PRECISE YELLOW TRAFFIC SIGNAL SOLUTIONS

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Abstract
This technical tutorial presents the science to substantially enhance state-of-the-art yellow change interval solutions by introducing a higher derivative of uniform vehicle motion. Presently, the highest derivative is acceleration, whereas the subsequent higher derivative is jerk (or jolt), describing the rate of change of acceleration over time. The inclusion of the kinematic variable jerk is required to curve fit true vehicle motion precisely mathematically. The presented enhanced kinematic equations of motion are derived, illustrated, and correlated to real-life test data.

In 2020 the Institute of Transportation Engineers (ITE) adopted the author's 2015 "An Extended Kinematic Equation" [1] in their "Guidelines for Determining Traffic Signal Change and Clearance Intervals," [2] but with errors. The ITE's recommended practices have worldwide coverage, and any misinformation has adverse effects on many. In the US, ITE's recommendations are used by several State Department of Transportations, local jurisdictions, and especially the Federal Highway Administration (FHWA) since they rely on the ITE's guidelines in their standards manual, the Manual of Uniform Traffic Control Devices (MUTCD). This tutorial includes corrections to the ITE's problems, but it mainly presents upgraded yellow change interval solutions with the common goal to provide a precise scientific foundation for a uniform traffic signal standard worldwide.

Primary Problem Statement
State-of-the-art kinematic traffic signal theory uses constant or *average* acceleration that does not correlate to recorded *instantaneous* vehicle dynamics test data. The inaccurate theories affect the entire yellow traffic signal system.

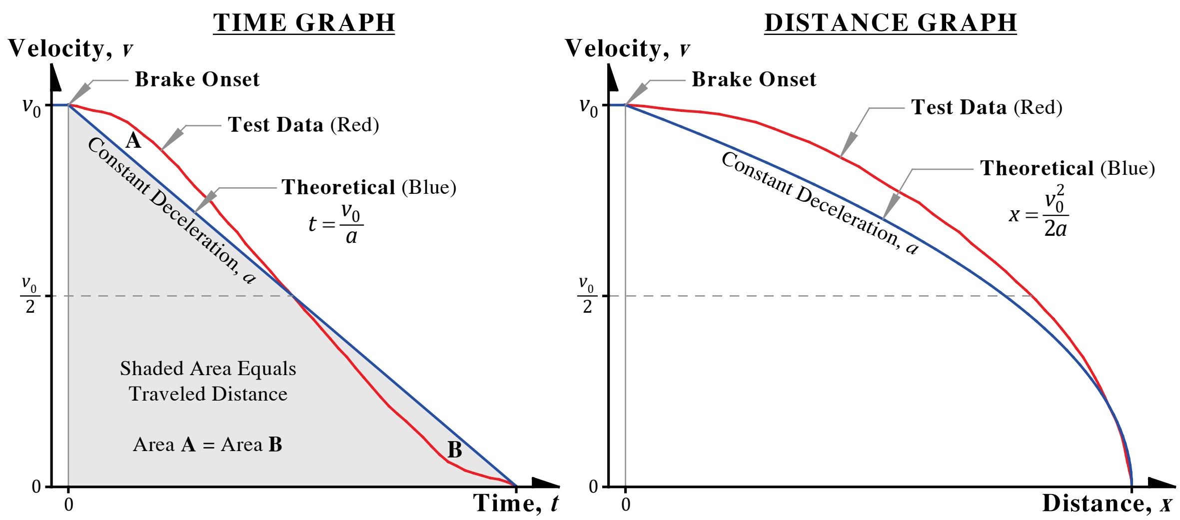

Figure 1 - Vehicle braking motion to stop plotted in time and distance, theoretical vs. recorded test data

Figure 1 presents time and distance graphs comparing plots of theoretical and recorded test data of a vehicle's braking motion to stop. The plots show significant differences because constant (average) deceleration is ONLY accurate at the initial (v_0) and end (zero) velocities. That includes the symmetrical test data (red) crossover point in the time graph at half the initial velocity ($v_0/2$) since the same data in the distance graph shows the difference.

The fundamental problem is the difference in slopes between the theoretical and recorded constant decelerations. The symmetrical test data in the time graph shows after the brake onset, an initial jerk increasing the deceleration, an intermediate constant deceleration, and an end jerk decreasing the deceleration before the final stop. Therefore, actual vehicle braking motion has three sections, an initial jerk, a constant deceleration, and an end jerk to stop. It is the kinematic equations for these three sections that become the source of the precise solutions.

Presently state-of-the-art yellow change interval theories do not consider the kinematic variable jerk, including the author's 2015 solutions. The jerk variable describes the comfortable transition between different states of motions, i.e., constant velocity to constant deceleration. Without including the kinematic variable jerk, transitions become instant, producing infinite jerks and uncomfortable motions. For example, the jerk describes how quickly and comfortably a vehicle's brakes are engaged and disengaged, i.e., via a brake pedal.

The instantaneous vehicle dynamics test data in Figure 1 was recorded using a Racelogic Video VBOX Lite GPS 10 Hz data logger recommended by Irish researcher Frank Cullinane. The data was part of a live demonstration presenting the author's extended kinematic equation at the Institute of Transportation Engineers (ITE) annual meeting in 2016. The data was also shared during ITE's comment period and the author's appeal in 2019 before the final release of their guidelines in 2020.

Secondary Problem Statement

Since 2015 the author and others have submitted a series of comments to the ITE regarding misunderstandings of the pioneering work by scientists Dr. Denos Gazis, Dr. Robert Herman, and Dr. Alexei A. Maradudin (GHM) published in 1960, "The Problem of the Amber Signal Light in Traffic Flow." [3] The others include Dr. Alexei A. Maradudin, Jay Beeber, Brian Ceccarelli, Joe Bahen, and Gary Biller.

The author's 2019 comments to the ITE [4] summarized issues and remarks since 2015 (hyperlinked on the last page). The summary's essential information related to this tutorial is as follows:

1. 2015 Comments to the ITE regarding yellow change intervals [5]
   a. 2014 working report "An investigation of the ITE formula and its use" [6] The mathematics of moving objects (aka kinematics) and how to illustrate uniform motion.
   b. The author's initial release of the Extended Kinematic Equation [7]
   c. The first revision of "The Problem with the ITE Formula’s Grade Implementation." [8]
   d. "An investigation of Dr. Liu’s universal change interval formula," [9]an analytical comparison of Dr. Liu’s 2002 yellow change interval solution with a modified version, GHM’s original kinematic equation, and the author's extended version.
2. 2016 presentation at the ITE's annual meeting with Racelogic test data
   a. Presentation of the Extended Kinematic Equation [10] (including links to test data, PDF 5.3MB)
3. 2018 email communication with the ITE
   a. Instantaneous versus uniform (average) motion including an Excel spreadsheet [11] (XLSX and PDF)
   b. Stopping on a grade and a precise and straightforward grade derivation [12]

Also, the author repeated the comments during the 2019 ITE appeal meeting in Washington DC (PowerPoint) [13] and followed up with a document titled "ITE's Fundamental Problems" [14] (PDF 10.0MB), stressing the issues.

Despite all efforts and information submitted to ITE, their latest guidelines, the April 2020 revision 2 of "Guidelines for Determining Traffic Signal Change and Clearance Intervals," [2] still contain misinformation and errors. It appears the guidelines were published prematurely without proper review.

The author submitted the complete original extended kinematic equation from January 13, 2015, [7] to the ITE in July 2015, and it included the effect from Earth's gravity stopping on a grade. However, the ITE's adoption of the author's equation presented in Figure 2 shows the equation in a half-done state, and it has either a typographical or a mathematical error since the Earth's gravitational constant is double in one instance than what it should be (marked with a red box). This error calculates excessive yellow change intervals for downhill grades, and a recommendation to apply the equation for uphill grades is a misunderstanding of a yellow traffic signal system and its limits.

Link to Figure 2 taken from ITE's March 3, 2020 webinar: https://youtu.be/fskIsOGdiUU?t=649 [15]

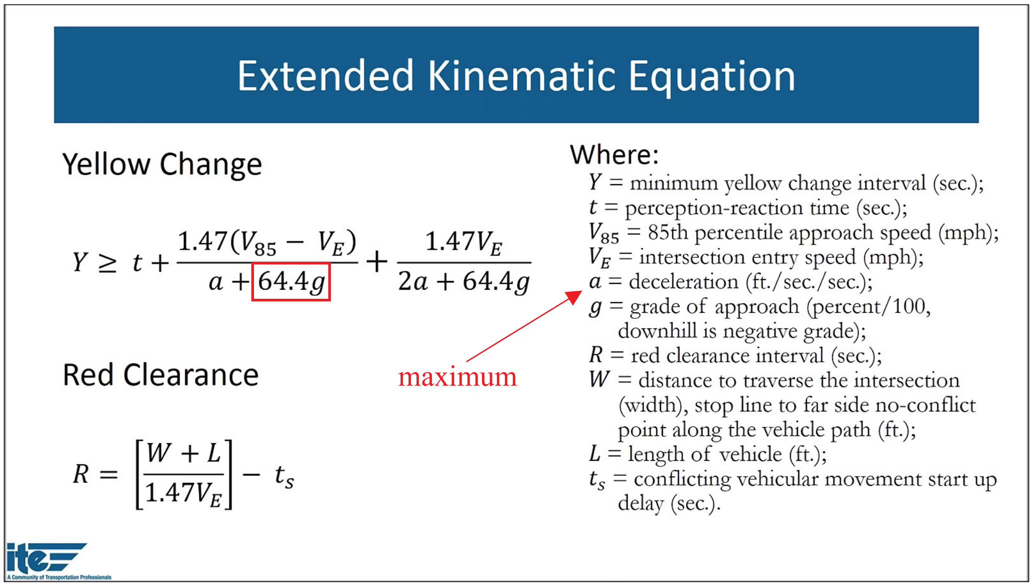

Figure 2 - The ITE's problematic adoption of the author's Extended Kinematic Equation

A corrected and finished form of the authors 2015 Extended Kinematic Equation as the "Yellow Change" using the ITE's variables and units from Figure 2, including the mph to ft/s conversion factor (1.47) and where the Earth's gravitational constant is 32.2 ft/s^2 presents as follows:

$$Y \geq t + \frac{1.47\,(V_{85} - ½\,V_E)}{a_{max} + 32.2g} \tag{1}$$

Equation (1) is the author's Extended Kinematic Equation in two spatial dimensions, where the grade of the approach is only valid for level or negative downhill grades ($g \leq 0$) since the deceleration is defined as a maximum safe and comfortable limit (a_{max}) at a level grade. The deceleration variable in Figure 2 does not show a maximum limit. However, GHM's original 1960 paper discusses the critical *maximum* safe and comfortable deceleration limit to stop before an intersection which becomes the source of their "critical distance" and ultimately their minimum yellow change interval. Unfortunately, the ITE still does not recognize the vital importance of GHM's minimum and maximum limits even though Figure 2 shows a redundant "minimum" yellow change interval.

Besides, the simplified and correct presentation of Equation (1) highlights the kinematic solution's vital foundation - the maximum deceleration variable (a_{max}). Therefore, as presented in the introductory problem statement, an incorrect deceleration variable in the equation's denominator affects the whole yellow traffic signal system. This tutorial presents the science and mathematics to derive an updated deceleration variable that precisely correlates to real-life vehicle motion.

Background

Sir Isaac Newton developed over three centuries ago the science and mathematics describing the motion of massless objects called calculus-based physics or kinematics. Newton’s motion equations have helped put humans on the moon and many other spectacular scientific and engineering achievements.

Through the automobile's invention over a century ago came the emergence of traffic control devices such as traffic signals. The yellow traffic signal indication first appeared between the green and red in 1920, and in 1960, Denos Gazis, Robert Herman, and Alexei A. Maradudin (GHM) pioneered a scientific solution for the yellow traffic signals in their paper, "The Problem of the Amber Signal Light in Traffic Flow." [3]

GHM’s 1960 solution's foundation is a minimum safe and comfortable distance to STOP, which they termed the "critical distance." GHM's GO solution, their minimum yellow change interval, describes a vehicle traveling the "critical distance" at a constant velocity to enter and travel straight through a level intersection. In 1965 the ITE adopted GHM's GO solution but presented it as the stopping time [16]. This error led to the belief that the yellow change interval is time wasted when it is the GO solution to enter and the end of the green interval.

In 2015 the author identified the internal braking distance within GHM’s "critical distance" as the STOP or GO boundary. The boundary defines a maximum safe and comfortable constant deceleration for vehicles and their occupants, including cargo, to come to a safe and comfortable stop before an intersection. This discovery resulted in the Extended Kinematic Equation, an adaptable linear yellow change interval solution applicable to any approach lane, including turning lanes where vehicle deceleration is required.

The Precise Solutions

This section expands upon the author's 2014 working report "An Investigation of the ITE formula and its use," [6] where chapter five presents motion in one spatial dimension and the kinematic equations applicable to traffic signal theory based on constant (average) acceleration. The report presents simplified methods where introductory algebra and basic geometry calculations suffice avoiding advanced mathematics for the derivations since plots of constant accelerations are linear in a velocity versus time graph. This tutorial continues on that path.

Figure 3 presents the new kinematic variable jerk's mathematical relationship to the variables acceleration, velocity, and distance (displacement). Jerk or jolt describes the rate of change of acceleration over time. Newton's calculus-based physics includes the mathematical calculus functions called differentiation and integration connecting all variables. Differentiation is a plotted mathematical function's slope describing the rate of change, and integration is the area under a plotted mathematical function. Jerk is the third derivative of distance and one derivative above acceleration, the top level of state-of-the-art traffic signal theory.

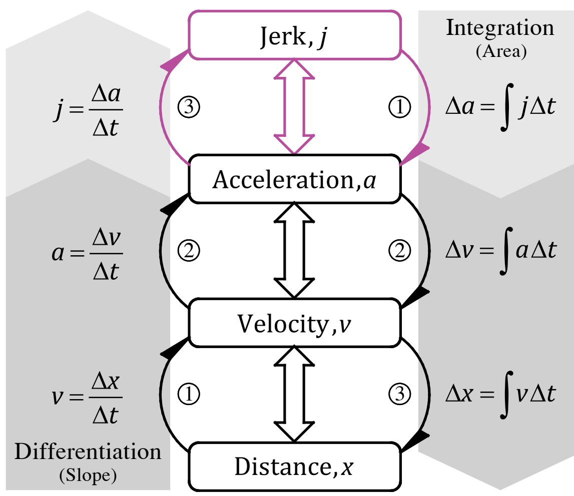

Figure 3 - Mathematical flow diagram of the state-of-the-art uniform motion variables plus jerk over time

Definition of Average and Instantaneous Motion

Uniform motion defines as the average motion between two instantaneous data samples in time, where the data can be any kinematic variable. For example, the definition of average velocity (v) is a change in position (Δx) over an elapsed time (Δt) and where the variables' subscripts denote the initial (n) to the end (n+1) data samples and "n" are integers defined in Equation (2):

$$v = \frac{\Delta x}{\Delta t} = \frac{x_{n+1} - x_n}{t_{n+1} - t_n} \tag{2}$$

Instantaneous motion defines the motion in a single data sample in time. For example, the definition of instantaneous velocity ($v(t)$) in one position and time ($x(t)$) and where ($\Delta t = t_{n+1} - t_n$) is as presented in Equation (3):

$$v(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \frac{dx(t)}{dt} \tag{3}$$

The Racelogic test instrumentation used by the author captures instantaneous vehicle motion sampled in time. The recorded vehicle motion data, the red plots in Figure 1, are instantaneous vehicle velocity sampled at 10 Hz or every 0.1 seconds.

Motion with a Constant Jerk

Figure 4 illustrates a *constant* jerk (j) plotted in a jerk versus time graph (top) and acceleration versus time graph (bottom). Newton's calculus-based physics connects the two graphs through mathematics called integration (area) and differentiation (slope). The acceleration versus time graph presents the jerk as a linear change of acceleration over time (slope), and the area under the plot represents the change of velocity (integration).

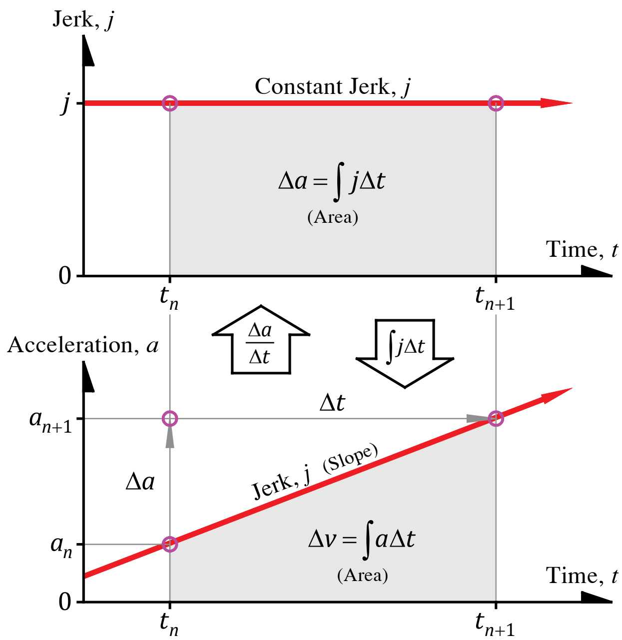

Figure 4 - Illustrations of a constant jerk (j) and its linear change of acceleration over time

The definition of a constant (average) jerk (j) is a linear change of acceleration (Δa) over an elapsed time (Δt) where the variables' subscripts denote the initial (n) to the end (n+1) values where "n" are integers as presented in Equation (4) and referenced in Figure 4:

$$Jerk, j = \frac{\Delta a}{\Delta t} = \frac{a_{n+1} - a_n}{t_{n+1} - t_n} \tag{4}$$

Rearranging Equation (4) yields the elapsed time (Δt) as presented in Equation (5):

$$\Delta t = \frac{\Delta a}{j} = \frac{a_{n+1} - a_n}{j} \tag{5}$$

As well as the end acceleration (a_{n+1}) from an initial acceleration (a_n) shown in Equation (6):

$$a_{n+1} = a_n + j\Delta t \tag{6}$$

Integration - Area Calculations

The shaded area in Figure 4 is per Newton's calculus-based physics "integration." In an acceleration versus time graph, the area between a motion plot and the time axis represents the change of velocity (Δv), and constant jerk produces a linear change of acceleration over time, as shown in Figure 4. The linear change allows for basic geometric area calculations. Hence, the change in velocity (Δv) and the area is simply the average "height" of the initial and end accelerations multiplied the "width," the elapsed time (Δt) as presented in Equation (7):

$$\Delta v = \frac{a_{n+1} + a_n}{2} \cdot \Delta t \tag{7}$$

Also, Equation (5) provides the elapsed time (Δt), and utilization of a conjugate rule produces Equation (8):

$$\Delta v = \frac{a_{n+1} + a_n}{2} \cdot \frac{a_{n+1} - a_n}{j} = \frac{a_{n+1}^2 - a_n^2}{2j} \tag{8}$$

Rearranging Equation (8) yields yet another end acceleration (a_{n+1}) from an initial acceleration (a_n) as presented in Equation (9):

$$a_{n+1}^2 = a_n^2 + 2j\Delta v \tag{9}$$

Accelerated Motion Including a Constant Jerk

A constant jerk describes a linear change of acceleration, as Figure 4 and top of Figure 5 graphs show. However, plotting constant jerk in a velocity versus time graph using Equation (8) involves nonlinear velocity changes presented at the bottom of Figure 5.

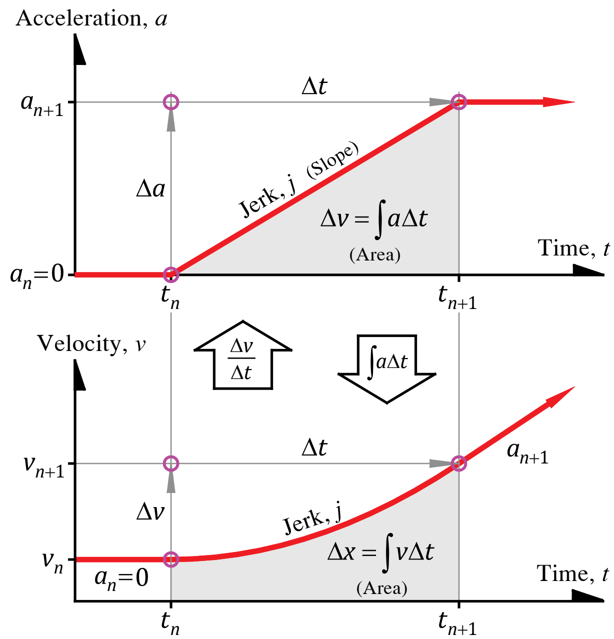

Figure 5 - Constant jerk plotted in both acceleration and velocity versus time graphs

The area under the plot in the velocity versus time graph of Figure 5 (bottom) requires advanced mathematics using the integration calculus function because of the plotted motion's nonlinearity and the lack of symmetry. However, for this tutorial, the test data's symmetry, as shown in Figure 1, mitigates the need for advanced mathematics.

Applying advanced mathematics, Appendix A presents the integrals that derive the kinematic equations from a constant jerk for acceleration, velocity, and traveled distance. Equation (10), taken from Appendix A, can be applied to calculate the area under the curve in a velocity versus time graph representing the traveled distance (Δx) as in Figure 5. Equation (10) calculates the traveled distance ending at position (x_{n+1}) starting from the initial position (x_n) with the initial velocity (v_n), and the initial acceleration (a_n) for a constant jerk (j) during the elapsed time ($\Delta t = t_{n+1} - t_n$) as follows:

$$x_{n+1} = x_n + \int_{t_n}^{t_{n+1}} v(t)dt = x_n + v_n\Delta t + \frac{a_n\Delta t^2}{2} + \frac{j\Delta t^3}{6} \tag{10}$$

The elapsed time ($\Delta t = t_{n+1} - t_n$) is, for instance, the sample rate used by data recorders such as the Racelogic VBOX video and GPS data loggers or the sample time used to plot the kinematic equations in an Excel spreadsheet. The variables presented in Equation (10) can also be converted from a single spatial dimension to three-dimensional space as vectors.

Stop Motion Including Symmetrical Jerks

Figure 6 presents an enhanced kinematic three-part vehicle stop motion model with symmetrical jerks ($\pm j$) to precisely correlate to test data plotted in red in Figure 1. As discussed, vehicle braking motion has three sections, an initial jerk, a constant instantaneous deceleration, and an end jerk to stop. These three sections are labeled 1, 2, and 3 in Figure 6.

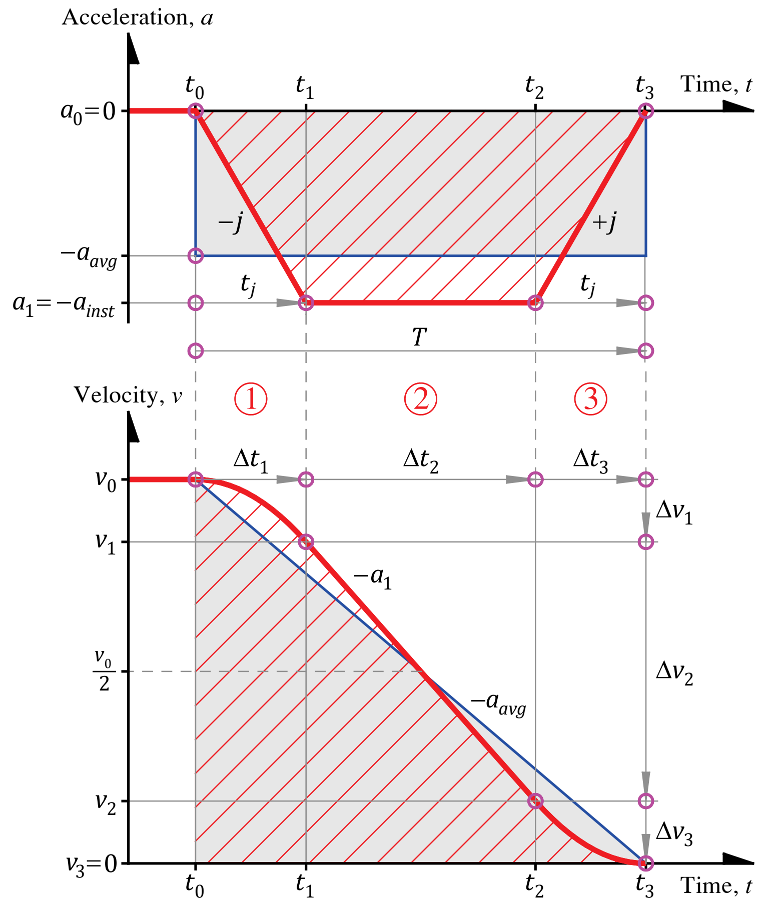

Figure 6 - Kinematic three-part vehicle stop motion model with symmetrical jerks

Figure 6 shows a comparison between the three-part model (red) and state-of-the-art uniform motion using average acceleration (blue) presented in two graphs. The top graph compares acceleration versus time, and the bottom graph compares velocity versus time. According to Newton's calculus-based physics, the areas under the two plotted comparisons represent average velocity in the top graph and traveled distance in the bottom graph. Since the comparisons' average velocities and traveled distances are equal yields, the shaded grey area and the hashed red area are equal in the top graph, and the symmetrical jerks produce equal areas in the bottom graph.

Comparison of Accelerations

Basic geometric analysis of the equal areas in the top acceleration versus time graph over the total time (T) yields the relationship between the state-of-the-art average acceleration (a_{avg}) and the new three-part model's constant instantaneous acceleration (a_{inst}) including the symmetrical jerk times (t_j), where ($t_j = t_1 - t_0 = t_3 - t_2$) and ($T = t_3 - t_0$) from Figure 6 produces the following:

$$-a_{avg}T = -a_{inst}(T - t_j) \tag{11}$$

Solving Equation (11) for the new symmetrical jerk stop motion model's acceleration (a_{inst}) yields the following relationship presented in Equation (12):

$$a_{inst} = \frac{a_{avg}}{\left(1 - \frac{t_j}{T}\right)} \tag{12}$$

Where $0 \leq 2t_j < T$

Equation (12) is valid for both accelerations and decelerations. Setting the symmetrical jerk times to zero ($t_j = 0$) yields infinite jerks, i.e., instant engagement and disengagement of a vehicle's brakes producing ($a_{inst} = a_{avg}$), the basis for state-of-the-art traffic signal theory. Furthermore, the two symmetrical jerk times must be less than the total time ($2t_j < T$).

Derivation of Stop Motion TIME Including Symmetrical Jerks

Figure 6, the bottom graph, presents the stop motion model's three-parts referenced to velocity and time. The model assumes a constant approach velocity (v_0) , constant symmetrical jerks ($\pm j$), and a constant (negative) instantaneous acceleration ($-a_{inst}$). The purpose for this exercise is to derive a time expression of the three-part stop motion model based on these input variables.

The bottom graph in Figure 6 defines the sum of the changes in velocities (Δv_n) across the stop motion model's three parts from the initial constant approach velocity (v_0) until the zero end stop velocity ($v_3 = 0$) producing the following:

$$v_3 = v_0 + \Delta v_1 + \Delta v_2 + \Delta v_3 = 0 \tag{13}$$

Figure 6 also defines the total stop motion time (T), and the model's three parts in time (Δt_n) as follows:

$$T = \Delta t_1 + \Delta t_2 + \Delta t_3 \tag{14}$$

The two symmetrical jerk times are equal ($\Delta t_1 = \Delta t_3$). Applying Equation (5) with the information from the model's first part (Δt_1) in Figure 6, it has a zero initial acceleration ($a_0 = 0$) at time (t_0) and an end instantaneous acceleration ($a_1 = -a_{inst}$) at the time (t_1) during a constant (negative) jerk ($j = -j$) yielding the following:

$$\Delta t_1 = \frac{a_1 - a_0}{j} = \frac{-a_{inst} - 0}{-j} = \frac{a_{inst}}{j} \tag{15}$$

Equation (15) provides the answer for the two equal jerk times ($\Delta t_1 = \Delta t_3$).

The remaining middle part (Δt_2) consist of the constant (negative) instantaneous acceleration ($-a_{inst}$) between the times (t_1) and (t_2) referenced in Figure 6. To solve for (Δt_2), the changes in velocities during the symmetrical initial and end jerks are needed, and the symmetry results in identical velocity changes ($\Delta v_1 = \Delta v_3$).

Equation (8) provides the needed solution where the initial constant (negative) jerk ($j = -j$) has a zero initial acceleration ($a_0 = 0$) at time (t_0) and an end instantaneous acceleration ($a_1 = -a_{inst}$) at the time (t_1) which presents as follows:

$$\Delta v_1 = \frac{a_1^2 - a_0^2}{2j} = \frac{(-a_{inst})^2 - 0}{-2j} = -\frac{a_{inst}^2}{2j} \tag{16}$$

Equation (16) provides the answers for the changes in velocities during the symmetrical initial (Δv_1) and end (Δv_3) jerks. Inserting the results from Equation (16) into Equation (13) yields a solvable expression for the middle part's change in velocity (Δv_2) referenced the total velocity changes across the three-part model as follows:

$$v_0 - \frac{a_{inst}^2}{2j} + \Delta v_2 - \frac{a_{inst}^2}{2j} = 0 \tag{17}$$

Solving for (Δv_2) yields:

$$\Delta v_2 = \frac{a_{inst}^2}{j} - v_0 \tag{18}$$

The definition of average acceleration (see Figure 3) rearranged to define the relationship between the negative instantaneous acceleration ($-a_{inst}$) and the change in velocity across the model's middle part (Δv_2) and combining with Equation (18) yields the following:

$$\Delta t_2 = \frac{\Delta v_2}{-a_{inst}} = \frac{1}{-a_{inst}}\left(\frac{a_{inst}^2}{j} - v_0\right) = \frac{1}{a_{inst}}\left(v_0 - \frac{a_{inst}^2}{j}\right) = \left(\frac{v_0}{a_{inst}} - \frac{a_{inst}}{j}\right) \tag{19}$$

Equation (14) defines the model's total stop motion time (T) accros all parts. Equation (15) provides the solution for the two equal jerk times ($\Delta t_1 = \Delta t_3$) and Equation (19) provides the solution for the remaining (Δt_2) producing a complete expression in time as follows:

$$T = \Delta t_1 + \Delta t_2 + \Delta t_3 = \frac{a_{inst}}{j} + \left(\frac{v_0}{a_{inst}} - \frac{a_{inst}}{j}\right) + \frac{a_{inst}}{j} \tag{20}$$

Simplification of Equation (20) presents the final stop motion time (T) and the solution is referenced to an initial constant approach velocity (v_0), a constant instantaneous acceleration or deceleration (a_{inst}) and constant symmetrical jerks (j):

$$T = \frac{v_0}{a_{inst}} + \frac{a_{inst}}{j} \tag{21}$$

Where $v_0 > \frac{a_{inst}^2}{j}$, $a_{inst} > 0$ and $j > 0$

Equation (21) is valid for both acceleration and deceleration motion profiles with symmetrical jerks where the constant velocity variable is either an initial velocity as in the derivation or an end velocity for acceleration from a standstill. The constant velocity (v_0) is defined with a minimum referencing the remaining back-to-back symmetrical jerk parts when the middle part, the constant instantaneous acceleration or deceleration, becomes zero. Besides, Equation (21) converts to the state-of-the-art theoretical version using constant average acceleration if the symmetrical jerks (j) are infinite.

Derivation of Stop Motion DISTANCE Including Symmetrical Jerks

The bottom graph in Figure 6 shows the three-part stop motion model with symmetrical jerks plotted in red, and the hashed red area between the stop motion plot and the time axis represents the traveled distance (Δx). The area calculations appear to be complex due to the curvature of the plotted stop motion, but the symmetry of the motion profile makes the derivation trivial requiring no advanced mathematics.

The distance traveled (Δx) is the average velocity (Δv) multiplied with the elapsed time (Δt) (See Figure 3). Equation (21) offers the elapsed time ($\Delta t = T$) and the average velocity is ($\Delta v = v_0/2$) yielding Equation (22):

$$\Delta x = \Delta v \Delta t = \frac{v_0}{2}\left(\frac{v_0}{a_{inst}} + \frac{a_{inst}}{j}\right) \tag{22}$$

Simplification of Equation (22) yields the final solution for the distance traveled presented in Equation (23):

$$\Delta x = \frac{v_0^2}{2a_{inst}} + \frac{v_0 a_{inst}}{2j} \tag{23}$$

Where $v_0 > \frac{a_{inst}^2}{j}$, $a_{inst} > 0$, and $j > 0$

Curve Fitted Test Data

The three-part stop motion model with symmetrical jerks, resulting in the time Equation (21) and the distance Equation (23), is the source to the precise solution. Figure 7 shows the equations plotted in yellow versus time and distance correlated to the red initially presented test data in Figure 1. The equation's input values for the variables were adjusted or "tuned" to match the test data to achieve curve fitting. Figure 7 demonstrates that the three-part stop motion model with symmetrical jerks correlates almost a hundred percent to the test data. The slight variations in the test data originate from GPS noise.

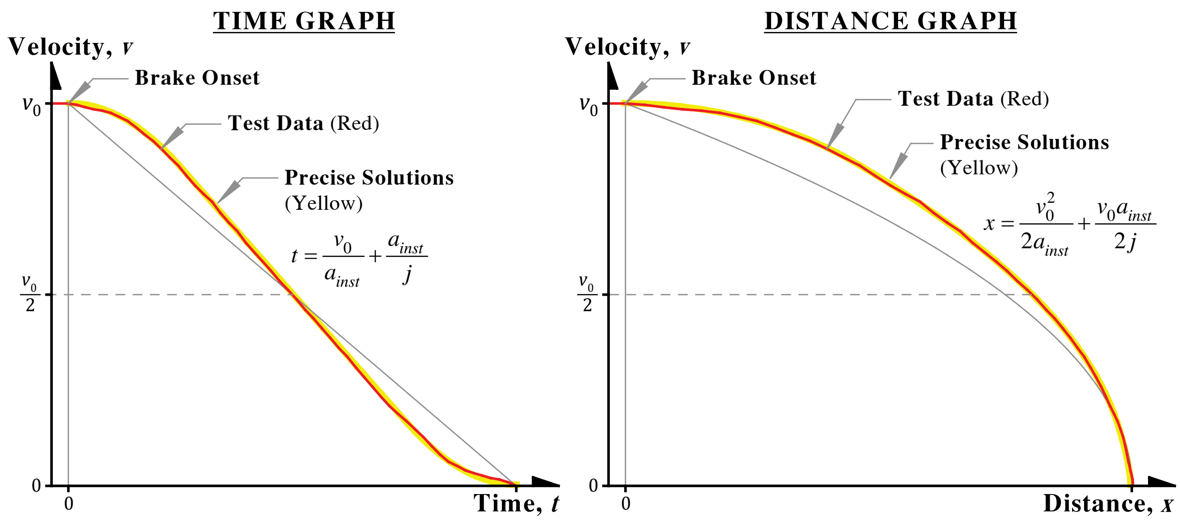

Figure 7 - Curve fitted test data using the three-part stop motion model with symmetrical jerks

The test data and the plotted equations in Figure 7 are available in an Excel spreadsheet (PDF) [17]:
https://jarlstrom.com/PDF/CurveFittingTrueKinematicMotionToVBOXDemoDATA1.3s_R0.xlsx

Average vs. Instantaneous Accelerations

Figure 8 illustrates the system errors caused by the inaccurate kinematic theories based on constant or *average* acceleration without jerk compared to the precise solutions correlated to actual vehicle motion using constant symmetrical jerks and *instantaneous* acceleration.

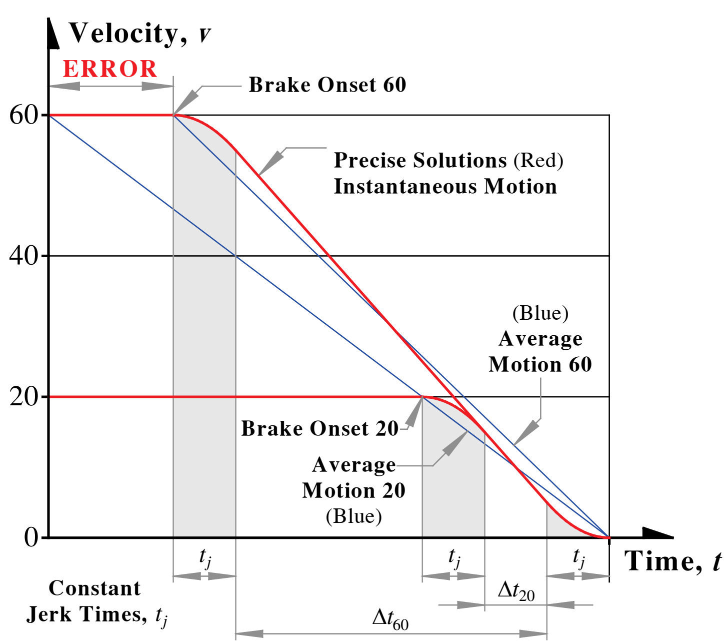

Figure 8 - Illustration of the state-of-the-art system error due to average motion

The Precise Yellow Traffic Signal Solutions

The precise solutions expand upon GHM's pioneering scientific foundation. The foundation for any traffic signal theory is GHM's "critical distance," which is a driver-vehicle's optional STOP distance when faced with a yellow traffic signal indication. Hence, the new equations derived from the precise three-part stop motion model enhances GHM's "critical distance" (x_c) as follows:

$$x_c = v_0 t_{PR} + \frac{v_0^2}{2a_{inst}} + \frac{v_0 a_{inst}}{2j} \tag{24}$$

Figure 9 presents the kinematic equations' evolution starting from GHM's 1960 solution ending with the author's two solutions, both the extended and the precise. The original equations, on the left side in Figure 9 and the enhanced precise versions on the right.

	Average Motion		Instantaneous Motion
1960	$x_c = t_{PR} v_0 + \frac{v_0^2}{2a_{avg}}$	2021	$x_c = t_{PR} v_0 + \frac{v_0^2}{2a_{inst}} + \frac{v_0 a_{inst}}{2j}$
	GHM's "Critical Distance"		
1960 1965	$Y \geq t_{PR} + \frac{1}{2}\left(\frac{v_0}{a_{avg}}\right)$	2021	$Y \geq t_{PR} + \frac{1}{2}\left(\frac{v_0}{a_{inst}} + \frac{a_{inst}}{j}\right)$
	GHM's & ITE's Kinematic Equation		
2015	$Y \geq t_{PR} + \frac{v_0 - ½v_1}{a_{avg}}$	2021	$Y \geq t_{PR} + \frac{v_0 - ½v_1}{a_{inst}} + \frac{a_{inst}}{2j}$
	Järlström's Extended Kinematic Equation		
2015	$Y \geq t_{PR} + \frac{1}{1+k}\left(\frac{v_0}{a_{avg}}\right)$ $k = \frac{v_1}{v_0}$	2021	$Y \geq t_{PR} + \frac{1}{1+k}\left(\frac{v_0}{a_{inst}} + \frac{a_{inst}}{j}\right)$ $k = \frac{v_1}{v_0}$
	Järlström's Precise Kinematic Equation		

Figure 9 - Evolution of the kinematic equations

Where the variables in Figure 9 are defined in US standard units and (SI units) as follows:

- x_C = GHM's critical distance - the minimum safe and comfortable stopping distance, ft (m)
- Y = Minimum duration of the yellow signal indication across the critical distance x_C, (s)
- t_{PR} = Maximum allocated driver-vehicle perception-reaction time, (s)
- v_0 = Maximum uniform initial/approach velocity, ft/s (m/s)
- v_1 = Maximum uniform intermediate/entry velocity, ft/s (m/s)
- k = The ratio between intermediate/entry and initial/approach velocity, no unit
- a_{avg} = Maximum uniform driver-vehicle safe and comfortable deceleration, ft/s^2 (m/s^2)
- a_{inst} = Maximum instantaneous driver-vehicle safe and comfortable deceleration, ft/s^2 (m/s^2)
- j = Maximum uniform jerk, ft/s^3 (m/s^3)

The precise kinematic equation evolved from the author's 2015 investigation of Dr. Chiu Liu's et al. 2002 [9] paper "Determination of Left-Turn Yellow Change and Red Clearance Interval," [18] and presented as follows:

$$Y \geq t_{PR} + \frac{1}{1 + \frac{v_1}{v_0}}\left(\frac{v_0}{a_{inst}} + \frac{a_{inst}}{j}\right) \tag{25}$$

The precise extended kinematic equation evolved from the author's 2015 version, and it presents as follows:

$$Y \geq t_{PR} + \frac{v_0 - ½\, v_1}{a_{inst}} + \frac{a_{inst}}{2j} \tag{26}$$

Figure 10 shows the two precise variable timing solutions, Equation (25) and Equation (26), in a yellow traffic signal system overview where the precise kinematic equation's nonlinear plotted solution shown in solid red and the extended linear version is plotted in dashed red. The linear version is an approximation that allows the capability to produce a nomogram. The system overview also shows the range of the variable entry velocity (v_1) (blue) and how it affects the yellow duration. The minimum entry velocity (v_1) is due to the two back-to-back symmetrical jerks where: $v_1 > a_{inst}^2/j$

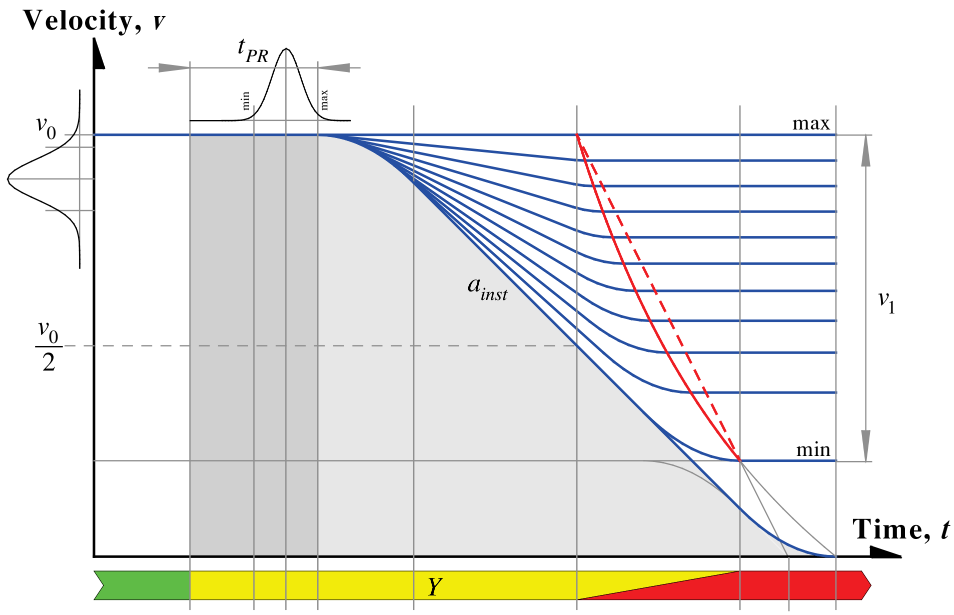

Figure 10 - Yellow traffic signal system overview of the two precise variable timing solutions

Precise Grade Implementation

The author presented a precise and straightforward grade derivation in 2015 [12]. Appendix B presents the safe and comfortable deceleration on a grade (g). Utilizing the safe and comfortable grade dependent deceleration (a_g) from appendix B, it is valid for ($0 < a_g \leq a_{inst} \leq a_{fmax}$) and where ($a_{inst}$) is the maximum safe and comfortable deceleration value *defined* at a level roadway grade ($g = 0$) and (G) is the Earth's gravitational constant:

$$a_g = fG_Z + G_X = \frac{fG}{\sqrt{1+g^2}} + \frac{gG}{\sqrt{1+g^2}} = \frac{a_{inst} + gG}{\sqrt{1+g^2}} \approx a_{inst} + gG \tag{27}$$

Hence, the precise grade-dependant deceleration (a_g) is as follows:

$$a_g = \frac{a_{inst} + gG}{\sqrt{1+g^2}} \tag{28}$$

Where $0 < a_g \leq a_{inst} \leq a_{fmax}$

The absolute maximum deceleration is defined by (a_{fmax}), the available friction (f) between a vehicle's tires and the roadway, i.e., during an emergency stop, as described in appendix B. Equation (28) precisely redefines the instantaneous deceleration to become roadway grade (g) dependent.

The precise kinematic equations, including grade, becomes:

The Precise Critical Distance

$$x_c = v_0 t_{PR} + \frac{v_0^2}{2a_g} + \frac{v_0 a_g}{2j} \tag{29}$$

The Precise Kinematic Equation

$$Y \geq t_{PR} + \frac{1}{1 + \frac{v_1}{v_0}} \left(\frac{v_0}{a_g} + \frac{a_g}{j} \right) \tag{30}$$

The Precise Extended Kinematic Equation

$$Y \geq t_{PR} + \frac{v_0 - ½\, v_1}{a_g} + \frac{a_g}{2j} \tag{31}$$

Where $a_g = \frac{a_{inst} + gG}{\sqrt{1+g^2}}$ and $0 < a_g \leq a_{inst} = maximum$ as presented in Equation (28)

Conclusion

This technical tutorial has presented the science to substantially enhance state-of-the-art yellow change interval solutions by introducing a higher derivative of uniform vehicle motion. The new kinematic variable jerk exposes the instantaneous acceleration making the yellow traffic signals and the overall system performance precise. As demonstrated, the mathematical model is almost identical to measured data proving the model's accuracy and descriptive equations, thus proving the model's validity and mathematical analysis as shown in Figure 7. However, the precise theories are not limited to traffic signal timing. They also apply to other kinematic equations, such as improved emergency stopping distance and sight stopping distance calculations.

The presented precise timing models require a proper understanding of the equations' input variables and their limits and tolerances, as shown in Figure 10. The next step is to determine the minimum and maximum motion parameters that allow all driver-vehicle combinations legally on the road efficient, safe, and comfortable motions through signalized intersections. Historically the legacy guidelines do not recommend the maximum limits to allow all driver-vehicle combinations to stop safely and comfortably with a margin of tolerance. The results of the guidelines' missing limits force many drivers to stop uncomfortably trying to fit improperly timed yellow signals or run the risk of being ticketed for a traffic violation through no fault of their own.

Past traffic signal research has typically observed vehicles through intersections remotely. However, these remote observations do not provide any information if the motion was comfortable or not. Test equipment such as the Racelogic GPS data loggers also incorporates video and audio, allowing motion comfort feedback from the vehicle's occupants to determine their maximum limits during turning maneuvers and stopping. The Federal Highway Administration's present research goal should be to find the limits to the precise kinematic solutions' input variables, to finally enable the MUTCD to provide a uniform US traffic signal standard.

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In memory of
Marianne & Roland Järlström, Sigge & Ethel Wennlöf, Lois & David Hodge,
Don Peyton, Gordon Long, and Chris Strahm.

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APPENDIX A

Calculus Integration

The following presents the steps to derive the kinematic equations of motion from a constant jerk referenced in time.

The integration power rule is defined as follows (for $n \neq -1$):

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad \text{(A1)}$$

For a defined *constant* jerk over time, $(j(t) = j)$, integrate once for acceleration (a) by applying the power rule presented in Equation (A1) and where the rule's constant "C" is used as the initial acceleration (a_n) at the time (t_n). The resulting Equation (A2) calculates the instantaneous end acceleration (a_{n+1}) at the time (t_{n+1}) over the elapsed time $(\Delta t = t_{n+1} - t_n)$ as follows:

$$a_{n+1} = a_n + \int_{t_n}^{t_{n+1}} j(t)dt = a_n + j\Delta t \qquad \text{(A2)}$$

Second, apply the integration power rule on Equation (A2) to derive the end velocity (v_{n+1}) from an initial velocity (v_n), and acceleration (a_n):

$$v_{n+1} = v_n + \int_{t_n}^{t_{n+1}} a(t)dt = v_n + a_n\Delta t + \frac{j\Delta t^2}{2} \qquad \text{(A3)}$$

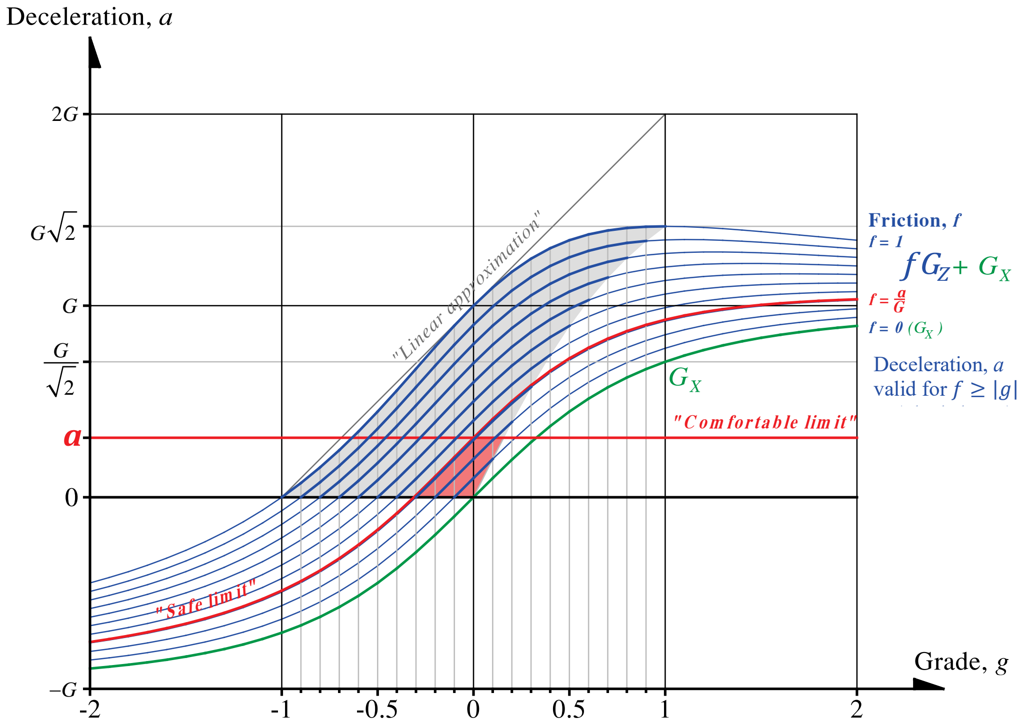
Third and final, repeat the integration rule on Equation (A3) to derive the traveled distance ending at position (x_{n+1}) from an initial position (x_n), an initial velocity (v_n), and an initial acceleration (a_n) yielding Equation (A4):

$$x_{n+1} = x_n + \int_{t_n}^{t_{n+1}} v(t)dt = x_n + v_n\Delta t + \frac{a_n\Delta t^2}{2} + \frac{j\Delta t^3}{6} \qquad \text{(A4)}$$

APPENDIX B

Safe and comfortable deceleration on a grade

Mats Järlström • Beaverton, Oregon, USA • March 28, 2015, • Rev. 6



Summary of equations and definitions:

$$G_X = \frac{gG}{\sqrt{1+g^2}} \approx gG \qquad G_Z = \frac{G}{\sqrt{1+g^2}} \approx G \qquad f = g = \frac{G_X}{G_Z} \qquad a = fG$$

Emergency stopping deceleration, a_{fmax} using maximum (static) friction coefficient, f_{max} on a grade, g, valid for $f_{max} \geq |g|$:

$$a_{fmax} = f_{max}G = f_{max}G_Z + G_X = \frac{f_{max}G}{\sqrt{1+g^2}} + \frac{gG}{\sqrt{1+g^2}} = \frac{G(f_{max}+g)}{\sqrt{1+g^2}} \approx G(f_{max}+g)$$

Safe and comfortable grade dependent deceleration, a_g is valid for $0 < a_g \leq a \leq a_{fmax}$ where (a) is the *defined* maximum safe and comfortable deceleration rate (i.e. $a = 10$ ft/s^2):

$$a_g = fG_Z + G_X = \frac{fG}{\sqrt{1+g^2}} + \frac{gG}{\sqrt{1+g^2}} = \frac{a+gG}{\sqrt{1+g^2}} \approx a + gG$$

For uphill and level grades ($g \geq 0$) the grade dependent deceleration, a_g is limited to the *defined* maximum safe and comfortable deceleration, a.

The weather dependent road/tire maximum friction coefficient, f_{max} and its corresponding maximum deceleration $a_{fmax} = f_{max}G$ is the overall limiting factor at any grade (i.e. emergency stopping).