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ANALYTICAL SOLUTION

1. ANALYTICAL SOLUTION FOR THE HEAT SOURCE FIELD

The strong form of the heat conduction equation in 1D is given by,

$$\frac{d}{dx}\left(Ak\frac{dT}{dx}\right) + s = 0, \qquad on\ \Omega$$

$$qn = -kn\frac{dT}{dx} = \bar{q} \qquad on\ \Gamma_q$$

$$T = \bar{T} \qquad on\ \Gamma_T$$

The strong form for the heat conduction is given by,

$$\nabla \cdot q - s = 0 \quad \dots\dots\dots\dots\dots (i)$$

Where the heat flux q is calculated as,

$$q = -\kappa\nabla T \quad \dots\dots\dots\dots\dots (ii)$$

In two dimensions, we have two flux components and two temperature gradient components. For isotropic materials in two dimensions, Fourier's law is given by,

$$\kappa\nabla^2 T + s = 0 \quad \dots\dots\dots\dots\dots (iii)$$

Where

$$\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = \nabla^T\nabla = \frac{d^2}{dx^2} + \frac{d^2}{dy^2}$$

The arbitrary solution for temperature is given by,

$$T(x,y) = 10\left(x - \frac{L}{2}\right)\left(x + \frac{L}{2}\right)\left(y - \frac{L}{2}\right)\left(y + \frac{L}{2}\right)$$

Differentiating with respect to x, we get,

$$\frac{\partial T}{\partial x} = 20x\left(y^2 - \frac{L^2}{4}\right)$$

$$\frac{d^2T}{dx^2} = 20\left(y^2 - \frac{L^2}{4}\right)$$

Differentiating with respect to y, we get,

$$\frac{\partial T}{\partial y} = 20y\left(x^2 - \frac{L^2}{4}\right)$$

$$\frac{d^2T}{dy^2} = 20\left(x^2 - \frac{L^2}{4}\right)$$

From equation (ii),

$$q = -\kappa\frac{\mathrm{dT}}{\mathrm{dx}} \dots\dots\dots (Equation\ for\ Heat\ Flux)$$

The flux vector $\bar{q}$ can be expressed in terms of two components: the component tangential to the boundary q_t and the component normal to the boundary q_n. The tangential component q_t does not contribute to the heat entering or exiting the control volume.

$$\vec{q} = q_x\vec{i} + q_y\vec{j}$$

Where $$\vec{n} = n_x\vec{i} + n_y\vec{j},$$

$$n_x{}^2 + n_y{}^2 = 1$$

The normal component q_n is given by the scalar product of the heat flux with the normal to the body:

$$q_n = \vec{q}.\vec{n} = q_x n_x + q_y n_y$$

$$\begin{vmatrix} q_x \\ q_y \end{vmatrix} = \begin{vmatrix} k_{xx} & k_{xy} \\ k_{xy} & k_{yy} \end{vmatrix} \begin{vmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial T}{\partial y} \end{vmatrix}$$

$$q = -k\left\{20x^2\left(y^2 - \frac{L^2}{4}\right) + 20y^2\left(x^2 - \frac{L^2}{4}\right)\right\}/r$$

$$\boxed{\mathbf{q = 20\,k\{\,(x^2+y^2)\frac{L^2}{4} - 2x^2y^2\}/r}}$$

This is the equation for the heat flux distribution over the body.

2. ANALYTICAL SOLUTION FOR THE HEAT FIELD

From (i), we have,

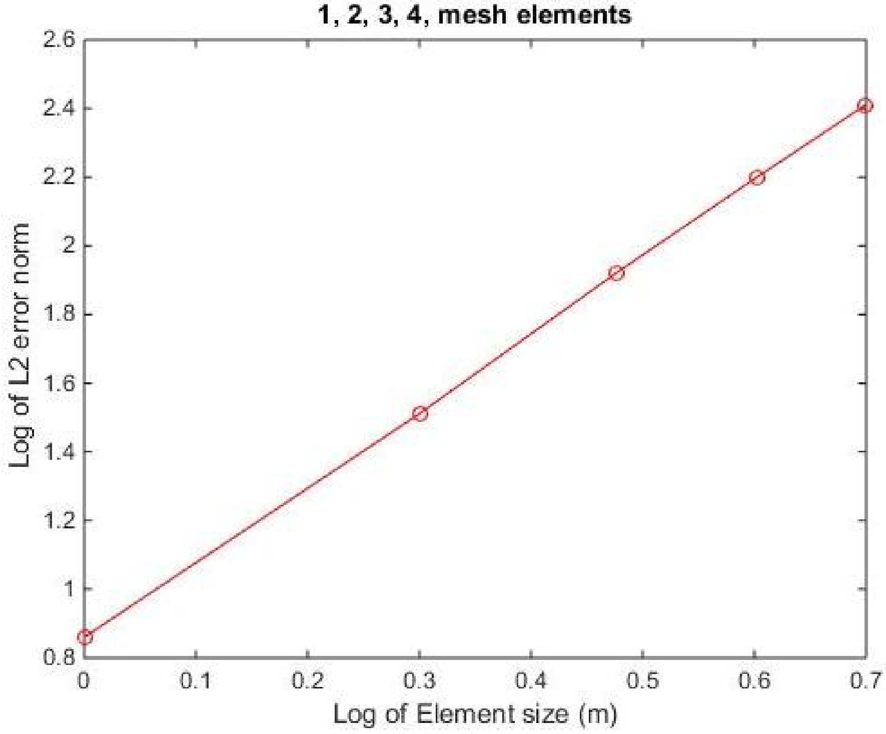
$$k\left(\frac{d^2T}{dx^2} + \frac{d^2T}{dy^2}\right) + s = 0$$

$$54 \times 20\left\{\left(x^2 - \frac{L^2}{4}\right) + \left(y^2 - \frac{L^2}{4}\right)\right\} + s = 0$$

Hence, the source variation is given as,

$$\boxed{\boldsymbol{s = -1080\left(x^2 + y^2 - \frac{L^2}{2}\right)}}$$

5. <u>PLOT L2 TEMPERATURE ERROR NORM VS MESH SIZE ON A LOG-LOG PLOT AND COMPUTE THE RATE OF CONVERGENCE</u>



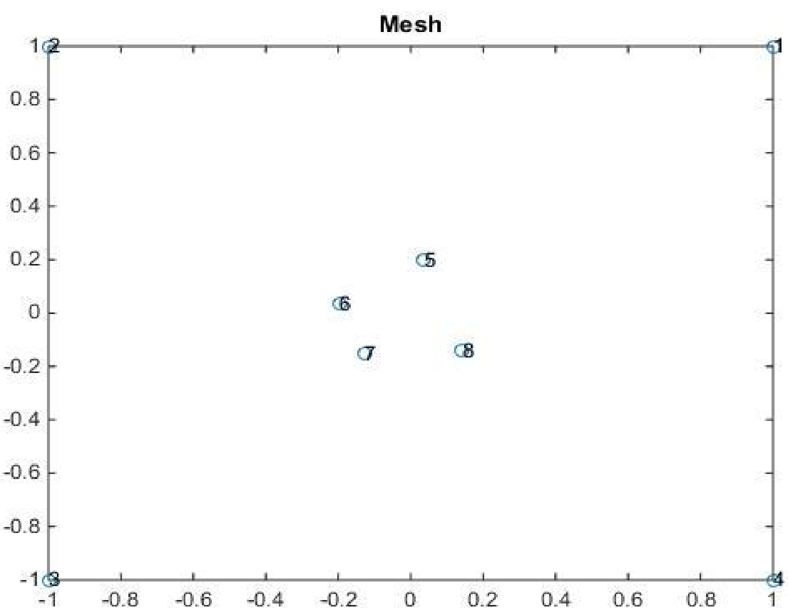
RATE OF CONVERGENCE ≅ 2

6. DEMONSTRATION OF CODE PASSING PATCH TEST

Generated mesh:

Total number of nodes: 8

Total number of elements: 4



The assumed temp solution is $T = 2 + 4x + 5y$

PATCH TEST PASS

```
%Nodal temperature values (FEM Solution)

d =
  11.000000000000002
   1.000000000000000
  -7.000000000000003
   3.000000000000001
   2.961499999999999
   1.153799999999999
   0.744199999999999
   2.141400000000000

%Nodal temperature values (Assumed analytical solution)

T =

  11.000000000000000
   1.000000000000000
  -7.000000000000000
   3.000000000000000
   2.961500000000000
   1.153800000000000
   0.744200000000000
   2.141400000000000
```

For the MATLAB code, see appendix.