Theory and Design of PID Controller for Nonlinear Uncertain Systems

Jinke Zhang and Lei Guo

***Abstract*—It is well-known that the classical proportional-integral-derivative (PID) controller plays a fundamental role in various engineering systems. However, up to now a theory that can explain the rationale why the linear PID can effectively deal with nonlinear uncertain dynamical systems and a method that can provide explicit design formula for the PID parameters are still lacking. This motivates our recent study on the theoretical foundation of the PID control. The main purpose of this letter is to extend the 1-D results to higher dimensional nonlinear uncertain systems and to improve the results significantly by a refined method. We will also consider a class of multi-agent uncertain nonlinear systems where each agent is controlled by a PID controller using its own regulation error. We will show that a parameter manifold can be constructed explicitly so that when the PID parameters are chosen from this manifold, the multi-agent systems will be globally stable and the tracking error of each agent will coverage to zero exponentially fast.**

***Index Terms*—PID control, uncertain system, output regulation, stability of nonlinear systems, agents-based systems.**

I. Introduction

It is well-known that the classical proportional-integral-derivative (PID) controller still plays a dominating role in engineering control systems by far. For example, it was reported that over 95% of process control loops are designed based on PID controller, see, e.g., [3], [4]. A more recent survey [5] published in 2017 shows that the impact rating of PID control is still much higher than the widely studied advanced control techniques, and that "we still have nothing that compares with PID" [5].

One may naturally wonder why the classical PID controller can achieve such an amazing success in practice. Some well-known partial answers include the following: 1) It does not rely on the precise mathematic models, and has an easy-to-use simple structure; 2) It not only can eliminate the steady state deviation via the integral action, but also can predict the future behavior through the derivative action; 3) The Newton's law which is widely used in modeling physical systems is of second order.

Of course, whether the PID controller can make the system to perform well depends on the choice of the three PID parameters, which is regarded as a complicated task. In fact, there are still lack of a theoretical foundation for the PID control of uncertain nonlinear systems, and most of the methods widely used in industrial processes are based on experience or experiments or both, including the Ziegler-Nichols rule and its variations, see [6]. There are also many other tuning or adaptation methods for the PID parameters design. However, except for some related studies for nonlinear systems (e.g., [7], [8]), most of the literature on PID control is focused on linear systems (e.g., [3], [4], [9] and [10]). Moreover, as pointed out in [11], "the PID controller is still poorly understood and, in particular, the PID control loops are poorly tuned in many applications".

In order to understand why the classical PID controller is so effective in dealing with real world systems, we need to take the non-linearity and uncertainty of the systems into account in our theoretical study. Moreover, a comprehensive understanding of the PID control may considerably improve its widespread practice, and so contribute to better product quality [3]. These are the primary motivations for the theoretical study of the PID controller.

To the best of authors' knowledge, almost all the existing theoretical results on the classical PID control are of local nature, it was not until recently that global results on stability and regulation theory were established for the closed-loop system of a class of second order nonlinear uncertain systems under the classical PID controller, see [1], [2] and [12]. For instance, it has been shown in [1] that a three-dimensional manifold has been constructed within which the PID parameters can be chosen arbitrarily to globally stabilize a class of one-dimensional second order nonlinear uncertain dynamical systems by using the information on the upper bounds of partial derivatives of the nonlinear dynamics.

In this letter, we will extend the one-dimensional result of [1] to higher dimensional uncertain systems and will improve the results of [2] significantly by a refined method. To be specific, we will construct a three-dimensional manifold for the high dimensional nonlinear uncertain systems by using

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The authors are with the Key Laboratory of Systems and Control, AMSS, Chinese Academy of Sciences, Beijing 100190, China, and also with the School of Mathematical Science, University of Chinese Academy of Sciences, Beijing 100049, China (e-mail: zhangjinke@amss.ac.cn; lguo@amss.ac.cn).

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the upper bounds of the partial derivatives of the uncertain functions, and will show that the three PID parameters can be chosen arbitrarily from this manifold to globally stabilize the system with exponential convergence rate of the regulation error. Moreover, we will also construct a manifold of the same form for a class of multi-agent uncertain nonlinear systems where each agent is controlled by a PID controller using its own regulation error. We will show that the closed-loop multi-agent system will be globally stabilized as long as the PID parameters are chosen from this manifold.

The rest of this letter is organized as follows. The problem formulation will be described in Section II. The main results will be presented in Section III, with their mathematical proofs given in Section IV. A simple simulation is given in Section V. And the auxiliary results will be placed in the Appendix.

II. Problem Formulation

Let us consider a moving body in $\mathbb{R}^n$, and denote $\boldsymbol{p}(t), \boldsymbol{v}(t), \boldsymbol{a}(t)$, as its position, velocity, acceleration at the time instant t, respectively. Assume that the external forces acting on the body consist of $\boldsymbol{f}$ and $\boldsymbol{u}$, where $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{p}, \boldsymbol{v})$ is a nonlinear function and $\boldsymbol{u}$ is the control force.

By Newton's second law, we can get the following equation

$$m\boldsymbol{a} = \boldsymbol{f}(\boldsymbol{p}, \boldsymbol{v}) + \boldsymbol{u}, \tag{1}$$

where $\boldsymbol{u}$ is the control input and m is the mass. Our control target is to design an output feedback controller to guarantee that the position will converge to a desired reference value $\boldsymbol{y}^* \in \mathbb{R}^n$ for any initial position and initial velocity.

In this letter, we use the classical PID controller as the control force

$$\boldsymbol{u}(t) = k_p\boldsymbol{e}(t) + k_i \int_0^t \boldsymbol{e}(s)\, ds + k_d\dot{\boldsymbol{e}}(t), \tag{2}$$

where $\boldsymbol{e}$ is the control error, defined by

$$\boldsymbol{e}(t) = \boldsymbol{y}^* - \boldsymbol{p}(t),$$

and where k_p, k_i, k_d are the controller parameters to be designed.

Without loss of generality, we assume that the body has the unit mass $m = 1$. Notice that $\boldsymbol{v} = \dot{\boldsymbol{p}}, \boldsymbol{a} = \ddot{\boldsymbol{p}}$, then (1) can be rewritten as

$$\ddot{\boldsymbol{p}} = \boldsymbol{f}(\boldsymbol{p}, \dot{\boldsymbol{p}}, t) + \boldsymbol{u}.$$

Denote $\boldsymbol{x}_1 = \boldsymbol{p}$ and $\boldsymbol{x}_2 = \dot{\boldsymbol{p}}$, then the state space equation of this basic mechanic system under PID control is

$$\begin{cases} \dot{\boldsymbol{x}}_1 = \boldsymbol{x}_2, \\ \dot{\boldsymbol{x}}_2 = \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2, t) + \boldsymbol{u}(t), \\ \boldsymbol{u}(t) = k_p\boldsymbol{e}(t) + k_i \int_0^t \boldsymbol{e}(s)\, ds + k_d\dot{\boldsymbol{e}}(t), \end{cases} \tag{3}$$

where $\boldsymbol{x}_1(0), \boldsymbol{x}_2(0) \in \mathbb{R}^n$ and $\boldsymbol{e}(t) = \boldsymbol{y}^* - \boldsymbol{x}_1(t)$.

Next, we define some notations that will be used throughout this letter:

$\|\boldsymbol{x}\|$ is the Eucidean norm of a vector $\boldsymbol{x}$.

Let P be an $m \times n$ matrix, define $\|P\| = \sup_{\boldsymbol{x}\in\mathbb{R}^n, \|\boldsymbol{x}\|=1} \|P\boldsymbol{x}\|$, which denotes the operator norm of the matrix P induced by the Euclidean norm.

Let $\boldsymbol{\Phi} : \mathbb{R}^n \to \mathbb{R}^m$, define $\frac{\partial \boldsymbol{\Phi}(\boldsymbol{x})}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial \Phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_m}{\partial x_1} & \cdots & \frac{\partial \phi_m}{\partial x_n} \end{bmatrix}$.

Let $\mathbb{R}^{3+} = (0, \infty) \times (0, \infty) \times (0, \infty)$.

In this letter, we will show that the three controller parameters k_p, k_i, k_d can be designed explicitly such that the position of the body tracks a given constant setpoint $\boldsymbol{y}^*$ with exponential convergence rate of the regulaltion error under the control law (2) for any initial position and velocity, as long as $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2, t)$ is a continuously differentiable function with known upper bounds for the partial derivatives with respect to the variables $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.

III. Main Results

To present the main theorem in this letter, we first introduce some basic definitions.

a). A function space:

$$\mathcal{F}_{L_1,L_2} = \Bigg\{ \boldsymbol{f} \in C^1(\mathbb{R}^{2n} \times \mathbb{R}^+ \to \mathbb{R}^n) \,\Big|\, \left\| \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}_1} \right\| \le L_1, \\ \left\| \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}_2} \right\| \le L_2,\ \forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^n,\ \forall t \in \mathbb{R}^+, \Bigg\}$$

where L_1 and L_2 are positive constants, and $C^1(\mathbb{R}^{2n} \times \mathbb{R}^+ \to \mathbb{R}^n)$ denotes the space of all functions from $\mathbb{R}^{2n} \times \mathbb{R}^+$ to $\mathbb{R}$ which are piecewise continuous in t, with continuous partial derivatives with respect to $(\boldsymbol{x}_1, \boldsymbol{x}_2)$.

We remark that the above definition for the class of uncertain functions is motivated by the investigation of maximum capability of the feedback mechanism in [13], where the Lipschitz constant turns out to be a suitable measure of the size of uncertain systems.

b). A three-dimensional parameter manifold:

$$\Omega_{pid} = \Bigg\{ (\bar{k}_p, k_i, \bar{k}_d) \in \mathbb{R}^{3+} \,|\, \bar{k}_p\bar{k}_d - k_i \\ > \sqrt{(\bar{k}_pL_1 + k_iL_2)(L_1 + \bar{k}_dL_2)} \Bigg\}$$

where $\bar{k}_p = k_p - L_1, \bar{k}_d = k_d - L_2$.

We now introduce a basic assumption:

$$\boldsymbol{f}(\boldsymbol{y}, 0, t) = \boldsymbol{f}(\boldsymbol{y}, 0, 0),\ \forall t \in \mathbb{R}^+,\ \forall \boldsymbol{y} \in \mathbb{R}^n,$$

which is automatically satisfied when $f(\cdot, \cdot, \cdot)$ is time-invariant.

Below is the main result of this letter.

Theorem 1: Consider the PID controlled system (3) with uncertain function $\boldsymbol{f} \in \mathcal{F}_{L_1,L_2}$. Then for any $L_1, L_2 > 0$, whenever the controller parameters (k_p, k_i, k_d) are taken from Ω_{pid}, the closed-loop system (3) will satisfy

$$\lim_{t\to\infty} \boldsymbol{x}_1(t) = \boldsymbol{y}^*,\ \lim_{t\to\infty} \boldsymbol{x}_2(t) = 0,$$

exponentially fast, for any initial value $\boldsymbol{x}_1(0), \boldsymbol{x}_2(0) \in \mathbb{R}^n$ and any vector-valued setpoint $\boldsymbol{y}^* \in \mathbb{R}^n$.

Remark 1: It is quite obvious that the 3-dimensional manifold Ω_{pid} is an open and unbounded set, which means that we may have much flexibility in the choice of the PID parameters. Of course, if we have more requirements on the control performance, the PID parameters may be further optimized within this set, which belongs to further investigation.

Next, we consider a special case where $n = 1$. We will demonstrate that the result in [1] can be refined by utilizing the same approach used in Theorem 1. This is the content of the following theorem.

Theorem 2: For any $L_1, L_2 > 0$, there exists a three dimensional manifold, defined by

$$\Omega_{pid} = \left\{ (\bar{k}_p, k_i, \bar{k}_d) \in \mathbb{R}^{3+} \,\middle|\, \bar{k}_p \bar{k}_d - k_i > L_2 \sqrt{k_i \bar{k}_d} \right\}$$

such that whenever the controller parameters (k_p, k_i, k_d) are taken from Ω_{pid}, the closed-loop system (3) will satisfy

$$\lim_{t\to\infty} x_1(t) = y^*, \quad \lim_{t\to\infty} x_2(t) = 0$$

exponentially fast, for any $f \in \mathcal{F}_{L_1,L_2}$, any initial value $x_1(0), x_2(0) \in \mathbb{R}$ and any constant setpoint $y^* \in \mathbb{R}$.

Furthermore, we point out that the PID parameters can be extended to matrices. A typical example is a kind of the multi-agent systems [14], in which each agent can be described as:

$$\begin{cases} \dot{x}_{1j} = x_{2j}, \\ \dot{x}_{2j} = f_j(\boldsymbol{x_1}, \boldsymbol{x_2}, t) + u_j(t), \\ u_j(t) = k_p^j e_j(t) + k_i^j \int_0^t e_j(s)\,ds + k_d^j \dot{e}_j(t). \end{cases} \tag{4}$$

To deal with this situation, we only need to take the PID parameters as positive definite diagonal matrices, expressed as

$$\boldsymbol{k_p} = \begin{pmatrix} k_p^1 & & & \\ & k_p^2 & & \\ & & \ddots & \\ & & & k_p^n \end{pmatrix}, \ \boldsymbol{k_i} = \begin{pmatrix} k_i^1 & & & \\ & k_i^2 & & \\ & & \ddots & \\ & & & k_i^n \end{pmatrix},$$

$$\boldsymbol{k_d} = \begin{pmatrix} k_d^1 & & & \\ & k_d^2 & & \\ & & \ddots & \\ & & & k_d^n \end{pmatrix}, \ k_p^j, k_i^j, k_d^j > 0, j = 1, 2, \cdots, n.$$

We will show that the minimal eigenvalues of $\boldsymbol{k_p}$, $\boldsymbol{k_d}$ and the maximum eigenvalue of $\boldsymbol{k_i}$ determine the range of PID parameters selection. The conclusion is as follows.

Theorem 3: For any $L_1, L_2 > 0$, there exists a three dimensional manifold $\Omega_{pid} \subset \mathbb{R}^3$, defined by

$$\Omega_{pid} = \Big\{ (\bar{\boldsymbol{k}}_{\boldsymbol{p}}, \|\boldsymbol{k_i}\|, \bar{\boldsymbol{k}}_{\boldsymbol{d}}) \in \mathbb{R}^{3+} \mid \bar{\boldsymbol{k}}_{\boldsymbol{p}} \bar{\boldsymbol{k}}_{\boldsymbol{d}} - \|\boldsymbol{k_i}\| > \sqrt{(\bar{\boldsymbol{k}}_{\boldsymbol{p}} L_1 + \|\boldsymbol{k_i}\| L_2)(L_1 + \bar{\boldsymbol{k}}_{\boldsymbol{d}} L_2)} \Big\}$$

where $\bar{\boldsymbol{k}}_{\boldsymbol{p}} = \min_j k_p^j - L_1$, $\bar{\boldsymbol{k}}_{\boldsymbol{d}} = \min_j k_d^j - L_2$, such that whenever the controller parameters (k_p, k_i, k_d) are taken from Ω_{pid}, the closed-loop system (3) will satisfy

$$\lim_{t\to\infty} \boldsymbol{x_1}(t) = \boldsymbol{y^*}, \quad \lim_{t\to\infty} \boldsymbol{x_2}(t) = 0,$$

exponentially fast, for any $\boldsymbol{f} \in \mathcal{F}_{L_1,L_2}$, any initial value $\boldsymbol{x_1}(0), \boldsymbol{x_2}(0) \in \mathbb{R}^n$ and any vector-valued setpoint $\boldsymbol{y}^* \in \mathbb{R}^n$.

Remark 2: Theorem 3 implies that uncoupled PID controllers can deal with coupled uncertain nonlinear systems. In some practical situations, the PID controllers are also designed in a similar spirit, see, e.g., [15].

IV. Proofs of the Main Results

A. Proof of Theorem 1

Proof: First, we introduce some notations.

Denote $\boldsymbol{x}(t) = \int_0^t \boldsymbol{e}(s)\,ds + \frac{f(y^*,0,0)}{k_i}$, $\boldsymbol{y}(t) = \boldsymbol{e}(t), \boldsymbol{z}(t) = \dot{\boldsymbol{e}}(t), \boldsymbol{g}(\boldsymbol{y}, \boldsymbol{z}, t) = -\boldsymbol{f}(\boldsymbol{y}^* - \boldsymbol{y}, -\boldsymbol{z}, t) + \boldsymbol{f}(\boldsymbol{y}^*, 0, t)$, then (3) goes over into

$$\begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{y}, \\ \dot{\boldsymbol{y}} = \boldsymbol{z}, \\ \dot{\boldsymbol{z}} = \boldsymbol{g}(\boldsymbol{y}, \boldsymbol{z}, t) - k_i \boldsymbol{x} - k_p \boldsymbol{y} - k_d \boldsymbol{z}. \end{cases} \tag{5}$$

By $f \in \mathcal{F}_{L_1,L_2}$, it is easy to see that $\boldsymbol{g} \in \mathcal{F}_{L_1,L_2}$ and $\boldsymbol{g}(0, 0, t) = 0$, $\forall t > 0$. Hence (0,0,0) is an equilibrium of (5).

Note that by using the mean value theorem of integral type (see, e.g., [16]), $\boldsymbol{g}(\boldsymbol{y}, \boldsymbol{z}, t)$ can be decomposed as:

$$\begin{aligned} \boldsymbol{g}(\boldsymbol{y}, \boldsymbol{z}, t) &= [\boldsymbol{g}(\boldsymbol{y}, 0, t) - \boldsymbol{g}(0, 0, t)] + [\boldsymbol{g}(\boldsymbol{y}, \boldsymbol{z}, t) - \boldsymbol{g}(\boldsymbol{y}, 0, t)] \\ &= \left\{ \int_0^1 \frac{\partial \boldsymbol{g}(\bar{\boldsymbol{y}}, 0, t)}{\partial \bar{\boldsymbol{y}}}\, d\lambda \right\} \boldsymbol{y} + \left\{ \int_0^1 \frac{\partial \boldsymbol{g}(\boldsymbol{y}, \bar{\boldsymbol{z}}, t)}{\partial \bar{\boldsymbol{z}}}\, d\lambda \right\} \boldsymbol{z} \\ &\triangleq b(\boldsymbol{y}, t)\boldsymbol{y} + a(\boldsymbol{y}, \boldsymbol{z}, t)\boldsymbol{z}, \end{aligned}$$

where $\bar{\boldsymbol{y}} = \lambda \boldsymbol{y}$ and $\bar{\boldsymbol{z}} = \lambda \boldsymbol{z}$.

Since $\boldsymbol{g} \in \mathcal{F}_{L_1,L_2}$., we can deduce the upper bounds of $\|b(\boldsymbol{y}, t)\|$ and $\|a(\boldsymbol{y}, \boldsymbol{z}, t)\|$ by using Schwarz inequality:

$$\begin{aligned} \|b(\boldsymbol{y}, t)\| &= \left\| \int_0^1 \frac{\partial \boldsymbol{g}(\bar{\boldsymbol{y}}, 0, t)}{\partial \bar{\boldsymbol{y}}}\, d\lambda \right\| \\ &\le \sqrt{\int_0^1 \left\| \frac{\partial \boldsymbol{g}(\bar{\boldsymbol{y}}, 0, t)}{\partial \bar{\boldsymbol{y}}} \right\|^2 d\lambda} \le L_1. \end{aligned}$$

Similarly, we can deduce $\|a(\boldsymbol{y}, \boldsymbol{z}, t)\| \le L_2$.

Hence, the closed-loop equation (5) can be rewritten as

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{y}} \\ \dot{\boldsymbol{z}} \end{bmatrix} = A(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, t) \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix}, \tag{6}$$

where

$$A(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, t) = \begin{bmatrix} 0 & I_n & 0 \\ 0 & 0 & I_n \\ -k_i I_n & -k_p I_n + b(\boldsymbol{y}, t) & -k_d I_n + a(\boldsymbol{y}, \boldsymbol{z}, t) \end{bmatrix}.$$

Similar to [17], we now proceed to show that the following quadratic form is indeed a Lyapunov function,

$$V(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) = [\boldsymbol{x}^\tau, \boldsymbol{y}^\tau, \boldsymbol{z}^\tau] P [\boldsymbol{x}^\tau, \boldsymbol{y}^\tau, \boldsymbol{z}^\tau]^\tau,$$

where the constant matrix P is

$$P = \frac{1}{2} \begin{bmatrix} \mu k_i I_n & k_i I_n & \varepsilon I_n \\ k_i I_n & (k_p + \mu k_d) I_n & \mu I_n \\ \varepsilon I_n & \mu I_n & I_n \end{bmatrix}, \tag{7}$$

μ is a constant defined by

$$\mu = \frac{2(\bar{k}_d \bar{k}_p + k_i) - L_1 L_2}{4\bar{k}_p + L_2^2},$$

and ε is a positive constant, which is small enough.

By Lemma 1 in the Appendix, we know that P is a positive definite matrix, hence $V(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ is a positive definite function which is radically unbounded in $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$.

Next, by simple calculations based on the definitions of P and $A(\boldsymbol{x},\boldsymbol{y},z,t)$ (denoted by $A(\cdot)$), it follows that the time derivative of $V(\boldsymbol{x},\boldsymbol{y},z)$ along the trajectories of (6), is given by

$$
\begin{aligned}
&\dot{V}(\boldsymbol{x},\boldsymbol{y},z)\\
&\quad= [\boldsymbol{x}^{\boldsymbol{\tau}},\boldsymbol{y}^{\boldsymbol{\tau}},\boldsymbol{z}^{\boldsymbol{\tau}}](PA(\cdot)+A(\cdot)^{\boldsymbol{\tau}}P)[\boldsymbol{x}^{\boldsymbol{\tau}},\boldsymbol{y}^{\boldsymbol{\tau}},\boldsymbol{z}^{\boldsymbol{\tau}}]^{\boldsymbol{\tau}}\\
&\quad= -k_i\varepsilon\boldsymbol{x}^{\boldsymbol{\tau}}\boldsymbol{x}+\boldsymbol{x}^{\tau}(-k_pI_n+b)\varepsilon\boldsymbol{y}+\boldsymbol{x}^{\tau}(-k_dI_n+a)\varepsilon\boldsymbol{z}\\
&\qquad-\boldsymbol{y}^{\boldsymbol{\tau}}\big[\mu(k_pI_n-\tfrac{b+b^{\boldsymbol{\tau}}}{2})-k_iI_n\big]\boldsymbol{y}+\boldsymbol{y}^{\boldsymbol{\tau}}(\mu a+b^{\boldsymbol{\tau}}+\varepsilon I_n)\boldsymbol{z}\\
&\qquad-\boldsymbol{z}^{\boldsymbol{\tau}}\big(-\mu I_n+k_dI_n-\tfrac{a+a^{\boldsymbol{\tau}}}{2}\big)\boldsymbol{z}\\
&\quad\le -k_i\varepsilon\|\boldsymbol{x}\|^2+\varepsilon\|\boldsymbol{x}^{\boldsymbol{\tau}}(b-k_pI_n)\boldsymbol{y}\|+\varepsilon\|\boldsymbol{x}^{\boldsymbol{\tau}}(a-k_dI_n)\boldsymbol{z}\|\\
&\qquad-(-k_i+\mu k_p)\|\boldsymbol{y}\|^2+\mu\|\boldsymbol{y}^{\boldsymbol{\tau}}\tfrac{b+b^{\boldsymbol{\tau}}}{2}\boldsymbol{y}\|\\
&\qquad+\|\boldsymbol{y}^{\boldsymbol{\tau}}(\mu a+b^{\boldsymbol{\tau}}+\varepsilon I_n)\boldsymbol{z}\|-(k_d-\mu)\|\boldsymbol{z}\|^2+\|\boldsymbol{z}^{\boldsymbol{\tau}}\tfrac{a+a^{\boldsymbol{\tau}}}{2}\boldsymbol{z}\|\\
&\quad\le -k_i\varepsilon\|\boldsymbol{x}\|^2+\varepsilon(k_p+L_1)\|\boldsymbol{x}\|\,\|\boldsymbol{y}\|+\varepsilon(k_d+L_2)\|\boldsymbol{x}\|\,\|\boldsymbol{z}\|\\
&\qquad-[-k_i+\mu(k_p-L_1)]\|\boldsymbol{y}\|^2+(\mu L_2+L_1+\varepsilon)\|\boldsymbol{y}\|\,\|\boldsymbol{z}\|\\
&\qquad-(-\mu+k_d-L_2)\|\boldsymbol{z}\|^2\\
&\quad= -[\|\boldsymbol{x}\|,\|\boldsymbol{y}\|,\|\boldsymbol{z}\|]Q[\|\boldsymbol{x}\|,\|\boldsymbol{y}\|,\|\boldsymbol{z}\|]^{\boldsymbol{\tau}},
\end{aligned}
$$

where Q is a symmetric matrix, expressed by

$$
Q=\begin{bmatrix} k_i\varepsilon & \frac{-(k_p+L_1)\varepsilon}{2} & \frac{-(k_d+L_2)\varepsilon}{2}\\ \frac{-(k_p+L_1)\varepsilon}{2} & -k_i+\mu\bar{k}_p & -\frac{\mu L_2+L_1+\varepsilon}{2}\\ \frac{-(k_d+L_2)\varepsilon}{2} & -\frac{\mu L_2+L_1+\varepsilon}{2} & -\mu+\bar{k}_d \end{bmatrix}. \tag{8}
$$

Let $B=\begin{bmatrix}1&0&0\\ \frac{k_p+L_1}{2k_i}&1&0\\ \frac{k_d+L_2}{2k_i}&0&1\end{bmatrix}$, then we can get

$$
BQB^{\boldsymbol{\tau}}=\begin{bmatrix}k_i\varepsilon & \\ & Q'-\varepsilon C\end{bmatrix},
$$

where $Q'=\begin{bmatrix}-k_i+\mu\bar{k}_p & -\frac{\mu L_2+L_1}{2}\\ -\frac{\mu L_2+L_1}{2} & -\mu+\bar{k}_d\end{bmatrix}$ and

$$
C=\begin{bmatrix}\frac{(k_p+L_1)^2}{4k_i} & \frac{(k_p+L_1)(k_d+L_2)+2k_i}{4k_i}\\ \frac{(k_p+L_1)(k_d+L_2)+2k_i}{4k_i} & \frac{(k_d+L_2)^2}{4k_i}\end{bmatrix}.
$$

To prove Q is positive definite, we only need to prove $Q'-\varepsilon C$ is positive definite. By using (20) and (21), we can get

$$
\begin{aligned}
&\lim_{\varepsilon\to0^+}-k_i+\mu\bar{k}_p-\frac{(k_p+L_1)^2}{2k_i}\varepsilon>0,\\
&\lim_{\varepsilon\to0^+}det(Q'-\varepsilon C)>0,
\end{aligned}
$$

which means $Q'-\varepsilon C$ is positive definite when ε is small enough. Then, we can deduce that Q is positive definite, which means the system converges exponentially fast. Hence, the proof of Theorem 1 is complete. ■

B. Proof of Theorem 2

Proof: First, we note that, from the basic assumption on $\boldsymbol{f}$, we can see that $b(\boldsymbol{y},t)$ is merely depends on $\boldsymbol{y}$, denoted henceforth by $b(\boldsymbol{y})$. We can deduce the conclusion by slightly modifying the Lyapunov function used in [1] as

$$
V(x,y,z)=[x,y,z]P_1[x,y,z]^{\tau}+\int_0^y(L_1-b(s))sds,
$$

where the constant matrix P_1 is

$$
P_1=\frac{1}{2}\begin{bmatrix}\mu_1k_i & k_i & \varepsilon_1\\ k_i & (\bar{k}_p+\mu_1k_d) & \mu_1\\ \varepsilon_1 & \mu_1 & 1\end{bmatrix}, \tag{9}
$$

μ_1 is a constant defined by $\mu_1=2(\bar{k}_d\bar{k}_p+k_i)/(4\bar{k}_p+L_2^2)$, and ε_1 is a positive constant, which is small enough.

Similar to the Lemma 1, we can prove that P_1 is positive definite, hence $V(x,y,z)$ is a positive definite function which is radically unbounded in x, y, z. Next, by simple calculations, we can get

$$
\begin{aligned}
&\dot{V}(x,y,z)\\
&\quad=[x,y,z](P_1A+A^{\tau}P_1)[x,y,z]^{\tau}+(L_1-b(y))yz\\
&\quad=-\varepsilon_1k_ix^2+\varepsilon_1(-k_p+b)xy+\varepsilon_1(-k_d+a)xz\\
&\qquad+[k_i+\mu_1(b-k_p)]y^2+(\mu_1a+\varepsilon_1)yz+(\mu_1-k_d+a)z^2\\
&\quad\le-\varepsilon_1k_i|x|^2+\varepsilon_1(k_p+L_1)|x||y|+\varepsilon_1(k_d+L_2)|x||z|\\
&\qquad+[k_i+\mu_1(L_1-k_p)]|y|^2+(\mu_1L_2+\varepsilon_1)|y||z|\\
&\qquad+(\mu_1-k_d+L_2)|z|^2\\
&\quad=-[|x|,|y|,|z|]Q_1[|x|,|y|,|z|]^{\tau},
\end{aligned}
$$

where Q_1 is a symmetric matrix, expressed as

$$
Q_1=\begin{bmatrix} k_i\varepsilon_1 & \frac{-(k_p+L_1)\varepsilon_1}{2} & \frac{-(k_d+L_2)\varepsilon_1}{2}\\ \frac{-(k_p+L_1)\varepsilon_1}{2} & -k_i+\mu_1\bar{k}_p & -\frac{\mu_1L_2+\varepsilon_1}{2}\\ \frac{-(k_d+L_2)\varepsilon_1}{2} & -\frac{\mu_1L_2+\varepsilon_1}{2} & -\mu_1+\bar{k}_d\end{bmatrix}.
$$

Similar to the prove of Q in Theorem 1, it is easy to verify that Q_1 is positive definite, which means the system converges exponentially fast. Hence, the proof of Theorem 2 is complete. ■

C. Proof of Theorem 3

Proof: Similar to the proof of Theorem 1, we only need to consider the following system

$$
\begin{bmatrix}\dot{\boldsymbol{x}}\\ \dot{\boldsymbol{y}}\\ \dot{\boldsymbol{z}}\end{bmatrix}=A(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},t)\begin{bmatrix}\boldsymbol{x}\\ \boldsymbol{y}\\ \boldsymbol{z}\end{bmatrix}, \tag{10}
$$

where

$$
A(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},t)=\begin{bmatrix}0 & I_n & 0\\ 0 & 0 & I_n\\ -\boldsymbol{k_i} & -\boldsymbol{k_p}+b(\boldsymbol{y},t) & -\boldsymbol{k_d}+a(\boldsymbol{y},\boldsymbol{z},t)\end{bmatrix}.
$$

Similar to the previous proof, we now proceed to show the following quadratic form is a Lyapunov function,

$$
V(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})=[\boldsymbol{x}^{\boldsymbol{\tau}},\boldsymbol{y}^{\boldsymbol{\tau}},\boldsymbol{z}^{\boldsymbol{\tau}}]P_2[\boldsymbol{x}^{\boldsymbol{\tau}},\boldsymbol{y}^{\boldsymbol{\tau}},\boldsymbol{z}^{\boldsymbol{\tau}}]^{\boldsymbol{\tau}},
$$

where the constant matrix P_2 is

$$
P_2=\frac{1}{2}\begin{bmatrix}\mu_2\boldsymbol{k_i} & \boldsymbol{k_i} & \varepsilon_2I_n\\ \boldsymbol{k_i} & \boldsymbol{k_p}+\mu_2\boldsymbol{k_d} & \mu_2I_n\\ \varepsilon_2I_n & \mu_2I_n & I_n\end{bmatrix}, \tag{11}
$$

μ_2 is a constant defined by

$$
\mu_2=\frac{2(\bar{\boldsymbol{k}}_{\boldsymbol{d}}\bar{\boldsymbol{k}}_{\boldsymbol{p}}+\|\boldsymbol{k_i}\|)-L_1L_2}{4\bar{\boldsymbol{k}}_{\boldsymbol{p}}+L_2^2},
$$

and ε_2 is a positive constant, which is small enough.

Firstly, we show that the matrix P_2 is positive definite. Similar to the proof of Lemma 1, we only need to show the constant matrix P^j is positive definite, $j = 1, 2, \cdots, n$.

$$P^j = \frac{1}{2}\begin{bmatrix} \mu_2 k_i^j & k_i^j & \varepsilon_2 \\ k_i^j & (k_p^j + \mu_2 k_d^j) & \mu_2 \\ \varepsilon_2 & \mu_2 & 1 \end{bmatrix}.$$

Similarly, we can prove that the following inequities

$$\mu_2 > 0, \tag{12}$$

$$\mu_2 < \bar{\boldsymbol{k}}_{\boldsymbol{d}}, \tag{13}$$

$$4(-\|\boldsymbol{k_i}\| + \mu_2 \bar{\boldsymbol{k}}_{\boldsymbol{p}})(-\mu_2 + \bar{\boldsymbol{k}}_{\boldsymbol{d}}) > (\mu_2 L_2 + L_1)^2, \tag{14}$$

$$-\|\boldsymbol{k_i}\| + \mu_2 \bar{\boldsymbol{k}}_{\boldsymbol{p}} > 0, \tag{15}$$

and P^j is positive definite.

Let $\bar{\boldsymbol{k}}_{\boldsymbol{i}} = \min_j k_i^j$.

Next, we consider the derivative of $V(\boldsymbol{x}, \boldsymbol{y}, z)$ along the trajectories of (10),

$$\begin{aligned}
&\dot{V}(\boldsymbol{x}, \boldsymbol{y}, z) \\
&= [\boldsymbol{x}^\tau, \boldsymbol{y}^\tau, z^\tau](P_2 A(\cdot) + A(\cdot)^\tau P_2)[\boldsymbol{x}^\tau, \boldsymbol{y}^\tau, z^\tau]^\tau \\
&= -\varepsilon_2 \boldsymbol{x}^\tau \boldsymbol{k_i} \boldsymbol{x} + \boldsymbol{x}^\tau(-\boldsymbol{k_p} + b)\varepsilon_2 \boldsymbol{y} + \boldsymbol{x}^\tau(-\boldsymbol{k_d} + a)\varepsilon_2 z \\
&\quad - \boldsymbol{y}^\tau\left[\mu_2(\boldsymbol{k_p} - \tfrac{b+b^\tau}{2}) - \boldsymbol{k_i}\right]\boldsymbol{y} + \boldsymbol{y}^\tau(\mu_2 a + b^\tau + \varepsilon_2 I_n)z \\
&\quad - z^\tau\left(-\mu_2 I_n + \boldsymbol{k_d} - \tfrac{a+a^\tau}{2}\right)z \\
&\le -\varepsilon_2 \bar{\boldsymbol{k}}_{\boldsymbol{i}}\|\boldsymbol{x}\|^2 + \varepsilon_2(\|\boldsymbol{k_p}\| + L_1)\|\boldsymbol{x}\|\|\boldsymbol{y}\| \\
&\quad + \varepsilon_2(\|\boldsymbol{k_d}\| + L_2)\|\boldsymbol{x}\|\|z\| - [-\|\boldsymbol{k_i}\| + \mu_2 \bar{\boldsymbol{k}}_{\boldsymbol{p}}]\|\boldsymbol{y}\|^2 \\
&\quad + (\mu_2 L_2 + L_1 + \varepsilon_2)\|\boldsymbol{y}\|\|z\| - (-\mu_2 + \bar{\boldsymbol{k}}_{\boldsymbol{d}})\|z\|^2 \\
&= -[\|\boldsymbol{x}\|, \|\boldsymbol{y}\|, \|z\|]Q_2[\|\boldsymbol{x}\|, \|\boldsymbol{y}\|, \|z\|]^\tau,
\end{aligned}$$

where Q_2 is a symmetric matrix, expressed by

$$Q_2 = \begin{bmatrix} \bar{\boldsymbol{k}}_{\boldsymbol{i}}\varepsilon_2 & \frac{-(\|\boldsymbol{k_p}\|+L_1)\varepsilon_2}{2} & \frac{-(\|\boldsymbol{k_d}\|+L_2)\varepsilon_2}{2} \\ \frac{-(\|\boldsymbol{k_p}\|+L_1)\varepsilon_2}{2} & -\|\boldsymbol{k_i}\| + \mu_2 \bar{\boldsymbol{k}}_{\boldsymbol{p}} & -\frac{\mu_2 L_2 + L_1 + \varepsilon_2}{2} \\ \frac{-(\|\boldsymbol{k_d}\|+L_2)\varepsilon_2}{2} & -\frac{\mu_2 L_2 + L_1 + \varepsilon_2}{2} & -\mu_2 + \bar{\boldsymbol{k}}_{\boldsymbol{d}} \end{bmatrix}. \tag{16}$$

Similar to the prove of Q in Theorem 1, it is easy to verify that Q_2 is positive definite, which means the system converges exponentially fast. Hence, the proof of Theorem 3 is complete. ■

V. Simulation

Example 1: We consider the following system:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = f(x_1, x_2) + u(t), \end{cases} \tag{17}$$

where $f(x_1, x_2) = a\sin x_1 + bx_2$, $|a| \le L_1$, $|b| \le L_2$, and a, b are unknown. In this case the PID parameter manifold is given by Ω_{pid} in Theorem 2. Assume that the initial points are $(x_1(0), x_2(0)) = (3, 2)$, and that the setpoint $y^* = 1$.

First, we try to illustrate that for a given triple (k_p, k_i, k_d) of PID parameters, how the control performance will depend on the upper bounds (L_1, L_2). For simplicity, we take $(k_p, k_i, k_d) = (10, 1, 2)$ and fix $L_2 = 1$, and only consider the influence of L_1. It is easy to verify that $(10, 1, 2) \in \Omega_{pid}$ for $L_1 = 1, 3, 7.5$, and that $(10, 1, 2) \notin \Omega_{pid}$ for $L_1 = 15, 20$.

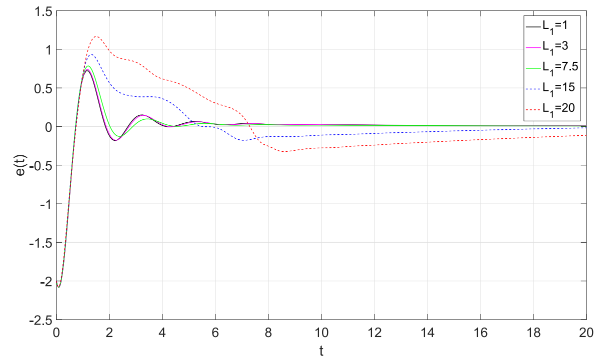

Fig. 1. Regulation errors corresponding to the increase of L_1.

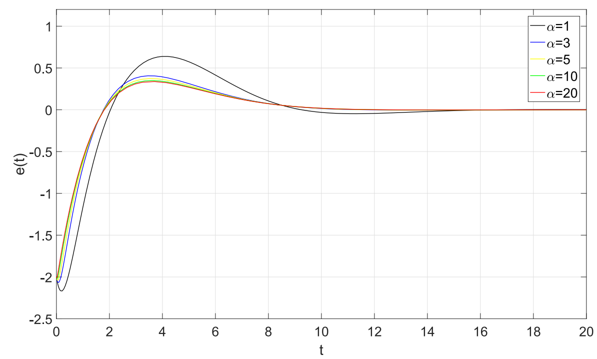

Fig. 2. Regulation errors corresponding to the increase of α.

Fig. 1 is constructed by taking the averaged regulation errors of 10 systems corresponding to randomly generated a in $[-L_1, L_1]$ and b in $[-1, 1]$. Fig. 1 shows that for the first three smaller values of L_1, the regulation error tend to zero quiet fast. However, for two larger values of L_1, the regulation performance become poor.

Next, we try to illustrate how the regulation performance depends on the gain of PID controller. It can be easily verified that for any positive L_1 and L_2, $(k_p, k_i, k_d) = \alpha(3, 1, 3) \in \Omega_{pid}$, $\forall \alpha \ge \max\{L_1, L_2, 1\}$. Let us take $L_1 = 1$ and $L_2 = 1$, and consider different values of α. Fig. 2 shows that the control performance will be improved by increasing the gain α.

VI. Conclusion and Future Works

It is well-known that one of the most basic and challenging issues in control theory is to establish the theoretical foundation of the widely used PID controller. In this letter, we have presented some mathematical results concerning global stabilization and asymptotic regulation of the PID controller for a class of nonlinear uncertain systems. We have shown that once the PID parameters are taken from a three-dimensional parameter manifold, constructed by using the knowledge about the upper bounds of the partial derivatives of the uncertain functions, the closed-loop control systems will be globally stable with regulation error converging to zero exponentially fast. This letter significantly improves the existing related results in the literature. For further investigations, it would be interesting to consider other more complicate situations, such as time-delay nonlinear system, nonlinear systems with dead-zone, general affine nonlinear systems [18], and so on.

Appendix

Lemma 1: The matrix P defined by (9) is positive definite.

Proof: Firstly, we show the fact that the matrix P is positive definite if and only if P_0 is positive definite, where the constant matrix P_0 is

$$P_0 = \frac{1}{2}\begin{bmatrix} \mu k_i & k_i & \varepsilon \\ k_i & (k_p + \mu k_d) & \mu \\ \varepsilon & \mu & 1 \end{bmatrix}.$$

By utilizing the Laplace Theorem, we can get $\left|\lambda I_{3n} - P\right| = \left|\lambda I_3 - P_0\right|^n$, which means the matrix P and P_0 share the same eigenvalues. Hence, we only need to prove the matrix P_0 is positive definite. To get the result, we now show that the following four inequalities hold,

$$\mu > 0, \tag{18}$$

$$\mu < \bar{k}_d, \tag{19}$$

$$4(-k_i + \mu\bar{k}_p)(-\mu + \bar{k}_d) > (\mu L_2 + L_1)^2, \tag{20}$$

$$-k_i + \mu\bar{k}_p > 0. \tag{21}$$

Note that, by the definition of Ω_{pid}, we have

$$\bar{k}_p\bar{k}_d - k_i > \sqrt{(\bar{k}_p L_1 + k_i L_2)(L_1 + \bar{k}_d L_2)},$$

thus

$$\bar{k}_p\bar{k}_d - k_i > 0,$$

$$(\bar{k}_p\bar{k}_d - k_i)^2 > (\bar{k}_p L_1 + k_i L_2)(L_1 + \bar{k}_d L_2). \tag{22}$$

Hence (19) is true since

$$\mu - \bar{k}_d = \frac{-2(\bar{k}_p\bar{k}_d - k_i) - L_1L_2 - L_2^2\bar{k}_d}{(4\bar{k}_p + L_2^2)} < 0.$$

Furthermore, by using (22), we can get

$$\begin{aligned} &4(-k_i + \mu\bar{k}_p)(-\mu + \bar{k}_d) - (\mu L_2 + L_1)^2 \\ &= -(4\bar{k}_p + L_2^2)\mu^2 + [4(\bar{k}_p\bar{k}_d + k_i) - 2L_1L_2]\mu - 4k_i\bar{k}_d - L_1^2 \\ &= \frac{[2(\bar{k}_p\bar{k}_d + k_i) - L_1L_2]^2 - (4k_i\bar{k}_d + L_1^2)(4\bar{k}_p + L_2^2)}{4\bar{k}_p + L_2^2} \\ &= \frac{4[(\bar{k}_p\bar{k}_d - k_i)^2 - (\bar{k}_p L_1 + k_i L_2)(L_1 + \bar{k}_d L_2)]}{4\bar{k}_p + L_2^2} > 0. \end{aligned}$$

Thus, we can see that (20) is valid, and consequently (21) follows from (19) and (20).

Additionally, from (21) and the fact $k_i, \bar{k}_d > 0$, we can deduce the inequality (18).

Next, it is easy to verify that P_0 is positive definite, since the following three inequality hold,

$$\mu k_i > 0,$$

$$\begin{vmatrix} \mu k_i & k_i \\ k_i & (k_p + \mu k_d) \end{vmatrix} > k_i(\mu k_p - k_i) > k_i(\mu\bar{k}_p - k_i) > 0,$$

and

$$\begin{aligned} \lim_{\varepsilon \to 0}&\begin{vmatrix} \mu k_i & k_i & \varepsilon \\ k_i & (k_p + \mu k_d) & \mu \\ \varepsilon & \mu & 1 \end{vmatrix} \\ &= k_i[(\mu k_p - k_i) + \mu^2(k_d - \mu)] > 0. \end{aligned}$$

The last inequality means $detP_0 > 0$ when the positive constant ε is small enough. ■

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