Secant Method

```
function x = mySecantRoot(f, x0, x1, N)
     y0 = f(x0);
     y1 = f(x1);
     for i=1:n
       x = x1 - (x1 - x0)* y1/ (y1 - y0);
       y = f(x);
       x0 = x1;
       y0 = y1;
       x1 = x;
       y1 = y;
     end
end
```

Regula Falsi Method (False position)- combination of bisection and secant

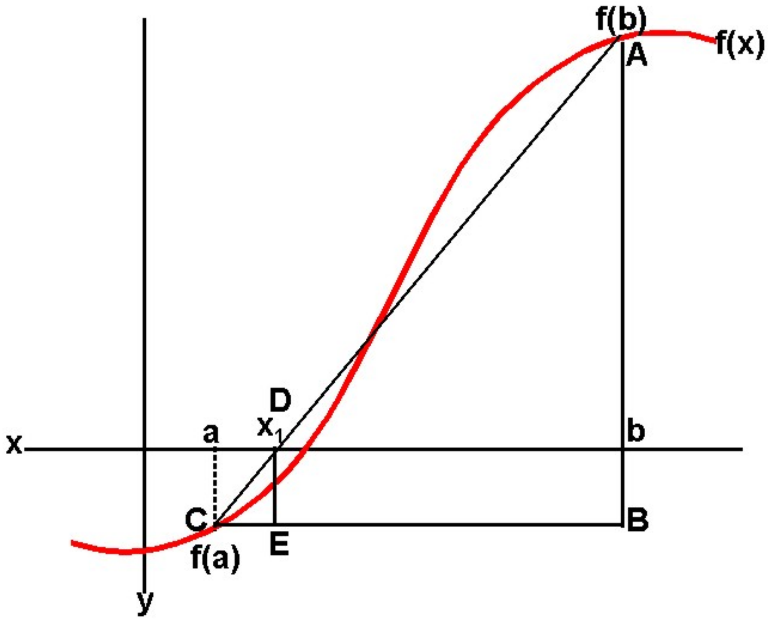
```
Inputs: a, b, MaxIter, Tolerance
Initialize: iteration = 0, xold = b (to start with)
   while (iteration <= MaxIter)
      compute f(a)
      compute f(b)

      x = a - ((f(a)*(b-a))/ (f(b) - f(a))
      Test convergence:
      if (|x - xold|/|x|) < Tolerance  then
         output x
         goto stop
      else
         output (iteration, a, b, x)
         xold = x
         compute f(x)
         if f(a) * f(x) > 0
              a = x
         else
              b = x
         end
```

```
      end
   end
stop
```

Regula-falsi method (Algorithm)

1. Find points a and b so that $a < b$ and $f(a)\,f(b) < 0$
2. Take the interval $[a, b]$, determine the next value x_1
3. If $f(x_1) = 0$ then x_1 is the exact root



4. else if $f(x_1)\,f(b) < 0$ then let $a = x_1$
5. else if $f(a)\,f(x_1) < 0$ then let $b = x_1$
6. Repeat steps 2-5 until $f(x_i) \leqslant$ Tolerance

Symbolic computations in Matlab

| syms x y (Declare variables used to be symbolic)

| fxy = cos(x)*sin(y) + 5*x^2*y (Define function)

| f = sin(x)+3*x^2

| subs(f, pi) (to find specific value of the function)

| g = exp(-y^2)

| h = compose(g,f) (Define $h(x) = g(f(x))$)

| k = f * g

| subs(k, [x,y], [0,1])

|f1=diff(f)

|k1x = diff(k,x)

|F=int(f),   Fd=int(f,0,2*pi)

Symbolic computation

Plot a symbolic function of one variable:

|ezplot(f), ezplot(g,-10,10)

Format a symbolic function of two variables

|ezsurf(k)

Simple algebra

poly = (x - 3)^5

polyex = expand(poly)

polysi = simplify(polyex)

solve(f), solve(g)

Linear Algebra

A=[1 3 -2 5; -1 -1 5 4; 0 1 -9 0]

u = [1 1 1 1]

z = A * u

B = [3 2 1; 7 6 5; 4 3 2]

B*B vs. B.* B

B^3 vs B.^3

Inverse of a matrix

C=randn(5,5)

inv(C)

Identity matrix I=eye(3) (3x3 identity matrix)

Norm of a matrix

If A is a $m \times n$ matrix, the norm of A is defined as

$$\|A\| = \max_{\|\boldsymbol{v}\|=1} \|A\boldsymbol{v}\|$$

Maximum is taken over all vectors with unit length. Norm of a matrix is defined as the largest factor by which it stretches or contracts a unit vector.

$$\|A\boldsymbol{v}\| \leqslant \|A\|\|\|\boldsymbol{v}\|\|$$

norm(A) norm of identity matrix is one.

size(A) dimensions of A; det(A) determinant; max(A) maximum of each column; min(A) minimum of each column; sum(A) sum of each column; mean(A) average of each column; diag(A) diagonal of each column; A' transpose of A.

$$
\begin{aligned}
A_{11}x_1 + A_{12}x_2 + \ldots + A_{1n}x_n &= b_1 \\
A_{21}x_1 + A_{22}x_2 + \ldots + A_{2n}x_n &= b_2 \\
A_{31}x_1 + A_{32}x_2 + \ldots + A_{3n}x_n &= b_3 \\
&\vdots \\
A_{n1}x_1 + A_{n2}x_2 + \ldots + A_{\mathrm{nn}}x_n &= b_n
\end{aligned}
$$

Matrix notation

$$
\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{\mathrm{nn}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

Augmented coefficient matrix

$$
[A|\boldsymbol{b}] = \left[\begin{array}{cccc|c} A_{11} & A_{12} & \cdots & A_{1n} & b_1 \\ A_{21} & A_{22} & \cdots & A_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{\mathrm{nn}} & b_n \end{array}\right]
$$

Uniqueness of Solution

A system of n linear equations in n unknowns has a unique solution, provided that the determinant of the coefficient matric is nonsingular ($\det(A) \neq 0$).

The rows and columns of a nonsingular (invertible) matix are linearly independent i.e. no row or column can be expressed as a linear combination of other rows or columns.

When the coefficient matrix is singular, the equations have an infinite number of solutions or no solutions at all.

$2x + y = 3; 4x + 2y = 6 \,(\text{infinite solutions})$

$2x + y = 3; 4x + 2y = 0 \,(\text{no solutions})$

Ill-Conditioning

$$\|A\| = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} A_{\mathrm{ij}}^2}, \qquad \|A\| = \max_{1 \leqslant i \leqslant n} \sum_{j=1}^{n} |A_{\mathrm{ij}}|$$

$$\mathrm{cond}(A) = \|A\| \, \|A^{-1}\|$$

When $\mathrm{cond}(A) \approx 1,$ the matrix is well-conditioned.

Condition number increases with the degree of ill-conditioning.

$2x + y = 3;\ 2x + 1.001y = 0$ $\qquad$ $\text{Solutions: } x = 1501.5,\ y = -3000$

$\det(A) = |A| = 0.002 \text{ is much smaller than the coefficients (Ill} - \text{conditioned)}$

$2x + y = 3;\ 2x + 1.002y = 0$ $\qquad$ $\text{Solutions: } x = 751.5,\ \ y = -1500$

0.1% change in coefficient produces 100% change in solution.

$$
\begin{array}{lrcl}
E_1: & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\
E_2: & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\
 & & \vdots & \\
E_n: & a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{\mathrm{nn}}x_n & = & b_n
\end{array}
$$

The coefficients are $a_{\mathrm{ij}}\,(i, j = 1, 2, \ldots, n)$ and b_i. We need to find unknowns

$x_1, x_2, \ldots, x_n$.

Three operations to simplify linear system:

- Equation E_i can be multiplied by any nonzero constant λ with the resulting equation replacing the original one. $(\lambda E_i) \longrightarrow (E_i)$
- Equation E_j can be multiplied by any constant λ and added to equation E_i with the resulting equation used in place of E_i. $(E_i + \lambda E_j) \longrightarrow (E_j)$
- Equations E_i and E_j can be transposed in order. $(E_i) \leftrightarrow (E_j)$.

Illustration

$$\begin{array}{lrcr} E_1: & x_1+x_2+0x_3+3x_4 & = & 4 \\ E_2: & 2x_1+x_2-x_3+x_4 & = & 1 \\ E_3: & 3x_1-x_2-x_3+2x_4 & = & -3 \\ E_4: & -x_1+2x_2+3x_3-x_4 & = & 4 \end{array}$$

Augmented matrix

$$\left[\begin{array}{rrrr|r} 1 & 1 & 0 & 3 & 4 \\ 2 & 1 & -1 & 1 & 1 \\ 3 & -1 & -1 & 2 & -3 \\ -1 & 2 & 3 & -1 & 4 \end{array}\right]$$

Use E_1 to eliminate x_1 from E_2, E_3 and E_4

Do

$(E_2 - 2E_1) \rightarrow (E_2)$; $(E_3 - 3E_1) \rightarrow (E_3)$; $(E_4 + E_1) \rightarrow (E_4)$

Augmented matrix

$R_2 \to R_2 - 2R_1; R_3 \to R_3 - 3R_1; R_4 \to R_4 - (-R_1)$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 4 \\ 2 & 1 & -1 & 1 & 1 \\ 3 & -1 & -1 & 2 & -3 \\ -1 & 2 & 3 & -1 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 4 \\ 0 & -1 & -1 & -5 & -7 \\ 0 & -4 & -1 & -7 & -15 \\ 0 & 3 & 3 & 2 & 8 \end{array}\right]$$

$R_3 \to R_3 - 4R_2; R_4 \to R_4 - (-3R_2)$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & 4 \\ 0 & -1 & -1 & -5 & -7 \\ 0 & 0 & 3 & 13 & 13 \\ 0 & 0 & 0 & -13 & -13 \end{array}\right]$$

Steps

1. If $a_{11} \neq 0$, we perform operations $(R_j - (a_{j1}/a_{11})R_1) \to (R_j)$ for each $j = 2, 3, \ldots, n$ to eliminate coefficient of x_1 in each of the j rows.

2. Sequential procedure for $i = 2, 3, \ldots, n-1$:

$(R_j - (a_{\mathrm{ji}}/a_{\mathrm{jj}})R_i) \to (R_j)$ for each $j = i+1, i+2, \ldots, n$

Backward substitution

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ & a_{22}^{(1)} & a_{23}^{(1)} & b_2^{(1)} \\ & & a_{33}^{(2)} & b_3^{(2)} \end{array}\right]$$

$$\begin{gathered} x_3 = b_3^{(2)} \,/\, a_{33}^{(2)} \\ x_2 = \left(b_2^{(1)} - a_{23}^{(1)} x_3\right) / \, a_{22}^{(1)} \\ x_1 = \left(b_1 - a_{12} x_2 - a_{13} x_3\right) / \, a_{11} \end{gathered}$$

General form

$$\begin{aligned} x_n &= b_n^{(n-1)} \,/\, a_{\mathrm{nn}}^{(n-1)} \\ x_i &= \left(b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{\mathrm{ij}}^{(i-1)} x_j\right) / \, a_{\mathrm{ii}}^{(i-1)} \\ \text{for } \ i &= \ n-1, n-2, \ldots, 1 \end{aligned}$$

Algorithm (Gaussian elimination with backward substitution)

Input: number of equations (unknowns) n; augmented matrix $A=[a_{\mathrm{ij}}]$, $1\leqslant i\leqslant n$ and $1\leqslant j\leqslant n+1$

Output: solution $x_1, x_2, \ldots, x_n$ or "no unique solution"

1. For $i=1,\ldots,n-1$ do steps $2-4$:
2. Find the smallest integer p with $i\leqslant p\leqslant n$ and $a_{\mathrm{pi}}\neq 0$.

   If no p found,then OUTPUT(no unique solution exists), STOP
3. If $p\neq i$ then perform $(R_p)\leftrightarrow(R_i)$
4. For $j=i+1,\ldots,n$ do:

   $m_{\mathrm{ji}}=a_{\mathrm{ji}}/a_{\mathrm{ii}}$;

   $(R_j-m_{\mathrm{ji}}R_i)\rightarrow(R_j)$
5. If $a_{\mathrm{nn}}=0$ then OUTPUT(no unique solution exists)
6. Set $x_n=a_{n,n+1}/a_{\mathrm{nn}}$
7. For $i=n-1,\ldots,1$ set $x_i=[a_{i,n+1}-\sum_{j=i+1}^{n}a_{\mathrm{ij}}x_j]/a_{\mathrm{ii}}$
8. OUTPUT$(x_1,\ldots,x_n)$, STOP