1 Optimization Problem

Problem is

$$\min_{x \in X} f_0(x), \tag{1}$$

$$f_i(x) \leq 0, \tag{2}$$

for $i = 1, \dots, m$, where $X \subseteq \mathbb{R}^n$ is a box:

$$\{x \mid x_j^{\min} \leq x_j \leq x_j^{\max}\}. \tag{3}$$

2 Approximate Problem

Suppose kth iterate is $x^{(k)}$. Then, for $i = 0, \dots, m$, replace $f_i(x)$ with

$$g_i(x) = \nabla f_i(x_0) \cdot (x - x^{(k)}) + \frac{\rho_i^2}{2} \left| \frac{x - x^{(k)}}{\boldsymbol{\sigma}} \right|^2. \tag{4}$$

And make a trust region T,

$$T = \{x \mid |x_j - x_j^{(k)}| \leq \sigma_j\}. \tag{5}$$

So that the new problem is

$$\min_{x \in T} g_0(x), \tag{6}$$

$$g_i(x) \leq 0. \tag{7}$$

3 Overall Scheme

For the kth iteration:

1. Solve approximate problem to find candidate $x^{(k+1)}$
2. Check conservative: $g_i(x^{(k+1)}) < f_i(x^{(k+1)})$.
   - If no, throw away candidate, double ρ_i for each non-conservative g_i, and solve approximate problem again.
3. Halve ρ (take bigger steps) and update σ (decrease σ_i if x_i oscillating, increase if monotonic i.e. heading somewhere else).

4 Solving approximate problem

4.1 Evaluating dual function

Lagrangian relaxation,

$$L(x,\lambda) = g_0(x) + \sum_{i=1}^{m} \lambda_i g_i(x) \tag{8}$$

$$= \left(\nabla f_0(x_0) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x_0)\right) \cdot (x - x^{(k)}) + \frac{1}{2}\left(\rho_0 + \sum_{i=1}^{m} \lambda_i \rho_i\right) \left|\frac{x - x^{(k)}}{\boldsymbol{\sigma}}\right|^2 \tag{9}$$

$$= \sum_{j=1}^{n} \left(\left((\nabla f_0(x_0))_j + \sum_{i=1}^{m} \lambda_i (\nabla f_i(x_0))_j\right)\left(x_j - x_j^{(k)}\right) + \frac{1}{2\sigma_j}\left(\rho_0 + \sum_{i=1}^{m} \lambda_i \rho_i\right)\left(x_j - x_j^{(k)}\right)^2\right). \tag{10}$$

Define dual function,

$$g(\lambda) = \min_{x \in T} L(x,\lambda) \tag{11}$$

$$= \sum_{j=1}^{n} g_j(\lambda), \tag{12}$$

where

$$g_j(\lambda) = \min_{|\delta_j| \le \sigma_j} \left(\left((\nabla f_0(x_0))_j + \sum_{i=1}^{m} \lambda_i (\nabla f_i(x_0))_j\right)\delta_j + \frac{1}{2\sigma_j}\left(\rho_0 + \sum_{i=1}^{m} \lambda_i \rho_i\right)\delta_j^2\right). \tag{13}$$

To evaluate, analytically minimize quadratic in $\delta_j = x_j - x_j^{(k)}$, snapping to bounds.

4.2 Maximizing dual function

The dual problem is,

$$\max_{\lambda \ge 0} g(\lambda). \tag{14}$$

5 Main Goals

- Support sparse Jacobians
- Support affine constraints
  - Does paper handle these? (First, find where paper handles box constraints.)
  - Maybe think of this as a more complicated X, rather than a simple f_i.