Graph Theory

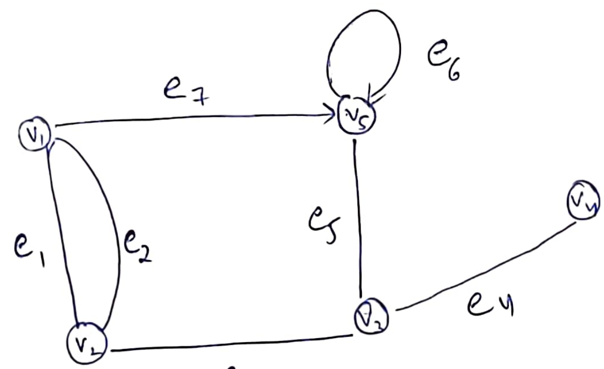
Definition: A graph is a pair (V, E) of ~~sto~~ Sets satisfying $E \subseteq V \times V$

Subset

V = Set of Vertices

E = Set of Edges.

A graph $G' = (V', E')$ is called a Subgraph of graph G if $V' \subseteq V$ & $E' \subseteq E$



$G = (V, E)$

$\left\{ \begin{array}{l} e_1 \,\&\, e_2 \text{ are Parallel edge} \\ e_6 \text{ is a loop.} \end{array} \right\}$

If there is an edge joining V_i & V_j then ~~tg~~ they are adjacent otherwise they are non-adjacent.

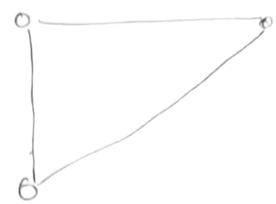
Two or moree edges with the same end points are Called Parallel edges.

A graph is simple if it has no loop or parallel edges.

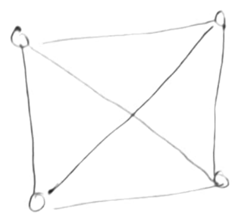
Complete Graph : A Graph is Complete if every pair of Vertices are adjacent.

K_1

K_2



K_3



K_4

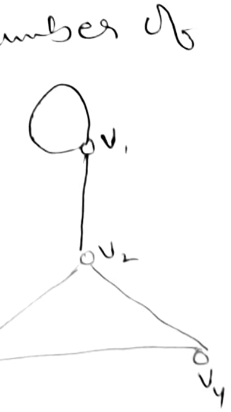
A complete graph with n vertices ~~have~~ is denoted by K_n

The number of edges in a complete graph with n vertices is $^nC_2 = \frac{n!}{(n-2)!\,2!} = \frac{n(n-1)}{2}$ edges.

A Graph is finite if both the number vertices (Cardinality) $|V|$ and number of edges (Cardinality of E) $|E|$ is infinite.

A Graph is infinite if $|V|$ or $|E|$ is infinite.

Def: The degree of a vertex v in a graph G, denoted by $d(v)$, is the number of edges of G incident to v



$d(v_3) = 2$ $\quad \sum_{i=1}^{4} d(v_i) = 10$

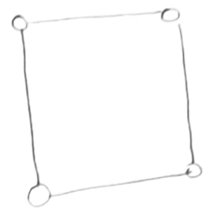
$d(v_4) = 2$ $\quad |E| = 5$

$d(v_2) = 3$ $\quad \Delta(G) =$ max - ~~deg~~ degree $= 3$

$d(v_1) = 3$ $\quad \delta(G) =$ min - degree $= 2$

A Graph G is Said to be k-regular, if $d(v) = k$ for all vertices $v \in V$.

A Complete Graph on n vertices is $(n-1)$-regular.



C_4: 2-regular Graph

Theorem: If a graph G with edges has vertices $v_1, v_2, \ldots, v_n$, then

$$\sum_{i=1}^{n} d(v_i) = 2|E|$$

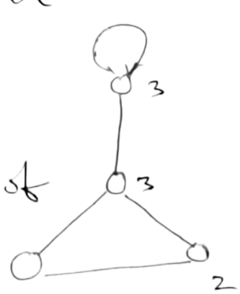
Proof: Every edge contributes two degree in the degree sum.

$$\sum_{i=1}^{n} d(v_i)$$

Corrolary: In a graph G, the average degree is $\frac{2|E|}{|V|}$

hence $\delta(G) \leq \frac{2|E|}{|V|} \leq \Delta(G)$

Theorem: In any Graph G the number of vertices with odd degree is even



Proof: Let $v_1, v_2, \ldots, v_n$ be vertices of G.

Suppose $v_1, v_2, \ldots v_t$ are odd vertices

$v_{t+1}, v_{t+2}, \ldots v_n$ are even vertices (in terms of degree)

$$d(v_1) + d(v_2) + \ldots + d(v_t) + d(v_{t+1}) + \ldots + d(v_n) = 2|E|$$

$$d(v_1) + d(v_2) + \ldots + d(v_t) = \underbrace{2|E|}_{\text{even}} - \underbrace{(d(v_{t+1}) + \ldots d(v_n))}_{\text{even}}$$

$$= \text{even}$$

$d(v_1), d(v_2), \ldots, d(v_t)$ are all odd

but the sum is even, So t must be even

$\Rightarrow$ # of vertices with odd degree is even **Proved**

Walk

Def: A walk in a Graph or multi Graph (Parallel edges & loops are allowed) is an alternating sequence of vertices & edges

$v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}\, e_n\, v_n$ begining and ending with vertices. such that $e_i = (v_{i-1}, v_i)$

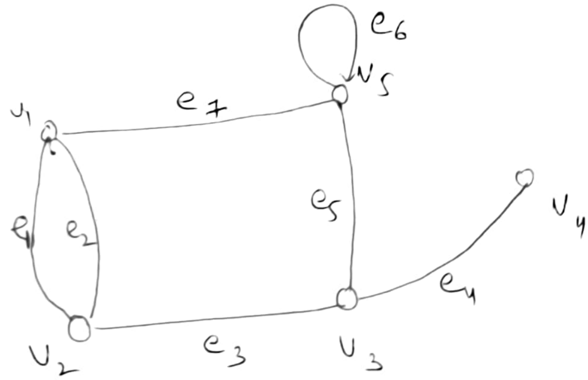
This walk joins v_0 and v_n; is called (v_0, v_n) walk

A walk is Closed if $v_0 = v_n$; otherwise open walk.

→ It is a trail if the edges are distinct.

→ It is a path if the vertices are distinct

→ If $v_0 = v_n$ but all othe vertices are distinct then it is a cycle



(v_1, v_5) walk: $v_1 e_2 v_2 e_3 v_3 e_4 v_4 e_4 v_3 e_5 v_5$

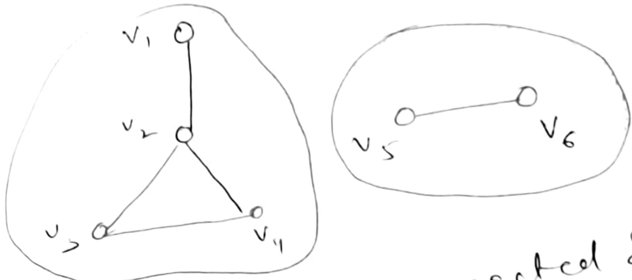
(v_1, v_5) Trail: $v_1\, e_2\, v_2\, e_3\, v_3\, e_5\, v_5$

(v_1, v_1) Cycle: $v_1\, e_2\, v_2\, e_3\, v_3\, e_5\, v_5\, e_7\, v_1$

(v_1, v_5) Path: $v_1\, e_2\, v_2\, e_3\, v_3\, e_5\, v_5$

A Graph is connected if every pair of vertices are joined by a path.

If a graph is not connected then it is called disconnected graph. and its maximal connected subgraphs are called components



G is disconnected & it has two components

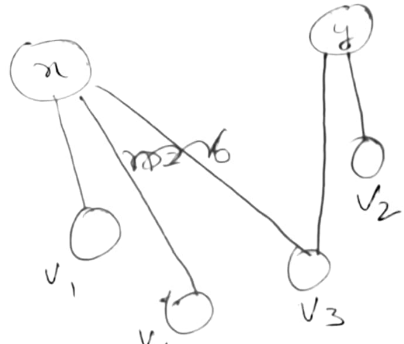
Theorem: Let G be a graph of order n (n vertices). If $d(u)+d(v) \geq n-1$ for every two non adjacent vertices u & v of G then G is connected.

Proof: We need to prove that every two vertices of G are connected by a path

Let $x, y \in V$ if $(x,y) \in E$, x & y are adjacent.

Assume that ~~x~~ $(x,y) \notin E$

$d(x) + d(y) \geq n-1$ implies there must be a vertex v_i that is adjacent to both x & y



$n = 6$

$d(x) + d(y) \geq 5$
$3 + 2$

Pigeonhole Principle: If $(n+1)$ pigeons are put in n holes then at least one hole contains more than or equal to 2 pigeons.

($x - v_i - y$: path of length 2) G is connected

Another Proof

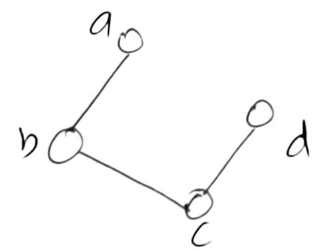
If G is a graph of order n with $\delta(G) \geq \frac{(n-1)}{2}$, then G is connected

Proof for every two non-adjacent vertices u & v

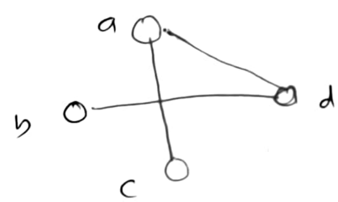
$$d(u) + d(v) \geq \frac{n-1}{2} + \frac{(n-1)}{2} = (n-1)$$

then G is connected.

Complement of a Graph G, denoted by $\bar{G}$ has the same set of vertices as G, but two vertices are adjacent in $\bar{G}$ iff they are non-adjacent in G.



G



$\bar{G}$

Theorem: If G is disconnected Graph then $\overline{G}$ is connected.

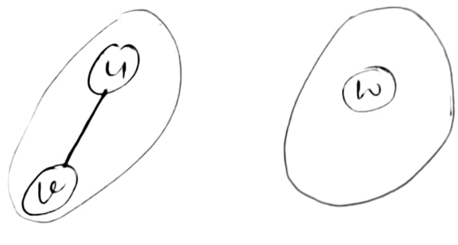
Proof: If u & v are two vertices of G $u, v \in V(G)$, we must find a path in $\overline{G}$ joining u & v.

Case 1 u & v are in two different components of G.

u & v are not adjacent in G, hence u & v are adjacent in $\overline{G}$

Case 2 u & v are in the same component

they may or may not be adjacent in G



Let w be a vertex in another component of G, then w is non adjacent to u & v

$\Rightarrow$ w is adjacent to u & v in $\overline{G}$

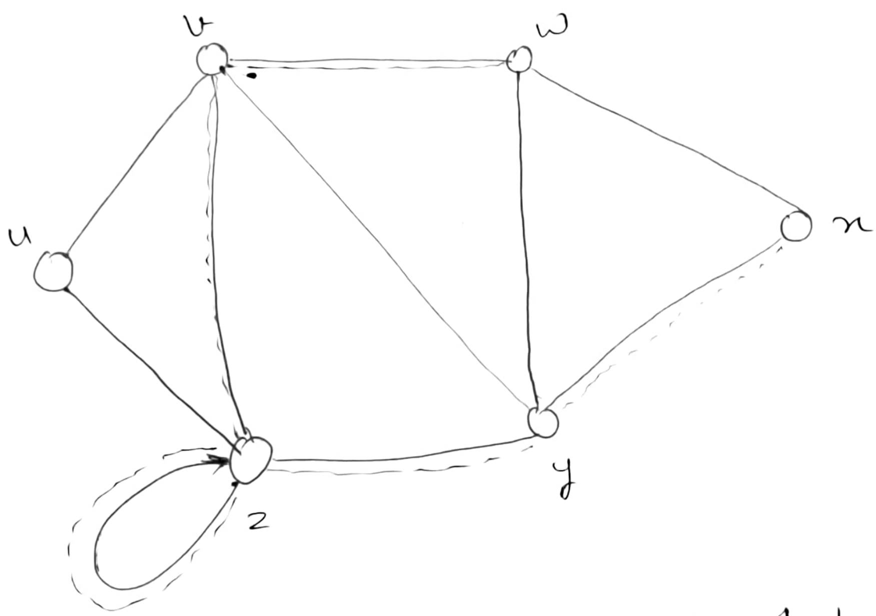
Thus there is a path uwv joining u & v in $\overline{G}$

hence $\overline{G}$ is a connected Graph.

Eulerian Trail

A trail is a walk in which all edges are distinct.

A Path is a walk in which all vertices & all edges are distinct



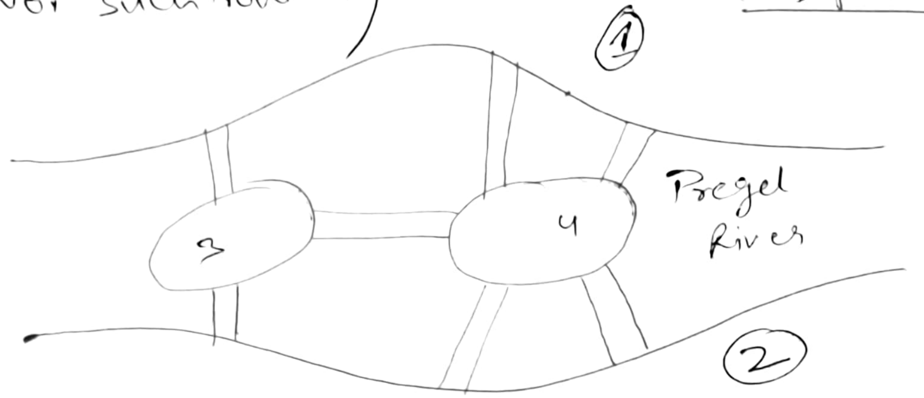
The ~~path~~ walk $x\,y\,z\,z\,v\,w$ is a trail of length 5

The walk $x\,y\,z\,v\,w$ is a Path of length 4

Eulerian Trail: is a closed trail which includes all the edges of Graph G

Konigsberg Seven Bridge Problem (1736) (Graph theory Started with this problem

(Euler proved that No such route is possible)

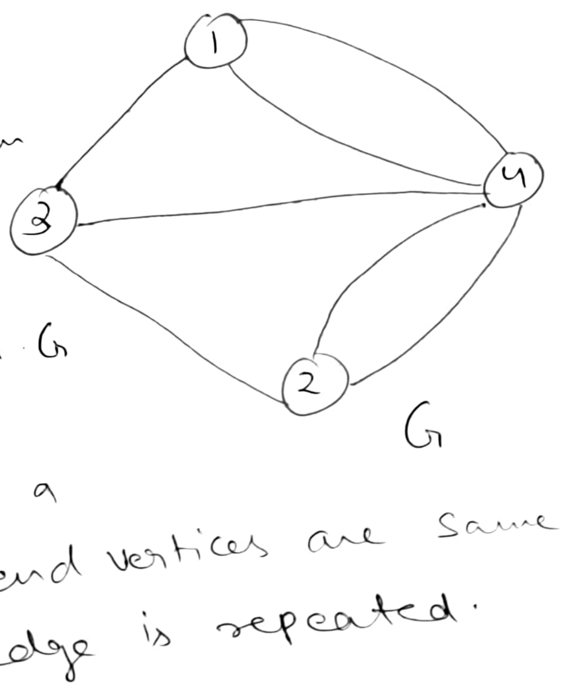


Attempt to find a route that would take them over each bridge exactly once and return to the Starting Point.

Euler Shows that no such route exists:

Represent the above scenario as a graph.

Finding a route in a KSB Problem is simmilar to find a Eulerian trail in Graph G



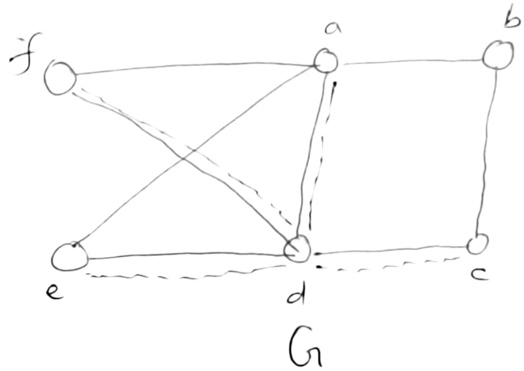
Multigraph

Eulerian Trail is a trail with end vertices are same i.e it is a loop in which no edge is repeated.

A Graph which have an Eulerian Trail is also known as Eulerian Graph.

Theorem: A Connected Graph/Multigraph G is Eulerian if and only if every vertex has even degree

Proof Let T be an Eulerian Trail ~~of~~ of Graph G.



T: a f d e a b c d a

Claim: Each occurrence of a vertex in T contributes 2 to the degree of the vertex

Thus degree of a vertex is sum of 2's only : So it is even.

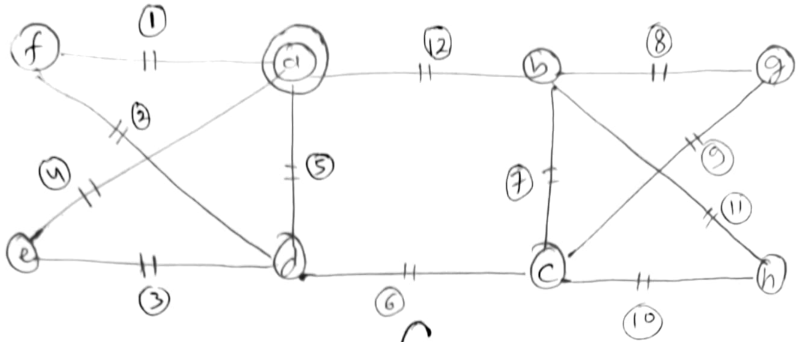
Algorithm to find Eulerian Trail

- Select a vertex & form an Eulerian Trail starting at that vertex.

Fleury's Algorithm

At each step, we move across an edge whose deletion does not result in more than one component unless we have no choice.

At the end of the algorithm there are no edges left and sequence of edges we moved across form a Eulerian Trail.



G

every vertex have even degree.

a, b, c, d have degree 4

f, e, g, h have degree 2

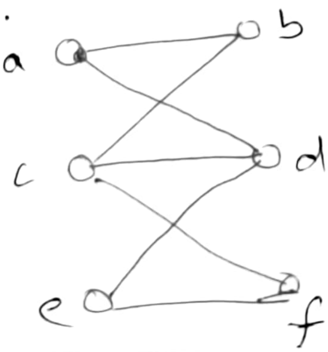
Eulerian Trail: a f d e a d c b g c h b a

~~A Hamiltonian path/Spanning path~~ in

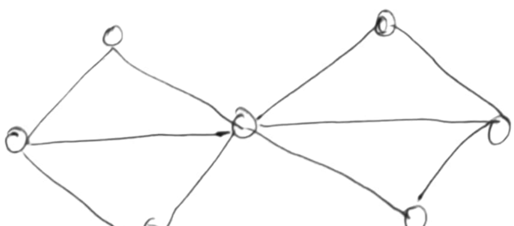
A Hamiltonian Cycle in a graph G is a cycle that includes all the vertices of the Graph

A Hamiltonian Path/Spanning path in a Graph G is a path that includes all the vertices of the Graph.

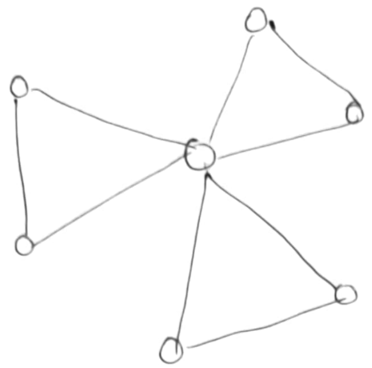
A Graph is Hamiltonian if it contains a hamiltonian cycle.



G2



G1



G_3

Hamiltonian cycle in G_2 is a b c f e d a

G_2 is a Hamiltonian Graph.

Unlike Eulerian Graph, there is no known Condition that can be used to characterize if a given graph is Hamiltonion.

Theorem: If G is a simple graph with $n (\geq 3)$ vertices and if degree of every vertex is $d(v) \geq \frac{n}{2}$ for each vertex v, then G is Hamiltonion. (This not a neccessary condition but a sufficient condition)

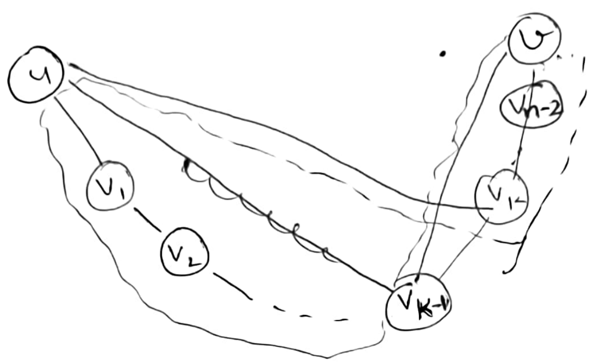
(due to P. Dirac)

Proof:

Let G be such a counter example to the theorem so that no graph on n vertices with more edges than G is also a counter example.

Let u & v be two nonadjacent vertices of G.

then there is a Hamiltonian path joining u & v in G



Let $d(u) = k \geq n/2$

Now we prove that if u is adjacent to v_k then

v_{k-1} can not be adjacent to v

If possible let v be adjacent to v_{k-1}

then we have the Hamiltonian cycle

$v\, v_{n-2} \cdots v_k\, u\, v_1, v_2 \cdots v_{k-1}\, v$

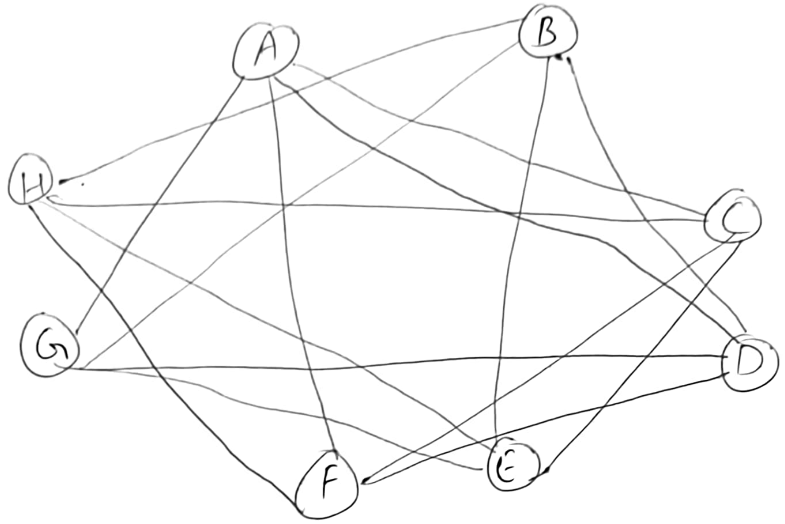
Since we assumed G is not Hamiltonian this is not possible, this implies v is not adjacent to v_{k-1}.

So v is not adjacent to atleast k of the $(n-1)$ vertices

Thus $d(v) \leq n-1-k \leq n-1-n/2 = \frac{n}{2} - 1$

This is a contradiction; Thus G must be Hamiltonian.

Ex According to Dirac's Theorem the following graph contains a Hamiltonian cycle. Find it.



$|V| = 8$

$d(v) = 4 \geq n/2 = \frac{8}{2} = 4$

Find the Hamiltonian cycle.

A C F H B E G D A

Theorem: Let G be a simple graph on n $(n \geq 3)$ vertices. Suppose $u, v \in V$ and such that $(u,v) \notin E$ and $d(u) + d(v) \geq n$. In this case G is Hamiltonian ~~if only if~~ iff $G + (u,v)$ is Hamiltonian.

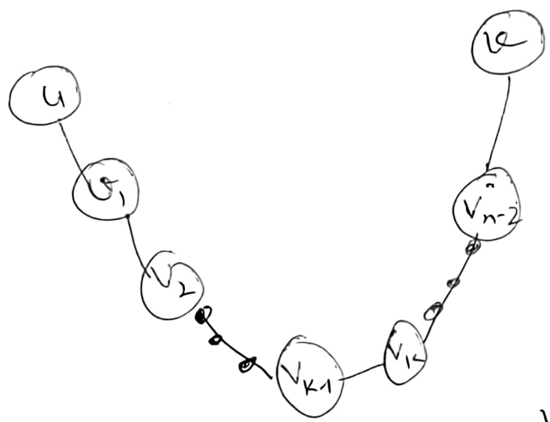
Proof G is hamiltonian then obviously $G+(u,v)$ is Hamiltonian.

Give that $G+(u,v)$ is hamiltonian

We need to prove that G is Hamiltonian.

Suppose G is not hamiltonian

then there is a hamitonian path or Spanning Path in G joining u and v



If u is adjacent to v_k then v can not be adjacent to v_{k-1}

let $d(u)=k$ so v is ~~to~~ not adjacent to atleas k of the $n-1$ vertices

Thus the $d(v) \leq n-1-k$

$$d(u)+d(v) \leq n-1-k+k$$
$$\leq n-1$$

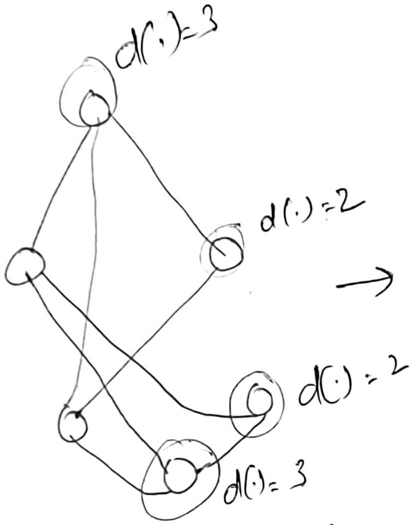
[because we have $d(u)+d(v) \geq n$]

which is a Contradiction

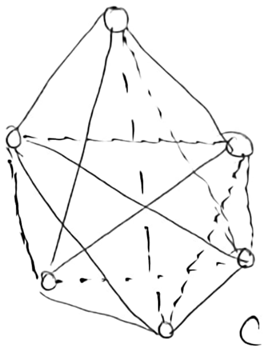
So G is hamiltonian

Closure $C(G)$ of a graph of order n is obtained from G by recussively joining pair of non adjacent vertices whose degree sum is atleast n untill no such pair exist

Ex

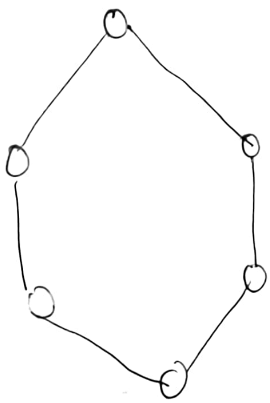


G $\quad n=6$

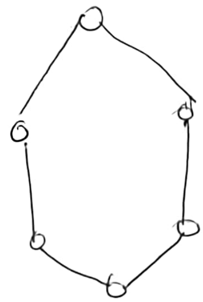


$C(G)$

Complete graph K_6



$G = C_6$



$C(G)$

Closure of Graph is Graph it self

Theorem : A Graph is hamiltonian iff it Closure is Hamiltonian.

Proof G is hamiltonian if G + (u, v) is Hamiltonian & $d(u) + d(v) \geq n$ $(u, v) \notin E$

To establish a graph is hamiltonian it is sufficient if we show that its Closure is Complete.

If the Closure of a graph is complete graph then the closure is hamiltonian.

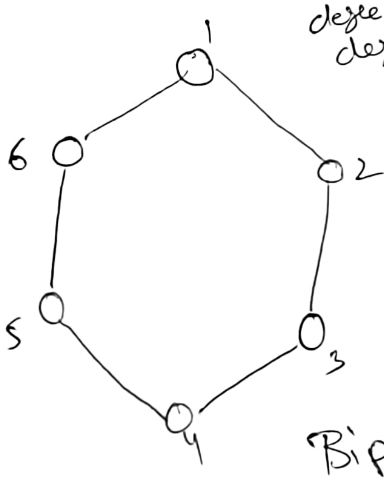
So G is hamiltonian.

Bipartite Graphs

A Bipartite Graph is a graph whose Vertices can be divided into two parts. A & B Such that every edge is Connect / joins ~~vertices~~ vertex in A to vertex in B.

$G = (A \cup B, E)$

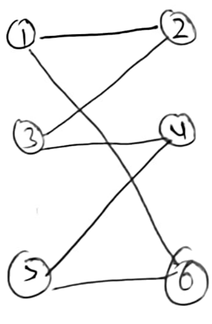
Example



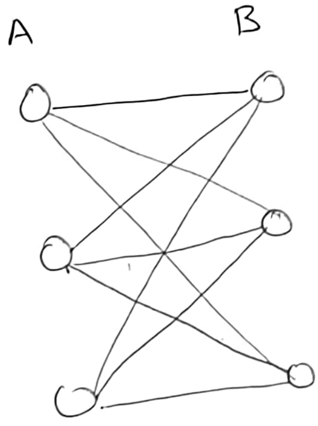
$A = \{1, 3, 5\}$

$B = \{2, 4, 6\}$

degree Sum of A = degree Sum of B



Bipartite.



$K_{3,3}$

Complete Bipartite

Ex Show that 4-regular Graph having 15 vertices cannot be a Bipartite

Graph is K-regular if every vertices have degee K

$$|V| = 15$$

$$V = A \cup B$$

Let $|A| = x$

$|B| = 15 - x$

$$\sum_{v \in A} d(v) = \sum_{v \in B} d(v)$$

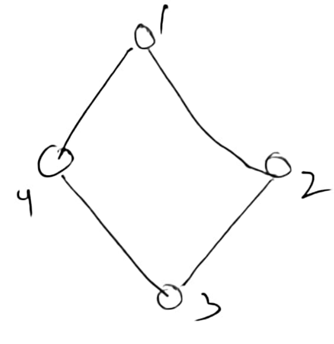
$$\Rightarrow 4x = (15-x)4$$

$$8x = 60$$

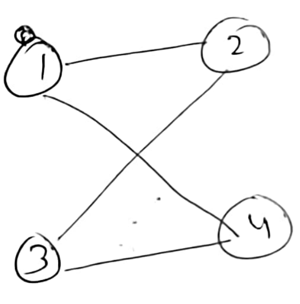
$x = \frac{60}{8}$, is not a integer.

So number of vertices in a set could not be fraction value. So we can say the ~~Graph~~ Graph is not a Bipartite.

Ex

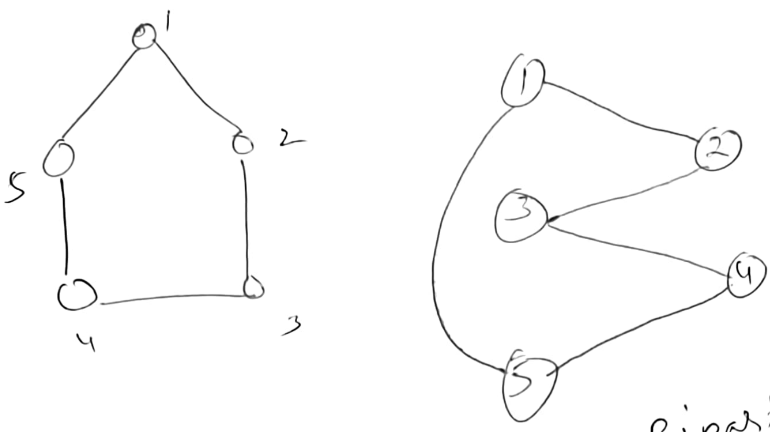


C_4



$K_{2,2}$

Cycle of odd length say 5



Not a Bipartite graph

Theorem

A Graph is Bipartite iff it does not have any odd cycle.

Proof $\Rightarrow$ Suppose $G = (A \cup B, E)$ is Bipartite

Suppose G has a odd cycle

$v_1, v_2, \ldots v_k v_1$ where k is odd

with loss of Generality assume that $v_1 \in A$, then

$v_i \in A$ if i is odd

$v_i \in B$ if i is even

then (v_1, v_k) ~~is~~ $\in E$ is an edge with both end points in A if k is odd, which is a contradiction ~~therefore~~

~~G is not a~~ G is not a Bipartite Graph. or k is not odd or the G has no odd cycles.

Let G be a graph with no odd cycles
with out loss of generality we assume that G is

Connected. ~~Let~~
Let $d(u,v)$ be the length of the ~~path~~ shortest path

between u & v

Pick any arbitrary vertex $u \in V$ and define the following

$A = \{u\} \cup \{w \mid d(u,w) \text{ is even}\}$

and $B = V \setminus A$

Claim: $G = (A \cup B, E)$ is Bipartite.

Proof by Contradiction

Suppose there exist an edge (v,w) with both v & $w \in A$

Then by Construction both $d(u,v)$ and $d(u,w)$ are

even

Let P_{uv} be the shortest path connecting u & v

P_{uw} " " " " u & w

Then the cycle

P_{uv}, (v,w), P_{wu}
has length

$\underbrace{d(u,v)}_{\text{even}} + \underbrace{d(u,w)}_{\text{even}} + \underbrace{1}_{\text{odd}} = \text{odd}$

So it is a Contradiction.

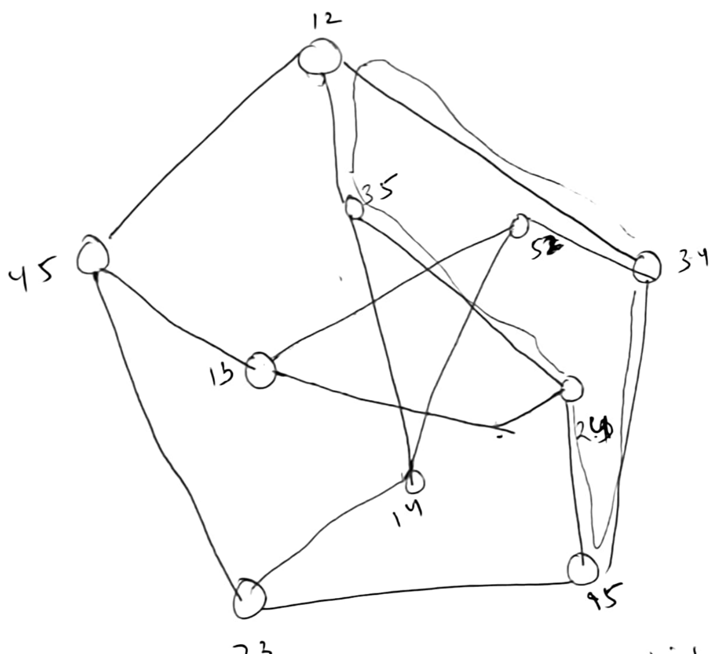
Hence G is Bipartite.

Ex

Petersen Graph

Let $S = \{1,2,3,4,5\}$

$V = \{12, 13, 14, 15, 23, 24, 25, 34, 35, 45\}$.



edges between pair of disjoint two element subsets.

Petersen graph is not bipartite

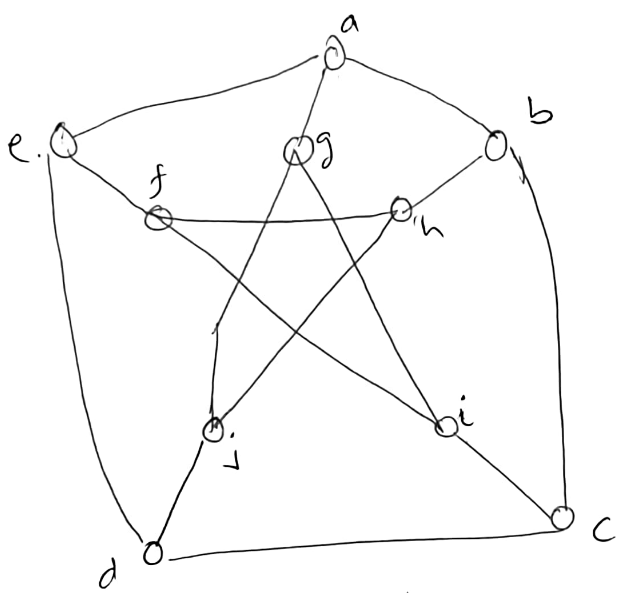
Diameter of a Graph:

Diameter of a Graph G

$$\text{diam}(G) = \max_{u,v \in V} d(u,v)$$

$d(u,v)$ is the length of the shortest path between 'u' & 'v'

Ex 1



Petersen Graph

$d(u,v) = 1$ if u & v are adjacent

~~$d(u,v)$ if u & v are non adjacent~~

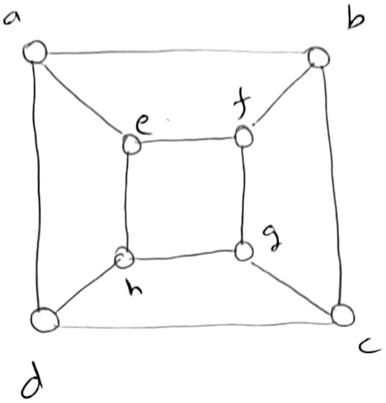
$d(u,v) = 2$ if u & v are non adjacent

In Petersen graph: Every pair of non adjacent vertices have a common neighbor.

⇒ Diameter of Petersen graph is 2

Ex 2

Cube Q_3



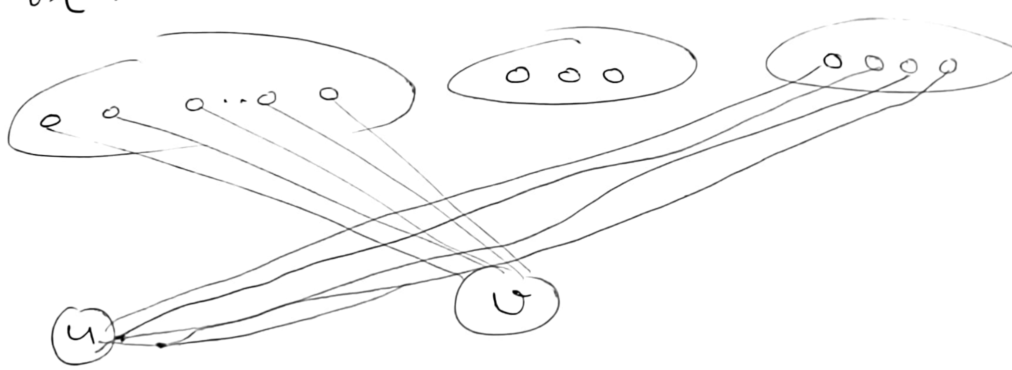
$d(b,d) = 2$ $\quad$ $\text{Diam}(Q_3) = 3$

$d(d,f) = 3$

Theorem : If G is a simple graph, then diameter of $G \geq 3$ implies $\text{diam}(\bar{G}) \leq 3$

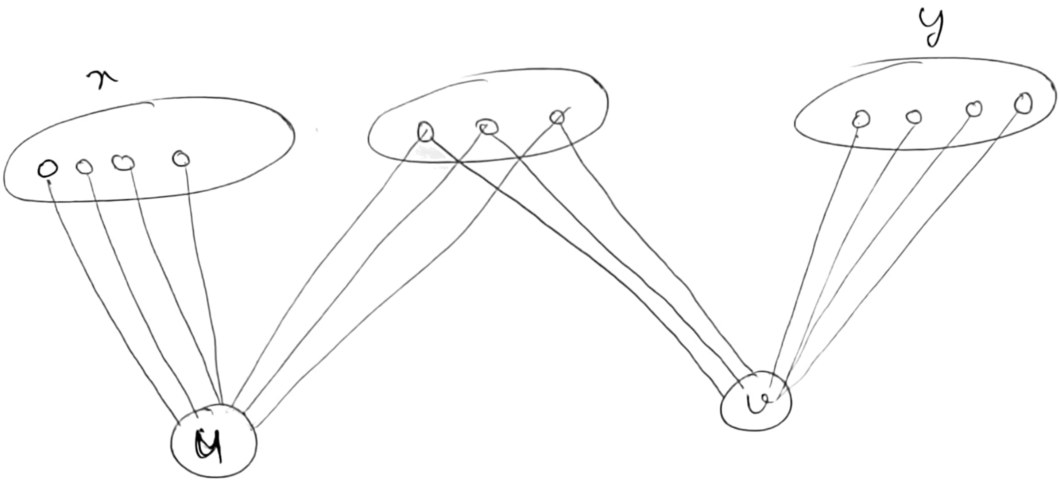
Proof : When $\text{diam}(G) \geq 3$, there are non adjacent vertices u & $v \in V$, with no common neighbor.

Every vertex $x \in V - \{u, v\}$ is adjacent to almost one of $\{u, v\}$ in G



G

??



$\bar{G}$

for every pair of vertices $x, y \in V - \{u, v\}$ there is a path of length at most 3 in $\bar{G}$ ~~com~~

Theorem if $\operatorname{diam}(G) \geq 4$, then $\operatorname{diam}(\bar{G}) \leq 2$

Proof Since $\operatorname{diam}(G) \geq 4$, there exist a pair of vertices (u, v) whose shortest distance is ≥ 4

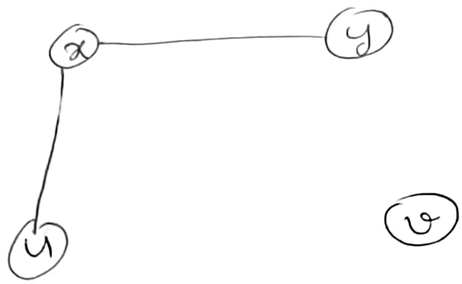
$d_G(u, v) \geq 4$

Suppose $\{x, y\}$ are two vertices $V - \{u, v\}$.

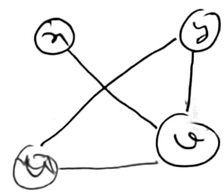
$x, y \in V - \{u, v\}$

we need to prove that $d_{\bar{G}}(x, y) \leq 2$

Case 1



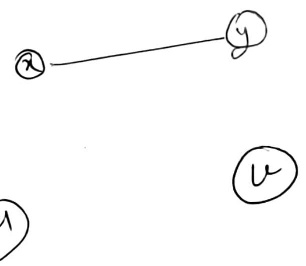
x is adjacent to u or v



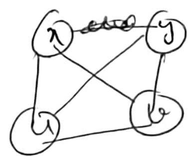
G (this possible in G)

$d_{\bar{G}}(x,y) = 2$

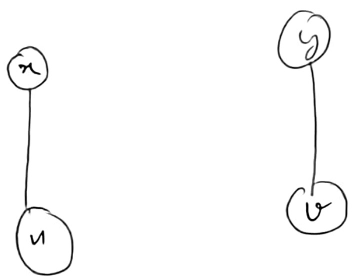
Case 2

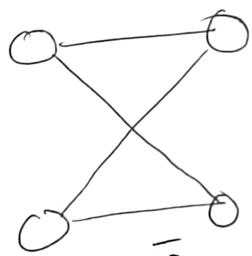


x, y are non adjacent to u or v



Case 3





$\bar{G}$

Assume $(x,u) \in E(G)$ & $(y,v) \in E(G)$

if $(x,y) \in E(G)$, then $d_G(u,v) = 3$

this contradicts the assumption $d_G(u,v) \geq 4$

therefore $(x,y) \notin E(G)$ and hence

$(x,y) \in E(\bar{G})$. So $d_{\bar{G}}(x,y) = 1$