

LexBFS and its applications

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Input : a graph G

Output : a BFS on \bar{G}

Time complexity : linear

What are we going to talk about ?

Our main subject will be LexBFS

- Introduced by Tarjan et al. in 1976
- Has known multiple applications since.
- We will mainly focus on a paper by Habib et al.

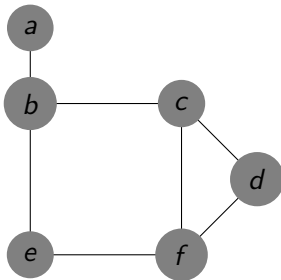
**Lex-BFS a partition refining technique, application to
transitive orientation and consecutive 1's testing**

Michel Habib, Ross Mac Connell, Christophe Paul, Laurent Viennot

What's in a graph ?

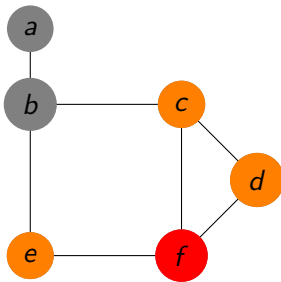
Graph

We only consider non-oriented, simple and **connected** graphs

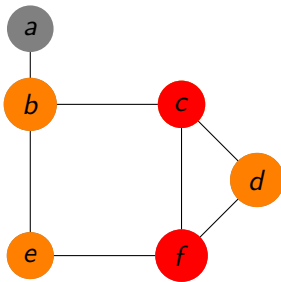


```
for i in [1, ..., n]:  
    if i == 1:  
        u = any vertex  
    else:  
        u = any unvisited marked vertex  
    visit(u)  
    for v in neighbours(u):  
        mark(v)
```

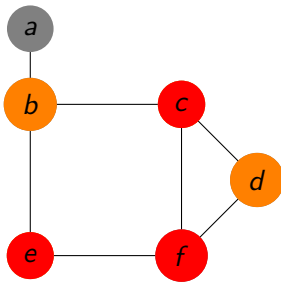
Example of generic search



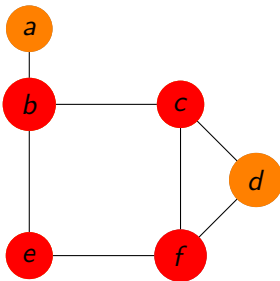
Example of generic search



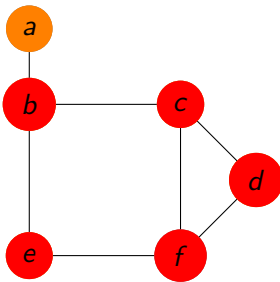
Example of generic search



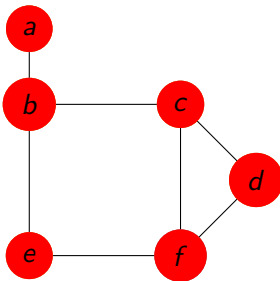
Example of generic search



Example of generic search



Example of generic search



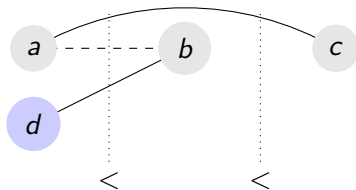
Another Characterization

Let's number vertices in the order they are visited.

Theorem

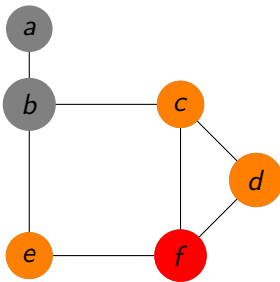
An ordering σ corresponds to a Generic Search if and only if

$$\forall a <_{\sigma} b <_{\sigma} c, ac \in E \text{ and } ab \notin E, \exists d <_{\sigma} b \text{ st } db \in E$$

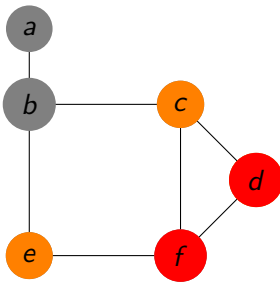


```
for i in [1, ..., n]:  
    if i == 1:  
        u = any vertex  
    else:  
        u = any unvisited vertex w/ max label  
    visit(u)  
    for v in neighbours(u):  
        label[v] = i
```

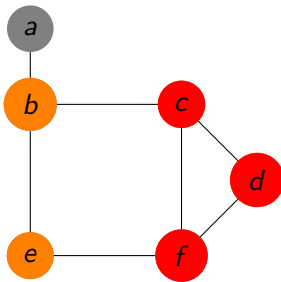
DFS example



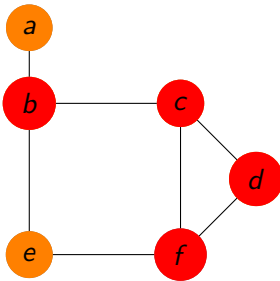
DFS example



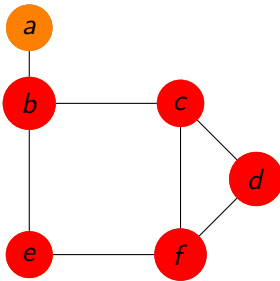
DFS example



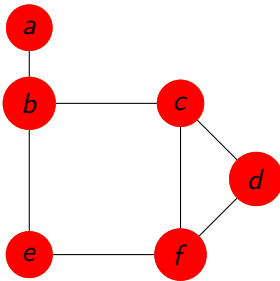
DFS example



DFS example



DFS example

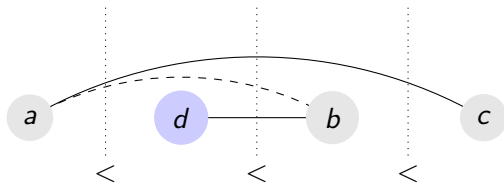


Another Characterization

Theorem

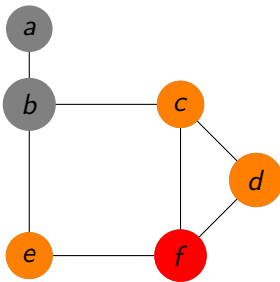
An ordering σ corresponds to a DFS if and only if

$$\forall a <_{\sigma} b <_{\sigma} c, ac \in E \text{ and } ab \notin E, \exists a <_{\sigma} d <_{\sigma} b \text{ st } db \in E$$

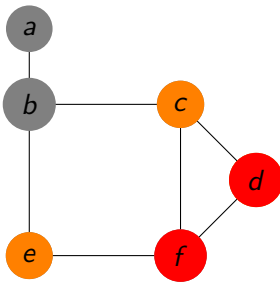


```
for i in [n, ..., 1]:  
    if i == n:  
        u = any vertex  
    else:  
        u = any unvisited vertex w/ max label  
    visit(u)  
    for v in neighbours(u):  
        if v has no label:  
            label[v] = i
```

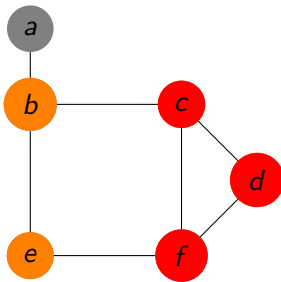
BFS example



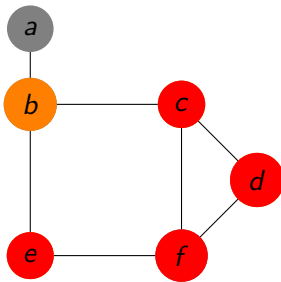
BFS example



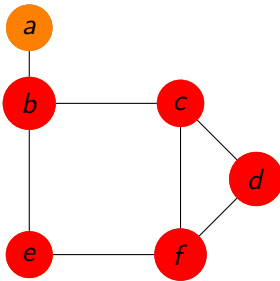
BFS example



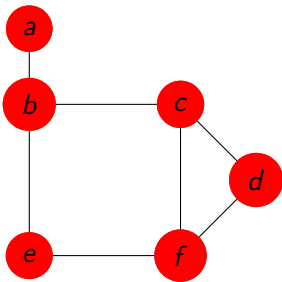
BFS example



BFS example



BFS example

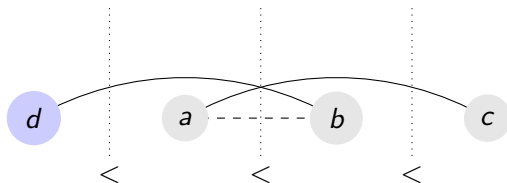


Another Characterization

Theorem

An ordering σ corresponds to a BFS if and only if

$$\forall a <_{\sigma} b <_{\sigma} c, ac \in E \text{ and } ab \notin E, \exists d <_{\sigma} a \text{ st } db \in E$$



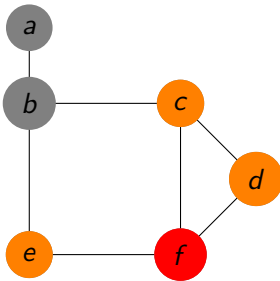
Let's rewrite BFS

```
for i in [n, ..., 1]:
    if i == n:
        u = any vertex
    else:
        u = any unvisited vertex
           w/ max first element of label
    visit(u)
    for v in neighbours(u):
        label[v].append(i)
```

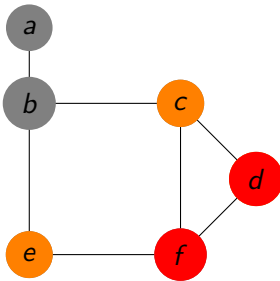
Here is LexBFS

```
for i in [n, ..., 1]:
    if i == n:
        u = any vertex
    else:
        u = any unvisited vertex
           w/ max lexicographical label
    visit(u)
    for v in neighbours(u):
        label[v].append(i)
```

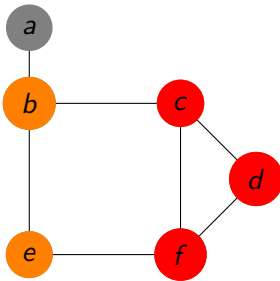
LexBFS example



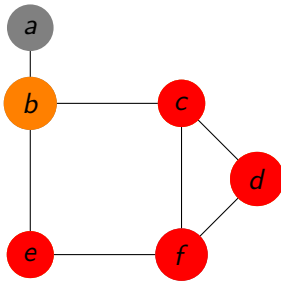
LexBFS example



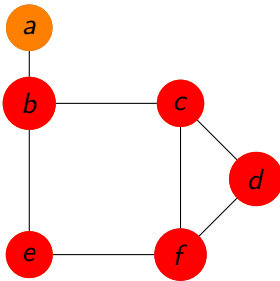
LexBFS example



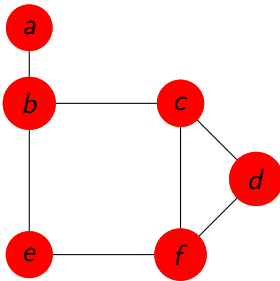
LexBFS example



LexBFS example



LexBFS example

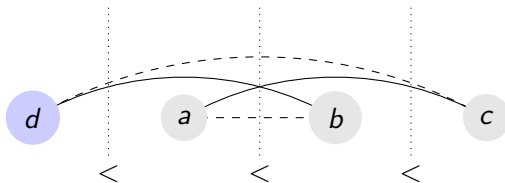


Another Characterization

Theorem

An ordering σ corresponds to a LexBFS if and only if

$\forall a <_{\sigma} b <_{\sigma} c, ac \in E \text{ and } ab \notin E, \exists d <_{\sigma} a \text{ st } db \in E \text{ and } dc \notin E$



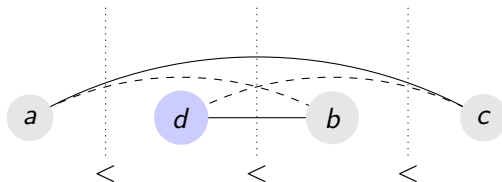
Another Characterization

Iterating on $[1, \dots, n]$ and prepending, we obtain LexDFS

Theorem

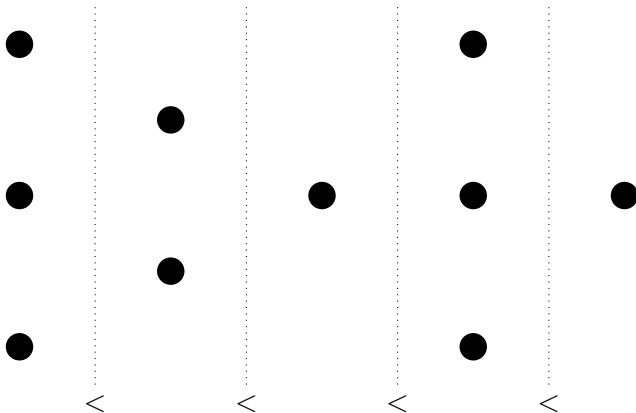
An ordering σ corresponds to a LexDFS if and only if

$$\forall a <_{\sigma} b <_{\sigma} c, ac \in E, ab \notin E, \exists a <_{\sigma} d <_{\sigma} b \text{ st } db \in E, dc \notin E$$



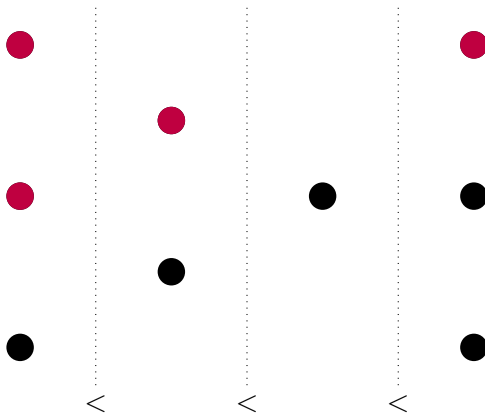
How to implement LexBFS: Pivoting

Let's order vertices by label :



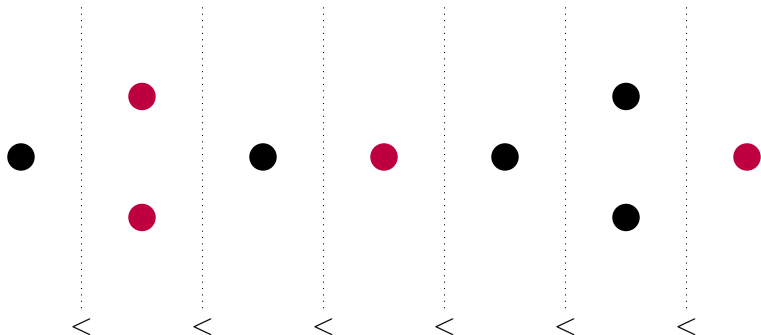
How to implement LexBFS: Pivoting

We remove one of the maximum vertex :



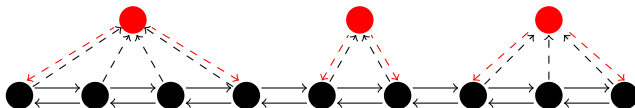
How to implement LexBFS: Pivoting

Now, we split each bucket



Data structure for pivoting

- Vertices in a doubly linked list
- Pointer from each vertex to its class
- Pointers from each class to its first and last vertex



Complexity

- Each step is $O(deg_v)$
- Thus, LexBFS is linear
- co-LexBFS is linear

LexDFS

As of 2019, no known linear-time for LexDFS

Chordal Graph

Chordal Graph

A graph is chordal if all cycles of four or more vertices have an edge that is not part of the cycle but connects two of its vertices.

Perfect Elimination Ordering (=PEO)

σ is a PEO if for each vertex v , v and its neighbours occurring after v in σ form a clique.

Chordal Graph

A graph is chordal if it admits a perfect elimination ordering.

Finding a Perfect Elimination Ordering

Main Theorem

If G is chordal, any LexBFS ordering taken backward is a PEO.

Proof:

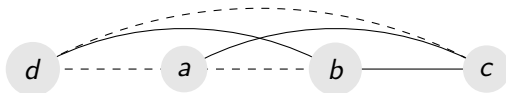
c : non simplicial vertex

Thus $\exists a <_{\sigma} b \in N(c)$ with $ab \notin E$

σ is a LexBFS thus $\exists d <_{\sigma} a$ with $db \in E, dc \notin E$

G is chordal, so $ad \notin E$

Ending the proof



- **Initialization:** (d, a, b) such that $da \notin E, ab \notin E, db \in E$
- **General Case:** $\exists d' <_{\sigma} d$ with $d'a \in E, d'b \notin E$ so $d'd \notin E$.
- Using the triple (d', d, a) , infinite chain.

Perfect Elimination co-Ordering is free

With our implementation, co-PEO is linear on co-chordal.

Chordal Recognition

For chordal recognition, need to verify if a given ordering is a PEO.

Verify if an ordering is a PEO

```
for x in vertices:
    RN[x] = neighborstotheright(x)
    parent[x] = leftmost(RN[x])
T = treefrompointers(parent)
for x in T in postorder:
    if (RN[x] \ x) not included in RN[parent[x]]
        return False
return True
```

Corollary

Recognizing chordal graphs can be done in linear time.

Easily adapted for running on the complementary :

- $\text{parent}[x] = \text{leftmost non-neighbor to the right of } x$
- check if $\text{RN}[\text{parent}[x]]$ is included in $\text{RN}[x]$

Corollary

Recognizing co-chordal graphs can be done in linear time.