Homework

Problems on UD

6.7

(1) $B \setminus (A \cap B)$
(2) $(A \cup B) \setminus (A \cap B)$
(3) $A \cap B \cap C$
(4) $(B \cap C) \setminus (A \cap B \cap C)$
(5) $((A \cap B) \cup (B \cap C) \cup (A \cap C)) \setminus (A \cap B \cap C)$

6.16

(a) Consider $\forall n \in A, n = x^2$, where $x \in Z$. Because x is an integer, then $x \times x$ is also a integer, namely x^2 is also an integer. Since $n = x^2$, then n is a integer, namely $n \in Z$. Now we have shown that all the elements in A are contained in B, namely $A \subseteq B$.
(b) Consider $\forall n \in A$, namely $\forall n \in R$. $\forall n \in A$, suppose $x = \frac{n}{2}$, since $n \in R$, then $\frac{n}{2} \in R$, namely $x \in R$, so $2x \in B$. Since $n = 2x$, so $n \in B$. Now we have proved $\forall n \in A, n \in B$, namely $A \subseteq B$.
(c) Consider $\forall (x, y) \in A$, namely where $y = \frac{5-3x}{2}$. Since $y = \frac{5-3x}{2}$, we can transform the equation into $2y + 3x = 5$. So we have $\forall (x, y) \in A, 2y + 3x = 5$, namely $(x, y) \in B$. So we have proved $\forall (x, y) \in A, (x, y) \in B$. Namely $A \subseteq B$.

6.17

(a) $A \subsetneq B$
Proof.
$\forall (x, y) \in A$, namely $xy > 0$. Because $x^2 + y^2 \geq xy$, so $x^2 + y^2 > 0$. So $\forall (x, y) \in A, x^2 + y^2 > 0$, namely $(x, y) \in B$. Now we have proved $\forall (x, y) \in A, (x, y) \in B$. Meanwhile, consider $(1, -1) \in B$, but it isn't contained in A, so $A \subsetneq B$.
(b) $A \subsetneq B$
Proof.
First, we have to show that if an element is contained in A, then it's contained in B. Consider A is $\emptyset$, then no element is contained in A, so the statement we have to show is a tautology. So $A \subset B$. Since every element contained in B can't be contained in A, then furthermore we have proved $A \subsetneq B$.

7.1

Suppose U is the universe.

(a)

Proof.

$$
\begin{aligned}
& x \in (A^c)^c \\
\leftrightarrow\ & x \in U \setminus A^c \\
\leftrightarrow\ & x \in U \setminus (U \setminus A) \\
\leftrightarrow\ & x \in A.
\end{aligned}
$$

(b)

Proof.

$$
\begin{aligned}
& x \in (A \cap (B \cup C)) \\
\leftrightarrow\ & x \in A \wedge (x \in B \vee x \in C) \\
\leftrightarrow\ & (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) \\
\leftrightarrow\ & x \in (A \cap B) \vee x \in (A \cap C) \\
\leftrightarrow\ & x \in (A \cap B) \cup (A \cap C)
\end{aligned}
$$

(c)

Proof.

$$
\begin{aligned}
& x \in X \setminus (A \cap B) \\
\leftrightarrow\ & x \in X \wedge x \notin (A \cap B) \\
\leftrightarrow\ & (x \in X \wedge x \in A \wedge x \notin B) \vee (x \in X \wedge x \in B \wedge x \notin A) \vee (x \in X \wedge x \notin A \wedge x \notin B) \\
\leftrightarrow\ & (x \in X \wedge x \notin A) \vee (x \in X \wedge x \notin B) \\
\leftrightarrow\ & x \in (X \setminus A) \vee x \in (X \setminus B) \\
\leftrightarrow\ & x \in ((X \setminus A) \cup (X \setminus B))
\end{aligned}
$$

(d)

Proof.

$$
\begin{aligned}
& A \subseteq B \\
\leftrightarrow\ & \forall x \in A \rightarrow x \in B \\
\leftrightarrow\ & \forall x \notin B \rightarrow x \notin A \\
\leftrightarrow\ & \forall x \in X \wedge x \notin B \rightarrow x \in X \wedge x \notin A \\
\leftrightarrow\ & \forall x \in (X \setminus B) \rightarrow x \in (X \setminus A) \\
\leftrightarrow\ & (X \setminus B) \subseteq (X \setminus A)
\end{aligned}
$$

(e)

Proof.

$$
\begin{aligned}
& A \cap B = B \\
\leftrightarrow\ & (\forall x \in A \wedge x \in B \rightarrow x \in B) \wedge (\forall x \in B \rightarrow x \in A \wedge x \in B) \\
\leftrightarrow\ & \forall x \in B \rightarrow x \in A \\
\leftrightarrow\ & B \subseteq A
\end{aligned}
$$

7.8

(a)

(ii)

(b)

(i)(iii)(iv)(v)

(c)

Proof.

$$
\begin{aligned}
& x \in (A \cap B) \setminus C \\
\leftrightarrow\ & x \in (A \cap B) \land x \notin C \\
\leftrightarrow\ & x \in A \land x \in B \land x \notin C \\
\leftrightarrow\ & (x \in A \land x \notin C) \land (x \in B \land x \notin C) \\
\leftrightarrow\ & x \in (A \setminus C) \land (x \in B \setminus C) \\
\leftrightarrow\ & x \in ((A \setminus C) \cap (B \setminus C))
\end{aligned}
$$

7.9

(a)

Proof.

$$
\begin{aligned}
& x \in (A \setminus B) \\
\leftrightarrow\ & x \in A \land x \notin B
\end{aligned}
$$

So $\forall x \in (A \setminus B) \to x \notin B$

Namely $A \setminus B$ and B are disjoint.

(b)

Proof.

$$
\begin{aligned}
& x \in (A \cup B) \\
\leftrightarrow\ & x \in A \lor x \in B \\
\leftrightarrow\ & (x \in A \land x \notin B) \lor (x \in A \land x \in B) \lor x \in B \\
\leftrightarrow\ & (x \in A \land x \notin B) \lor x \in B \\
\leftrightarrow\ & x \in (A \setminus B) \lor x \in B \\
\leftrightarrow\ & x \in ((A \setminus B) \cup B)
\end{aligned}
$$

7.10

Disproof.

By counterexample:

Suppose $A = \{1,2,3\}, B = \{1\}, C = \{1,2\}$

Such A, B and C satisfy $A \cup B = A \cup C$, but obviously $B \neq C$

7.11

Counterexample:

Suppose $A = \{1,2,3,4\}, B = \{1,2,3\}, Y = \{1\}$

Such A, B and Y satisfy $A \cap Y = B \cap Y$, but obviously $A \neq B$

8.1

(a)
$\cup_{n=1}^{\infty} A_n = [0,1)$
$\cup_{n=1}^{\infty} B_n = [0,1]$
$\cup_{n=1}^{\infty} C_n = (0,1)$
(b)
$\cap_{n=1}^{\infty} A_n = \{0\}$
$\cap_{n=1}^{\infty} B_n = \{0\}$
$\cap_{n=1}^{\infty} C_n = \emptyset$
(c)
No, because $0 \in N$, but n can't be 0.

8.4

Proof.
If there are sets in $\{A_n : n \in \mathbb{Z}^+\}$ such that the sets disjoint, then $\cap_{n=1}^{\infty} A_n = \emptyset$. Then obviously $\cap_{n=1}^{\infty} A_n \subset \cap_{n=1}^{\infty} B_n$.
If all the sets in $\{A_n : n \in \mathbb{Z}^+\}$ intersect with each other, we use contradictory. Suppose $A_n \subset B_n$ for all $n \in \mathbb{Z}^+$ and $\cap_{n=1}^{\infty} A_n \not\subset \cap_{n=1}^{\infty} B_n$, so there is an element in $\cap_{n=1}^{\infty} A_n$ that isn't in $\cap_{n=1}^{\infty} B_n$. Since it's in $\cap_{n=1}^{\infty} A_n$, then it must be an element of all A_k, where $k \in \mathbb{Z}^+$, because $A_n \subset B_n$ for all $n \in \mathbb{Z}^+$, so the element must be in all B_j, where $j \in \mathbb{Z}^+$, furthermore, it must be in $\cap_{n=1}^{\infty} B_n$. But we suppose it shouldn't be, so it contradicts. Then we have proved $\cap_{n=1}^{\infty} A_n \subset \cap_{n=1}^{\infty} B_n$.

8.7

(a)
By contradictory.
According to the definition, the element in $\cap_{\alpha \in I} A_\alpha$ must be in every A_α where $\alpha \in I$. Suppose $\cap_{\alpha \in I} A_\alpha \neq \emptyset$, namely there are elements in $\cap_{\alpha \in I} A_\alpha$, and they have to be in all the A_α where $\alpha \in I$, but for $\alpha \in I$, $A_\alpha = \emptyset$, namely there are no elements in them.Contradicts. So we have proved if $A_\alpha = \emptyset$ for some $\alpha \in I$, then $\cap_{\alpha \in I} A_\alpha = \emptyset$.
(b)
Suppose $A_k = X, k \in I$, then according to the definition, $\cup_{\alpha \in I} A_\alpha = \cup_{\alpha \in I \setminus \{k\}} A_\alpha \cup A_k$, namely $\cup_{\alpha \in I \setminus \{k\}} A_\alpha \cup X = X$.
(c)
If $B \subseteq A_\alpha$ for every $\alpha \in I$, then $B \cap A_\alpha = B$ for every $\alpha \in I$. So $\cap_{\alpha \in I}(B \cap A_\alpha) = B$, namely $B \cap (\cap_{\alpha \in I} A_\alpha) = B$. So $B \subseteq \cap_{\alpha \in I} A_\alpha$.

8.8

$A = \mathbb{Z}$

Proof.

$\quad \cap_{n \in \mathbb{Z}^+} (R \setminus \{-n, -n+1, ..., 0, ..., n-1, n\})$
$\leftrightarrow \cap_{n \in \mathbb{Z}^+} (R \cap \{-n, -n+1, ..., 0, ...n-1, n\}^c)$
$\leftrightarrow R \cap \mathbb{Z}^c$
$\leftrightarrow R \setminus \mathbb{Z}$
So $A = R \setminus \cap_{n \in Z^+} (R \setminus \{-n, -n+1, ..., 0, ..., n-1, n\})$
$\qquad = R \setminus (R \setminus \mathbb{Z})$
$\qquad = \mathbb{Z}$

8.9

$A = \{x : x = 2n, n \in \mathbb{Z}\}$

Proof.

$\quad Q \setminus \cap_{n \in \mathbb{Z}} (\mathbb{R} \setminus \{2n\})$
$\leftrightarrow Q \setminus \cap_{n \in \mathbb{Z}} (\mathbb{R} \cap \{2n\}^c)$
$\leftrightarrow Q \cap (\cap_{n \in \mathbb{Z}} (\mathbb{R} \cap \{2n\}^c))^c$
$\leftrightarrow Q \cap \cup_{n \in \mathbb{Z}} (\mathbb{R} \cap \{2n\}^c)^c$
$\leftrightarrow Q \cap \cup_{n \in \mathbb{Z}} (\mathbb{R}^c \cup \{2n\})$
$\leftrightarrow Q \cap \cup_{n \in Z} \{2n\}$
$\leftrightarrow \{x : x = 2n, n \in \mathbb{Z}\}$

8.11

(a)

$A_1 = \{1\}, A_2 = \{2\}, A_3 = \{3\} ..., A_n = \{n\}.$

(b)

If $A_\alpha \neq A_\beta$, then $A_\alpha \cap A_\beta = \emptyset$.

(c)

If $A_\alpha = A_\beta$, then $A_\alpha \cap A_\beta \neq \emptyset$.

(d)

Yes.

(e)

Yes.

(f)

Yes.

(g)

No.

9.2

(a)

$x \in (\mathrm{P}(A) \cup \mathrm{P}(B))$
$\leftrightarrow x \in \mathrm{P}(A) \vee x \in \mathrm{P}(B)$.
If $x \in \mathrm{P}(A)$, so $x \subseteq A$, then $x \subseteq (A \cup B)$, namely $x \in \mathrm{P}(A \cup B)$.
In the same way, we can prove if $x \in \mathrm{P}(B)$, then $x \in \mathrm{P}(A \cup B)$.
So $x \in (\mathrm{P}(A) \cup \mathrm{P}(B)) \rightarrow x \in \mathrm{P}(A \cup B)$.
Namely $\mathrm{P}(A) \cup \mathrm{P}(B) \subseteq \mathrm{P}(A \cup B)$.

(b)

$A = \{1\}, B = \{2\}, \mathrm{P}(A) \cup \mathrm{P}(B) = \{\{1\}, \{2\}, \emptyset\}, \mathrm{P}(A \cup B) = \{\{1, 2\}, \{1\}, \{2\}, \emptyset\}$

9.4

$\rightarrow$:
$\forall X \subseteq A$, namely $X \in \mathrm{P}(A)$, because $A \subseteq B$, so $X \subseteq B$, namely $X \in \mathrm{P}(B)$. So if $A \subseteq B$, then $\forall X \in \mathrm{P}(A) \rightarrow X \in \mathrm{P}(B)$, namely $\mathrm{P}(A) \subseteq \mathrm{P}(B)$.
$\leftarrow$:
Consider the contrapositive:If $A \nsubseteq B$, then $\mathrm{P}(A) \nsubseteq \mathrm{P}(B)$. Obviously that if A isn't contained in B, then there must exist a subset of A which contains an element that doesn't belong to B, so it can't be a subset of B. So we find a subset of A that isn't a subset of B, thus have proved $\mathrm{P}(A) \nsubseteq \mathrm{P}(B)$.

9.12

(a)

Proof.
$\leftarrow$:
Obviously that for all nonempty sets A, B, C and D, if $A = C$ and $B = D$, we have $A \times B = C \times D$.
$\rightarrow$:
Considering A, B, C and D are all nonempty sets
$A \times B = C \times D$
$\leftrightarrow (\forall (x, y) \in A \times B \rightarrow (x, y) \in C \times D) \wedge (\forall (x, y) \in C \times D \rightarrow (x, y) \in A \times B)$
$\leftrightarrow ((\forall x \in A \rightarrow x \in C) \wedge (\forall y \in B \rightarrow y \in D)) \wedge ((\forall x \in C \rightarrow x \in A) \wedge (\forall y \in D \rightarrow y \in B))$
$\leftrightarrow ((\forall x \in A \rightarrow x \in C) \wedge (\forall x \in C \rightarrow x \in A)) \wedge ((\forall y \in B \rightarrow y \in D) \wedge (\forall y \in D \rightarrow y \in B))$
$\leftrightarrow ((A \subseteq C) \wedge (C \subseteq A)) \wedge ((B \subseteq D) \wedge (D \subseteq B))$
$\leftrightarrow (A = C) \wedge (B = D)$
$Q.E.D$

(b)

Without loss of generality,suppose $A = \emptyset$, then without $B \subseteq D$, we can still have $A \times B \subseteq C \times D$, because $\forall (x, y) \in A \times B \to (x, y) \in C \times D$ will be a tataulogy as there are no elements in A hence no elements in $A \times B$. Then the proof will be invalid.

9.13

Yes.

Proof.

$\to$:

If $A \subseteq C$ and $B \subseteq D$, then

$$\begin{aligned} & \forall (x, y) \in A \times B \\ \leftrightarrow\ & x \in A \land y \in B \\ \to\ & x \in C \land x \in D \\ \leftrightarrow\ & (x, y) \in C \times D \end{aligned}$$

So $A \times B \subseteq C \times D$.

$\leftarrow$:

Consider the contrapositive: If $A \nsubseteq C \lor B \nsubseteq D$, then $A \times B \nsubseteq C \times D$.

Without loss of generality, we suppose $A \nsubseteq C$, so $\exists x_{\in} A \land x_0 \notin C$, consider (x_0, y) for every $y \in B$, obviously $(x_0, y) \in A \times B$, but $(x_0, y) \notin B \times D$, so $A \times B \nsubseteq C \times D$.

9.14

(a)

True.

Proof.

$$\begin{aligned} & \forall (x, y) \in A \times (B \cup C) \\ \leftrightarrow\ & x \in A \land y \in B \cup C \\ \leftrightarrow\ & x \in A \land (y \in B \lor y \in C) \\ \leftrightarrow\ & (x \in A \land y \in B) \lor (x \in A \land y \in C) \\ \leftrightarrow\ & ((x, y) \in A \times B) \lor ((x, y) \in A \times C) \\ \leftrightarrow\ & (x, y) \in (A \times B) \cup (A \times C) \end{aligned}$$

So $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(b)

True.

Proof.

$$\begin{aligned} & \forall (x, y) \in A \times (B \cap C) \\ \leftrightarrow\ & x \in A \land y \in B \cap C \\ \leftrightarrow\ & x \in A \land (y \in B \land y \in C) \\ \leftrightarrow\ & (x \in A \land y \in B) \land (x \in A \land y \in C) \\ \leftrightarrow\ & ((x, y) \in A \times B) \land ((x, y) \in A \times C) \end{aligned}$$

$\leftrightarrow (x, y) \in (A \times B) \cap (A \times C)$
So $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

9.16

(a)

Proof.
If $(a, b) = (x, y)$, namely $\{\{a\}, \{a, b\}\} = \{\{x\}, \{x, y\}\}$
So the elements of the two sets must be all equal.
So $\{a\} = \{x\} \wedge \{a, b\} = \{x, y\}$ or $\{a\} = \{x, y\} \wedge \{a, b\} = \{x\}$.
If the latter part of the statement is true, then $\{a\} = \{x, y\}$, oviously $\{a\} \neq \{x, y\}$ because the number of elements in them are not even equal. So the first part of the statement is true, namely $\{a\} = \{x\} \wedge \{a, b\} = \{x, y\}$ is true.
Now that $\{a\} = \{x\}$, so $a = x$. Furthermore, $\{a, b\} = \{x, y\} \rightarrow (a = x \wedge b = y) \vee (a = y \wedge b = x)$. Since $a = x$, then $b = y$.

(b)

Proof.
If $a \in A$ and $b \in B$, then $\{a\} \subseteq A, \{a\} \in \mathrm{P}(A)$, namely $\{a\} \in \mathrm{P}(A \cup B); \{a, b\} \subseteq A \cup B, \{a, b\} \in \mathrm{P}(A \cup B)$. So $\{\{a\}, \{a, b\}\} \subseteq \mathrm{P}(A \cup B)$, namely $\{\{a\}, \{a, b\}\} \in \mathrm{P}(\mathrm{P}(A \cup B))$. So $(a, b) \in \mathrm{P}(\mathrm{P}(A \cup B))$.

(c)

Proof.
$\quad A \subseteq C \wedge B \subseteq D$
$\leftrightarrow \forall x \in A, \forall y \in B, \{x\} \subseteq A \rightarrow \{x\} \subseteq C \rightarrow \{x\} \subseteq C \cup D, \{x, y\} \subseteq A \cup B \rightarrow \{x, y\} \subseteq C \cup D$
$\leftrightarrow \forall \{\{x\}, \{x, y\}\} \subseteq \mathrm{P}(A \cup B) \rightarrow \{\{x\}, \{x, y\}\} \subseteq \mathrm{P}(C \cup D)$
$\leftrightarrow \forall \{\{x\}, \{x, y\}\} \in \mathrm{P}(\mathrm{P}(A \cup B)) \rightarrow \{\{x\}, \{x, y\}\} \in \mathrm{P}(\mathrm{P}(C \cup D))$
$\leftrightarrow \forall (x, y) \in A \times B \rightarrow (x, y) \in C \times D.$
So if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.