Homework

Problems on UD

13.3

(a)
No.
Because $\forall x \in \mathbb{R}, \exists y_1, y_2, y_1 \neq y_2, x^2 + y_1^2 = 4, x^2 + y_2^2 = 4$, namely $\forall x \in \mathbb{R}, \exists y_1, y_2, y_1 \neq y_2, (x, y_1) \in f, (x, y_2) \in f$.
(b)
No.
Because consider $x_0 = 0 \in \mathbb{R}$, whereas $f(x_0) = \frac{1}{x0+1} \notin \mathbb{R}$.
(c)
Yes.
Because **(i)** $\forall (x, y) \in \mathbb{R}^2, x + y \in \mathbb{R}$, namely $\forall (x, y) \in \mathbb{R}^2, \exists f(x, y) \in \mathbb{R}$. **(ii)** $\forall (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, if $(x_1, y_1) = (x_2, y_2)$, namely $x_1 = x_2, y_1 = y_2$, then $x_1 + y_1 = x_2 + y_2$, namely $f(x_1, y_1) = f(x_2, y_2)$.
(d)
Yes.
Because **(i)** $\forall [a, b]$, where $a, b \in \mathbb{R}, a \leq b$, we have $f([a, b]) = a \in \mathbb{R}$ **(ii)** $\forall [a_1, b_1], [a_2, b_2]$, if $[a_1, b_1] = [a_2, b_2]$, namely $a_1 = a_2, b_1 = b_2$, then $f([a_1, b_1]) = a_1 = a_2 = f([a_2, b_2])$.
(e)
Yes.
Because **(i)** $\forall (n, m) \in \mathbb{N} \times \mathbb{N}, m \in \mathbb{N}$, thus $n \in \mathbb{R}$, so $\forall (n, m) \in \mathbb{N} \times \mathbb{N}, \exists f(n, m) = m \in \mathbb{R}$. **(ii)** $\forall (n_1, m_1), (n_2, m_2) \in \mathbb{N} \times \mathbb{N}, f(n_1, m_1) = m_1, f(n_2, m_2) = m_2$, so if $(n_1, m_1) = (n_2, m_2)$, namely $m_1 = m_2, n_1 = n_2$, then $f(n_1, m_1) = f(n_2, m_2)$.
(f)
Yes.
Because **(i)** Since $0 \in \mathbb{R}$, so $\forall x \in R, f(x) = 0$ or $f(x) = x, 0 \in \mathbb{R}$ and $x \in \mathbb{R}$, namely $\forall x \in \mathbb{R}, \exists f(x) \in \mathbb{R}$. **(ii)** $\forall x_1, x_2 < 0$, if $x_1 = x_2 = k$, then $f(x_1) = f(x_2) = k$. $\forall x_1, x_2 > 0$, if $x_1 = x_2$, then $f(x_1) = f(x_2) = 0$. If $x_1 = 0, x_2 = 0$, then $f(x_1) = f(x_2) = 0$.
(g)
No
Because consider $6 \in Q$, then since $6 = 2 \times 3, f(6) = 7$, again since $6 = 3 \times 2, f(6) = 5$. So there are two different $f(6)$, which contradicts with the definition.
(h)

Yes.

Because **(i)** every circle has a circumference and **(ii)** every circle only has one definite circumference.

(i)

Yes.

Because **(i)** every polynominal with real coefficients is derivable and **(ii)** once the polynominal is definite, then its only derivative it's definite and only.

(j)

Yes.

Because **(i)** every polyminal is integrable, and **(ii)** its definite integral on $[0, 1]$ is definite and only.

13.4

Proof.

(i) $\forall \mathrm{A} \in \mathrm{P}(\mathbb{R})$, there are two cases. **(a)** If $\mathrm{A} \cap \mathbb{N} = \emptyset$, namely all the numbers in A are not natural numbers, then $f(\mathrm{A}) = -1 \in \mathbb{Z}$. **(b)** If $\mathrm{A} \cap \mathbb{N} \neq \emptyset$, namely there are natural numbers in A, then according to Well-ordering principle of $\mathbb{N}$, every nonempty subset of the natural numbers contains a minimum, and $\mathrm{A} \cap \mathbb{N}$ is a nonempty subset of $\mathbb{N}$, so there exists $min(\mathrm{A} \cap \mathbb{N}) \in \mathbb{N}$, thus $min(\mathrm{A} \cap \mathbb{N}) \in \mathbb{Z}$. So $\forall \mathrm{A} \in \mathrm{P}(\mathbb{R}), \mathrm{A} \cap \mathbb{N} \neq \emptyset$, then $\exists f(\mathrm{A}) = min(\mathrm{A} \cap \mathbb{N}) \in \mathbb{Z}$.

(ii)

From the definition, we can see that if $\mathrm{A}_1 = \mathrm{A}_2$, then there are two cases. **(a)** $\mathrm{A}_1 = \mathrm{A}_2, \mathrm{A}_1 \cap \mathbb{N} = \emptyset, \mathrm{A}_2 \cap \mathbb{N} = \emptyset$, so $f(\mathrm{A}_1) = f(\mathrm{A}_2) = -1$. **(b)** $\mathrm{A}_1 = \mathrm{A}_2, \mathrm{A}_1 \cap \mathbb{N} \neq \emptyset, \mathrm{A}_2 \cap \mathbb{N} \neq \emptyset$, so $f(\mathrm{A}_1) = min(\mathrm{A}_1 \cap \mathbb{N}), f(\mathrm{A}_2) = min(\mathrm{A}_2 \cap \mathbb{N})$, because $\mathrm{A}_1 = \mathrm{A}_2$, so $\mathrm{A}_1 \cap \mathbb{N} = \mathrm{A}_2 \cap \mathbb{N}$, so $min(\mathrm{A}_1 \cap \mathbb{N}) = min(\mathrm{A}_2 \cap \mathbb{N})$, namely $f(\mathrm{A}_1) = f(\mathrm{A}_2)$.

13.5

(a)

Yes, it is.

Proof.

(i) Consider an arbitrary set $A, \forall x \in X, x$ is either contained in A or not, so $\forall x, \exists \chi_A(x) = 1$ or $\chi_A(x) = 0$.

(ii) Consider an arbitrary set $A, \forall x_1, x_2 \in X$, if $x_1 = x_2$, then either both x_1 and x_2 are contained in A, or neither of them are contained in A, so $\chi_A(x_1) = \chi_A(x_2)$.

(b)

Domain:X

Range:$\{0, 1\}$

13.7

(i) To prove $ran(f) \subseteq \mathbb{R} \setminus \{\frac{1}{2}\}$.

$\forall x \in \mathbb{R} \setminus \{\frac{3}{2}\}, y = \frac{x-5}{2x-3} \in ran(f)$, and obviously $y \in \mathbb{R}$, os $ran(f) \subseteq \mathbb{R}$. Now we have to

show $y \neq \frac{1}{2}$. Suppose $\exists y_0 \in ran(f), y_0 = \frac{1}{2}$. So $\exists x_0 \in \mathbb{R} \setminus \{\frac{3}{2}\}, \frac{x-5}{2x-3} = \frac{1}{2}$, furthermore we have $2x - 10 = 2x - 3$, then $-10 = -3$. So $y \neq \frac{1}{2}$, namely $\forall y \in ran(f), y \in \mathbb{R} \setminus \{\frac{1}{2}\}$, then $ran(f) \subseteq \mathbb{R} \setminus \{\frac{1}{2}\}$.
(ii) To prove $\mathbb{R} \setminus \{\frac{1}{2}\} \subseteq ran(f)$.
$\forall y \in \mathbb{R} \setminus \{\frac{1}{2}\}$, let $x = \frac{3y-5}{2y-1}$, since $y \neq \frac{1}{2}$, we can see that $x \in \mathbb{R}$. Now we have to show that $x \neq \frac{3}{2}$, suppose $x = \frac{3}{2}$, then $\frac{3y-5}{2y-1} = \frac{3}{2}$, so $6y - 10 = 6y - 3$, then $-10 = -3$. So $x \neq \frac{3}{2}$, namely $x \in dom(f)$ and $f(x) = \frac{\frac{3y-5}{2y-1}-5}{2\frac{3y-5}{2y-1}-3} = y$. So $\mathbb{R} \setminus \{\frac{1}{2}\} \subseteq ran(f). Q.E.D.$

13.11

No.
Because consider $A = \{1, 2, 3\}, B = \{4, 5\}$, we can define $f = \{(1, 4), (2, 4), (3, 5)\}$ as a function from A to B, then the relation $\{(y, x) : (x, y) \in f\} = \{(4, 1), (4, 2), (5, 3)\}$, and obviously the relation contradicts with the second property of a function because $4 \in B, (4, 1), (4, 2) \in \{(y, x) : (x, y) \in f\}$, but $1 \neq 2$. So $\{(y, x) : (x, y) \in f\}$ isn't necessarily a function from B to A.

13.13

The identity function i_X.

14.8

(a)
Not an injection, not a surjection.
$f(-1) = f(1) = \frac{1}{2}$.
$ran(f) = (0, 1]$.
(b)
Not an injection, not a surjection.
$f(0) = f(2\pi) = 0$.
$ran(f) = [-1, 1]$.
(c)
Not an injection, but a surjection.
$f(10, 2) = f(4, 5) = 20$.
(d)
Not an injection, but a surjection.
$f((2, 1), (10, 1)) = f((4, 1), (5, 1)) = 21$.
It's the scalar product of two 2-dimensional vectors.
(e)
Not an injection, not a surjection.
$f((0, 0), (1, 0)) = f((0, 0), (-1, 0)) = 1$.

$ran(f) = [0, +\infty)$.
It's the distance of two points in the 2-dimensional plane.

(f)

An injection and a surjection.

(g)

An injection and a surjection.

(h)

Not a injection and not a surjection.

Consider $B = \{1, 2\}, A_1 = \{1, 3\}, A_2 = \{1, 3, 4\}, f(A_1 \cap B) = f(A_2 \cap B)$, but $A_1 \neq A_2$.
$ran(f) = B$.

(i)

An injection but not a surjection.

$ran(f) = (0, +\infty)$.

14.12

$f(x) = c + \frac{d-c}{b-a}(x - a)$.

Proof.

injective:

We have to show that $\forall x_1, x_2 \in [a, b], y_1 = f(x_1) = c + \frac{d-c}{b-a}(x_1 - a), y_2 = f(x_2) = c + \frac{d-c}{b-a}(x_2 - a), y_1, y_2 \in [c, d]$, if $y_1 = y_2$, then $x_1 = x_2$. Because $y = c + \frac{d-c}{b-a}(x - a)$, so $x = \frac{ad-bc}{d-c} + \frac{b-a}{d-c}y$. Suppose $x_1 \neq x_2$, so $\frac{ad-bc}{d-c} + \frac{b-a}{d-c}y_1 \neq \frac{ad-bc}{d-c} + \frac{b-a}{d-c}y_2$, then $y_1 \neq y_2$, which contradicts with our hypothesis. So we have prove that f is an injection.

surjective:

We only have to show that $[c, d] \subseteq ran(f). \forall y \in [c, d], x = \frac{ad-bc}{d-c} + \frac{b-a}{d-c}y$, so $x \in [a, b]$ and $f(x) = c + \frac{d-c}{b-a}(\frac{ad-bc}{d-c} + \frac{b-a}{d-c}y - a) = y$. So $y \in ran(f)$. Thus we have prove $[c, d] \subseteq ran(f)$, and it is obviously that $ran(f) \subseteq [c, d]$, so $ran(f) = [c, d]$, so f is surjective.

14.13

Yes, it's a function, and it's onto, but it isn't one-to-one.

Proof.

Prove function:

(i) For every real-valued function f defined on $[0, 1]$, since it's real-valued and is defined on $[0, 1]$, so $\exists f(0) \in \mathbb{R}$. Namely $\forall f \in F([0, 1]), \exists \phi(f) \in \mathbb{R}$.
(ii) $\forall f_1, f_2 \in F([0, 1])$, if $f_1 = f_2$, then $f_1(0) = f_2(0)$, namely $\phi(f_1) = \phi(f_2)$.

Prove onto:

We just have to show that $\mathbb{R} \subseteq ran(\phi)$. It is obviously that $ran(\phi)$ is the set of value at 0 of all the real-valued functions defined on $[0, 1]. \forall x \in \mathbb{R}$, then there must exist a function defined on $[0, 1]$ that $f(0) = x$. So $\forall x \in \mathbb{R}, x \in ran(\phi)$. Namely $\mathbb{R} \subseteq ran(\phi)$. Now we proved that ϕ is onto.

Prove not one-to-one:

Counterexample: $f_1 = \sqrt{x} + \sqrt{1-x}$, $f_2 = \sqrt{x} - \sqrt{1-x} + 2$. $f_1, f_2 \in F([0,1])$, obviously $f_1 \neq f_2$, but $\phi(f_1) = \phi(f_2) = 1$.

14.15

Proof.

(i) $\forall x \in \mathbb{R}$, then $\exists f(x) \in \mathbb{R}$, and $f(x) \cdot f(x) \in \mathbb{R}$, namely $\forall x \in \mathbb{R}, \exists (f \cdot f)(x) = f(x) \cdot f(x) \in \mathbb{R}$.

(ii) $\forall x_1, x_2 \in \mathbb{R}$, suppose that $x_1 = x_2$, since f is function, so $f(x_1) = f(x_2)$, so $f(x_1) \cdot f(x_1) = f(x_2) \cdot f(x_2)$, then $(f \cdot f)(x_1) = (f \cdot f)(x_2)$.

(a)

Yes, there exists.

Example: $f(x) = e^x$.

(b)

No.

$ran(f \cdot f) = |ran(f)|$

15.1

(a)

$(f \circ g)(x) = \frac{1}{1+x^2}$ domain: $\mathbb{R}$ range: $(0,1]$

$(g \circ f)(x) = \frac{1}{(1+x)^2}$ domain: $(-\infty, -1) \cup (-1, +\infty)$ range: $(0, +\infty)$

(b)

$(f \circ g)(x) = x$ domain: $[0, +\infty)$ range: $[0, +\infty)$

$(g \circ f)(x) = x$ domain: $\mathbb{R}$ range: $\mathbb{R}$

(c)

$(f \circ g)(x) = \frac{1}{x^2+1}$ domain: $\mathbb{R}$ range: $(0,1]$

$(g \circ f)(x) = \frac{1}{x^2} + 1$ domain: $(-\infty, 0) \cup (0, +\infty)$ range: $(1, +\infty)$

(d)

$(f \circ g)(x) = |x|$ domain: $\mathbb{R}$ range: $[0, +\infty)$

$(g \circ f)(x) = |x|$ domain: $\mathbb{R}$ range: $[0, +\infty)$

15.6

(a)

$(f \circ g)(x) = x$

$(g \circ f)(x) = x$

(b)

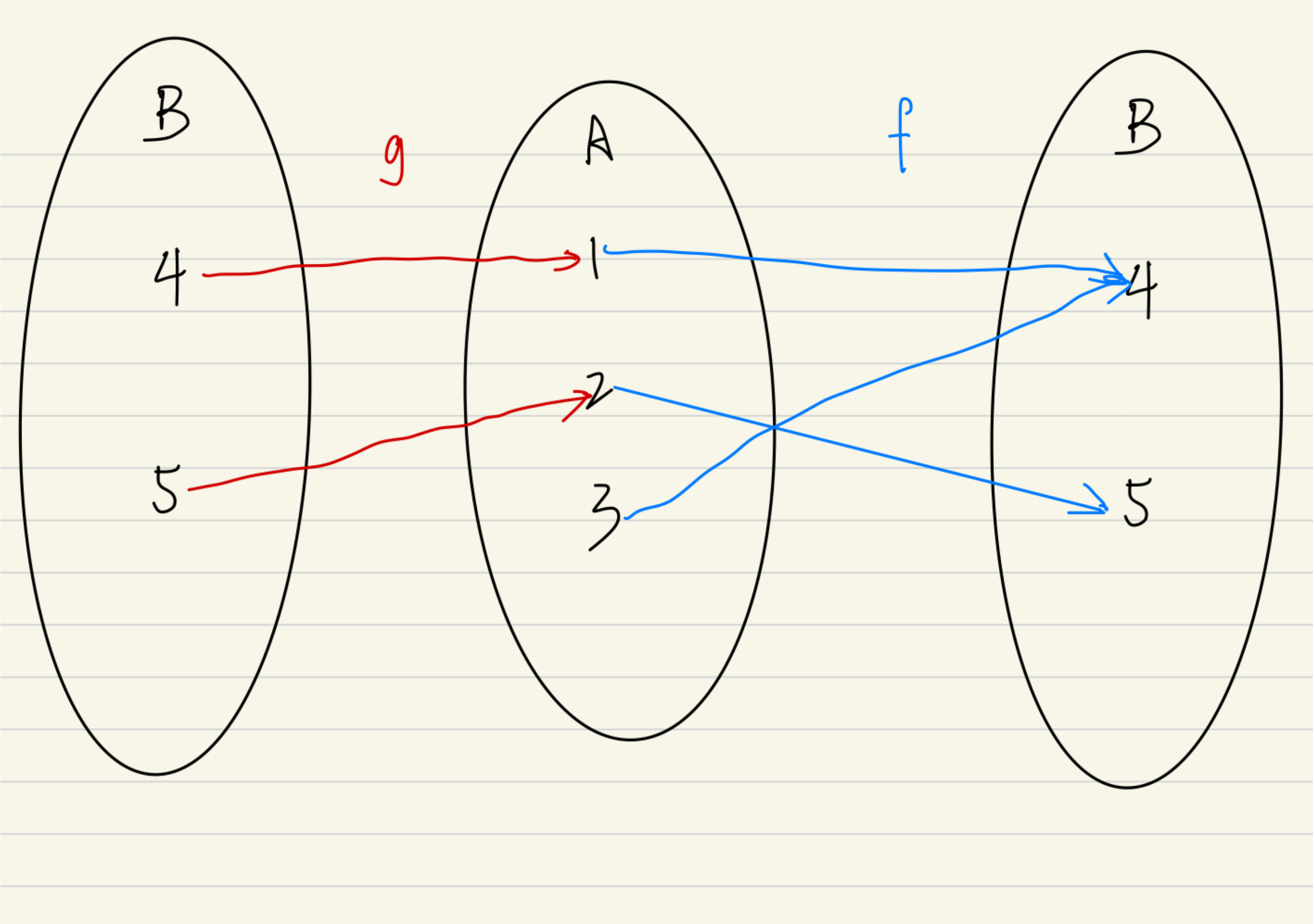
$\forall f : A \to B, g : B \to A$, if $f = g^{-1}, g = f^{-1}$, then $f \circ g = i_B, g \circ f = i_A$.

It derives from ***Theorem 15.8***

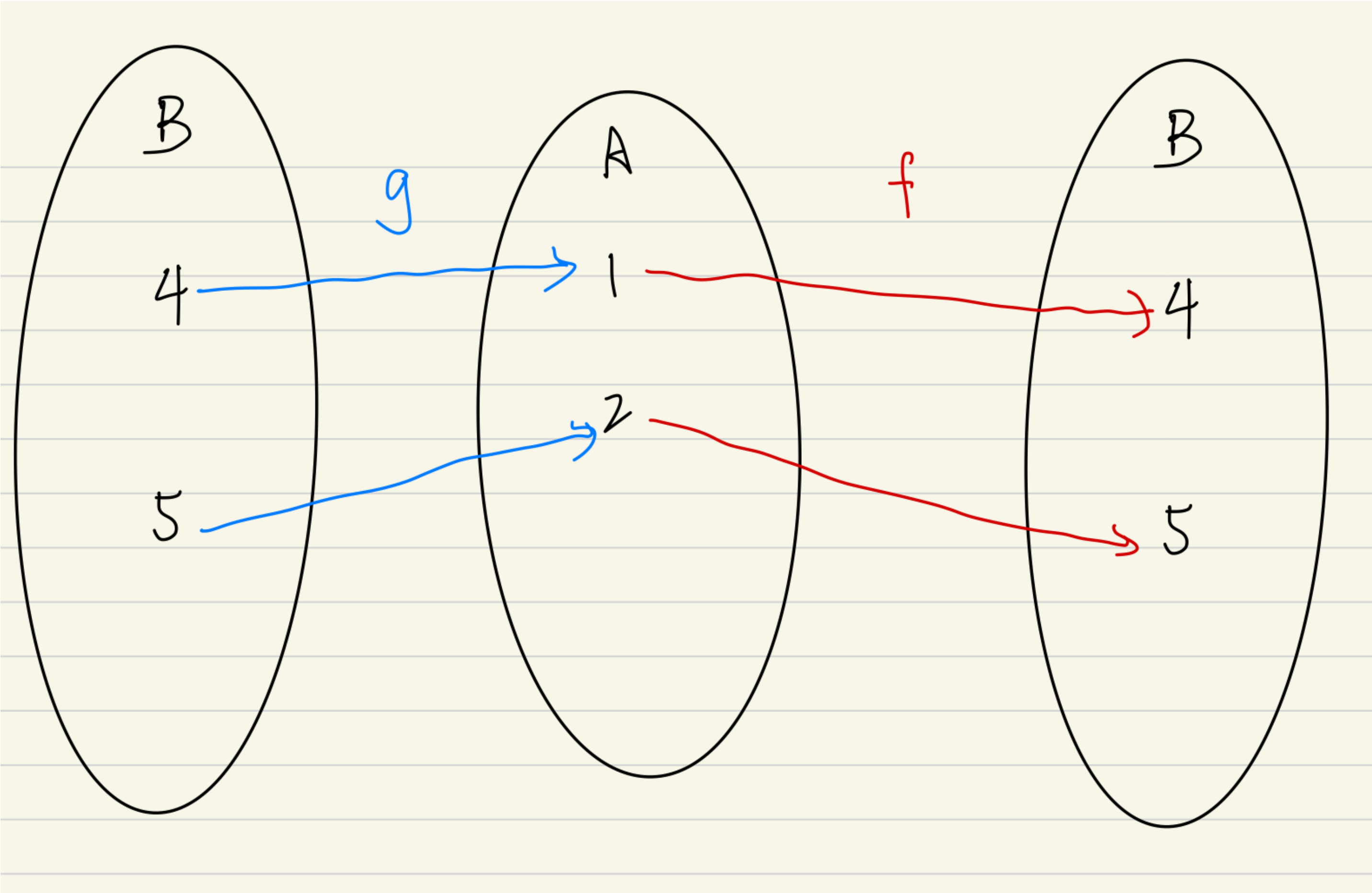
15.7

(a)

(i) $f(x) = -(x-2)^2 + 5$ $g(x) = x - 3$



(ii) $f(x) = x + 3$ $g(x) = x - 3$



(iii) Impossible.

(b)
$A=\{1,2,3\}, B=\{4,5\}, f(x)=-(x-2)^2+5, g(x)=x-3$
Because in this example, f isn't bijective, whereas in ***Theorem15.4 (iv)*** , we declare that f must be a bijection and then the theorem is valid.
(c)
$A=\{4,5\}, B=\{1,2,3\}, f(x)=x-3, g(x)=-(x-2)^2+5$
Because in this example, f isn't bijective, whereas in ***Theorem15.4 (iv)*** , we declare that f must be a bijection and then the theorem is valid.
(d)
f must be onto, but it needn't be one-to-one.
(e)
f must be one-to-one, but it needn't be onto.

15.11

Proof.
Because $f \circ g_1 = f \circ g_2$, so $\forall x \in B, (f \circ g_1)(x) = (f \circ g_2)(x)$. Suppose $\forall x \in B, g_1(x) = \alpha_1, g_2(x) = \alpha_2$. From $\forall x \in B, (f \circ g_1)(x) = (f \circ g_2)(x)$ we can know that $\forall x \in B, f(\alpha_1) = f(\alpha_2)$. Because f is bijective, so $f(\alpha_1) = f(\alpha_2) \leftrightarrow \alpha_1 = \alpha_2$. So $\forall x \in B, \alpha_1 = \alpha_2$, namely

$g_1(x) = g_2(x)$. Since $\forall x \in B, g_1(x) = g_2(x)$, then $g_1 = g_2 . Q.E.D.$
If $g_1 \circ f = g_2 \circ f$ and f is bijective, must $g_1 = g_2$.

15.12

Yes, it is.
If f is one-to-one, the eqiuvalence class of a point $a \in A$ is $E_a = \{a\}$

15.13

No.
$f(x) = x$ is such a function.

15.14

(a)
Proof.
Prove it a function:
$\forall (x, y) \in A \times C$, so $x \in A$ and $y \in C$, since f and g are functions, there exists $f(x) \in B$ and $g(y) \in D$. So there exists $(f(x), g(y)) \in B \times D$. So we have proved that $\forall (x, y) \in A \times C, \exists H(x, y) = (f(x), g(y)) \in B \times D$.
And $\forall (x_1, y_1), (x_2, y_2) \in A \times C$, if $(x_1, y_1) = (x_2, y_2)$, namely $x_1 = x_2$ and $y_1 = y_2$, then $f(x_1) = f(x_2), g(y_1) = g(y_2)$, namely $(f(x_1), g(y_1)) = (f(x_2), g(y_2))$. So we have proved that $\forall (x_1, y_1), (x_2, y_2) \in A \times C$, if $(x_1, y_1) = (x_2, y_2)$, then $H(x_1, y_1) = H(x_2, y_2)$.
In conclusion, H is a function.
Prove it one-to-one:
We have to show that If $H(x_1, y_1) = H(x_2, y_2)$, then $(x_1, y_1) = (x_2, y_2)$, namely $x_1 = x_2, y_1 = y_2$.
If $H(x_1, y_1) = H(x_2, y_2)$, then $(f(x_1), g(y_1)) = (f(x_2), g(y_2))$, namely $f(x_1) = f(x_2), g(y_1) = g(y_2)$, since f and g are one-to-one, so $x_1 = x_2, y_1 = y_2$, namely $(x_1, y_1) = (x_2, y_2)$. Thus, we haave proved that if $H(x_1, y_1) = H(x_2, y_2)$, then $(x_1, y_1) = (x_2, y_2)$, namely H is one-to-one.
(b)
Proof.
We have to show that if $ran(f) = B, ran(g) = D$, then $ran(H) = B \times D$.
To show that, we just have to show $B \times D \subseteq ran(H)$ as $ran(H) \subseteq B \times D$ is obvious.
Consider $\forall (\alpha, \beta) \in B \times D$, namely $\alpha \in B, \beta \in D$. Since $ran(f) = B, ran(g) = D$, so there exist $x \in A$ and $y \in C$ such that $f(x) = \alpha, g(y) = \beta$. Since $(f(x), g(y)) = H(x, y)$, so $(f(x), g(y)) \in ran(H)$, namely $(\alpha, \beta) \in ran(H)$. Thus we have showed that $\forall (\alpha, \beta) \in B \times D, (\alpha, \beta) \in ran(H)$, namely $B \times D \subseteq ran(H)$. So we have proved that if $ran(f) = B, ran(g) = D$, then $ran(H) = B \times D$.

15.15

Not a function:
Consider it when $A \cap C \neq \emptyset$, so $\exists x_0 \in A \cap C$, we have $H(x_0) = f(x_0)$ and $H(x_0) = g(x_0)$, if $f(x_0) \neq g(x_0)$, then obviously H isn't a function.
Is a function:
Consider $f \neq g$. If $A \cap C = \emptyset$, then $\forall x \in A \cup C, x \in A$ or $x \in C$. If $x \in A, H(x) = f(x)$, if $x \in C, H(x) = g(x)$. So $\forall x \in A \cup C, \exists H(x) \in B \cup D$ and if $x_1 = x_2$, then $H(x_1) = H(x_2)$. So H is a function.
If $A \cap C = \emptyset$, then H is a function.

15.20

(a)
Consider $\forall x_1, x_2 \in A_1$, because $A_1 \subset A$, so $x_1 \in A, x_2 \in A$. Because f is one-to-one, so if $f(x_1) = f(x_2)$, then $x_1 = x_2$. So $\forall x_1, x_2 \in A_1$, if $F(x_1) = F(x_2)$, namely $f(x_1) = f(x_2)$, then $x_1 = x_2$. So we have proved $f|_{A1}$ is one-to-one.
(b)
Because $F : A_1 \to B, \forall x \in A_1, F(x) = f(x)$. So obviously $ran(F) \subseteq ran(f)$. Because $f|_{A1}$ is onto, so $ran(F) = B$. So $B \subseteq ran(f)$. Because $ran(f) \subseteq B$, so $ran(f) = B$, namely f is onto.

16.19

(i) Because f is onto, so $\forall X \in \{f^{-1}(\{b\}) : b \in B\}, X \neq \emptyset$.
(ii) $\cup_{b \in B} f^{-1}(\{b\}) = f^{-1}(B) = A$.
(iii) $b_1, b_2 \in B$, suppose $f^{-1}(\{b_1\}) \cap f^{-1}(\{b_2\}) \neq \emptyset$, but $f^{-1}(\{b_1\}) \neq f^{-1}(\{b_2\})$, so $b_1 \neq b_2$. Consider $x_0 \in f^{-1}(\{b_1\}) \cap f^{-1}(\{b_2\})$, so $f(x_0) = b_1$ and $f(x_0) = b_2$, but $b_1 \neq b_2$, thus contradicts with definition of function. So if $f^{-1}(\{b_1\}) \cap f^{-1}(\{b_2\}) \neq \emptyset$, then $f^{-1}(\{b_1\}) = f^{-1}(\{b_2\})$.

16.20

(a)
No, it needn't.
(b)
Proof.
If $f(A_1) = f(A_2)$, suppose $A_1 \neq A_2$, without loss of generality, consider it when $\exists a \in A_1, a \notin A_2$. Because f is bijective, so f is one-to-one, then if $f(x) = f(a), x = a$. Because $a \in A_1, a \notin A_2$, so $f(a) \in f(A_1), f(a) \notin f(A_2)$, so $f(A_1) \neq f(A_2)$. Contradict. So if $f(A_1) = f(A_2)$, then $A_1 = A_2$. I just used one-to-one.

16.21

(a)

No, it needn't.

(b)

Proof.

If $f^{-1}(B_1)=f^{-1}(B_2)$, suppose $B_1\neq B_2$. Without loss of generality, suppose that $\exists b\in B_1, b\notin B_2$. Because f is bijective, so f is onto, then $\forall y\in Y\leftrightarrow\exists x\in X$, such that $y=f(x)$. Consider b, because $b\in B_1, b\notin B_2$, so there exists $a_1\in X$ such that $b=f(a_1)$ and $f(a_1)\notin B_2$. So $a_1\in f^{-1}(B_1)$ and $a_1\notin f^{-1}(B_2)$, so $f^{-1}(B_1)\neq f^{-1}(B_2)$. Contradict. So if $f^{-1}(B_1)=f^{-1}(B_2)$, then $B_1=B_2$. I just used onto.

16.22

(a)

Yes, it must.

(b)

Proof.

$\forall x\in X$, if $x\notin A_1\cap A_2$, then $\chi_{A_1\cap A_2}(x)=0$. Because $x\notin A_1\cap A_2$, so χ_{A_1} and χ_{A_2} can't be 1 at the same time, namely $\chi_{A_1}\cdot\chi_{A_2}=0$.

$\forall x\in X$, if $x\in A_1\cap A_2$, then $\chi_{A_1\cap A_2}(x)=1$. Because $x\in A_1\cap A_2$, namely $x\in A_1$ and $x\in A_2$, so $\chi_{A_1}=1, \chi_{A_2}=1, \chi_{A_1}\cdot\chi_{A_2}=1$.

So we have showed that in any cases, $\chi_{A_1\cap A_2}=\chi_{A_1}\cdot\chi_{A_2}$.

(c)

For all $x\in X$, consider it in the following four cases:

(1) $x\in A_1, x\in A_2$**:**

In this case, $\chi_{A_1}=1, \chi_{A_2}=1, \chi_{A_1\cap A_2}=1, \chi_{A_1\cup A_2}=1$. So $\chi_{A_1}+\chi_{A_2}-\chi_{A_1\cap A_2}=\chi_{A_1\cup A_2}=1$.

(2) $x\in A_1, x\notin A_2$**:**

In this case, $\chi_{A_1}=1, \chi_{A_2}=0, \chi_{A_1\cap A_2}=0, \chi_{A_1\cup A_2}=1$. So $\chi_{A_1}+\chi_{A_2}-\chi_{A_1\cap A_2}=\chi_{A_1\cup A_2}=1$.

(3) $x\notin A_1, x\in A_2.$**:**

In this case, $\chi_{A_1}=0, \chi_{A_2}=1, \chi_{A_1\cap A_2}=0, \chi_{A_1\cup A_2}=1$. So $\chi_{A_1}+\chi_{A_2}-\chi_{A_1\cap A_2}=\chi_{A_1\cup A_2}=1$.

(4) $x\notin A_1, x\notin A_2$**:**

In this case, $\chi_{A_1}=0, \chi_{A_2}=0, \chi_{A_1\cap A_2}=0, \chi_{A_1\cup A_2}=0$. So $\chi_{A_1}+\chi_{A_2}-\chi_{A_1\cap A_2}=\chi_{A_1\cup A_2}=0$.

In conclusion, we have showed that in all possible cases, $\chi_{A_1}+\chi_{A_2}-\chi_{A_1\cap A_2}=\chi_{A_1\cup A_2}$, thus it's proved.

(d)

$\chi_{X\setminus A_1}\cdot\chi_{X\setminus A_2}=\chi_{X\setminus A_1\cup A_2}$