1 Lecture 8

1.1 Dimensional analysis of the dye cloud (?)

We continue with the **Richardson's law** in case of steady wind. Here, the traveled distance along the wind is proportional to time, so the cross-section radius of a cloud (smoke,virus...) grows as distance to the power of 1.5 :

$$\Delta x \propto t^{1.5} \;\Rightarrow\; r^2 \propto t^3$$

Hence, the squared radius goes with the cube of time, and the concentration of the dye decreases inversely proportional to the distance cubed:

$$n \propto \frac{1}{r^2} \propto t^{-3}$$

attach pic of the cloud

If you walk through the cloud, the exposure e_n to the concentration n of the dye, is given as the integral of the concentration in the time

$$e_n = \int_{t_{in}}^{t_{out}} n\, dt$$

and the time spent in the cloud scales as the diameter (walking perendicularly through the cloud). So the exposure scales as time to -1.5:

$$h\tau \propto t^{-3} \cdot \frac{t^{1.5}}{v} \propto t^{-1.5} \;;\; \tau \propto \frac{r}{v}$$

1.2 Scalars in fully developed turbulence

We consider two walls: one cold (blue) and one warm (yellow). The temperature field will be similar to the picture: [1]

Figure 1: Scalarfield transported by a turbulent flow (numerical simulation results)

In the domain, there are regions of almost constant temperature, and other areas with huge temperature drop. These **temperature fronts** have a ΔT of the same order of magnitude of the

[1] Falkovich, Gawe¸dzki, & Vergassola: Particles and fields in fluid turbulence - Rev. Mod. Phys., Vol. 73, No. 4, October 2001

source ΔT (yellow-blue). Also, the temperature fronts have **fractal** morphology and topology. The *fractal dimension* of these curves will be related to the *structure function scaling exponents*.
missing idea
The structure function $S_p(\mathbf{a})$ (of the order of p) is defined as

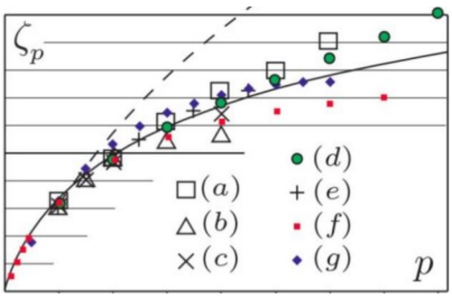
$$\langle |T(\mathbf{r}) - T(\mathbf{r} + \mathbf{a}?)|^p \rangle_{\mathbf{r}}$$

Here we have averaged the temperature over the spatial coordinate $\mathbf{r}$. In the case of turbulence, witin inertial range [2] , we expect the structure function $S_p(\mathbf{a})$ to be a power law of $\mathbf{a}$. This means $S_p(\mathbf{a}) \propto \mathbf{a}^{\xi}$, where $\xi = \xi_p$ is a function of p.

In the case of *Gaussian self-affine* function [3] ,the structure scaling exponent would be a linear function $\xi_p = H \cdot p$, where H is the **Hurst exponent**: here some steps are missing

$$\langle |x(t) - x(t + \tau)| \rangle \propto \tau^H$$

So it corresponds to p=1 which case?
If the statistics is non-Gaussian, it may be a non-linear function. In this case we say that the temperature **field is intermittent**. In the case of passive scalars in Kolmogorov turbulence, the field is very intermittent, and the graph would look like that (here zeta stands for what we have been denoting with xi):



1.3 Intermittency and the multiplicative cascade add ideas from paper

Let us consider $S_p(\mathbf{a})$ with a very big p. In this case, only the biggest $\Delta T(r)$ values contribute significantly to $S_p(\mathbf{a})$. Indeed, $S_p(\mathbf{a})$ was defined as the average of $\Delta T(r)^p$, so if p is big, those $\Delta T(r)$ -s which are not the biggest ones, once we take them to the power of p become much smaller than those which are the biggest ones. the subordinates need some de-entanglement

Then $S_p(\mathbf{a})$ can be estimated as the maximal $\Delta T(r)^p$, multiplied by the probability of meeting a maximal $\Delta T(r)$ at scale a. If we break the whole field into small boxes of size a, this probability would be given as the number of boxes covering the mature fronts, and this number scales according to the fractal (box-counting) dimension df of the mature fronts:
check formula

$$Hp : p \gg 1 \ \Rightarrow \ \langle \Delta T^p \rangle_a \approx \frac{\overbrace{N_{mt}}^{\approx a^{-d_f}}}{\underbrace{N}_{\approx a^{-d}}} (\Delta T_{max})^p \approx a^{d-d_f} (\Delta T)^p$$

where $d - d_t = \xi_p$ is the co-dimension. [4]

[2] i.e. for scales lying between the energy input scale (big scale) and the dissipation scale (small)
[3] i.e. when the statistics of $\Delta T \equiv T(\mathbf{r}) - T(0)$ is gaussian. probability remark?
[4] for $p = 2 \ \Rightarrow \ \xi = \frac{2}{3}$ check - Obukhov, 1938 - Corrsin,1941

So we see that the *structure function scaling exponent* is equal to the co-dimension $d - d_f$ of the mature fronts, and will no longer depend on p. This dependence is valid assuming there are always mature fronts [5] .

Another model predicts that the temperature drop over mature fronts starts decreasing slightly with by? decreasing scale a. This leads to a logarithmic dependance [6] of ξ_p on p .

In either case, the *temperature field is very intermittent*, that means that the dependance of ξ_p on p is very non-linear. The degree of intermittency is sometimes measured by the so-called anomalous saling exponent strangeletter $= 2\xi_2 - \xi_7$ can't read subscripts.

Intermittency has usually origins in the **multiplicative cascade** when, e.g. the vortex is divided into two smaller vortices with slight imbalance in energy: one get $0.5 - \varepsilon$ of the initial energy, and the other get $0.5 + \varepsilon$. If we go down the cascade, these *factors get multiplied* so that at the **smallest scale**, the energy of the weakest and strongest **vortices becomes very different**:

attach pic of multiplicative cascade

1.4 Homework:

Retrive data about temperature as a function of time at a certain oceanic arbitrary spot from the link.

Figure 2: https://apps.aims.gov.au/ts-explorer

Based on the downloaded data, determine the structure functions, and if these appear to be *power laws* (over a certain range of the values for a, then determine $\xi_{11}\xi_{21}\xi_?$? It will be convenient to find it by making a plot with both axes in log-scale $ln(s)$ *versus* $ln(\tau)$ and then by finding the slope of the tangent

1.5 Mature fronts

The reason of the emergence of the mature fronts is the following: we consider two lines, initially far away(spatially and also very different in temperature). We measure the time needed to bring those lines to have a contact in some point. This *time interval is finite*: biggest vortex

[5] a front is said to be mature when the temperature drop is on the order of the global temperature differences

[6] Jaan Kalda and Aleksandr Morozenko 2008 New J. Phys. 10 093003
https://iopscience.iop.org/article/10.1088/1367-2630/10/9/093003/pdf

will decrease the distance twice within its rotation time, the next generation will decreas the distance again twice in some points within its own rotation time (which is $2^{\frac{2}{3}}$ times smaller than the rotation time of the previous bigger vortex), et cetera... To have a graphical intuition, see the figure below.

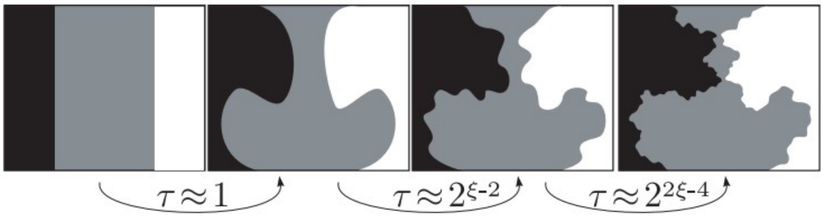


Figure 3: Kolmogorov turbulence with $\xi = \frac{4}{3}$. Simplified scheme of the formation of tracer discontinuities. Characteristic time of eddies of size a scales as $\tau \approx a^{2-\xi}$. Due to the combined effect of large and small eddies, low- and high-density regions (black and white, respectively) are brought into contact within a finite time.

Since those τ values for a geometric progression, it has a *finite sum*: in few points, the lines are brought together within a finite time.

1.5.1 Is the temperature a passive scalar? It depends on the scale

Most often can be considered as a passive scalar: indeed we need to compare the buoyancy force (due to the density difference) and the dynamic pressure.

$$\frac{F}{V} = \Delta\rho g \ll$$

$$\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \rho\big(\frac{\partial}{\partial t} + \mathbf{v}\nabla\big)\mathbf{v} + \nabla p = 0$$

The $\rho\mathbf{v}\nabla\mathbf{v}$ term (mass times acceleration per volume) has magnitude on the order of $\frac{\rho v^2}{a}$. If this is much bigger than the buoyancy force per volume (due to heated liquid being less dense), $\Delta\rho g$ then we can consider the temperature as having only negligible effect on the fluid flow. [a]

[a] check! This is a good Hp. in case of small scales, where the limit of the ratio surface/volume goes to infinite

$$\lim_{a\to 0}\frac{S}{V} = +\infty$$

Instead, on big scales, where this ratio goes to zero, the buoyancy force plays a significant role, e.g. in meteorology, where

$$\lim_{a\to +\infty}\frac{S}{V} = 0^+$$

1.6 A passive vector field in Magneto-Hydro-Dynamics

We can express the Omh's law in differential form:

$$\mathbf{j} = \delta\big(\mathbf{E} + \mathbf{v}\times\mathbf{B}\big)$$

and from here we take a curl:

$$\frac{1}{2}\nabla\times\mathbf{j} = \nabla\times\mathbf{E} + \nabla\times\mathbf{v}\times\mathbf{B}$$

We substitute here the curlo of E from the Faraday's law and j from the Ampère's circulation theorem:

$$\nabla\times\mathbf{E} = \frac{\partial\mathbf{B}}{\partial t} \;\Rightarrow\; \nabla\times\mathbf{B} = \mu_0\mathbf{j} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}$$

We take a curlo of this equation to obtain:

$$\mu_0 \nabla \times \mathbf{j} = \nabla \times \left[\nabla \times \mathbf{B}\right] - \frac{\mu_0}{c^2}\frac{\partial}{\partial t}\nabla \times \mathbf{E}$$

We plug this into the previous equation and we get check, some parts collides between my notes and the summary

$$-\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \left[\nabla \times \mathbf{B}\right] + \overset{???}{\cancel{\frac{1}{c^2}\frac{\partial^2 \mathbf{B}}{\partial t^2}}} = \frac{1}{\delta \mu_0}\nabla \times \left[\nabla \times \mathbf{B}\right]$$

—The curl of curl of $\mathbf{B}$ is .... and we neglect the second time derivative of $\mathbf{B}$ for non-relativistic speeds. So,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{v} \times \mathbf{B} + \frac{1}{\delta \mu_0}\nabla \mathbf{B}$$

This describes two effects:

- The first term at the R.H.S. describes the fact that magnetic flux is conserved across material loops
- The second term describes the **skin effect**:

$$\mathbf{v} = 0 \;\Rightarrow\; \frac{\partial \mathbf{B}}{\partial t} = \frac{1}{\delta \mu_0}\nabla \mathbf{B}$$

1.6.1 Digression on skin effect

Wiki-def: Skin effect is the tendency of an alternating electric current (AC) to become distributed within a conductor such that the **current density is largest near the surface of the conductor** and decreases exponentially with greater depths in the conductor. The electric current flows mainly at the surface of the conductor, between the outer surface and a level called the skin depth. Skin depth *depends on the frequency* of the alternating current; as frequency increases, current flow moves to the surface, resulting in less skin depth. *Skin effect reduces the effective cross-section of the conductor and thus increases its effective resistance.* Skin effect is caused by opposing eddy currents induced by the changing magnetic field resulting from the alternating current. At 60 Hz in copper, the skin depth is about 8 mm. At high frequencies the skin depth becomes much smaller. In a good conductor, skin depth is proportional to square root of the resistivity. This means that better conductors have a reduced skin depth. Skin depth also varies as the inverse square root of the permeability of the conductor.

Litz wires: A type of cable called litz wire (from the German Litzendraht, braided wire) is used to mitigate the skin effect for frequencies of a few kilohertz to about one megahertz. It consists of a number of *insulated wire strands* woven together in a carefully designed pattern, so that the overall magnetic field acts equally on all the wires and causes the total *current to be distributed equally* among them. With the skin effect having little effect on each of the thin strands, the bundle does not suffer the same increase in AC resistance that a solid conductor of the same cross-sectional area would due to the skin effect.