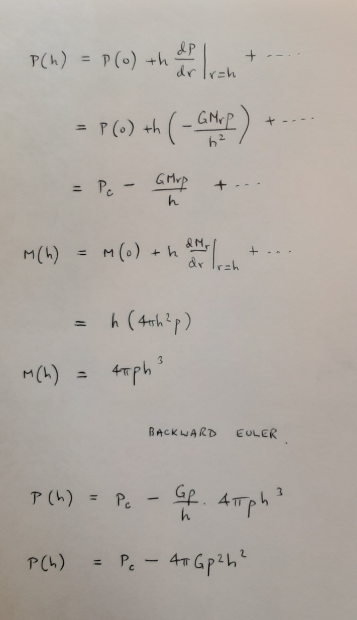


The third derivative of is zero since the second derivative does not depend on . So the series terminates.

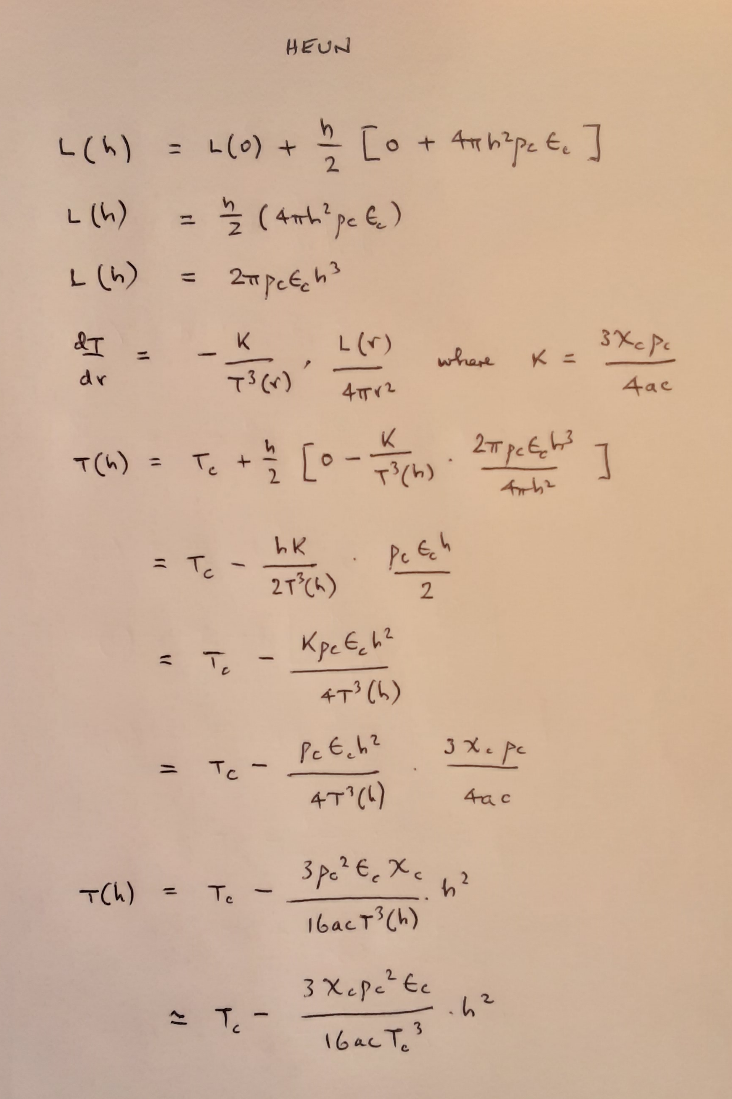
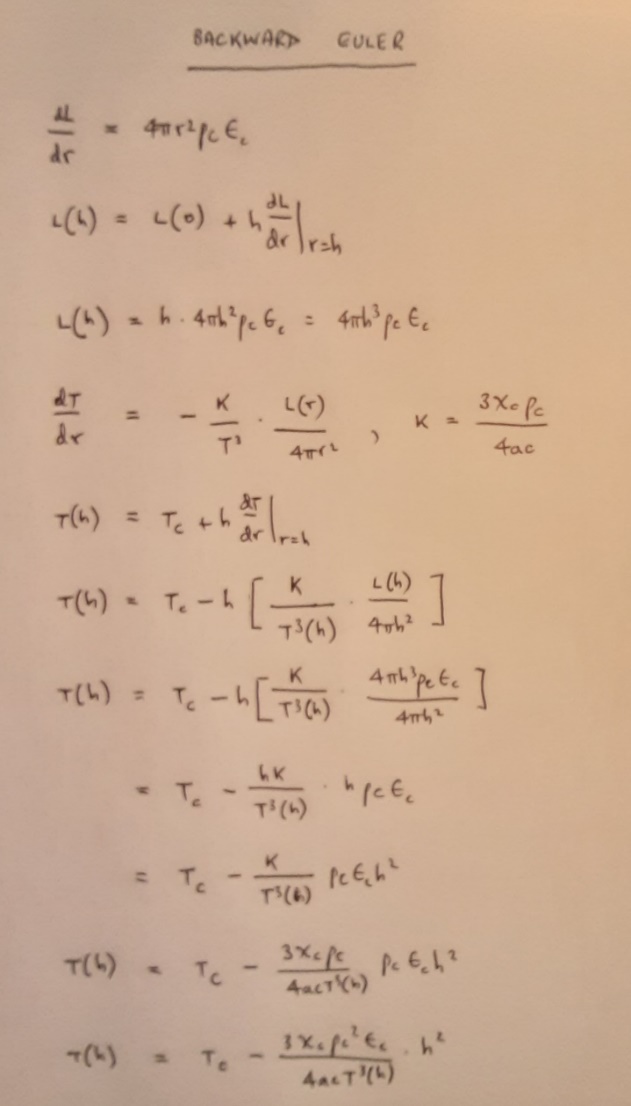


## Backward Euler and Heun Methods

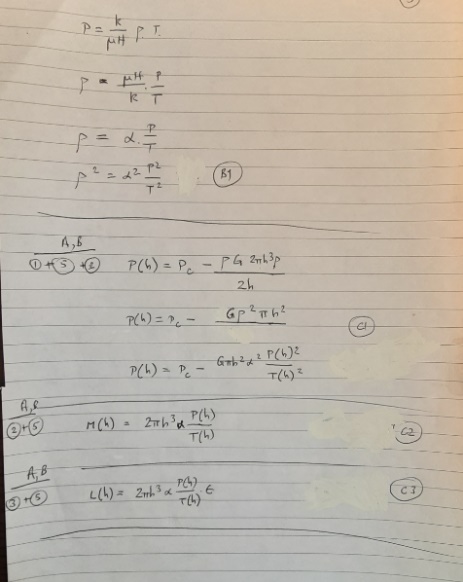
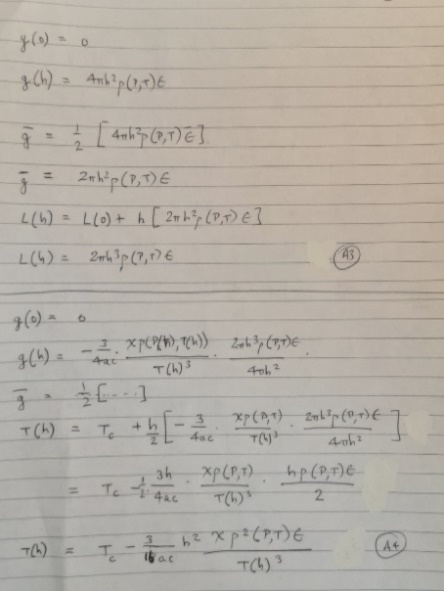
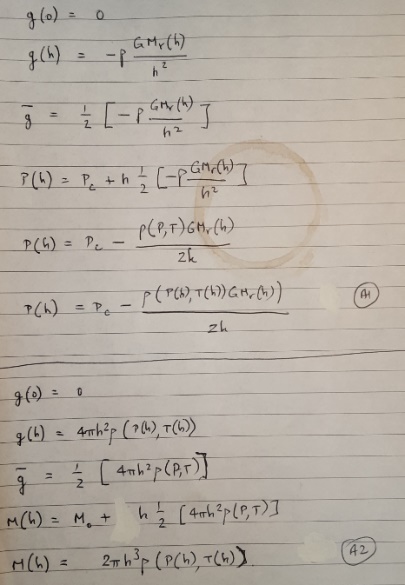
Pressure and mass fraction :

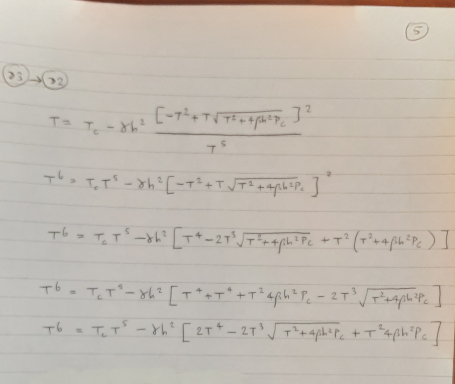
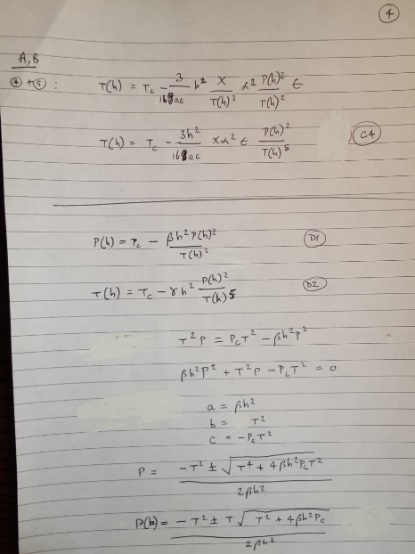


Temperature and Luminosity :



## Heun’s Method With Varying Density





## Code Results

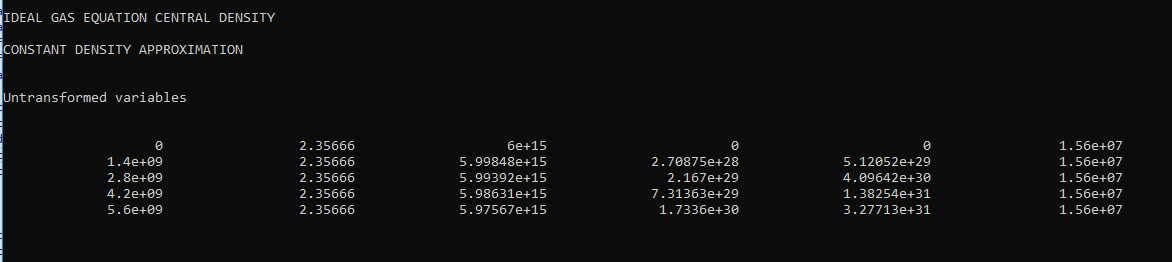
### Central Solution Start, Ideal Gas Density (constant)

X=0.98, Y=0.02, Z=0.0

Central density = ideal gas density

Constant density

Columns are , , , .



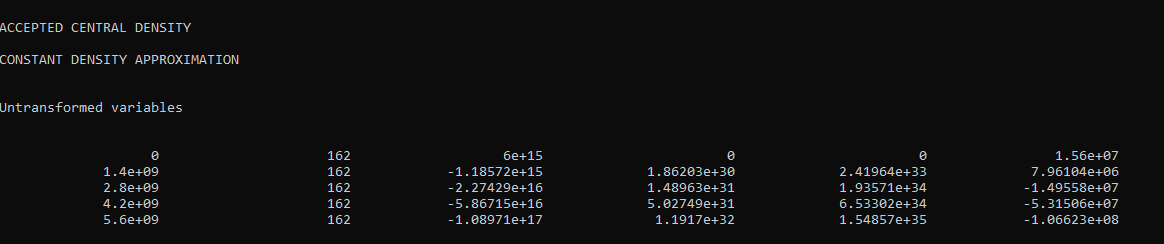
### Central Solution Start, Accepted Gas Density (constant)

X=0.98, Y=0.02, Z=0.0

Central density = accepted density (162g/cm3).

Constant density

Columns are , , , .



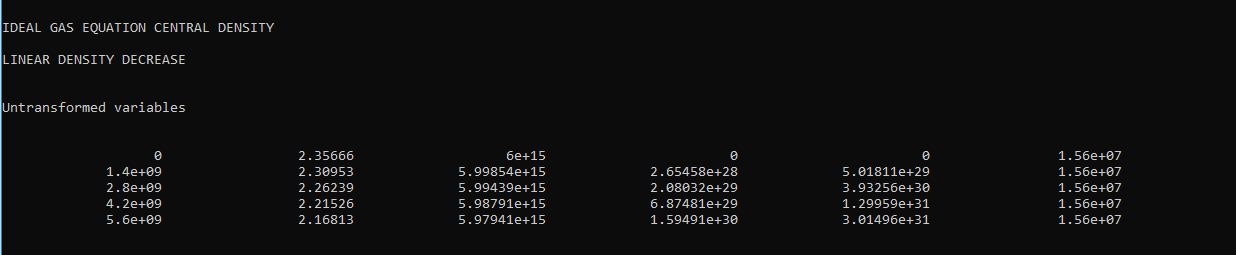
### Central Solution Start, Ideal Gas Density (decreasing)

X=0.98, Y=0.02, Z=0.0

Central density = ideal gas density

Decreasing density (2% for each step for 5 steps)

Columns are , , , .



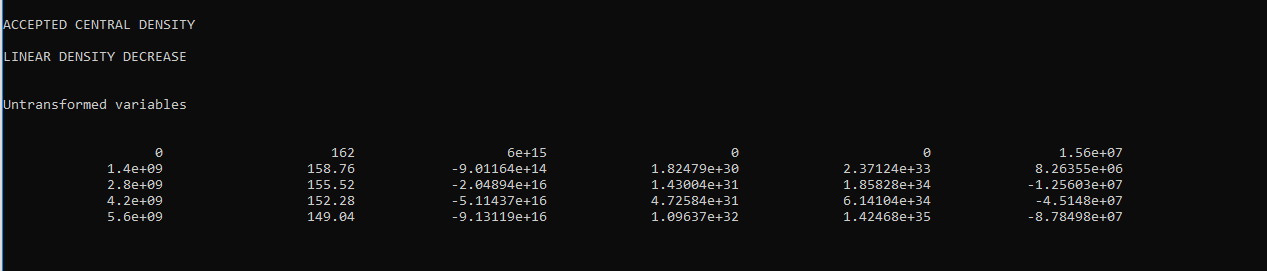
### Central Solution Start, Accepted Density (decreasing)

X=0.98, Y=0.02, Z=0.0

Central density = accepted density (162 g/cm3).

Decreasing density (2% for each step for 5 steps).

Columns are , , , .



### The Role of the Density

The temperature variation is very volatile as can be seen in the outputs above. An ideal gas equation density makes the temperature virtually constant. An accepted value of 162g/cm3 leads to negative temperatures. Looking at the values of the temperatures subtracted from , with ideal gas density :

~ 15 K

A temperature drop of 15K over 1/50th of the radius is not going to lead to a sensible solution. The linear density version is a distraction because it only reduces the subtracted value further (unless later integrations somehow reverse the trend). So it seems that a value intermediate between the ideal gas value and the accepted value might give a better solution. A good solution would include a sensible temperature reduction and a calculated zero pressure near the actual solar radius.

But what is the supposed solar radius ? The model is meant to be for an initial main sequence sun with hardly any helium. The current solar radius is cms. An initial sum corresponds to Table 28.3 in Schwarzschild’s book. This has a radius more-or-less the same as at present.

### What are Givens and what are Calculated Values ?

The only givens are the mass and chemical composition. For the case of an initial main sequence star, the chemical composition is taken to be constant as the star will be composed entirely of hydrogen. The radius and total luminosity will be calculated values, along with the central pressure and central temperature. Pressure and temperature will be assumed to be zero at the surface.

# **C++ CODE FOR STELLAR STRUCTURE CALCULATION**

The code below is applicable to any star whose physics follows the equations of section X.X. This means stars whose….This is generally between X and Y solar masses.

