problem 3

a) if all $u_j = u$ $j = 1 \dots k$

$$\frac{dx_j}{dt} = u_{j-1} x_{j-1}(t) - u_j x_j(t) \Rightarrow \frac{d f(n,t)}{dt} = u f(n-1,t) - u f(n,t) \quad (1)$$

$$\rightarrow \frac{df(0,t)}{dt} = -u f(0,t) \Rightarrow \frac{1}{f(0,t)} \frac{d f(0,t)}{dt} = -u \Rightarrow$$

$$\int \frac{df(0,t)}{f(0,t)} = -\int u\,dt \Rightarrow \ln f(0,t) = -ut + \cancel{C_0}^{\,0} \Rightarrow \boxed{f(0,t) = e^{-ut}}$$

due to non-simultaneity

$$\rightarrow \frac{df(1,t)}{dt} = u f(0,t) - u f(1,t) \Rightarrow \frac{df(1,t)}{dt} + u f(1,t) = u f(0,t)$$

$$\Rightarrow \frac{df(1,t)}{dt} + u f(1,t) = u e^{-ut} \xrightarrow{\times e^{ut}} \frac{df(1,t)}{dt} e^{ut} + u e^{ut} f(1,t) = u$$

$$\int \left[e^{ut} \frac{d f(1,t)}{dt} + u e^{ut} f(1,t) \right] = \int u$$

$$f(1,t)\, e^{ut} = ut + c_1 \xrightarrow{x_1(0) = f(1,0) = 0 = c_1} \boxed{f(1,t) = ut\, e^{-ut}}$$

$$\rightarrow \int \left[\frac{df(2,t)}{dt} e^{ut} + u e^{ut} f(2,t) \right] = \int u^2 t \Rightarrow f(2,t)\, e^{ut} = \frac{u^2 t^2}{2} \Rightarrow$$

$$\boxed{f(2,t) = e^{-ut} \frac{u^2 t^2}{2}}$$

$$\rightarrow \int \left[\frac{df(3,t)}{dt} e^{ut} + u e^{ut} f(3,t) \right] = \int \frac{u^3 t^2}{2} \Rightarrow f(3,t)\, e^{ut} = \frac{u^3 t^3}{6}$$

$$\boxed{f(3,t) = e^{-ut} \frac{u^3 t^3}{6}}$$

(2

$$\rightarrow \int\left[\frac{df(4,t)}{dt}e^{ut}+ue^{ut}f(4,t)\right]=\int\frac{u^4t^3}{6} \Rightarrow f(4,t)e^{ut}=\frac{u^4t^4}{24}$$

$$\Rightarrow \boxed{f(4,t)=\frac{e^{-ut}u^4t^4}{24}}$$

we can show that $f(n,t)=\frac{e^{-ut}u^nt^n}{n!}$ for all n without pluging in each number, by showing the equation (1) is also satisfies for $n+1$.

$\boxed{\text{for } n+1} \rightarrow$

$$\frac{df(n+1,t)}{dt}+uf(n+1,t)=uf(n,t)=\frac{e^{-ut}u^{n+1}t^n}{n!} \xrightarrow{\times e^{ut}}$$

$$\int\left[\frac{df(n+1,t)}{dt}e^{ut}+ue^{ut}f(n+1,t)\right]=\int\frac{u^{n+1}t^n}{n!} \Rightarrow f(n+1,t)e^{ut}=\frac{u^{n+1}t^{n+1}}{(n+1)\,n!}+C_{n+1}$$

$$\xrightarrow{f(n+1,0)=C_{n+1}=0} \quad f(n+1,t)=e^{-ut}\frac{(ut)^{n+1}}{(n+1)!}$$

$$\Rightarrow \forall\, n\geq 0 \quad f(n,t)=e^{-ut}\frac{(ut)^n}{n!}$$

and since here we limite equation (1) to be correct for $0<n<k$

$$\Rightarrow \forall\, 0<j<k \quad f(j,ut)=e^{-ut}\frac{(ut)^j}{j!}$$

$\boxed{b}$ if $ut \ll 1 \Rightarrow e^{-ut}$ very small $\Rightarrow x_j=f(j,ut)\propto(ut)^j$

or $x_j=f(j,ut)\propto t^j$