Critical Phenomena, Conformal field theory and the MERA.

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Critical Phenomena

- Critical phenomena can be best explained through an example of the Ising Model given by the hamiltonian

$$H = -\sum_{j=1}^{N}\left(B\sigma_j^Z + J\sigma_j^X\sigma_{j+1}^X\right)$$

where σ^k are pauli operators. B is the external magnetic field and J is the exchange coupling. We will consider only the ferromagnetic case when $J > 0$.

- When $B \neq 0$ at low temperatures all spins tend to align and have the same value resulting a non zero total magnetisation $M =< \sigma_i >$.
- At high temperatures the spins are disordered and M vanishes for vanishing B.

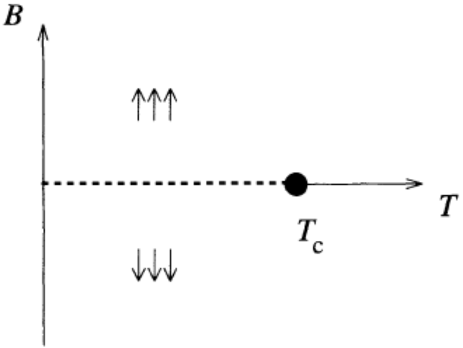


Figure: We can see that M changes sign at B=0. This is a first order phase transition. The critical point is located at $T = T_c$ and B = 0.

- At $T > T_c$ there could be clusters of spin of the same orientation. The length of these clusters is called correlation length ξ.
- The correlation length increases as T goes down and eventually it diverges at $T = T_c$. It sets a scale across which long range interactions become important.

Critical Exponents

- We can now define critical exponents (α, β, γ) for models near the critical point.

$$
\begin{aligned}
C &= -\frac{T}{N}\frac{\partial^2 F}{\partial T^2} \approx |t|^{-\alpha}, h = 0 \\
M &= -\frac{1}{N}\frac{\partial F}{\partial B} \approx (-t)^{\beta}, t < 0, h = 0 \\
\chi &= \frac{\partial M}{\partial \beta} \approx |t|^{-\gamma}
\end{aligned}
$$

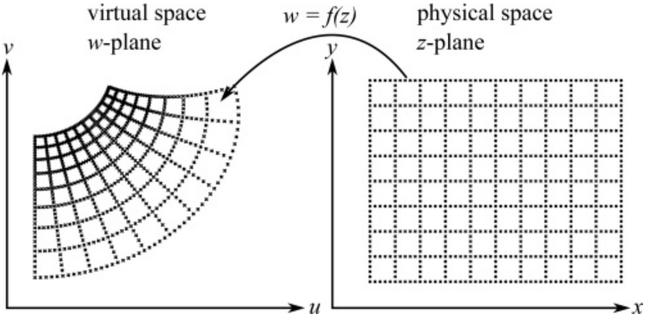
where $t = \frac{T - T_C}{T_c}$. These critical exponents determine the behaviour of many body systems close to the critical point.

Universality

- So far, the results we obtained are for the Ising model on a hypercubic lattice.
- One could also think of other models on different lattices with next-nearest neighbour interactions with different couplings.
- The value of critical exponents is found to be the same when the model is defined on hexagonal lattice. Independence of critical exponents on microscopic details is called universality.
- Since, near criticality we can describe the system by a small number of parameters, such systems must also have scale invariance.

Conformal Field theory

- Conformal transformations are coordinate transformations that leave the metric invariant upto a scale factor.
- Conformal transformations are also angle preserving maps.



- Conformal field theories are useful in condensed matter physics as they can describe continuous phase transitions in terms of a CFT that describes low energy physics.
- A conformal field theory and be completely described by the central charge, scaling dimensions and OPE coefficients. These are referred to as "conformal data".
- So, we see that the behaviour of both critical systems and conformal field theories only depends on a few parameters. We will make an analogy between these in the next few slides.

- For a CFT in two spacetime dimensions, fields transform under a scaling factor λ or rotations of angle θ.

$$z \to \lambda z, \varphi_\alpha(0) \to \lambda^{-\Delta_\alpha} \varphi_\alpha(0)$$
$$z \to e^{i\theta} z, \varphi_\alpha(0) \to e^{-i\theta S_\alpha} \varphi_\alpha(0)$$

- Here Δ_α and S_α are scaling dimensions and conformal spins respectively.

Central Charge

- The central charge c arises when we consider the Operator Product Expansion (OPE) of the energy momentum tensor.

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-2)^2} + \frac{\partial T(w)}{(z-w)} + \ldots$$

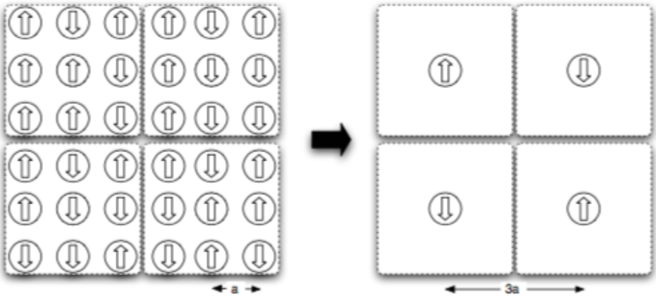
- c depends on the specific model.
- One can think of the central charge as a trace anomaly that arises when we define a CFT on a curved two dimensional manifold. The curvature induces a macroscopic scale in the system and the trace of the stress tensor, instead of vanishes is proportional to the curvature of the system.
- Models with the same central charge belong to the same universality class.

Analogy between critical systems and CFT

- Cardy, Blote and Nightingale discovered in 1986 (Phys, Rev Lett 56,742) that the energy and momenta of quantum critical spin chains is
$$E_\alpha = \frac{2\pi}{N}\left(\Delta_\alpha - \frac{c}{12}\right) + \mathscr{O}(N^{-x}), P_\alpha = \frac{2\pi}{N} S_\alpha$$
- Comparing this to the operator state correspondence that establishes that for each scaling operator φ_α there is an eigenstate $|\varphi_\alpha\rangle$ of the hamiltonian on a circle with energy and momentum
$$E_\alpha^{CFT} = \frac{2\pi}{L}\left(\Delta_\alpha - \frac{c}{12}\right), P_\alpha = \frac{2\pi}{L} S_\alpha$$
- Thus one can estimate scaling dimensions and central charge from energy and momentum computed on the lattice.

Renormalisation Group

- The idea of renormalisation in condensed matter systems was introduced by Kadanoff in 1966 where he proposed "block spin renormalisation".



- We can do these coarse graining transformations more explicitly. Consider a partition function

$$Z = Tr_s e^{-\mathscr{H}(s)}$$

- We can now define a projection operator that performs these coarse graining transformations

$$T(s'; s_1, ...s_9) = 1\, if \sum_i s_i > 0$$

Now the new coarse grained hamiltonian is defined as

$$e^{-\mathscr{H}'} = Tr_s T(s'; s_1, ..s_9) e^{-\mathscr{H}(s)}$$

- Once we have the coarse grained hamiltonian we can rewrite the partition function in terms of the new hamiltonian.
- This block spin procedure can be done exactly for a few models, but in general other renormalization schemes are necessary to study systems, on of which involves the MERA.
- For systems with scale invariance this procedure can be particularly very useful.

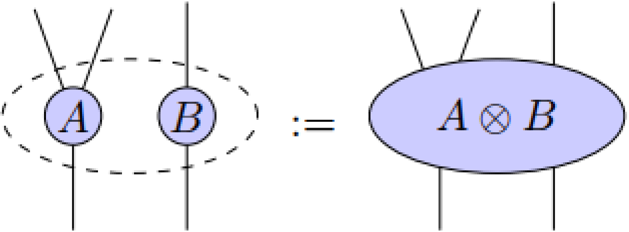
Tensor Networks

- Tensor networks are powerful tools to analyse low energy properties of quantum many body systems on a lattice.
- In tensor network notation a single tensor is simply represented by a geometric shape with legs sticking out of it, each corresponding to an index.

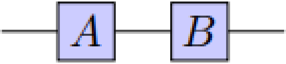
$$R^{\rho}{}_{\sigma\mu\nu} \Longrightarrow$$

Some Tensor Operations

- Tensor product $A^{\mu\nu}_{\rho} \otimes B^{\alpha}_{\beta}$



- Contraction $A^{\alpha}_{\beta} B^{\beta}_{\gamma}$



The MERA

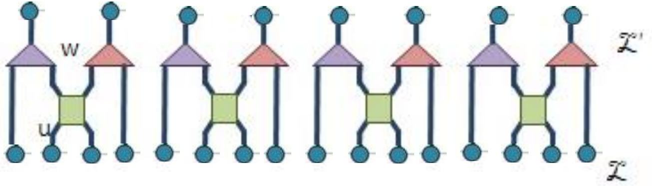
- Tensor networks can be used to represent wave functions of states on a lattice.
- There are various types of tensor networks that have been developed over the past which include Matrix Product States (MPS), Multiscale Entanglement Renormalization Ansaz (MERA), Tensor Network Renormalization (TNR), etc.
- For 1-d critical systems it turns out that the MERA can be used to efficiently represent states.

- Consider a 1-D Lattice $\mathscr{L}$, with each site characterised by a Hilbert space V. We can introduce a coarse graining transformation U which maps the lattice $\mathscr{L}$ to $\mathscr{L}'$

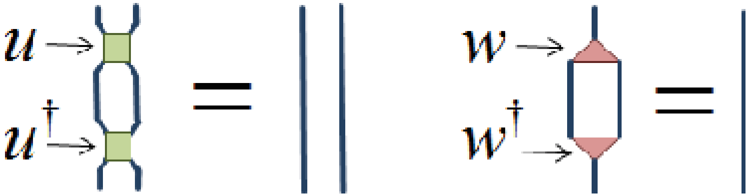
$$U^\dagger : \mathscr{L} \to \mathscr{L}^{'}.$$

- Thus a MERA defines coarse-graining transformations that leads to real space renormalization scheme called Entanglement Renormalization.

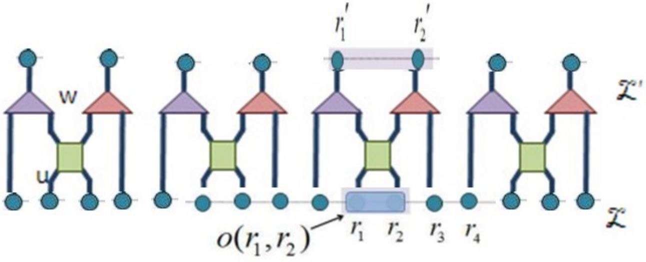
- The transformation U can be decomposed into two local transformations known as isometries w and disentanglers u.
$$u^\dagger : V^{\otimes 2} \to V^{\otimes 2}, w^\dagger : V^{\otimes 2} \to V^{\otimes 1}$$



Where u and w satisfy the constraint.

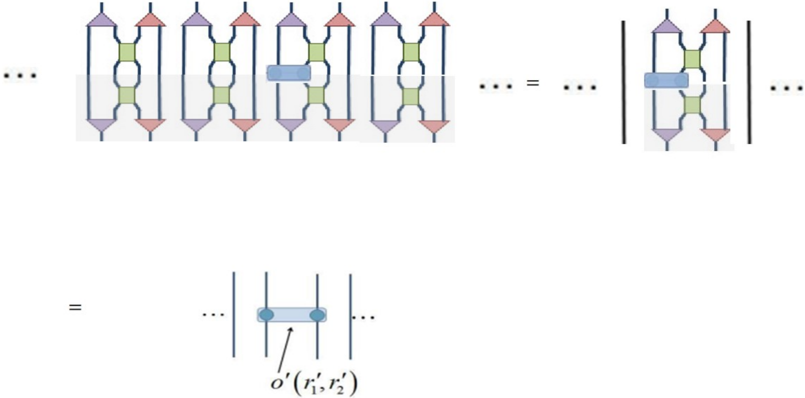


A feature of these coarse graining transformations is that it preserves locality.



Under these transformations a two site operator $o(r_1, r_2)$ becomes

$$U^\dagger o(r_1, r_2) U = \ldots \otimes \mathbb{1}' \otimes o'(r_{1'}, r_{2'}) \otimes \mathbb{1}' \otimes \ldots$$



When evaluating the two site operator, notice that the constraints on isometries and disentanglers annihilate to identity.

At this point it is useful to introduce Ascending superoperators $(\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3, \mathscr{A}_4)$.

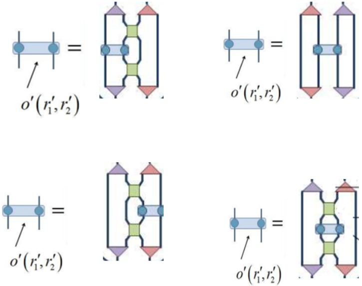


Figure: The specific choice of an ascending operator depends on the position in the lattice. We can now implement these coarse graining steps T times to get a sequence coarser lattices and thus get coarse grained operators.

$$o^{[0]} \xrightarrow{\mathscr{A}^{[0]}} o^{[1]} \xrightarrow{\mathscr{A}^{[1]}} o^{[2]} \ldots \xrightarrow{\mathscr{A}^{[T-1]}} o^{[T]}$$

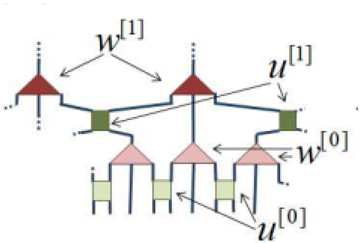
- So far we have considered Entanglement Renormalization (RE) transformations which generate RG flow.
- We can also consider the inverse RG flow. For a quantum state $|\Psi^{[T]}\rangle$ defined on a lattice $\mathscr{L}^{[T]}$, we can use the transformation $U^{[T-1]}$ to get a state $|\Psi^{[T-1]}\rangle$, and repeat coarse graining steps to obtain $|\Psi^{[0]}\rangle$.

$$|\Psi^{[0]}\rangle = U^{[0]} U^{[1]} ... U^{[T-1]} |\Psi^{[T]}\rangle$$

  A multiscale entanglement renormalization ansatz (MERA) is the class of states $|\Psi^{[0]}\rangle$ that can be defined by some choice of $(U^{[0]}, U^{[1]} ... U^{[T-1]}$ and a state $|\Psi^{[T]}\rangle$.
- We assume that the number of layers is chosen as $T =\approx \log N$ so the state on $\mathscr{L}^{[T]}$ has a small number of parameters.

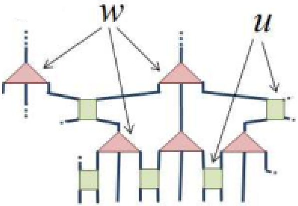
Translation Invariance

- A MERA of N lattice sites has $T \approx \log N$ layers and $O(N)$ different tensors. If we choose all the tensors to be the same, then the computation cost can be reduced to $O(\log N)$.
- However, as we have seen before the structure of coarse-graining is not homogeneous. So a MERA is non-translation invariant.
- That being said, the violations of translation invariance becomes negligible once the tensors are properly optimised.



Scale invariance

- Scale invariance can be easily enforced by choosing all disentanglers and isometries to be copies of a single pair u and w.



Entanglement Entropy

- The entanglement entropy S_L of a block of L contiguous sites can be seen to scale as the logarithm of L.

$$S_L \approx \frac{c}{3} \log L$$

- This can be proven by by studying minimally connected regions and geodesic paths in the (discrete) holographic geometry generated by the tensor network (arXiv:1106.1082).

Extraction of Central charge

- Since the entanglement entropy of the MERA scales as the log of L we can use it to compute the central charge.
- For a properly optimized MERA we can easily compute the fixed point two site density matrix and then compute the Von Neumann Entropy.

$$S(\rho) = -tr(\rho \log(\rho))$$

Then, the central charge is given by

$$S(\rho) - S(\rho^{(1)}) = \frac{c}{3}(\log 2 - \log 1)$$

Here $\rho^{(1)}$ is the one site reduced density matrix that can be obtained by tracing over the other site of the two site reduced density matrix.

Extraction of conformal dimensions

- A few slides ago we had defined the Ascending Superoperator that performs coarse graining transformation.
- Then, the conformal dimensions can be obtained by

$$S(\phi_\alpha) = \lambda_\alpha \phi_\alpha$$

where

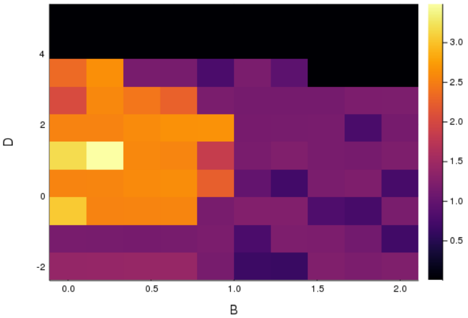
$$\Delta_\alpha = -\log_2 \lambda_\alpha$$

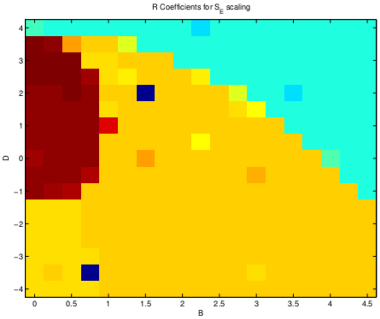
Thus we can see that the conformal spins can be obtained by diagonalizing the ascending superoperator.

Bilinear-Biquadratic spin 1 chain $\beta = 1$

$$h_{i,i+1} = S_i.S_{i+1} + \beta(S_i.S_{i+1})^2 + B\,S_i^x + D\,(S_i^z)^2$$

$$H = \Sigma h_i$$





Some Open Questions and future work

- Optimization of the MERA.
- Extraction of conformal spins.
- Find new ways to extract conformal data (Koo Saleur Formula).