Continuum mechanics Homework 2

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Are given the equation of motion by the linear mapping $\chi((X))$

$$\mathbf{x} = \chi(\mathbf{X}) = (aX_1 + bX_2 + cX_3)\mathbf{e_1} + (dX_1 + eX_2 + fX_3)\mathbf{e_2} + (gX_1 + hX_2 + lX_3)\mathbf{e_3}$$

and its coefficients a=-1; b=1; c=0; d=3; e=3; f=0; g=0; h=0; l=1; (Version n.3)
It is also given the stress tensor $\mathbf{T}$ [MPa]

$$\mathbf{T} = \begin{pmatrix} 140 & 160 & 300 \\ 160 & 300 & 460 \\ 300 & 460 & 600 \end{pmatrix}$$

1 Ex 1: Deformation gradient F tensor

We explicit the deformation gradient tensor $\mathbf{F}$

$$\mathbf{F} = F(i,j) = \frac{\partial x_i}{\partial X_j} = \begin{pmatrix} \alpha & \beta & \kappa \\ \beta & \alpha+\beta & \beta+\kappa \\ \kappa & \beta+\kappa & \alpha+\beta+\kappa \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2 Ex 2: 2^{nd} Piola-Kirchhoff stress tensor S

We can compute the 2^{nd} Piola-Kirchhoff stress tensor $\mathbf{S}$ by

$$\mathbf{S} = \mathbf{F^{-1}} \cdot \mathbf{T^0} = J\,\mathbf{F^{-1}} \cdot \mathbf{T} \cdot (\mathbf{F^{-1}})^{\mathbf{T}} = \begin{pmatrix} -100 & 160 & 440 \\ 160 & -420 & -1360 \\ 440 & -1360 & -3600 \end{pmatrix}$$

Where:
$\mathbf{F^{-1}}$ is the inverse of the tensor $\mathbf{F}$

$$\mathbf{F^{-1}} = \begin{pmatrix} -0.5 & 0.16667 & 0 \\ 0.5 & 0.16667 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathbf{T^0}$ is the 1^{st} Piola-Kirchhoff stress tensor

$$\mathbf{T^0} = J\,\mathbf{T}\,(\mathbf{F^{-1}})^{\mathbf{T}} = \begin{pmatrix} 260 & -580 & -1800 \\ 180 & -780 & -2760 \\ 440 & -1360 & -3600 \end{pmatrix}$$

3 Ex 3: Compute von Mises stress for the tensor $\mathbf{T}$

As first we compute the trace of the 2^{nd} order tensor $\mathbf{T}$ as the sum of its eigen-values:

$$tr(\mathbf{T}) = \sum_{i=1}^{3} \lambda_i = 1057 + 41 - 58 = 1040$$

Then we compute the hydrostatic $\mathbf{T}_{hyd}$ and deviatoric $\mathbf{T}_{dev}$ components of $\mathbf{T}$ as:

$$T_{hyd}(i,j) = \frac{tr(\mathbf{T})\delta_{ij}}{3} = \begin{pmatrix} 346.67 & 0 & 0 \\ 0 & 346.67 & 0 \\ 0 & 0 & 346.67 \end{pmatrix}$$

And we compute the deviatoric component by:

$$\mathbf{T}_{dev} = \mathbf{T} - \mathbf{T}_{hyd} = \begin{pmatrix} -206.67 & 160 & 300 \\ 160 & -46.667 & 460 \\ 300 & 460 & 253.33 \end{pmatrix}$$

Then it follows the computation of von Mises stress :

$$\sigma_{vM} = \sqrt{1.5\,\mathbf{T}_{dev} : \mathbf{T}_{dev}} = \sqrt{1.5 \sum_{i,j=1}^{3} [T_{dev}(i,j)]^2} \Rightarrow$$

$\Rightarrow\ \sigma_{vM} = 1070.1$ [MPa]

4 Ex 4: Compute strain tensor ϵ

We are given the value of Young modulus $E = 70$[GPa] and the Poisson coefficient $\nu = 0.3$. We can compute the strain tensor ϵ by

$$\epsilon(i,j) = \frac{1}{E}[(1+\nu)\,\sigma_{ij} - \nu\,\delta_{ij}\,\sigma_{kk}] \ \Rightarrow\ \epsilon = \frac{1}{E}[(1+\nu)\mathbf{T} - \nu\,\mathbf{I}\,tr(\mathbf{T})]$$

$$\epsilon = \begin{pmatrix} -1.8571 & 2.9714 & 5.5714 \\ 2.9714 & 1.1143 & 8.5429 \\ 5.5714 & 8.5429 & 6.6857 \end{pmatrix}$$

And we can decompose ϵ in its hydrostatic and deviatoric components:

$$\epsilon_{hyd}(i,j) = \frac{tr(\epsilon)\delta_{ij}}{3} = \begin{pmatrix} 1.981 & 0 & 0 \\ 0 & 1.981 & 0 \\ 0 & 0 & 1.981 \end{pmatrix}$$

$$\epsilon_{dev} = \epsilon - \epsilon_{hyd} = \begin{pmatrix} -3.8381 & 2.9714 & 5.5714 \\ 2.9714 & -0.86667 & 8.5429 \\ 5.5714 & 8.5429 & 4.7048 \end{pmatrix}$$

4.1 Show that it holds the relation $T_{dev}(i,j) = 2\,G\,\epsilon_{dev}(i,j)$

We compute the shear modulus in function of Young modulus and Poisson coefficient $G = \frac{E}{2\,(1+\nu)} = 26.92$[GPa]

We verify the given relation by computing $T_{dev} - 2\,G\,\epsilon_{dev}$ and we see that we get a zero matrix. Indeed the following terms are equal:

$$\mathbf{T}_{dev} = \begin{pmatrix} -206.67 & 160 & 300 \\ 160 & -46.667 & 460 \\ 300 & 460 & 253.33 \end{pmatrix}$$

$$2\,G\,\epsilon_{dev} = \begin{pmatrix} -206.67 & 160 & 300 \\ 160 & -46.667 & 460 \\ 300 & 460 & 253.33 \end{pmatrix}$$

5 Stiffness tensor computation and component explicitation

We compute the tensor $\mathbf{C}$ by the expression:

$$C(i,j,k,l) = \frac{E}{1+\nu}[0.5(\delta_{ik}\,\delta_{jl} + \delta_{jk}\,\delta_{il}) + \frac{\nu}{1-2\,\nu}\delta_{ij}\,\delta_{kl}]$$

In particular, the "slice" of 4^{th} order tensor $\mathbf{C}$ that we are looking for is

$$C(:,:,2,3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 26.923 \\ 0 & 26.923 & 0 \end{pmatrix}$$

In particular $C(2,3,2,3) = 26.923$

6 Appendix: Cmd Window and Script

6.1 Results from Matlab command window:

```
T =
   140   160   300
   160   300   460
   300   460   600
Ex1: compute deformation gradient tensor
F =
    -1     1     0
     3     3     0
     0     0     1
--------------------------------------------------------------------------
Ex2:
invF =
         -0.5      0.16667            0
          0.5      0.16667            0
            0            0            1
J =
    -6
------------------------
First PK stress tensor
T0 =
         260         -580        -1800
         180         -780        -2760
         440        -1360        -3600
Second PK stress tensor
S =
        -100          160          440
         160         -420        -1360
         440        -1360        -3600
--------------------------------------------------------------------------
Ex3:Von Mises stress computation
eigvec =
       0.4743      0.80989      0.34514
      0.60606     -0.58473      0.53924
     -0.63854     0.046586      0.76818
eigval =
      -59.437            0            0
            0       41.739            0
            0            0       1057.7
trT =
        1040
T_hyd =
       346.67            0            0
            0       346.67            0
            0            0       346.67
T_dev =
      -206.67          160          300
          160      -46.667          460
          300          460       253.33
```

```
sigma_vm =
       1070.1
--------------------------------------------------------------------------
Ex4: Strain tensor Epsilon computation by Hooke's law
E =
    70
nu =
          0.3
Eps =
      -1.8571       2.9714       5.5714
       2.9714       1.1143       8.5429
       5.5714       8.5429       6.6857
Show the relation between deviators T^_ij = 2 G Eps^_ij
T_dev =
      -206.67          160          300
          160      -46.667          460
          300          460       253.33
Eps_hyd =
        1.981            0            0
            0        1.981            0
            0            0        1.981
Eps_dev =
      -3.8381       2.9714       5.5714
       2.9714     -0.86667       8.5429
       5.5714       8.5429       4.7048
G =
       26.923
2 G Eps_dev=
ans =
      -206.67          160          300
          160      -46.667          460
          300          460       253.33
TestVar =
     0
The relation between deviators is verified
--------------------------------------------------------------------------
Ex5: Stiffness tensor computation
C(:,:,1,1) =
       94.231            0            0
            0       40.385            0
            0            0       40.385
C(:,:,2,1) =
            0       26.923            0
       26.923            0            0
            0            0            0
C(:,:,3,1) =
            0            0       26.923
            0            0            0
       26.923            0            0
C(:,:,1,2) =
            0       26.923            0
```

```
       26.923            0            0
            0            0            0
C(:,:,2,2) =
       40.385            0            0
            0       94.231            0
            0            0       40.385
C(:,:,3,2) =
            0            0            0
            0            0       26.923
            0       26.923            0
C(:,:,1,3) =
            0            0       26.923
            0            0            0
       26.923            0            0
C(:,:,2,3) =
            0            0            0
            0            0       26.923
            0       26.923            0
C(:,:,3,3) =
       40.385            0            0
            0       40.385            0
            0            0       94.231
--------------------------------------------------------------------------
C(2,3,2,3) is equal to
ans =
       26.923
--------------------------------------------------------------------------
Hello World
```

6.2 Matlab script:

```
clc; clear all; %ver3
format shortG %cut decimals as needed
format compact %cmd window compact out
%axis equal
a=-1; b=1; c=0; d=3; e=3; f=0; g=0; h=0; l=1;
aa=140; bb=160; kk=300; %alpha,beta,kappa
dash="------------------------------------------------------------------------";
T=[aa bb kk ; bb aa+bb bb+kk ; kk bb+kk aa+bb+kk]
disp("Ex1: compute deformation gradient tensor")
F=[ a b c; d e f ; g h l]
disp(dash)
disp("Ex2:")%ex2:compute 2nd piola-kirchhoff stress tensor S
invF=inv(F)
J=det(F)
disp("------------------------")
disp("First PK stress tensor")
T0=J * T * invF' %first piola-kirch stress tens
disp("Second PK stress tensor")
S=invF *T0 %second p-k

disp(dash) %ex3 Von Mises
disp('Ex3:Von Mises stress computation')
[eigvec,eigval]=eig(T)
trT=trace(T)
T_hyd=zeros(3,3);
for i=1:3
T_hyd(i,i)=trT/3;
end
T_hyd
T_dev=T-T_hyd

temp=0;
for i=1:3 %computation of sigma prime
    for j=1:3
        temp=temp+T_dev(i,j)^2;
    end
end
sigma_vm=(3/2*temp)^0.5

disp(dash) %ex4
disp("Ex4: Strain tensor Epsilon computation by Hooke's law")
E=70 %[GPa]
nu=0.3
Eps=[(1+nu)*T-nu*eye(3)*trT]/E
disp("Show the relation between deviators T^_ij = 2 G Eps^_ij")
T_dev
for i=1:3
Eps_hyd(i,i)=trace(Eps)/3;
end
Eps_hyd
```

```
Eps_dev=Eps-Eps_hyd
G=E/2/(1+nu)
disp("2 G Eps_dev=")
2*G*Eps_dev
TestVar=det(T_dev-2*G*Eps_dev)
if TestVar<10^(-9)
   disp("The relation between deviators is verified")
end

disp(dash) %ex5
disp("Ex5: Stiffness tensor computation")

%ijkq=2323
I=eye(3);
for i=1:3
    for j=1:3
        for k=1:3
            for q=1:3
C(i,j,k,q)=E/(1+nu)*(0.5*( I(i,k)*I(j,q)+I(j,k)*I(i,q))+(nu/(1-2*nu)*I(i,j)*I(k,q)));
            end
        end
    end
end
C
disp(dash)
disp("C(2,3,2,3) is equal to")
C(2,3,2,3)
disp(dash)
disp("Hello World")
```