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Article *in* IEEE Transactions on Circuits and Systems I Fundamental Theory and Applications · April 1994

DOI: 10.1109/81.273922 · Source: IEEE Xplore

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Transactions Briefs

On the Properties of Gram Matrix

V. Sreeram and P. Agathoklis

Abstract— **The Gram matrix of the system is very useful in system identification [5] model-reduction applications [10]. The computation of this matrix involves evaluation of scalar product of repeated integrals and can be computed in frequency domain as shown in [2], [10], [15], [16]. In this paper new properties of the Gram matrix are presented. Based on these properties new techniques for the computation of Gram matrix and the characteristic equation of the system are presented. The new techniques presented are shown to be elegant and easier to use than the existing techniques.**

I. Introduction

The Gram matrix [5] has many applications namely system identification [5], [6], [8], [11], [13], [14], modeling of power density spectrum [7], and model reduction [2,10]. The Gram matrix contains elements which are scalar products of repeated integrals of impulse response of the system. In system identification application [5], the elements of this matrix can be generated experimentally, whereas in model-reduction application the elements of this matrix has to be computed from the mathematical model of the original system. This involves evaluation of integrals of the form

$$I_{j,k} = \int_0^\infty f_j(t) f_k(t) dt$$
$$\text{where} f_1(t) = h(t) \text{and} f_{n+1}(t) = \int_\infty^t f_n(\tau) d\tau$$

$h(t)$ being the impulse response of the system. These integrals can be computed in frequency domain as suggested in [2], [10]. However, the methods proposed in [2], [10] involve writing special purpose functions for computing the Gram matrix. The existing functions in matlab cannot be used directly to determine the Gram matrix of a given system.

In this paper, two new properties of Gram matrix are presented. These properties are the following: (1) the Gram matrix of a system is related to the observability, controllability and cross Gramians, and (2) the characteristic equation of the system can be easily obtained from the characteristic Gram matrix of the system.

Based on these properties, new techniques for the computation of Gram matrix and the characteristic equation of the system are presented. The techniques presented are very elegant and can be easily implemented using standard functions in matlab. The technique proposed for the computation of Gram matrix and the characteristic equation of the system are easier to use than the frequency domain computation techniques of [2], [10], [15], [16] and the well known technique of [5]–[7], respectively.

Manuscript received January 25, 1993; revised August 16, 1993. This work was supported by ARC for the first author and Micronet for the second author.
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IEEE Log Number 9216709.

II. Preliminaries

In this section, we define the following terms: Gram matrix, reciprocal system, impulse-response Gramian, Markov parameters and time moments for linear time invariant continuous system. We also list the properties of Gram matrix available in the literature.

Consider a stable single-input, single-output continuous system described by the following minimal realization:

$$\dot{x}(t) = Ax(t) + bu(t) \tag{1}$$
$$y(t) = cx(t) \tag{2}$$

where $x(t) \in R^n$.

Definition 1: The Gram matrix [5]–[7] for the above system is defined as follows:

$$G_r = [g_{ij}] = \begin{bmatrix} <y_1, y_1> & <y_1, y_2> & \dots & <y_1, y_r> \\ <y_1, y_2> & <y_2, y_2> & \dots & <y_2, y_r> \\ \vdots & \vdots & & \vdots \\ <y_1, y_r> & <y_2, y_r> & \dots & <y_r, y_r> \end{bmatrix} \tag{3}$$

where g_{ij} is the inner product of y_i and y_j, i.e.,

$$g_{ij} = <y_i, y_j> = \int_0^\infty y_i . y_j dt$$

$$\text{and} y_1(t) = h(t) = ce^{At}b \tag{4}$$
$$y_{r+1}(t) = \int_\infty^t y_r(\alpha) d\alpha r = 1, 2, \dots, p \tag{5}$$

The lowest order matrix G_{n+1} which is singular is designated "characteristic Gram matrix." The diagonal cofactors of this matrix have the following property.

Lemma 1 [5]: The characteristic equation of the system is

$$\sum_{i=1}^{n+1} \sqrt{(\Delta_{ii})} \lambda^{n-i+1} = 0$$

where Δ_{ii} are the diagonal cofactors of the characteristic Gram matrix.

Lemma 2 [5]: If the system order is n, then the Gram matrix in (3) is nonsingular if r is less than n and singular if r is greater than n.

Due to the above properties (Lemma 1 and 2), the Gram matrix is extremely useful in finding good pole locations for reduced-order model as suggested in [2].

Definition 2 [19]: If a system,$\{A, b, c\}$ ((1)–(2)) is represented by the transfer function

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \dots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}$$

then the reciprocal system is given by the transfer function

$$\hat{G}(s) = s^{-1} G(s^{-1})$$

The reciprocal system can also be represented by the realization, $\{\hat{A}, \hat{b}, \hat{c}\}$ where

$$\hat{A} = A^{-1}, \hat{b} = A^{-1} b, \text{and} \hat{c} = -c.$$

Definition 3 [1]: The impulse-response Gramian for the above system is defined as follows:

$$W_{ir} = [g_{ij}] = \begin{bmatrix} < z_1, z_1 > & < z_1, z_2 > & \dots & < z_1, z_n > \\ < z_1, z_2 > & < z_2, z_2 > & \dots & < z_2, z_n > \\ \vdots & \vdots & & \vdots \\ < z_1, z_n > & < z_2, z_n > & \dots & < z_n, z_n > \end{bmatrix}$$

where g_{ij} is the inner product of z_i and z_j, i.e.,

$$g_{ij} = < z_i, z_j > = \int_0^\infty z_i . z_j dt$$

and

$$z_1(t) = h(t) = ce^{At}b$$
$$z_{r+1}(t) = \frac{dz_r(t)}{dt} r = 1, 2, \dots, n-1$$

Lemma 3 [17], [18]: The Gram matrix of the system $\{A, b, c\}$ ((1)–(2)) is equal to the impulse-response Gramian [1] of the reciprocal system, $\{\hat{A}, \hat{b}, \hat{c}\}$.

Definition 4: The Markov parameters (m_i) and time moments (t_i) of the system (1)–(2) are given by

$$m_i = cA^{i-1}b, i = 1, 2, \dots \tag{6}$$
$$t_i = cA^{-i}b, i = 1, 2, \dots \tag{7}$$

III. Main Results

In this section, two new properties of Gram matrix are presented. The first property shows the connection between the Gram matrix and the other Gramians [3], [12]. The second property gives a simple way of computing the characteristic equation of the system.

Theorem 1: If W_c, W_o, and W_{co} are the controllability [9], [12], the observability [9], [12], and the cross Gramians [3] of the system $\{A, b, c\}$ then the Gram matrix, G_n of the system is given by

$$(i) G_n = \mathcal{O} W_c \mathcal{O}^T$$
$$(ii) G_n = \mathcal{C}^T W_o \mathcal{C}$$
$$(ii) G_n = \mathcal{O} W_{co} \mathcal{C}$$

$$\text{where} \mathcal{O} = \begin{bmatrix} c \\ cA^{-1} \\ \vdots \\ cA^{-n+1} \end{bmatrix} \tag{8}$$
$$\text{and} \mathcal{C} = \left[b, A^{-1}b, \dots, A^{-n+1}b\right]$$

Proof: To prove (i) consider the impulse response of the system $\{A, b, c\}$ given by

$$h(t) = ce^{At}b$$

From (4) and (5), we have

$$y_1(t) = h(t) = ce^{At}b$$
$$y_2(t) = \int_\infty^t h(\alpha) d\alpha = cA^{-1}e^{At}b$$
$$\vdots = \vdots$$
$$y_{r+1}(t) = cA^{-r}e^{At}b \tag{9}$$

From (9) and (3), the Gram matrix can be written as

$$G_n = \begin{bmatrix} c \\ cA^{-1} \\ \vdots \\ cA^{-n+1} \end{bmatrix} \int_0^\infty e^{At}bb^Te^{A^Tt}dt$$
$$. \left[c^T, A^{-T}c^T, \dots, (A^{-T})^{n-1}c^T\right]$$
$$= \mathcal{O} W_c \mathcal{O}^T$$

where W_c is the controllability Gramian given by

$$W_c = \int_0^\infty e^{At}bb^Te^{A^Tt}dt$$

Similarly, one can show (ii) and (iii).

Corollary If the n-th order minimal realization of the system ((1)-(2)) is of the form:

$$A_o = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_n \\ 1 & & & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & 1 & 0 \end{bmatrix} \tag{10}$$
$$b_o = [m_1 \quad t_1 \quad \dots \quad t_{n-1}]^T \tag{11}$$
$$c_o = [1 \quad 0 \quad \dots \quad 0] \tag{12}$$

where m_1 and $t_i, i = 1, 2, \dots, n-1$ are respectively the Markov parameters and time moments of the system, then the Gram matrix is equal to the controllability Gramian of the system realization $\{A_o, b_o, c_o\}$.

Proof: From theorem 1, we have

$$G_n = \mathcal{O}_o W_c^o \mathcal{O}_o^T$$
$$= \mathcal{O}_o \left\{ \int_0^\infty e^{A_ot}b_ob_o^Te^{A_o^Tt}dt \right\} \mathcal{O}_o^T$$

$$\text{Since} \mathcal{O}_o = \begin{bmatrix} c_o \\ c_oA_o^{-1} \\ \vdots \\ c_oA_o^{-n+1} \end{bmatrix} = I$$

the Gram matrix is equal to the controllability Gramian for the realization, $\{A_o, b_o, c_o\}$. Hence

$$G_n = W_c^o = \int_0^\infty e^{A_ot}b_ob_o^Te^{A_o^Tt}dt \tag{13}$$

Remarks:

1) By duality one can show that for a n-th order realization

$$A_c = A_o^T, b_c = c_o^T, \text{and} c_c = b_o^T$$

the Gram matrix of the system is equal to the observability Gramian given by

$$G_n = W_o^c = \int_0^\infty e^{A_c^Tt}c_c^Tc_ce^{A_ct}dt \tag{14}$$

of the above realization.

2) Note that the observability Gramian, W_o^c, (14), and controllability Gramian, W_c^o, (13), can be obtained by solving the following equations:

$$A_o^TW_o^c + W_o^cA_o = -c_o^Tc_o \tag{15}$$
$$A_cW_c^o + W_c^oA_c^T = -b_cb_c^T \tag{16}$$

Theorem 1 and its corollary can be easily used to compute the Gram matrix in time domain. This has applications in model reduction [2] wherein it is required to compute the Gram matrix from the mathematical model of the system. In [2], [10], [15], [16], the Gram matrix is computed using frequency domain techniques and in [18] it is computed using impulse-response Gramians (lemma 3).

Given the transfer function $G(s)$, of the system, one can write the system matrix A_o ((10)) directly from the denominator coefficients of $G(s)$. The elements of the input vector b_o ((11)) can be obtained immediately from the Markov parameters and the time moments of the system, $G(s)$. The output vector c_o follows directly from (12)). Once the realization $\{A_o, b_o, c_o\}$ is obtained for the transfer function $G(s)$, the Gram matrix can be easily computed by solving the Lyapunov equation (16) for the controllability Gramian.

Theorem 2: If the characteristic Gram matrix of the system, G_{n+1}, can be partitioned as

$$G_{n+1} = \begin{bmatrix} G_n & \mathbf{g}_{1,n+1} \\ \mathbf{g}_{1,n+1}^T & g_{n+1,n+1} \end{bmatrix}$$

where $G_n \in R^{n\times n}$, $\mathbf{g}_{1,n+1} \in R^{n\times 1}$ and $g_{n+1,n+1}$ is a scalar, then

$$\begin{bmatrix} \hat{a}_n \\ \hat{a}_{n-1} \\ \vdots \\ \hat{a}_1 \end{bmatrix} = -G_n^{-1}\mathbf{g}_{1,n+1} \tag{17}$$

In the above equation, $\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n$ are the coefficients of the characteristic equation, $\hat{a}(s) = s^n + \hat{a}_1 s^{n-1} + \hat{a}_2 s^{n-2} + \ldots + \hat{a}_n$, of the reciprocal system, $\{\hat{A}, \hat{b}, \hat{c}\}$.

Proof: To prove the above theorem, it is enough if we show that

$$-G_n \begin{bmatrix} \hat{a}_n \\ \hat{a}_{n-1} \\ \vdots \\ \hat{a}_1 \end{bmatrix} = \mathbf{g}_{1,n+1}$$

Consider now

$$\begin{aligned}
RHS = \mathbf{g}_{1,n+1} &= \int_0^\infty \begin{bmatrix} y_1(t)y_{n+1}(t) \\ y_2(t)y_{n+1}(t) \\ \vdots \\ y_n(t)y_{n+1}(t) \end{bmatrix} dt \\
&= \int_0^\infty \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} y_{n+1}(t)dt \\
&= \int_0^\infty \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} cA^{-n}e^{At}bdt \\
&= \int_0^\infty \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} c\hat{A}^n e^{At}bdt \\
&= \int_0^\infty \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} c \\
&\quad \cdot \left[-\hat{a}_n I - \hat{a}_{n-1}\hat{A} - \ldots - \hat{a}_1\hat{A}^{n-1}\right] e^{At}bdt \\
&= \int_0^\infty \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} c \\
&\quad \cdot \left[-\hat{a}_n I - \hat{a}_{n-1}A^{-1} - \ldots - \hat{a}_1 A^{-(n-1)}\right] e^{At}bdt \\
&= \int_0^\infty \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} [-\hat{a}_n y_1 - \hat{a}_{n-1}y_2 - \ldots - \hat{a}_1 y_n]dt \\
&= \int_0^\infty \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} [y_1(t) \quad y_2(t) \quad \ldots \quad y_n(t)]dt \\
&\quad \cdot \begin{bmatrix} -\hat{a}_n \\ -\hat{a}_{n-1} \\ \vdots \\ -\hat{a}_1 \end{bmatrix} \\
&= -G_n \begin{bmatrix} \hat{a}_n \\ \hat{a}_{n-1} \\ \vdots \\ \hat{a}_1 \end{bmatrix} \\
&= LHS
\end{aligned}$$

Remarks:

1) Equation (17) is similar to that of Yule-Walker equations used for estimation of AR model parameters from the correlation matrix [4].
2) The characteristic equation of the original system, $\{A, b, c\}$ can be easily obtained [19] using the following formula:

$$a(s) = s^n \hat{a}(s)$$

Corollary: The first *n* rows of the last column of the characteristic Gram matrix is given by

$$\mathbf{g}_{1,n+1} = \begin{bmatrix} g_{1,n+1} \\ g_{2,n+1} \\ \vdots \\ g_{n,n+1} \end{bmatrix} = \begin{bmatrix} t_1 t_n \\ t_2 t_n \\ \vdots \\ t_{n-1}t_n \\ t_n t_n/2 \end{bmatrix} - \begin{bmatrix} g_{2n} \\ g_{3n} \\ \vdots \\ g_{nn} \\ 0 \end{bmatrix}$$

where $t_i, i = 1, 2, \ldots, n$ are the time moments of the system.

The proof of the above can be easily obtained using integration by parts as shown in [15].

Remarks:

1) Theorem 2 can be used to find the characteristic equation of the reciprocal system and subsequently (using definition 2) the characteristic equation of the original system.
2) The corollary implies that two set of time moments can be obtained from a given characteristic Gram matrix.

Theorem 2 and its corollary has applications in system identification [5], [6] wherein the characteristic Gram matrix is obtained experimentally and is required to obtain the characteristic equation of the system. Although, one can obtain the characteristic equation from lemma 1, theorem 2 is elegant and is easier to use than lemma 1.

3.1. Extensions to Discrete-time Case

The properties of Gram matrix derived in the paper (Theorems 1 and 2) can be easily extended to discrete-time systems. Note that for discrete-time systems the Gram matrix and the impulse response Gramian are equal from their definitions. The theorem 1 is shown to be true for impulse-response Gramian in discrete-time case [20]. Therefore it is also true for Gram matrix in discrete-time systems. Theorem 2 for discrete-time systems can be easily shown using Cayley-Hamilton theorem.

IV. Example

Consider the third-order system described by

$$G(s) = \frac{8s^2 + 6s + 2}{s^3 + 4s^2 + 5s + 2}$$

A minimal realization for the above system in the form given by (10)–(12) is

$$A_o = \begin{bmatrix} -4 & -5 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, b_o = \begin{bmatrix} 8 \\ -1 \\ -0.5 \end{bmatrix}$$

$$\text{and} c_o = [1 \quad 0 \quad 0]$$

The Gram matrix of $G(s)$ is given by

$$G_n = W_c = \begin{bmatrix} 9.2222 & -0.5 & -1.1944 \\ -0.5 & 0.6944 & -0.125 \\ -1.1944 & -0.125 & 0.4514 \end{bmatrix}$$

The characteristic Gram matrix of $G(s)$ is given by

$$G_{n+1} = \begin{bmatrix} 9.2222 & -0.5 & -1.1944 & -0.6250 \\ -0.5 & 0.6944 & -0.125 & -0.8264 \\ -1.1944 & -0.125 & 0.4514 & -0.2812 \\ -0.6250 & -0.8264 & -0.2812 & 2.6684 \end{bmatrix}$$

The coefficients of the characteristic polynomial of the reciprocal system are:

$$\begin{bmatrix} \hat{a}_3 \\ \hat{a}_2 \\ \hat{a}_1 \end{bmatrix} = -G_n^{-1}\mathbf{g}_{1,n+1} = \begin{bmatrix} 0.5 \\ 2.0 \\ 2.5 \end{bmatrix}$$

The characteristic polynomials of the reciprocal system and original system are

$$\hat{a}(s) = s^3 + 2.5s^2 + 2.0s + 0.5$$
$$a(s) = s^n\hat{a}(s^{-1}) = s^3 + 4.0s^2 + 5.0s + 2.0$$

respectively.

V. Conclusion

In this paper, new properties of the Gram matrix have been presented. It has been shown that the Gram matrix of a system is related to the observability, controllability, and the cross Gramians. It has also been shown that the Gram matrix can be computed by the solution of Lyapunov equations. A simple relation between the characteristic equation of a reciprocal system and the characteristic Gram matrix has also been presented. These properties can be extremely useful in time-domain computation of Gram matrices, model reduction and system identification applications. Since this method involves only matrix manipulations, it is easier to use in matlab than the frequency domain techniques [2], [10], [15], [16].

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