

LexBFS and its applications

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Overview

1 Graph Searches

What's in a graph ?

Graph

We consider non-oriented, simple and **connected** graphs

INCLUDE GRAPHS EXAMPLES AND COUNTEREXAMPLES

Generic Search

```
for i in [1, ..., n]:  
    if i == 1:  
        u = any vertex  
    else:  
        u = any unvisited marked vertex  
    visit(u)  
    for v in neighbours(u):  
        mark(v)
```

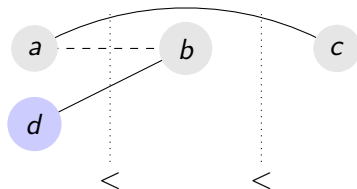
Another Characterization

Let's number vertices in the order they are visited.

Theorem

An order σ corresponds to a Generic Search if and only if

$$\forall a <_{\sigma} b <_{\sigma} c, ac \in E \text{ and } ab \notin E, \exists d <_{\sigma} b \text{ st } db \in E$$



DFS

INCLUDE GRAPH EXAMPLE

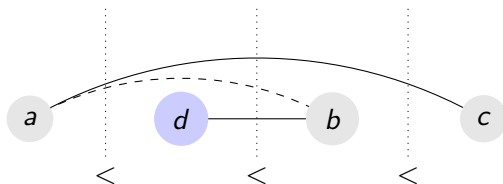
```
for i in [1, ..., n]:  
    if i == 1:  
        u = any vertex  
    else:  
        u = any unvisited vertex w/ max label  
    visit(u)  
    for v in neighbours(u):  
        label[v] = i
```

Another Characterization

Theorem

An order σ corresponds to a DFS if and only if

$$\forall a <_{\sigma} b <_{\sigma} c, ac \in E \text{ and } ab \notin E, \exists a <_{\sigma} d <_{\sigma} b \text{ st } db \in E$$



BFS

```
for i in [n, ..., 1]:  
    if i == n:  
        u = any vertex  
    else:  
        u = any unvisited vertex w/ max label  
    visit(u)  
    for v in neighbours(u):  
        if v has no label:  
            label[v] = i
```

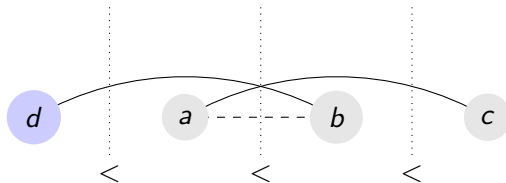
INCLUDE GRAPH EXAMPLE

Another Characterization

Theorem

An order σ corresponds to a BFS if and only if

$$\forall a <_{\sigma} b <_{\sigma} c, ac \in E \text{ and } ab \notin E, \exists d <_{\sigma} a \text{ st } db \in E$$



Let's rewrite BFS

```
for i in [n, ..., 1]:  
    if i == n:  
        u = any vertex  
    else:  
        u = any unvisited vertex  
           w/ max first element of label  
    visit(u)  
    for v in neighbours(u):  
        label[v].append(i)
```

Here is LexBFS

```
for i in [n, ..., 1]:
    if i == n:
        u = any vertex
    else:
        u = any unvisited vertex
            w/ max lexicographical label
    visit(u)
    for v in neighbours(u):
        label[v].append(i)
```

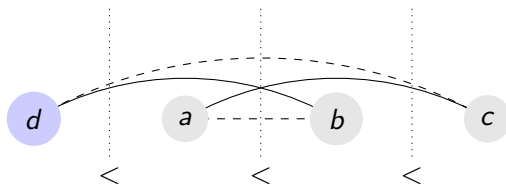
INCLUDE GRAPH EXAMPLE

Another Characterization

Theorem

An order σ corresponds to a LexBFS if and only if

$\forall a <_{\sigma} b <_{\sigma} c, ac \in E \text{ and } ab \notin E, \exists d <_{\sigma} a \text{ st } db \in E \text{ and } dc \notin E$



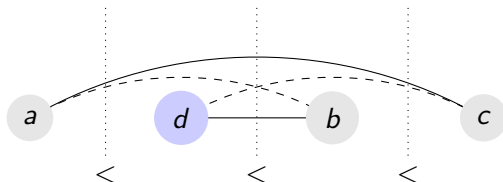
Another Characterization

Iterating on $[1, \dots, n]$ and prepending, we obtain LexDFS

Theorem

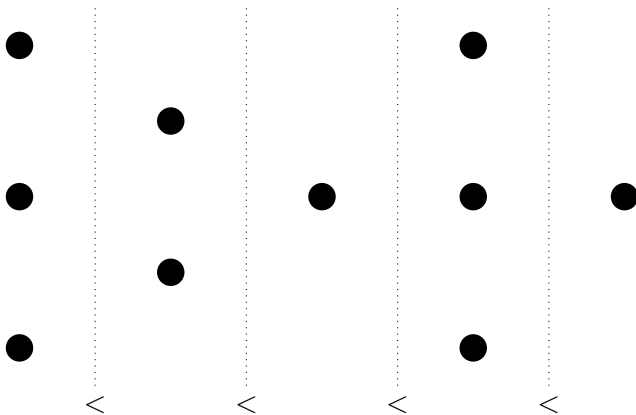
An order σ corresponds to a LexDFS if and only if

$$\forall a <_{\sigma} b <_{\sigma} c, ac \in E, ab \notin E, \exists a <_{\sigma} d <_{\sigma} b \text{ st } db \in E, dc \notin E$$



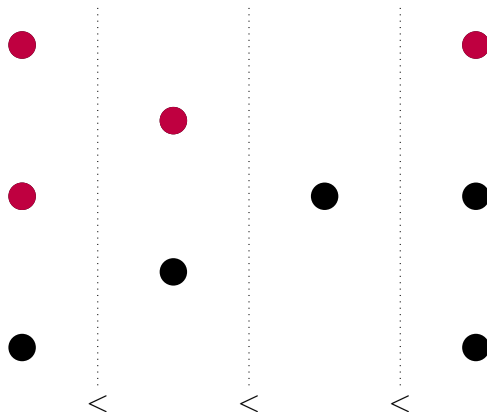
How to implement LexBFS: Pivoting

Let's order vertices by label :



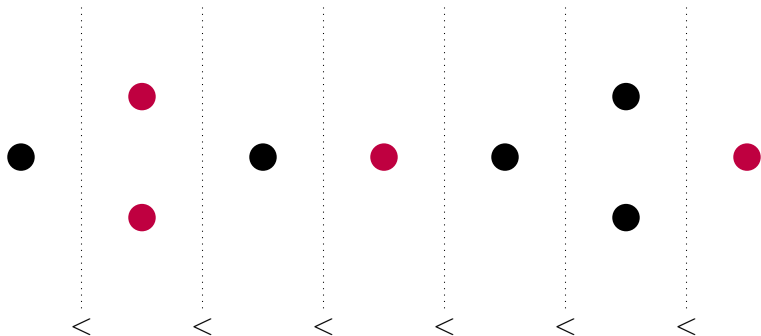
How to implement LexBFS: Pivoting

We remove one of the maximum vertex :



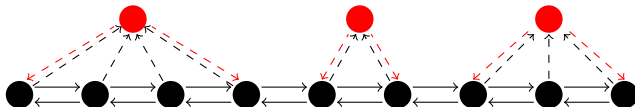
How to implement LexBFS: Pivoting

Now, we split each bucket



Data structure for pivoting

- Vertices in a doubly linked list
- Pointer from each vertex to its class
- Pointers from each class to its first and last vertex



Some remarks

Complexity

- Each step is $O(deg_v)$
- Thus, LexBFS is linear
- co-LexBFS is linear

LexDFS

As of 2019, no known linear-time for LexDFS

Chordal Graph

Chordal Graph

A graph is chordal if all cycles of four or more vertices have an edge that is not part of the cycle but connects two of its vertices.

Perfect Elimination Ordering (=PEO)

σ is a PEO if for each vertex v , v and its neighbours occurring after v in σ form a clique.

Chordal Graph

A graph is chordal if it admits a perfect elimination ordering.

Finding a Perfect Elimination Ordering

Main Theorem

If G is chordal, any LexBFS ordering taken backward is a PEO.

Proof:

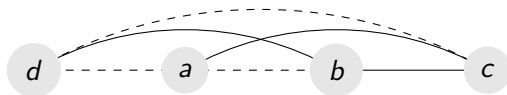
c : non simplicial vertex

Thus $\exists a <_{\sigma} b \in N(c)$ with $ab \notin E$

σ is a LexBFS thus $\exists d <_{\sigma} a$ with $db \in E, dc \notin E$

G is chordal, so $ad \notin E$

Ending the proof



- **Initialization:** (d, a, b) such that $da \notin E$, $ab \notin E$, $db \in E$
- **General Case:** $\exists d' <_{\sigma} d$ with $d'a \in E$, $d'b \notin E$ so $d'd \notin E$.
- Using the triple (d', d, a) , infinite chain.

Some remarks

Perfect Elimination co-Ordering is free

With our implementation, co-PEO is linear on co-chordal.

Chordal Recognition

For chordal recognition, need to verify if a given ordering is a PEO.