EECS 114:
Engineering Data Structures and Algorithms
Lecture 10

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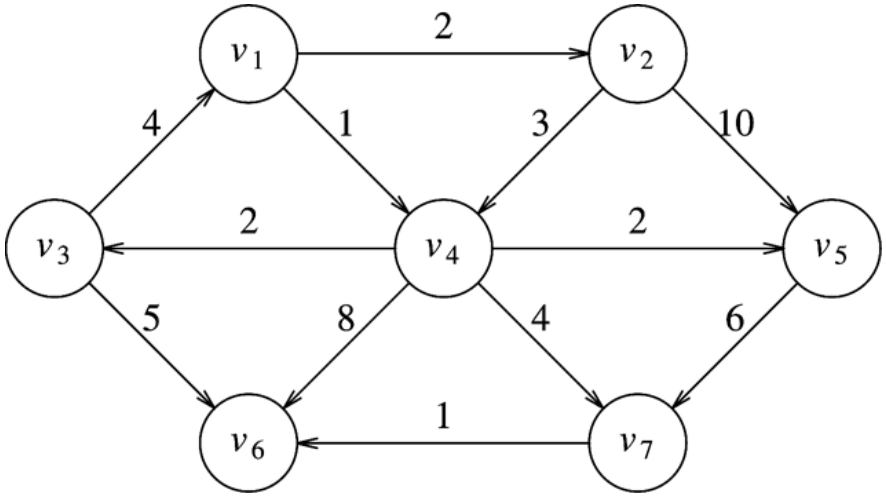
Graph

- A Graph $G = (V,E)$
  - A set V of vertices and a set E edges
- A graph is a way representing connections/relationships between pairs of objects in V
- Each edge is a pair (v,w), where $v,w \in V$
- Edges are either directed or undirected
  - Directed referred to as ordered
    - Directed graphs are called Digraphs
    - Directed graphs with no cycles(acyclic) are called DAGs
  - Undirected referred to as unordered

Graphs



Graph

- An edge (v,v) is a loop
- Vertices can have incoming and outgoing edges
  - indegree – number of incoming edges of a vertex v
  - outdegree – number of outgoing edges of a vertex v
- A path in a graph is a sequence of vertices w_1, w_2, w_3, …, w_N such that $(w_i, w_{i+1}) \in E$ for $1 \leq i < N$
  - A **simple path** is a path where all vertices are distinct except possibly first/last
  - **path length:** $N-1$ for N vertices
- Edges can have an associated cost called the **weight**.

Graphs - Definitions

- An undirected graph is connected if there is a path from every vertex to every other vertex
- A directed graph that is connected is called strongly connected.
- Weakly connected - If directed graph is not strongly connected, but removing direction makes graph connected
- Complete graph – edge between every pair of vertices.

Graph Representations

- Adjacency matrix – $|V|^2$ matrix, with **1** if there is an edge between two vertices, **0** otherwise.
  - Good for dense graphs.
  - Wasted space if not dense.
  - If graph is sparse, adjacency list is better.

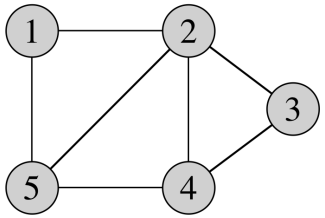
$$|E| = \Theta(|V|^2)$$

- Adjacency list – for each vertex, store a list of vertices that share an edge
  - Example…

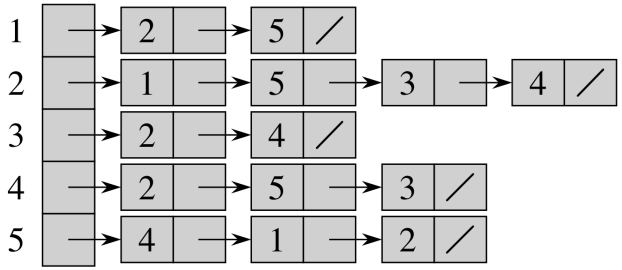
Adjacency list

1	2, 4, 3
2	4, 5
3	6
4	6, 7, 3
5	4, 7
6	(empty)
7	6

Undirected Graph



(a)

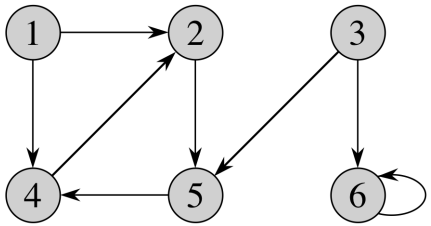


(b)

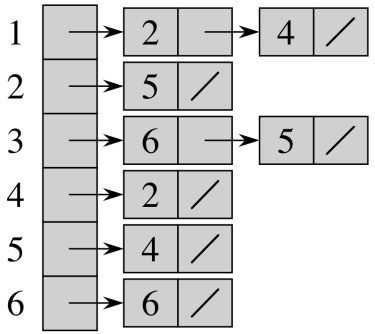
	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

(c)

Directed Graph



(a)

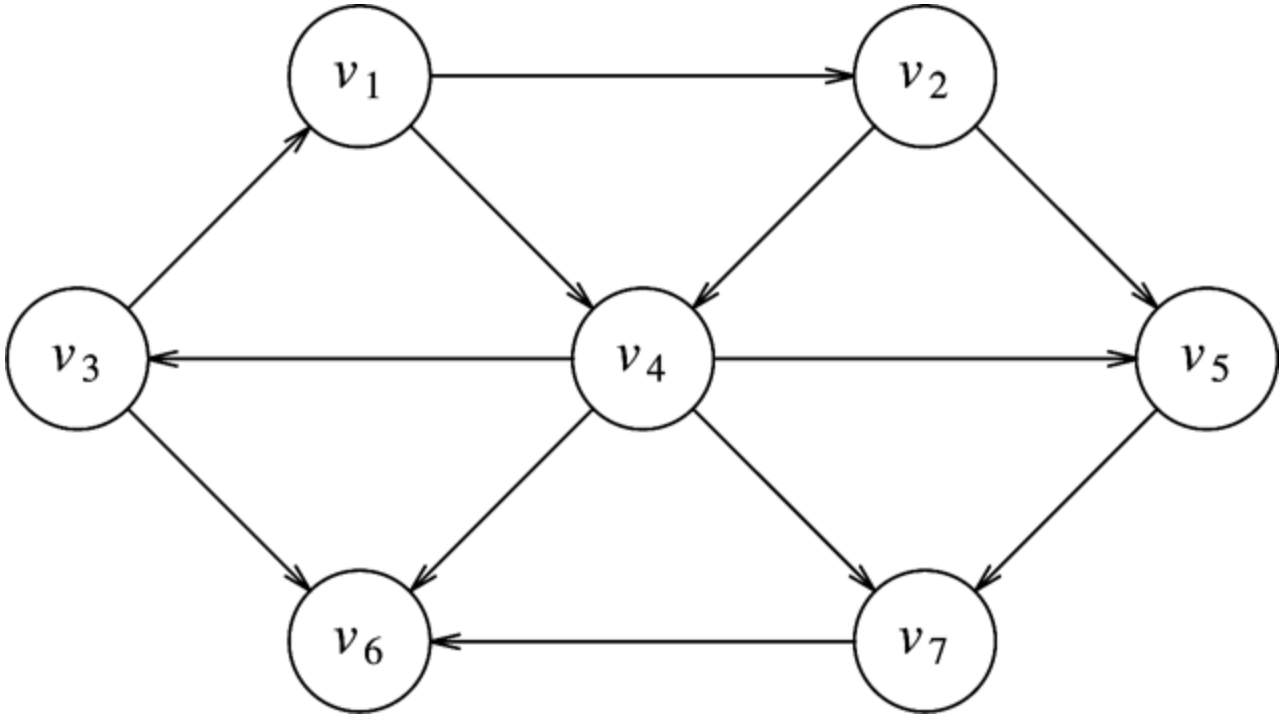


(b)

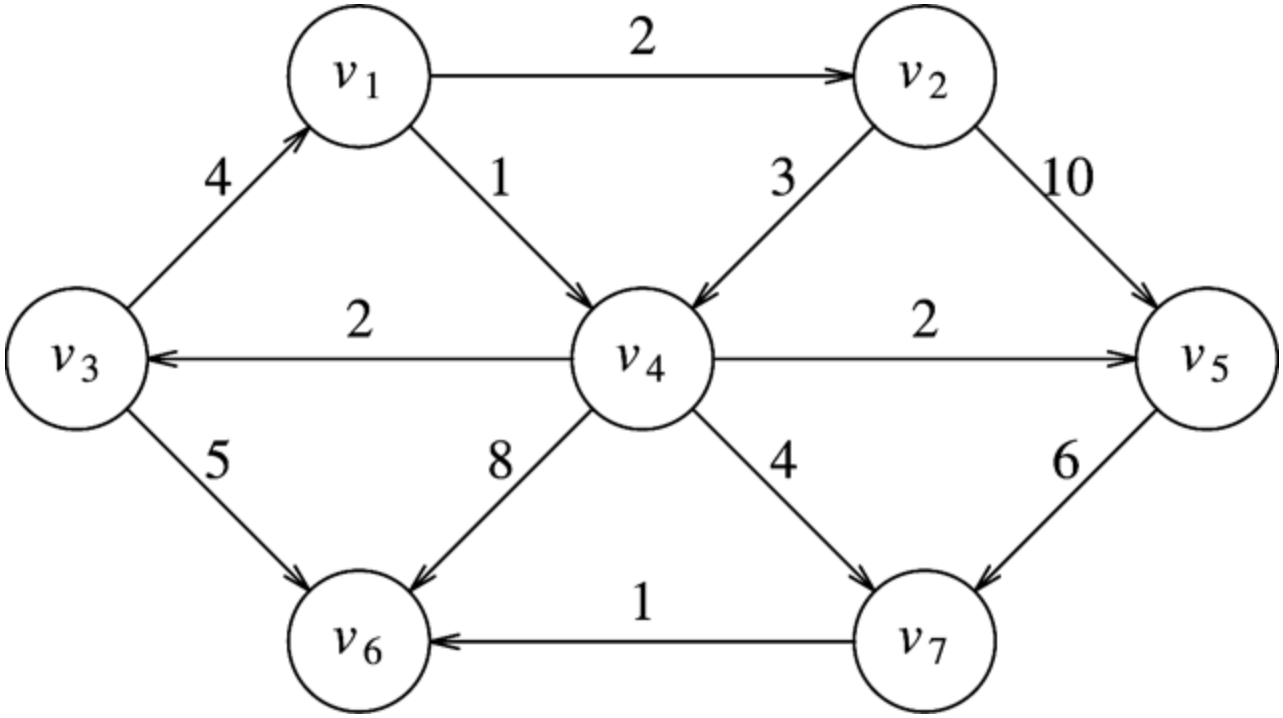
	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1

(c)

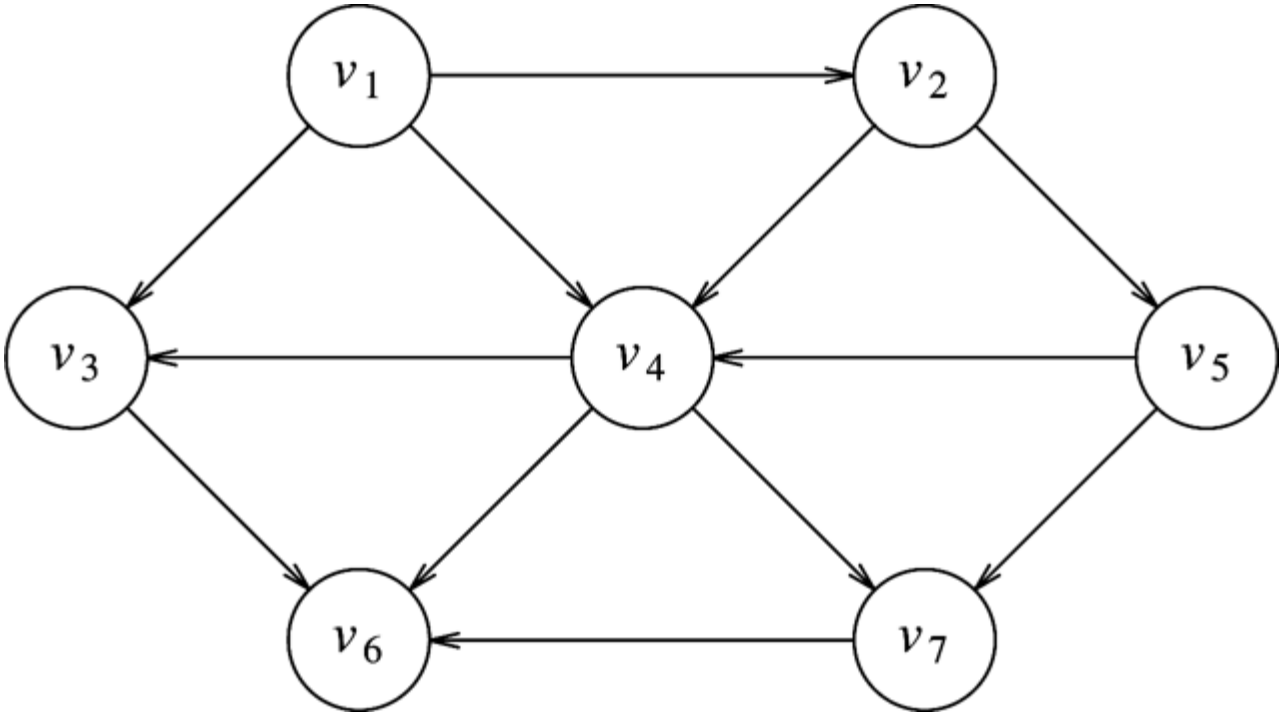
Directed Graph – Unweighted



Directed Graph - Weighted



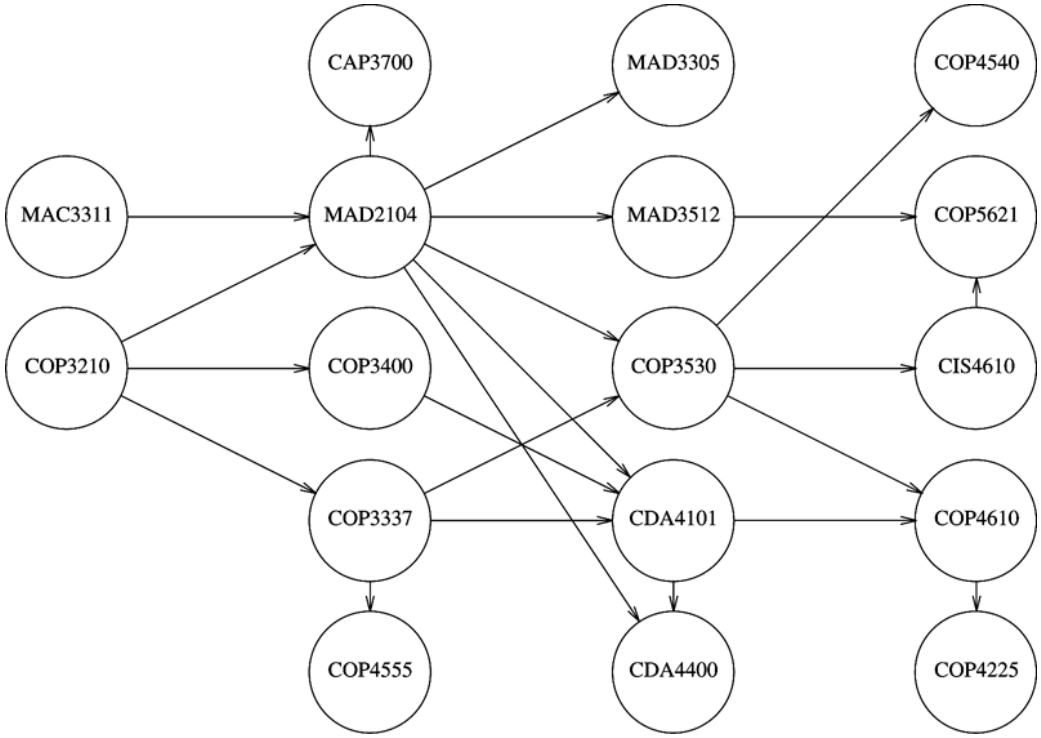
Directed Acyclic Graph



Topological Sort

- Linear ordering of vertices for a DAG (Directed Acyclic Graph).
- Every vertex comes before all nodes to which it has outgoing edges.
- Every DAG has at least 1 topological sort
  - 1 indicates a Hamiltonian path
  - 2 or more, no Hamiltonian path
  - Hamiltonian path – every vertex is visited exactly once.
- Topological Sort - used in scheduling jobs or tasks

Scheduling

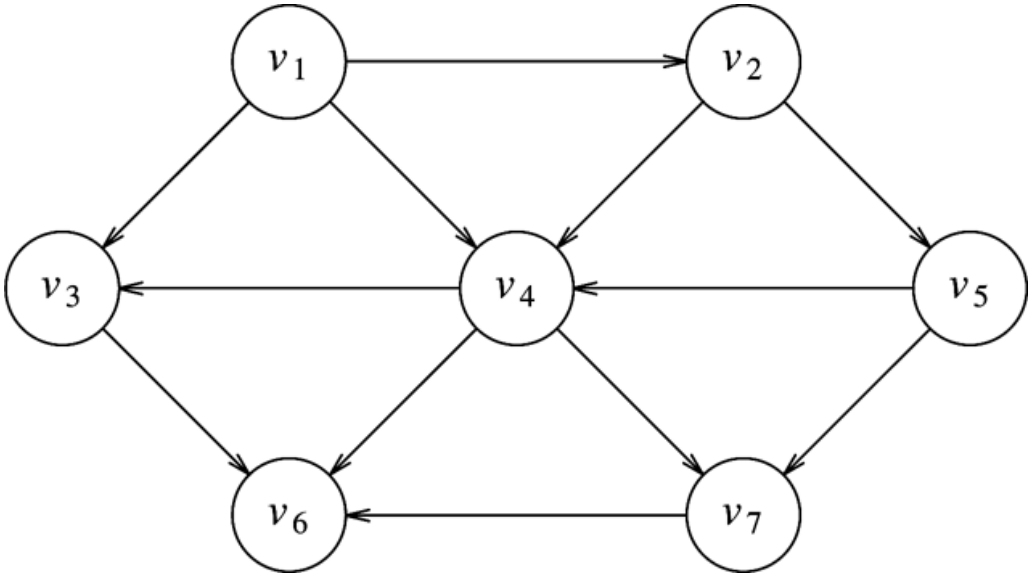

CAP3700
MAD3305
COP4540
MAC3311
MAD2104
MAD3512
COP5621
COP3210
COP3400
COP3530
CIS4610
COP3337
CDA4101
COP4610
COP4555
CDA4400
COP4225

Topological Sort

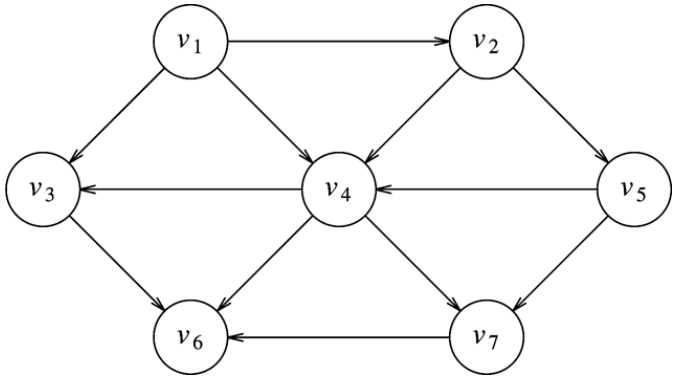
- **Simple algorithm:**
  1. Find an initial vertex v with no incoming edges
  2. Print v
  3. Remove v from graph
  4. Remove $v's$ edges from graph, updating effected vertices
  5. Iterate(steps 1-4) over the remaining graph

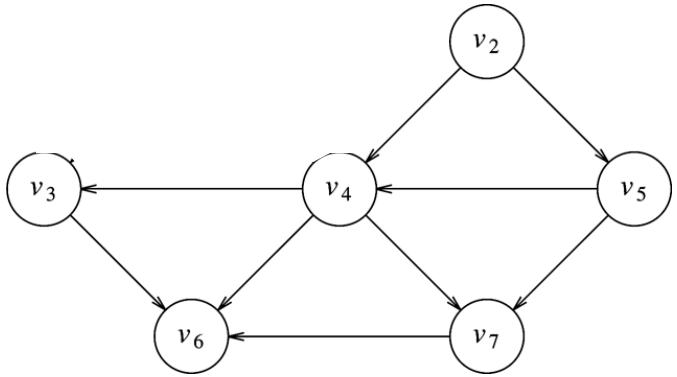
  Running Time : $O(|V|^2)$
  - Finding a vertex with in-degree zero, linear scan of V
  - There are $|V|$ calls to do this
- Speed up:
  - Use a Queue to store vertices with in-degree zero
  - Only have to remove the front of Queue, skip linear scan of V
  - **Running Time:** $O(|E| + |V|)$

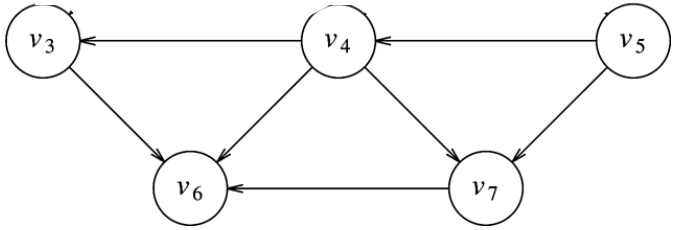
Topological Sort

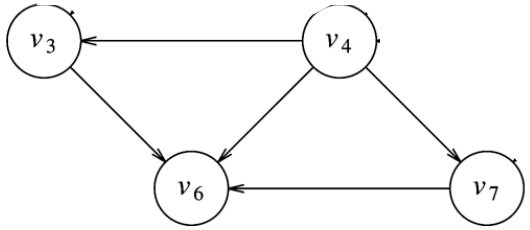


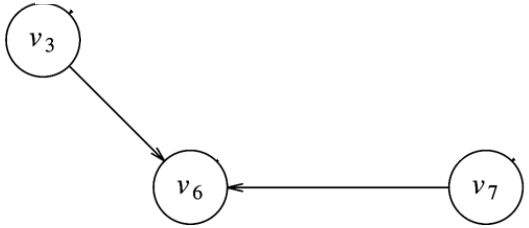
v1, v2, v5, v4, v3, v7, v6

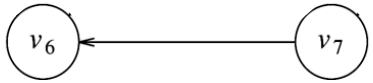










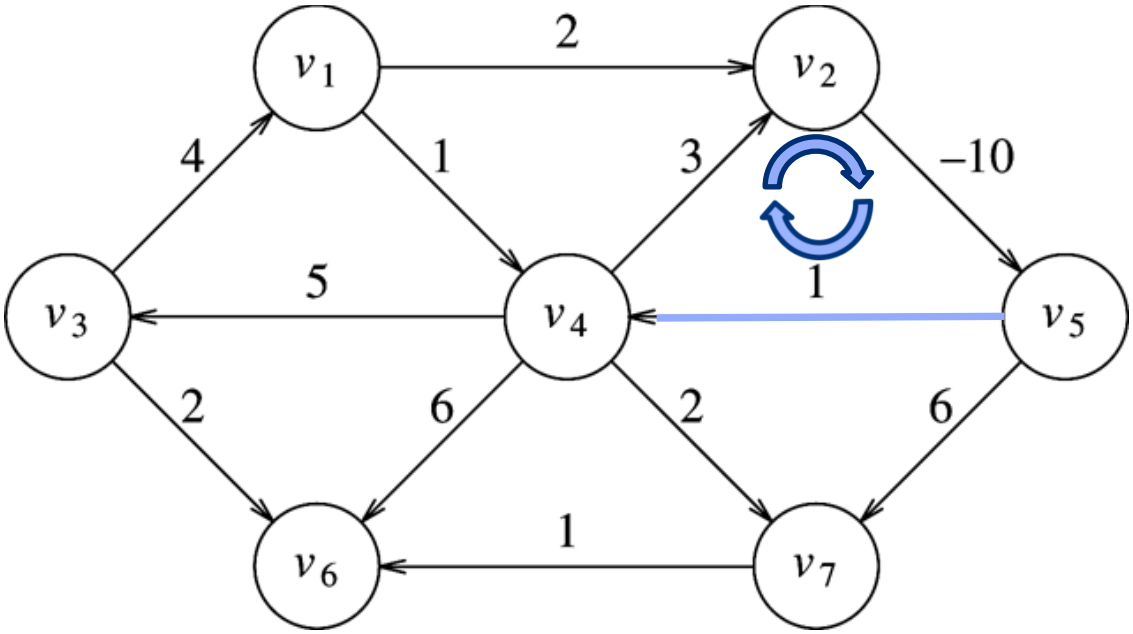


v1, v2, v5, v4, v3, v7, v6

Shortest Path

- **Problem:** find the path between two vertices that minimizes the number of edges or minimizes the sum of the weights of the edges.
  - Unweighted Graphs – number of edges
  - Weighted Graphs – sum of the costs
- Referred to as *single-pair shortest path* problem
- Used in network routing, flight scheduling, everyday life.
- **B**readth **F**irst **S**earch – Find shortest path in unweighted graph.

Negative Cost Cycle

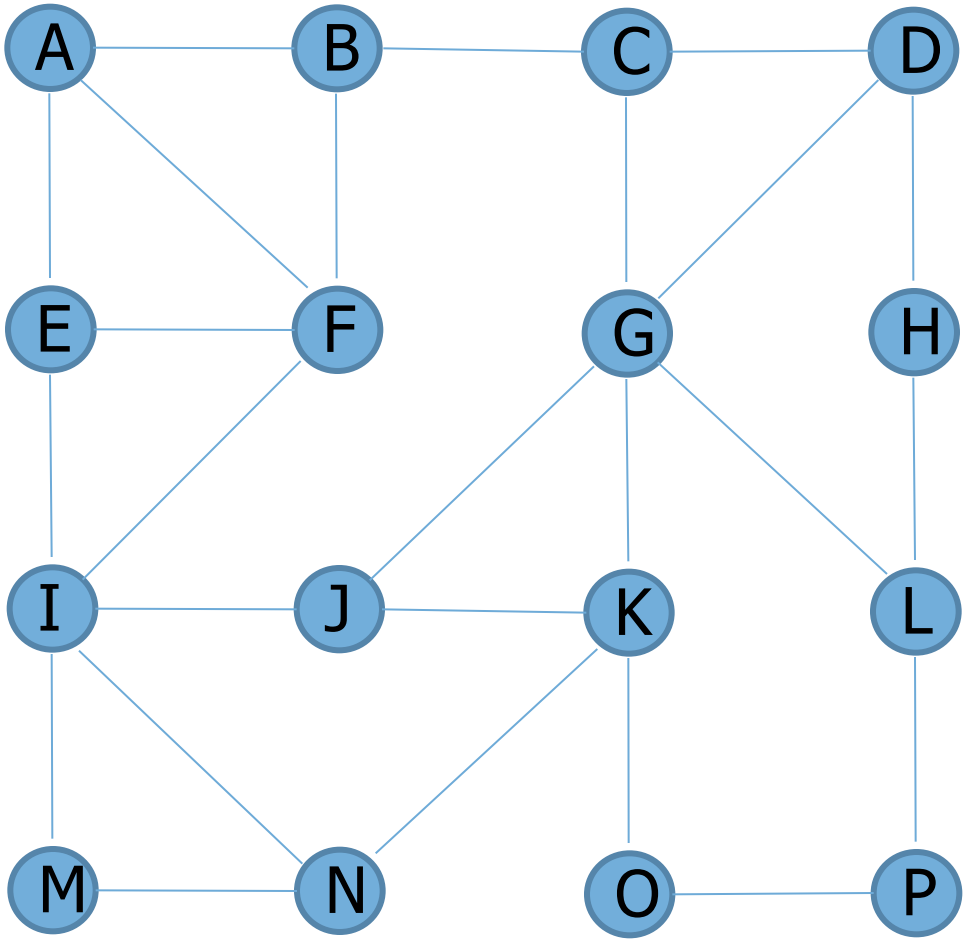


Shortest Path NOT defined

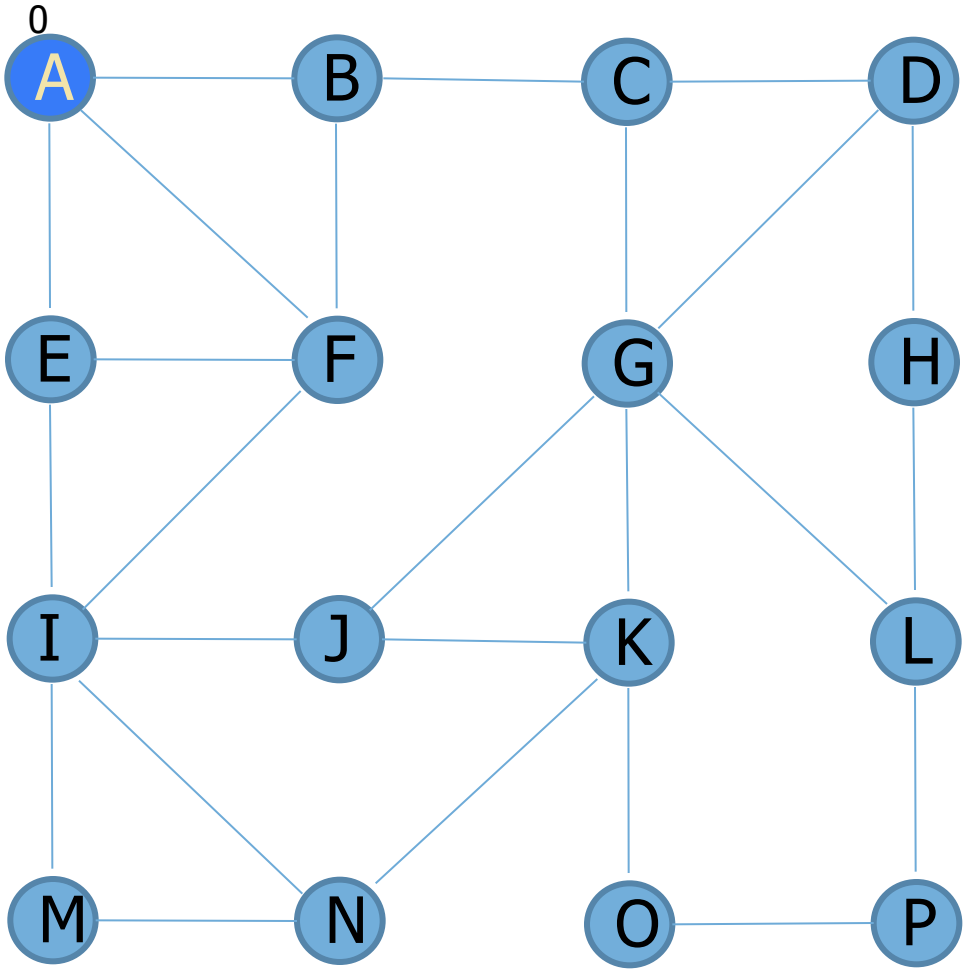
BFS - Breadth First Search

- Process vertices in layers
  - s – source vertex
  - d – destination vertex
  - Vertices **1** edge away from vertex s, then **2** edges away from s, etc.
- Each vertex d holds information:
  - $Distance(s,d)$, number of edges in path from s to d
  - $Parent(d)$, for retrieving path

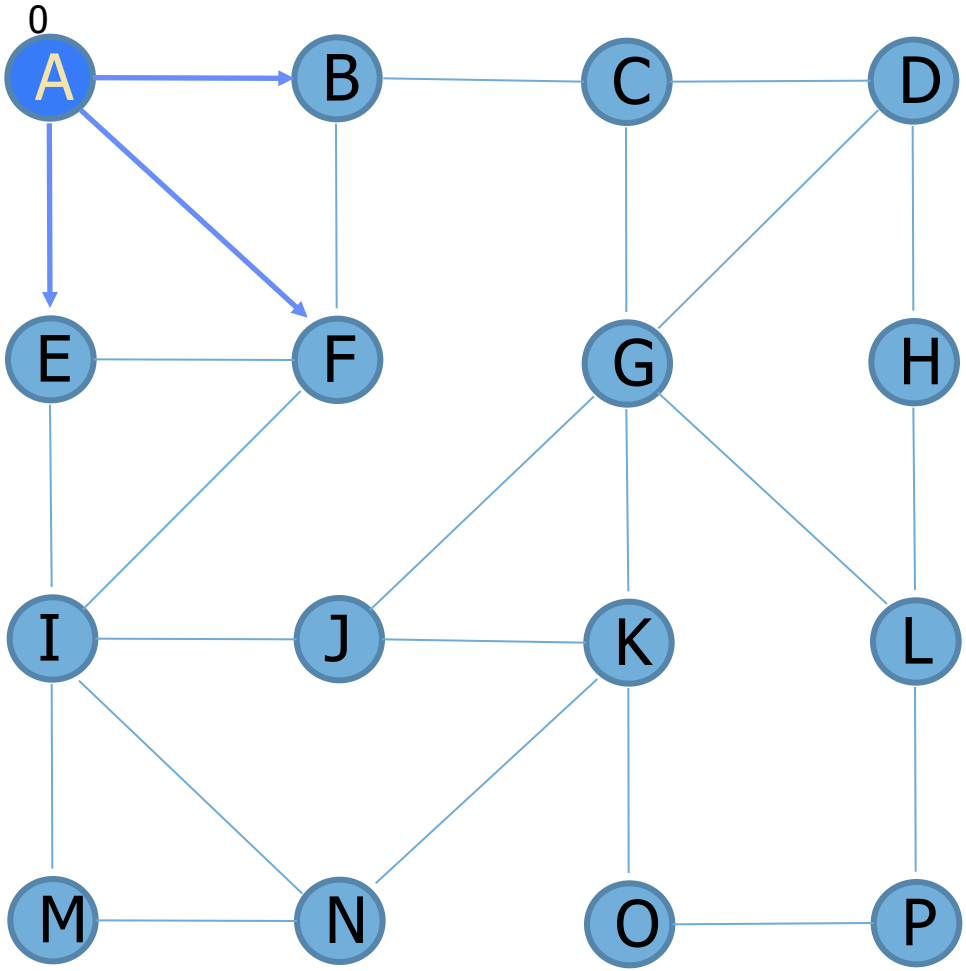
Breadth First Search



A visited

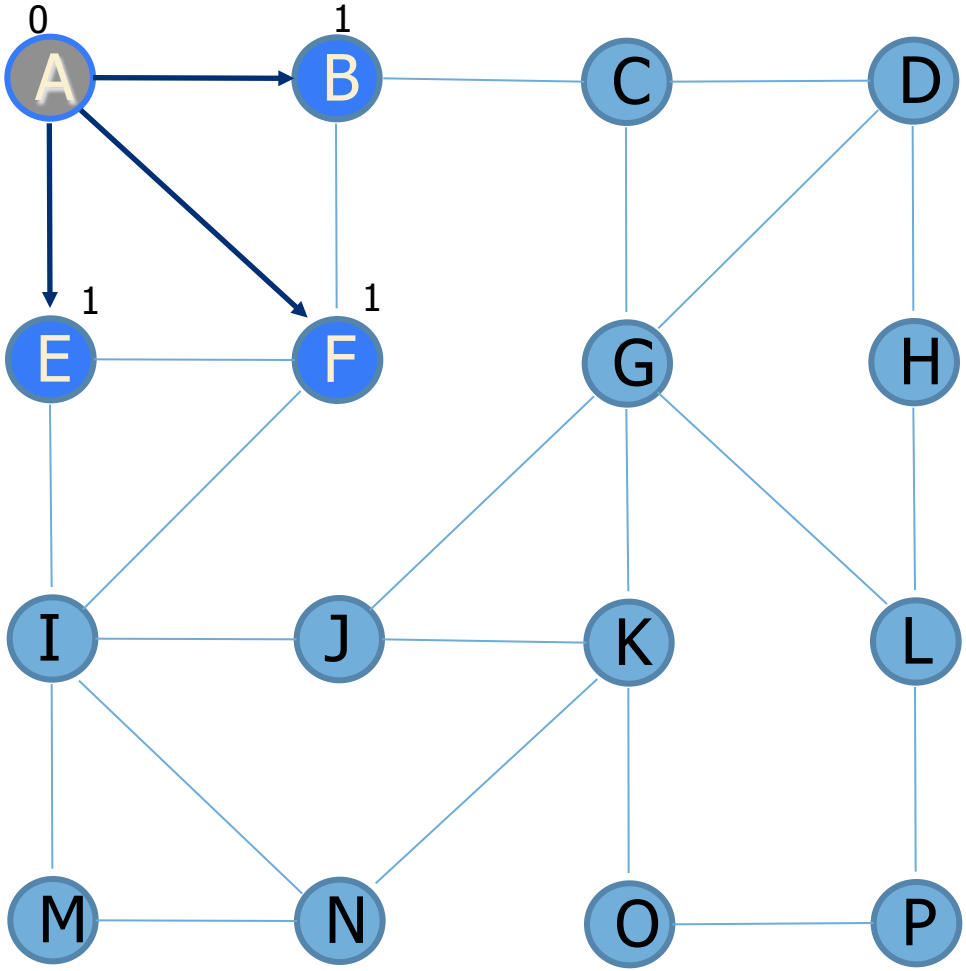


Explore A's neighbors



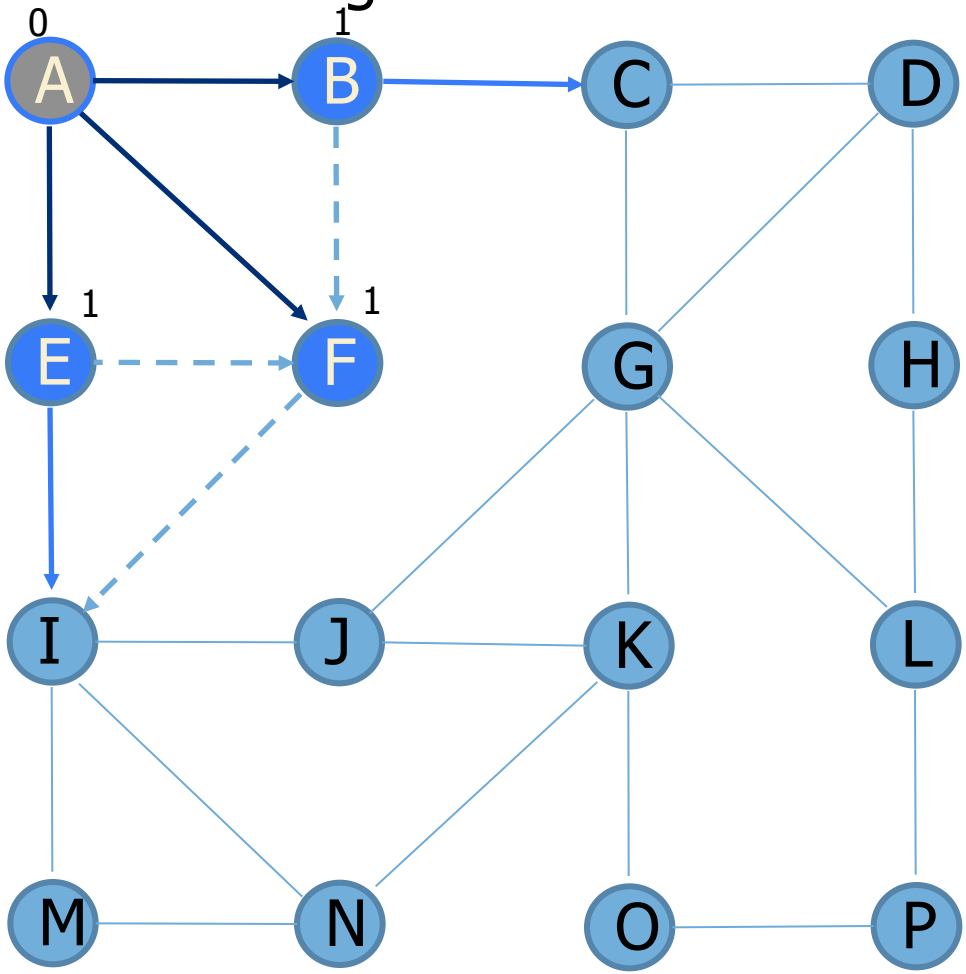
A is processed

B, E, F visited



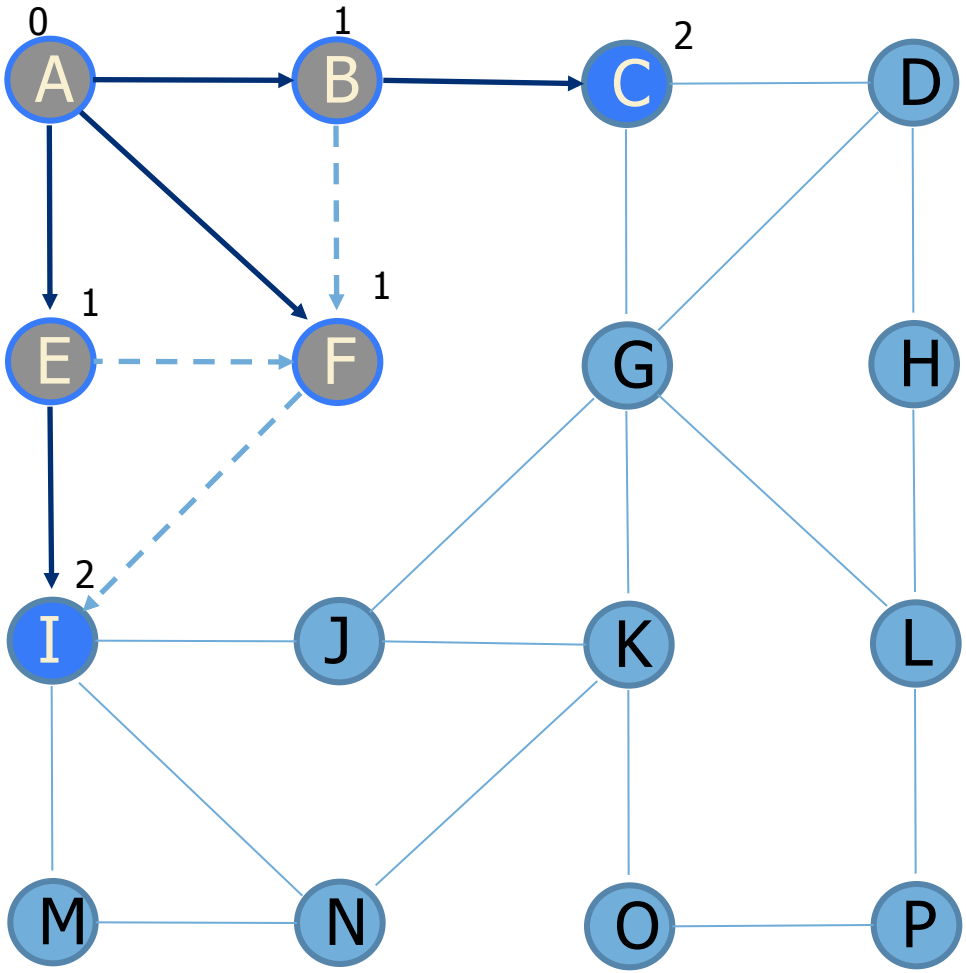
Explore B's neighbors

Explore E then F's neighbors

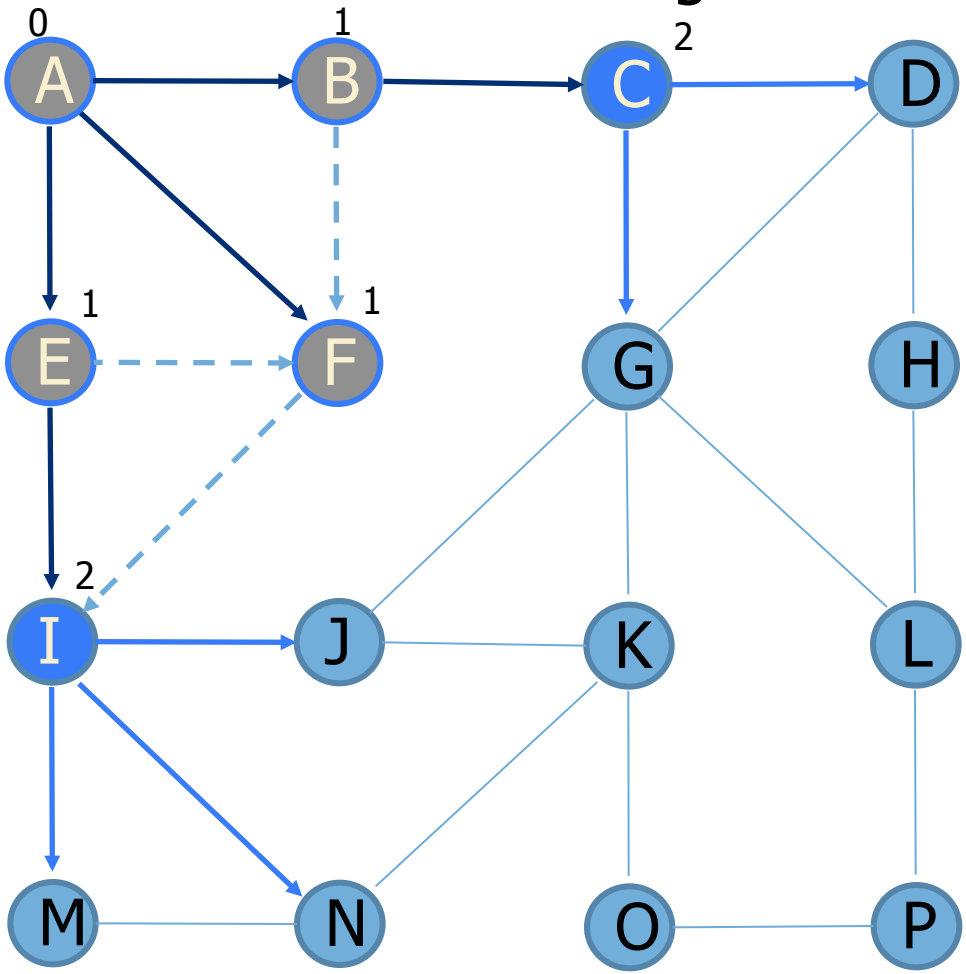


C, I visited

B,E,F are processed

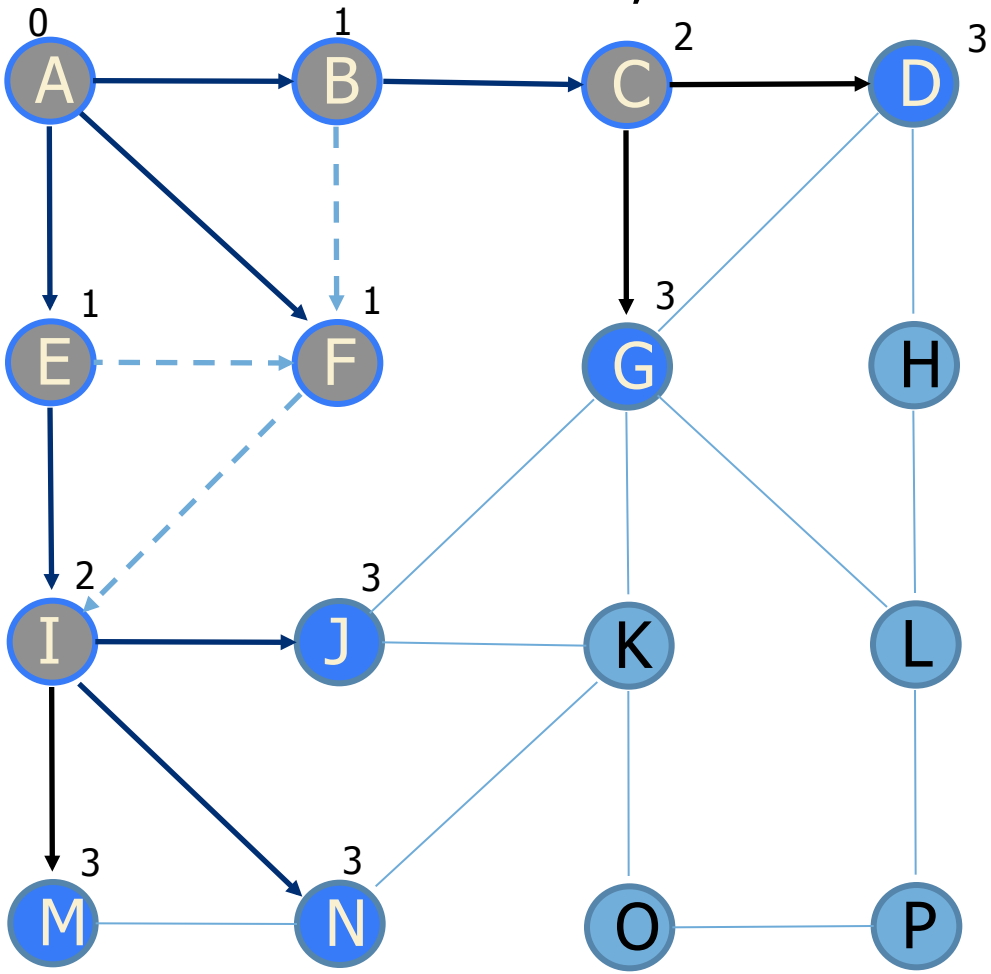


Explore C's neighbors



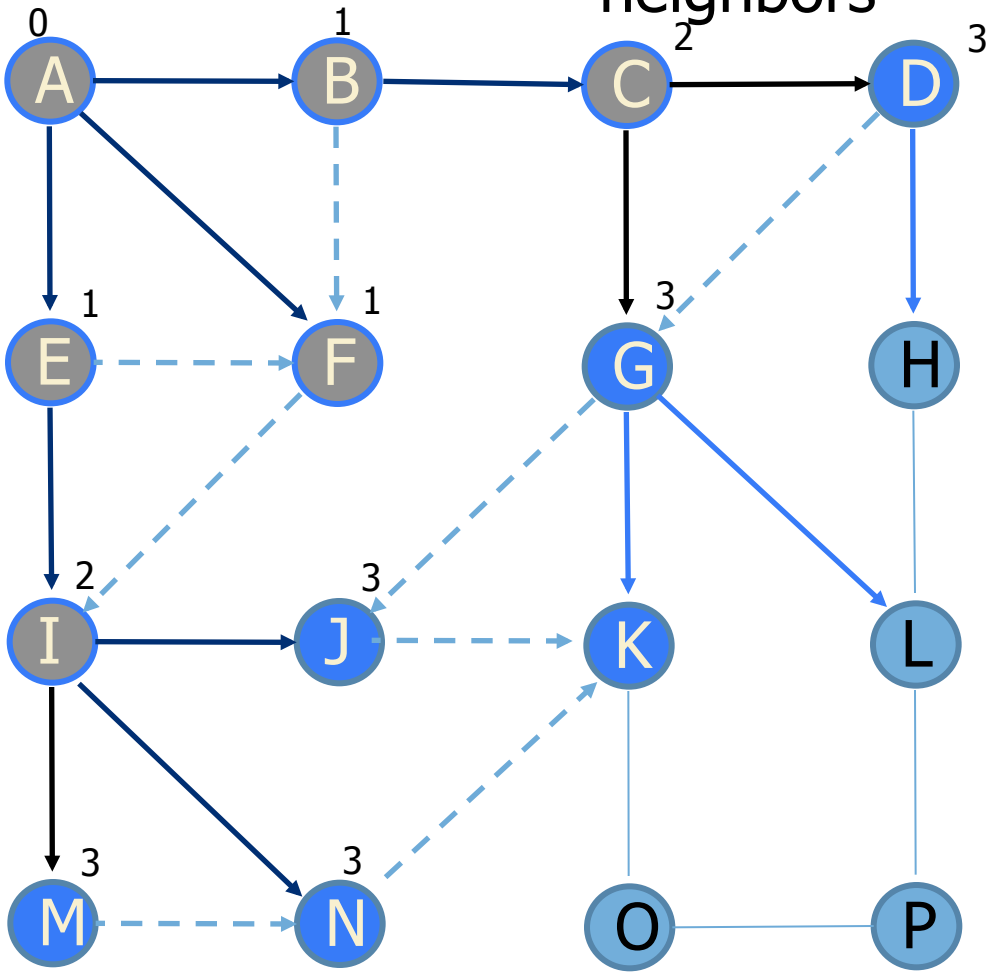
Explore I's neighbors

C processed
D,E visited



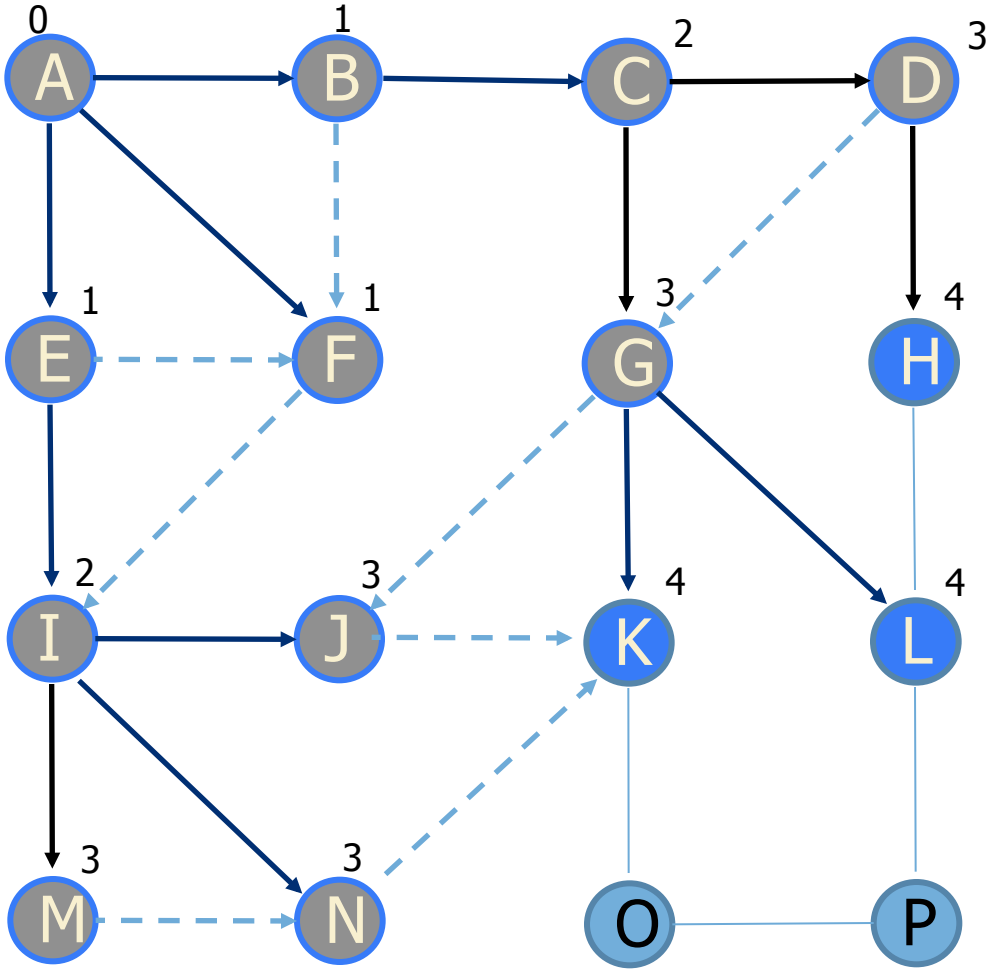
I processed
J,M,N
visited

Explore
D's,G's
neighbors



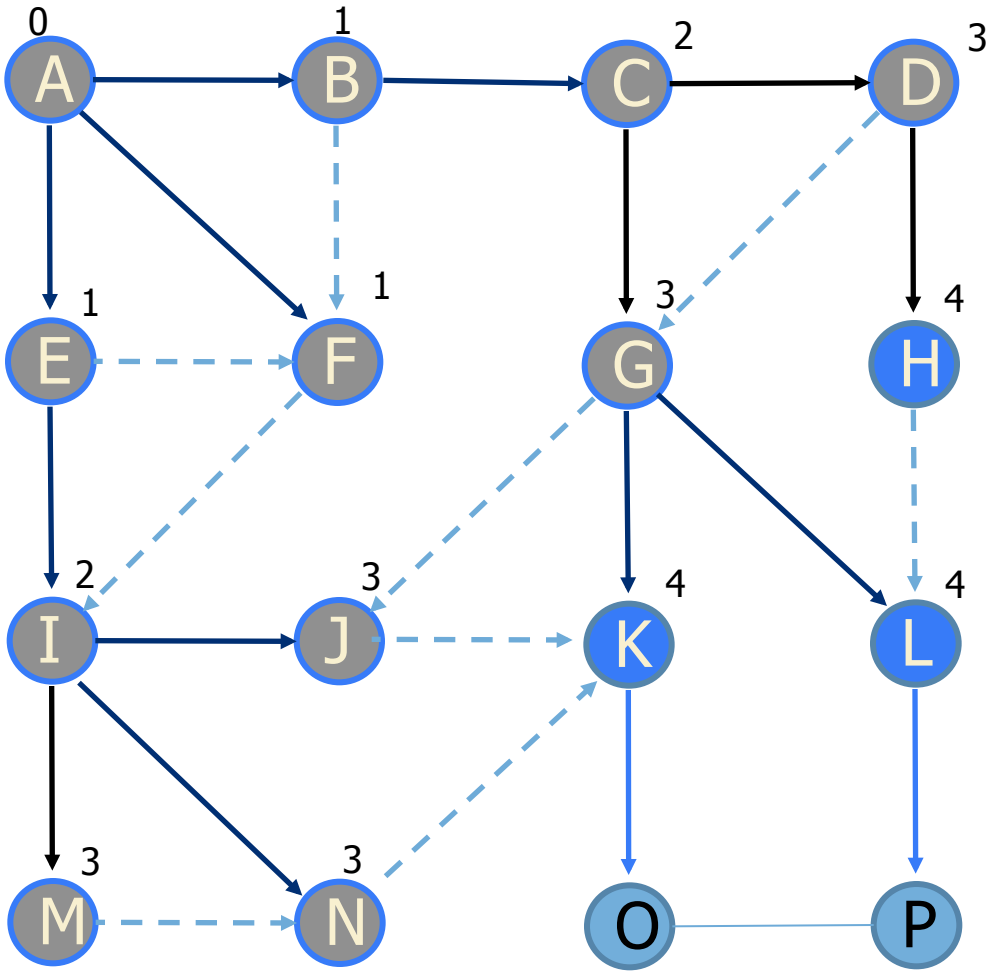
Explore
J's, M's,
N's
neighbors

D, G
processed

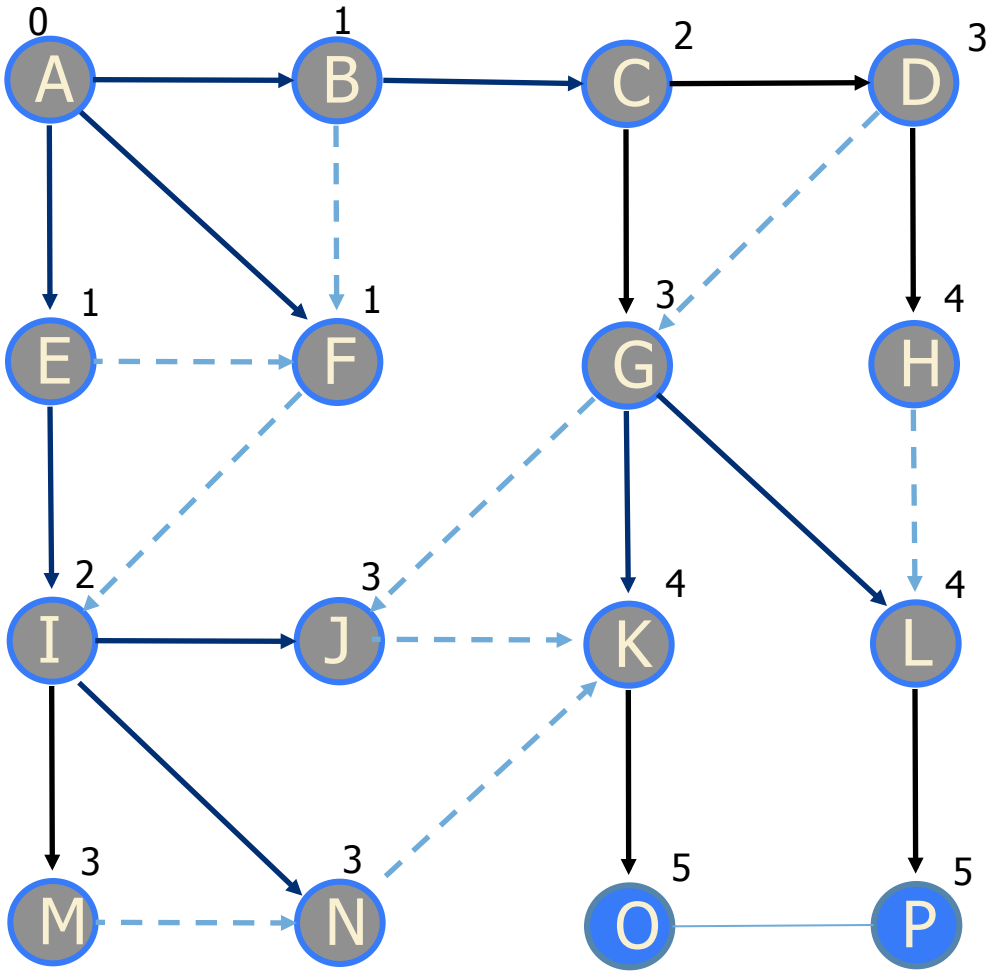


H, K, L
visited

J, M, N
processed

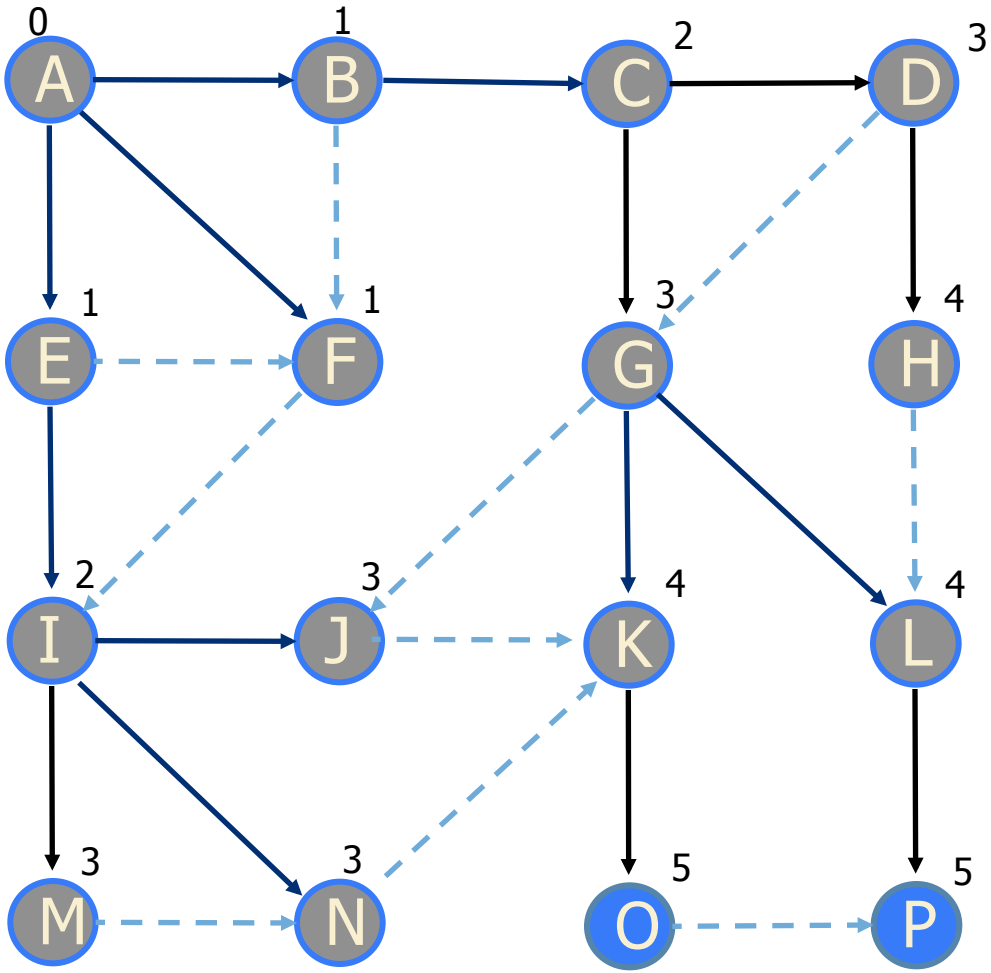


Explore
H, K, L
neighbors



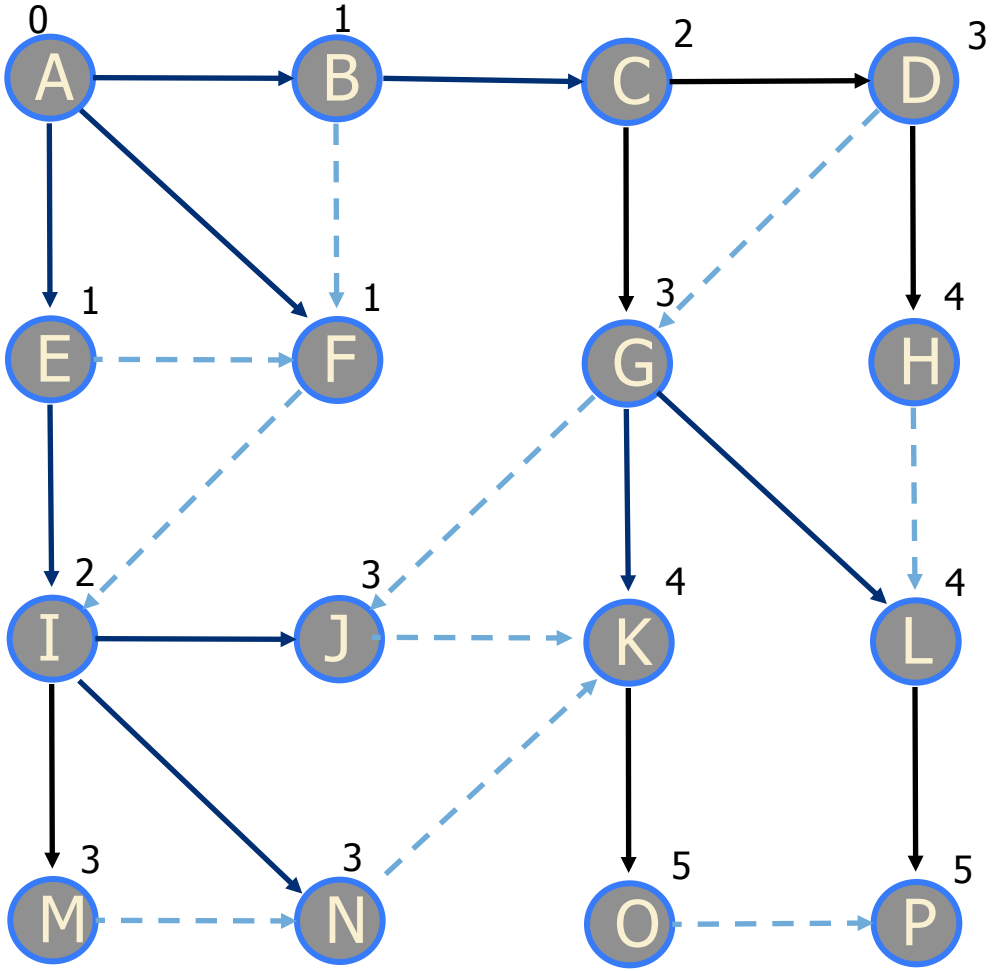
H, K, L processed

O,P visited



Explore O's neighbor

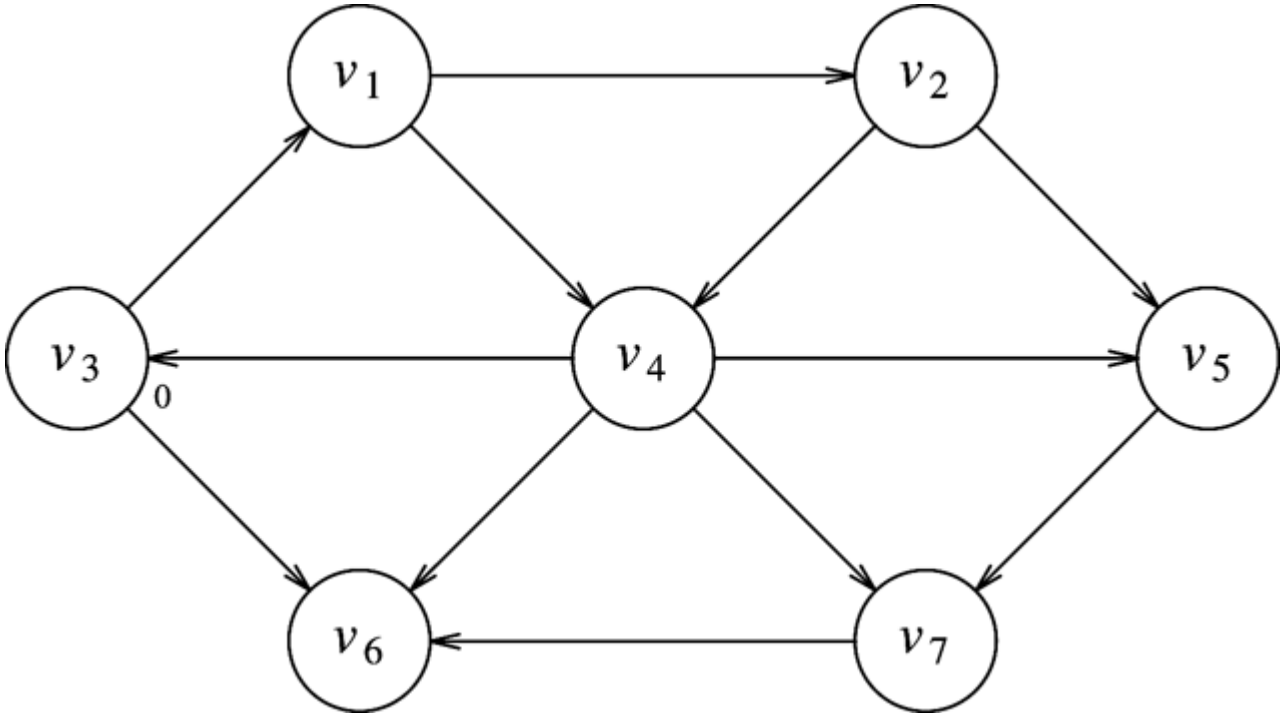
BFS



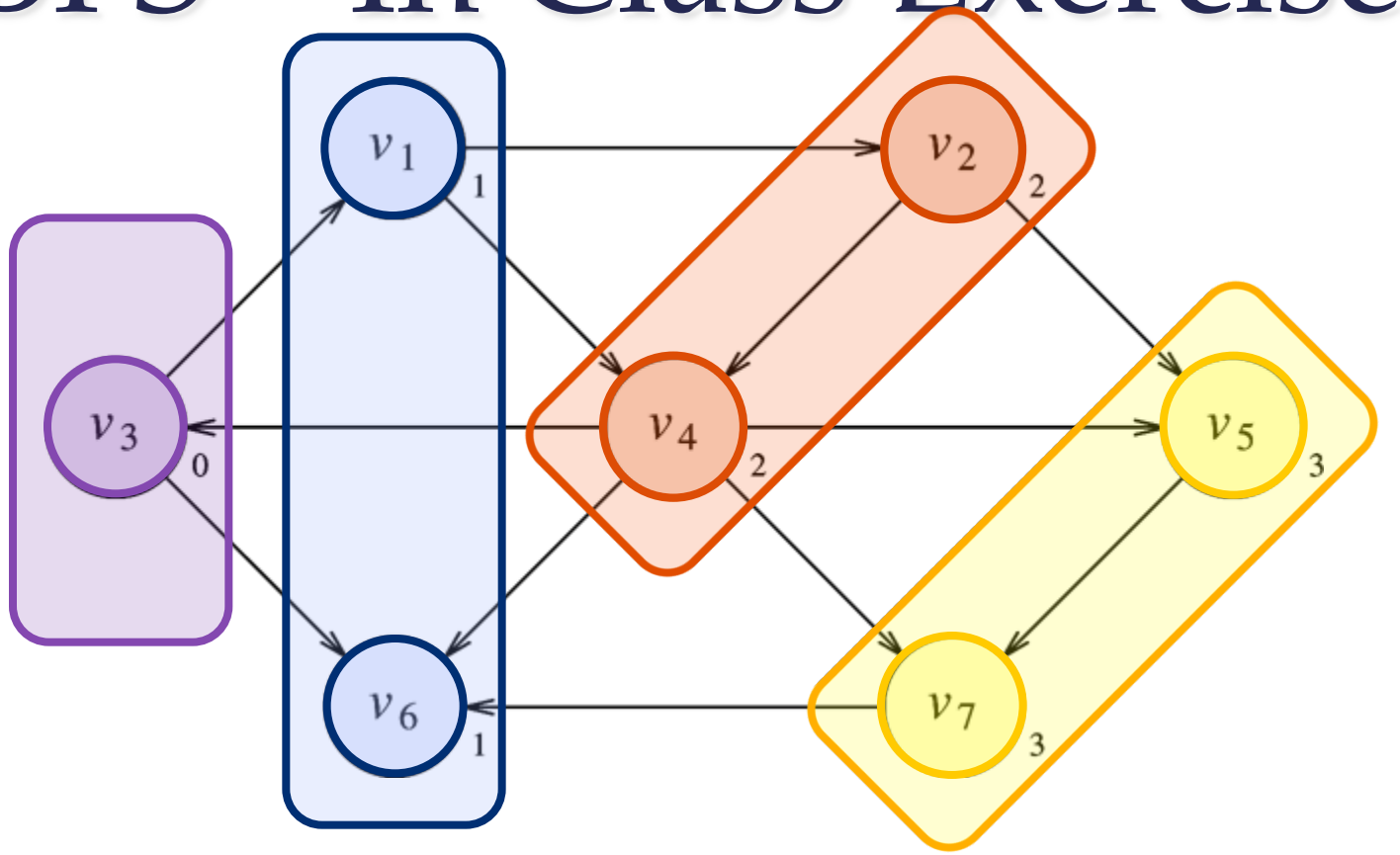
Done

Running Time = $O(|E| + |V|)$

BFS – In Class Exercise

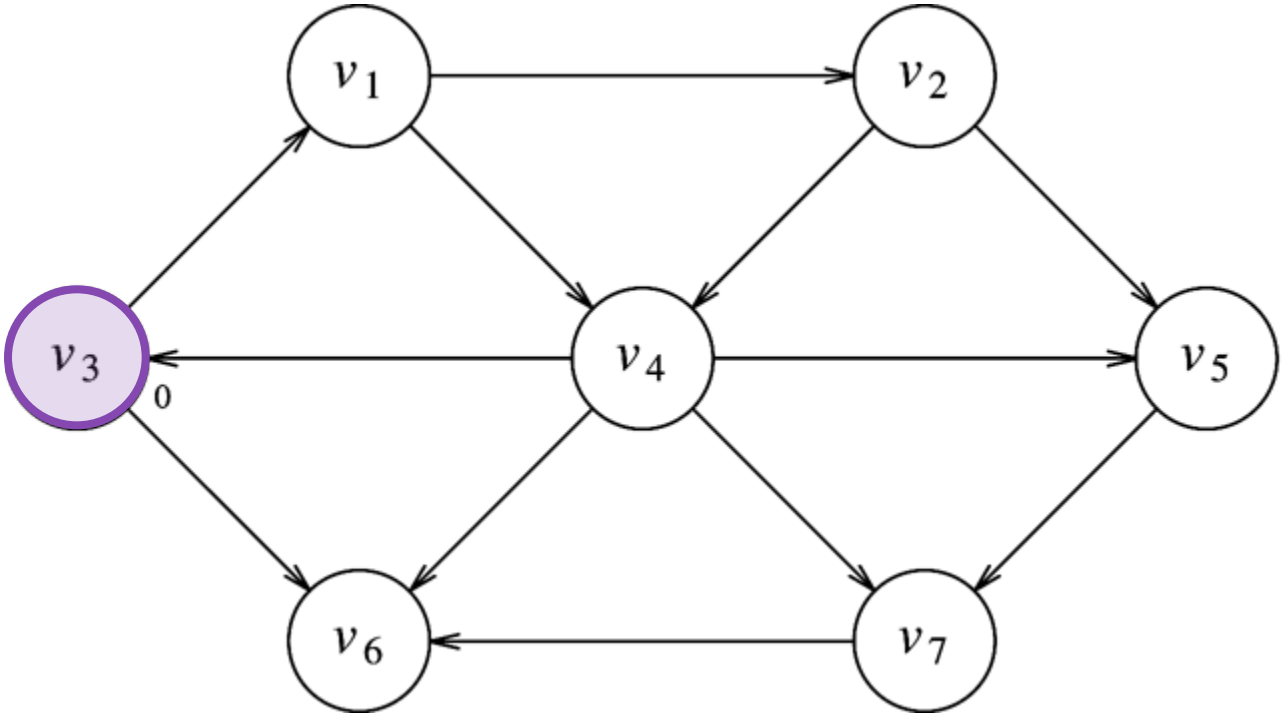


BFS – In Class Exercise

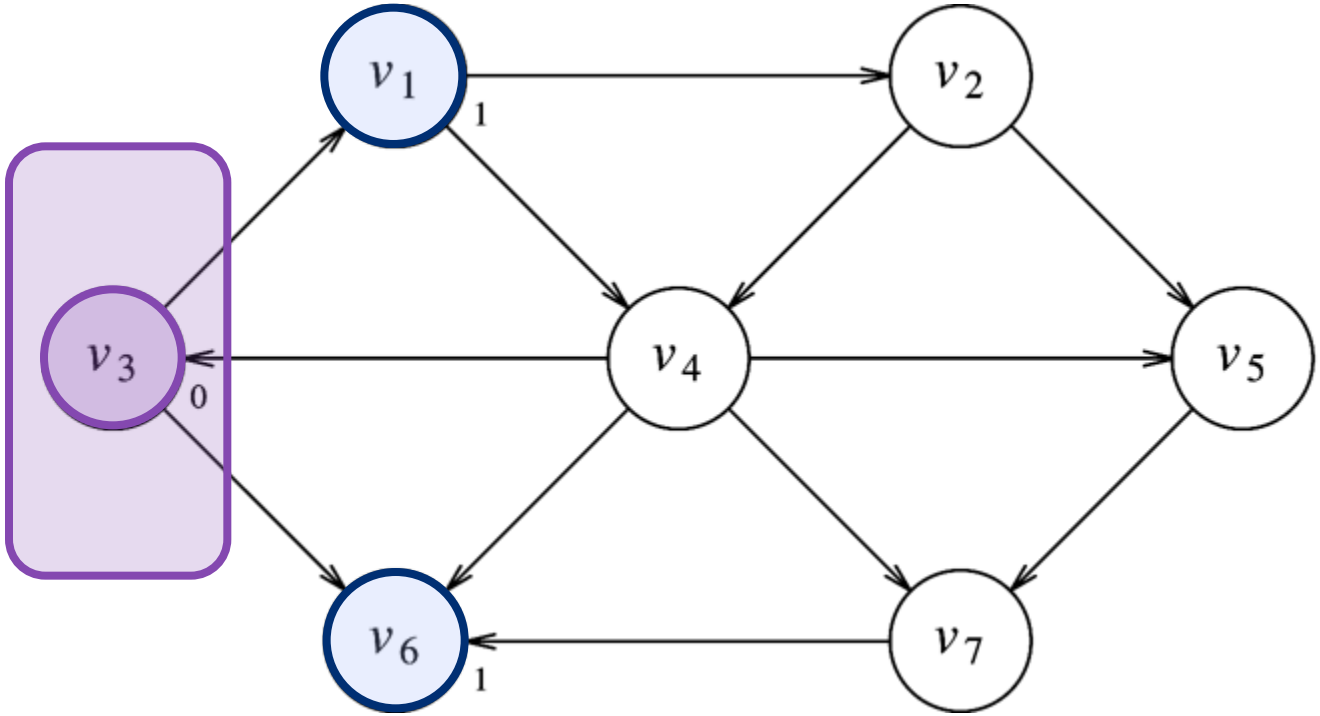


Running Time = $O(|E| + |V|)$, every V is explored, all E in worst case.

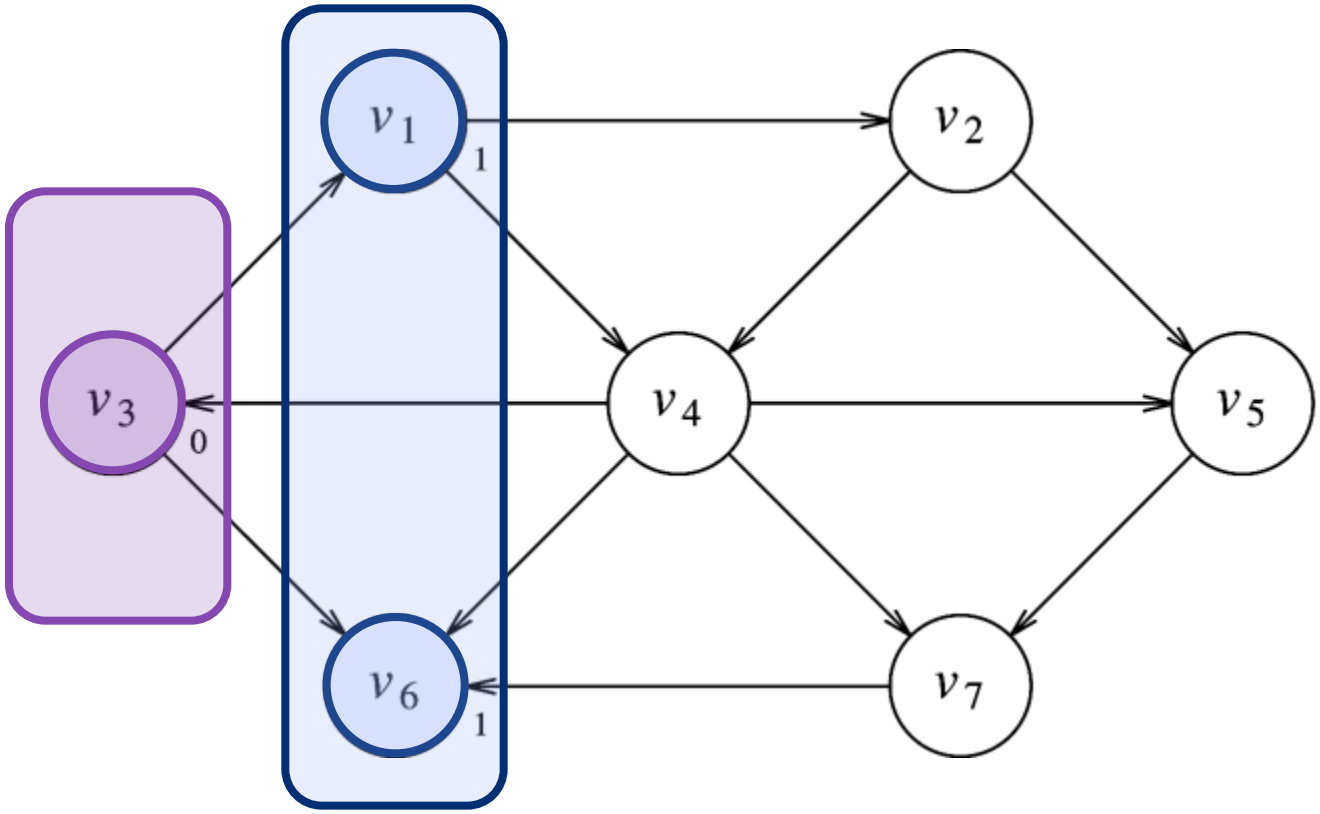
BFS – In Class Exercise



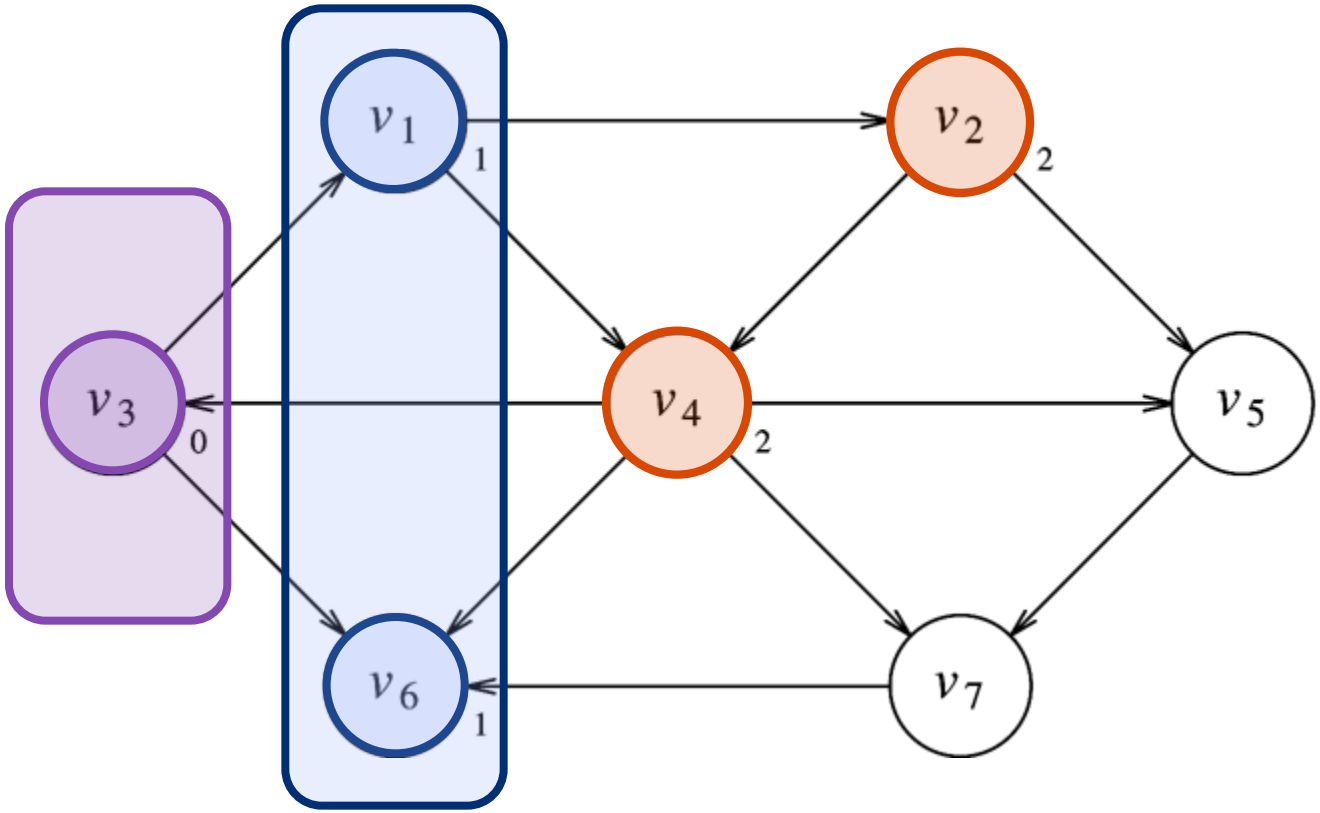
BFS – In Class Exercise



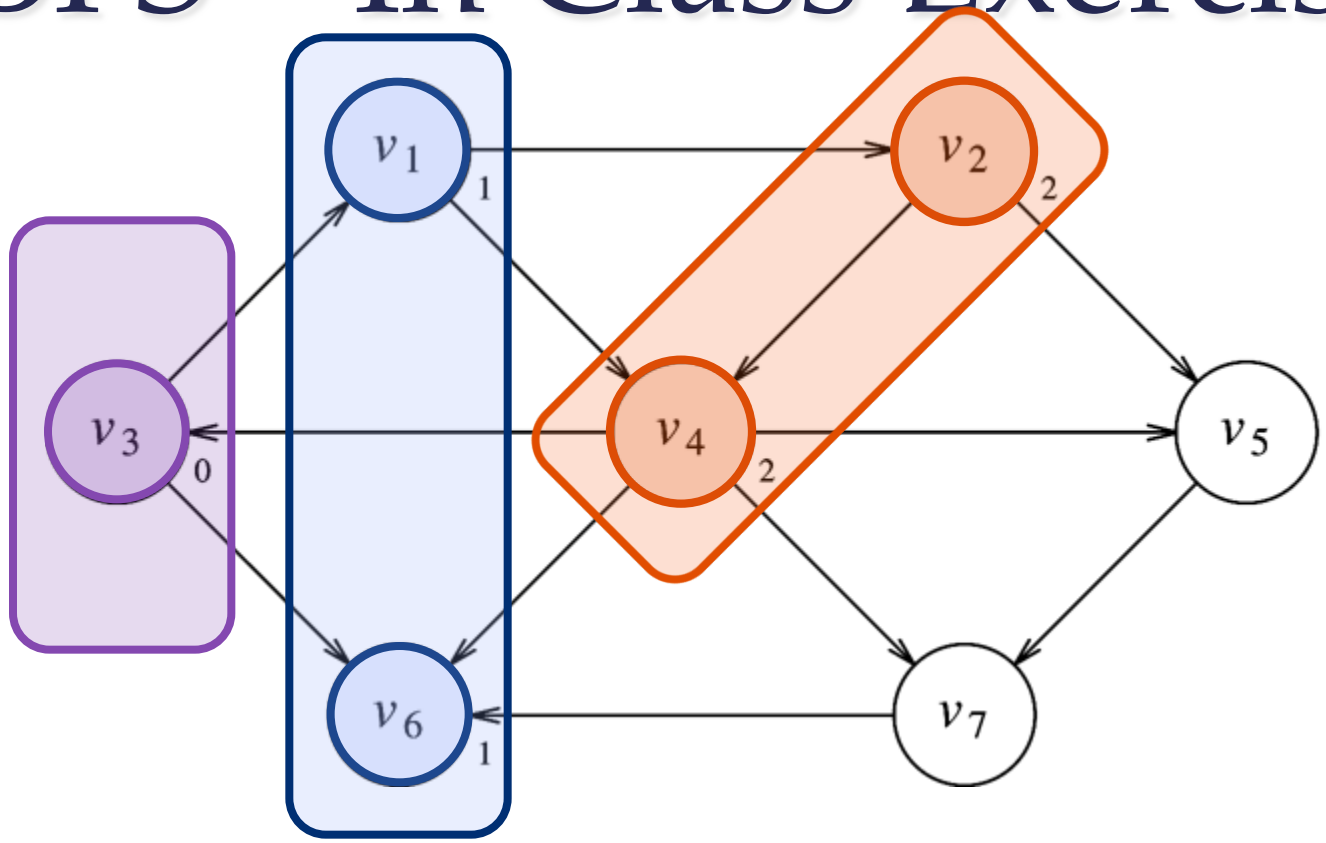
BFS – In Class Exercise



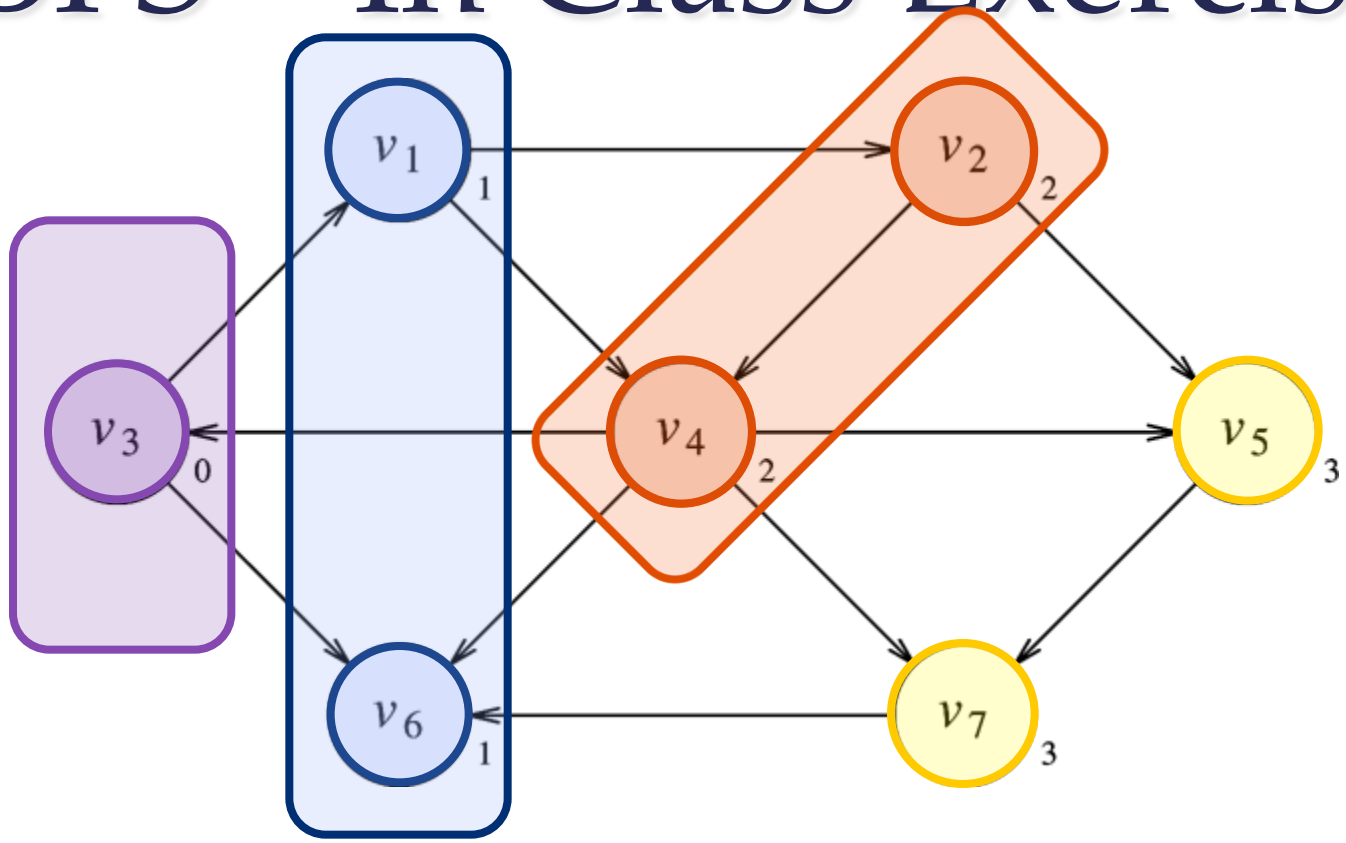
BFS – In Class Exercise



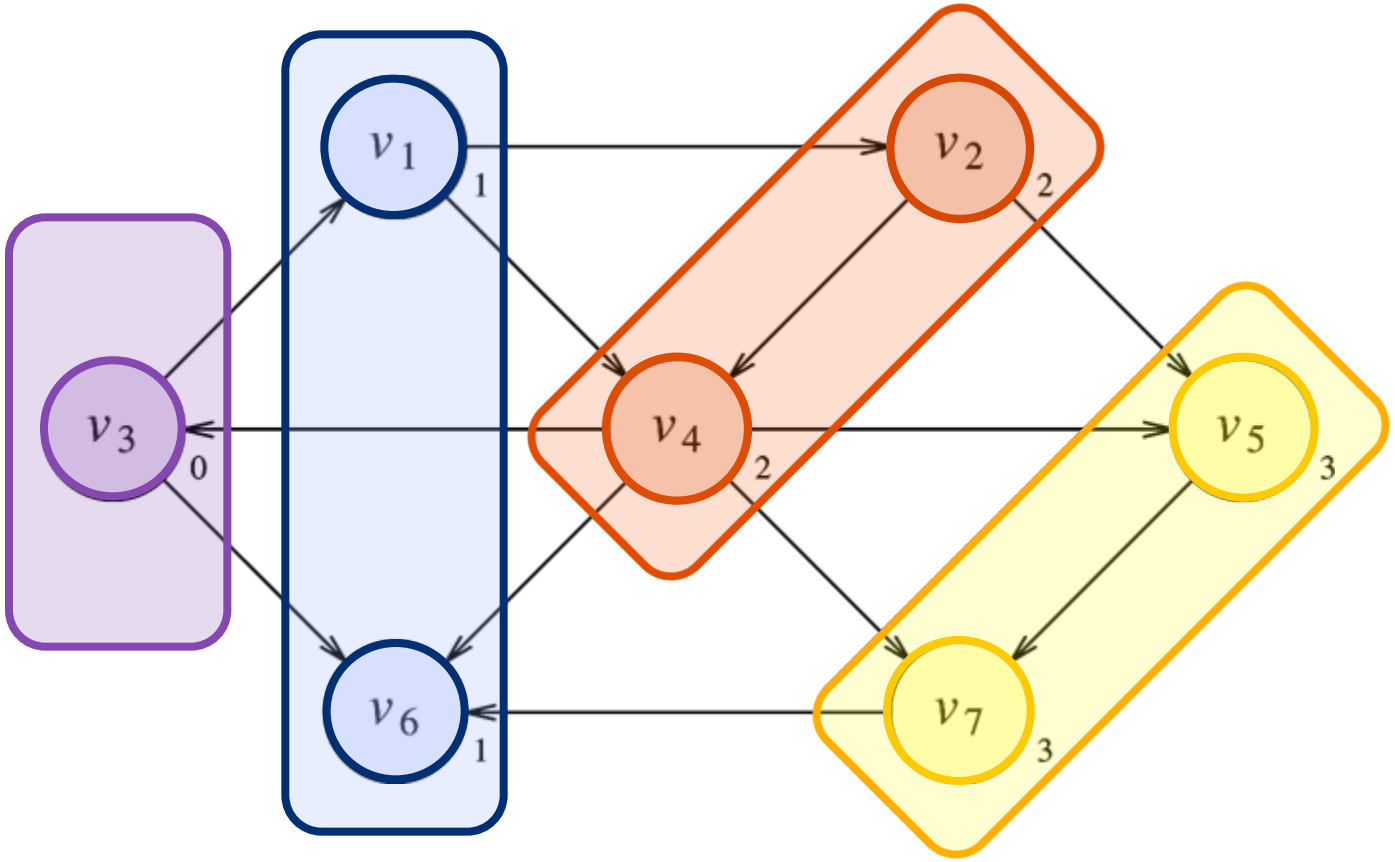
BFS – In Class Exercise



BFS – In Class Exercise

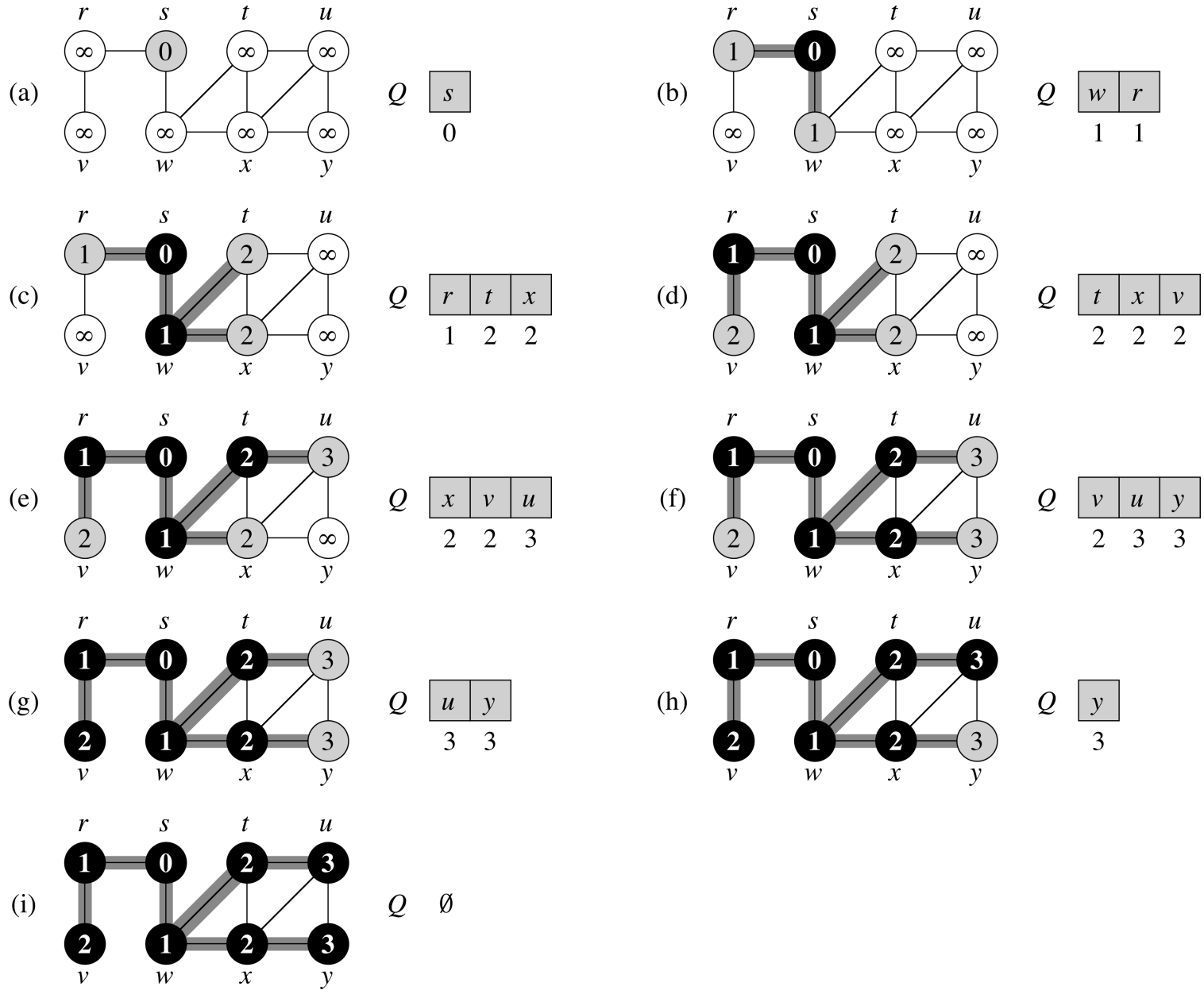


BFS – In Class Exercise



Running Time = $O(|E| + |V|)$

```
BFS(G, s)
 1  for each vertex u ∈ G.V − {s}
 2       u.color = WHITE
 3       u.d = ∞
 4       u.π = NIL
 5  s.color = GRAY
 6  s.d = 0
 7  s.π = NIL
 8  Q = ∅
 9  ENQUEUE(Q, s)
10  while Q ≠ ∅
11       u = DEQUEUE(Q)
12       for each v ∈ G.Adj[u]
13            if v.color == WHITE
14                 v.color = GRAY
15                 v.d = u.d + 1
16                 v.π = u
17                 ENQUEUE(Q, v)
18       u.color = BLACK
```


(a)
r s t u
∞ 0 ∞ ∞
∞ ∞ ∞ ∞
v w x y
Q
s
0
(b)
r s t u
1 0 ∞ ∞
∞ 1 ∞ ∞
v w x y
Q
w r
1 1
(c)
r s t u
1 0 2 ∞
∞ 1 2 ∞
v w x y
Q
r t x
1 2 2
(d)
r s t u
1 0 2 ∞
2 1 2 ∞
v w x y
Q
t x v
2 2 2
(e)
r s t u
1 0 2 3
2 1 2 ∞
v w x y
Q
x v u
2 2 3
(f)
r s t u
1 0 2 3
2 1 2 3
v w x y
Q
v u y
2 3 3
(g)
r s t u
1 0 2 3
2 1 2 3
v w x y
Q
u y
3 3
(h)
r s t u
1 0 2 3
2 1 2 3
v w x y
Q
y
3
(i)
r s t u
1 0 2 3
2 1 2 3
v w x y
Q
∅

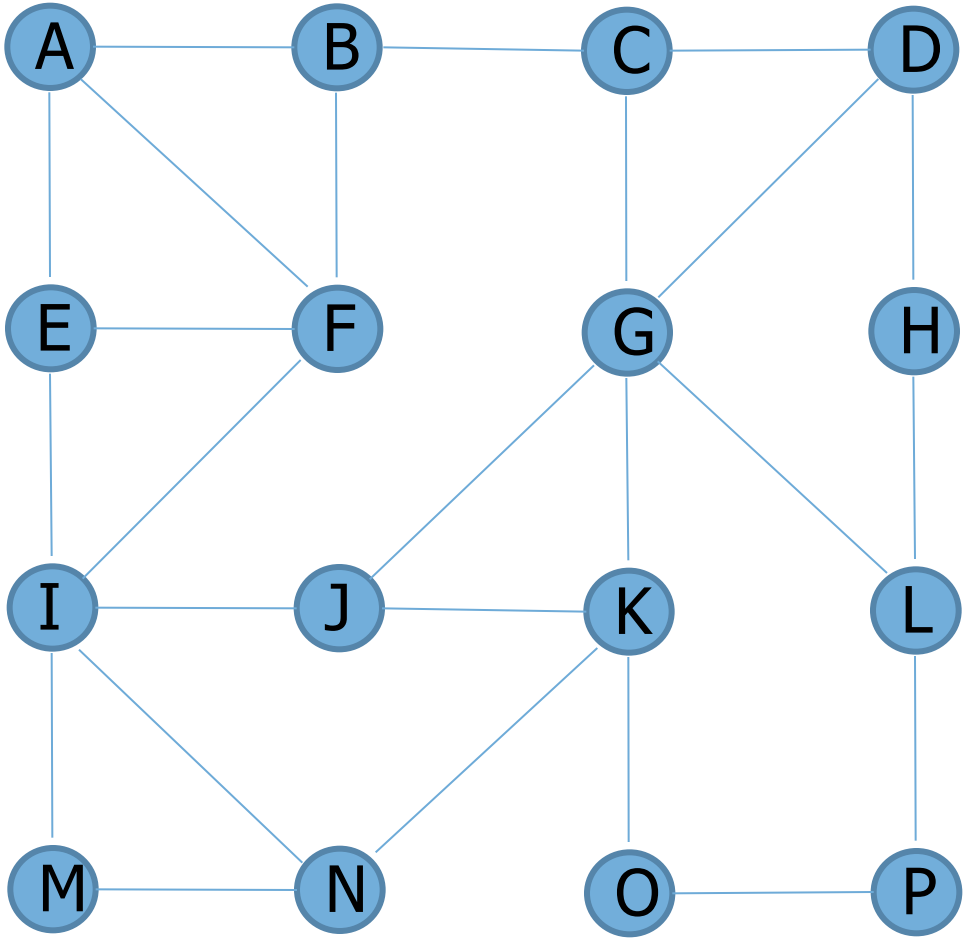
```
BFS(V, E, s)
  for each u ∈ V − {s}
        u.d = ∞
  s.d = 0
  Q = ∅
  ENQUEUE(Q, s)
  while Q ≠ ∅
        u = DEQUEUE(Q)
        for each v ∈ G.Adj[u]
              if v.d == ∞
                    v.d = u.d + 1
                    ENQUEUE(Q, v)
```

DFS - Depth First Search

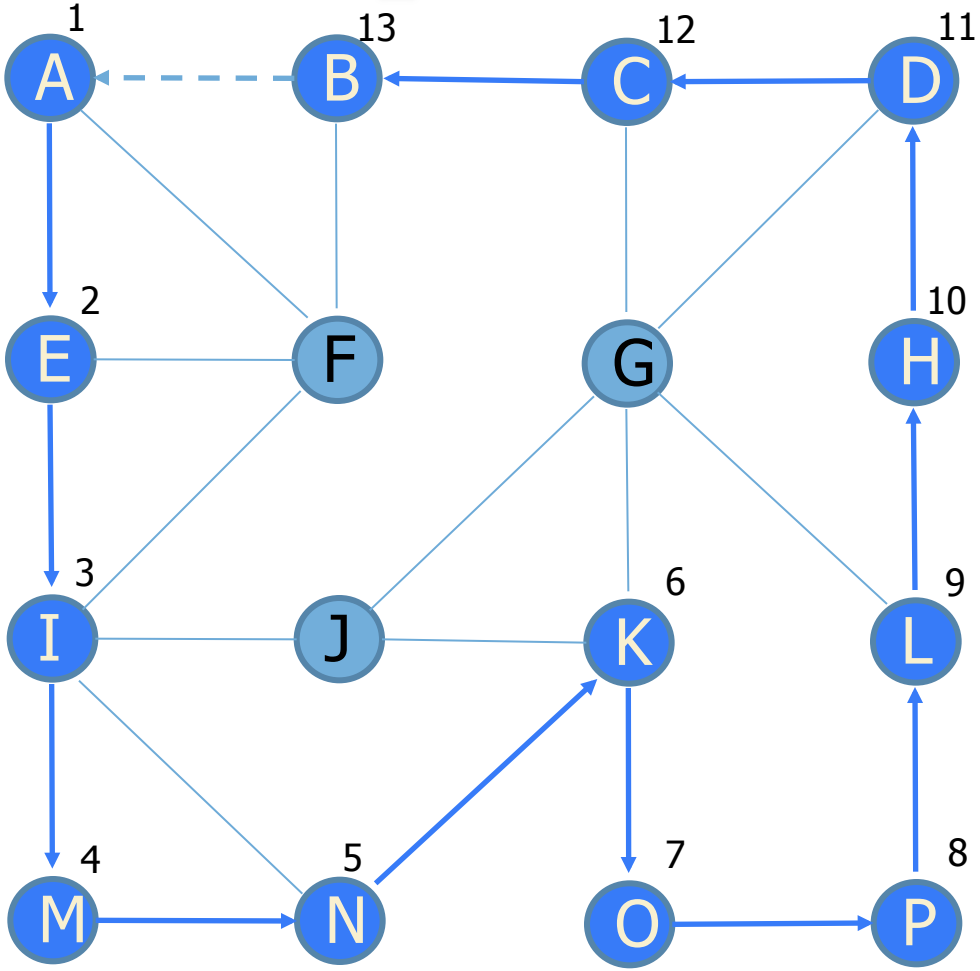
Generalization of a preorder traversal.

1. Start at an arbitrary vertex s.
2. Traverse (visit) as many descendents of s as possible without visiting a previously visited vertex.
3. Once a previously visited vertex is encountered, back out until a vertex v_i with unexplored edges is found.
4. Depth first search on $v_{i.}$
5. Repeat (steps 3-4) until all vertices visited.

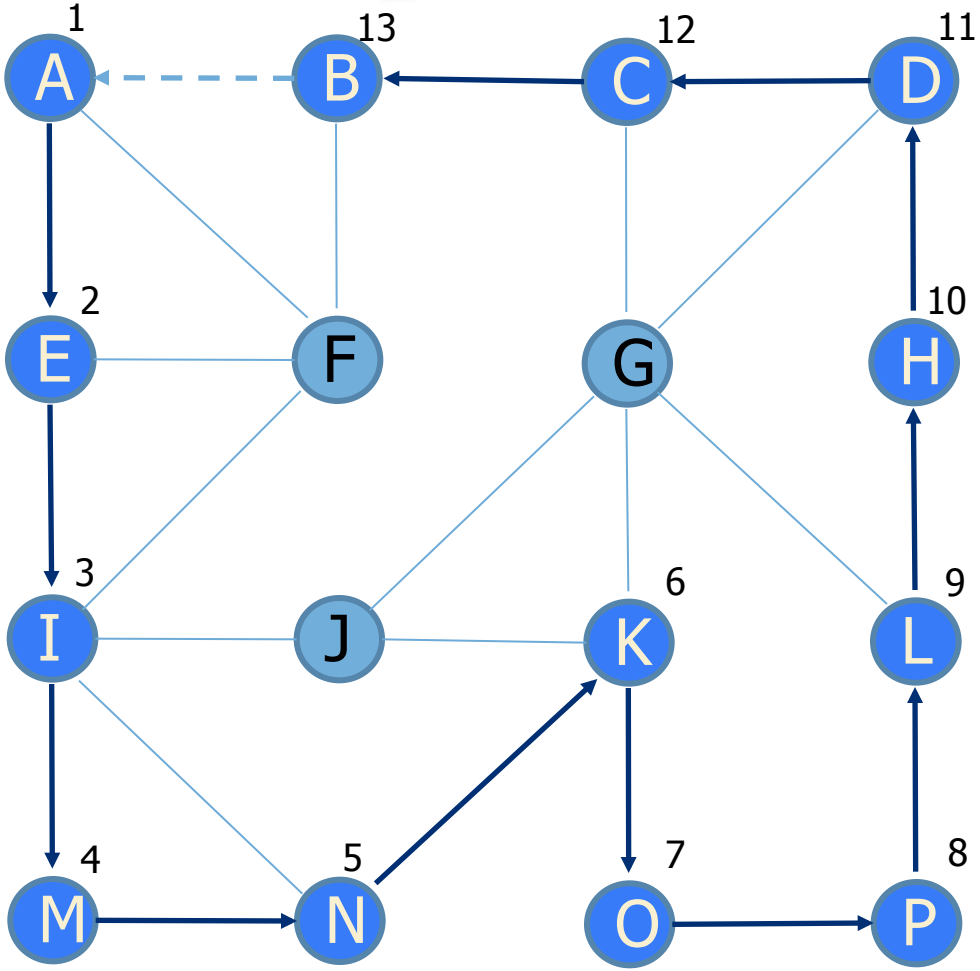
DFS - Depth First Search



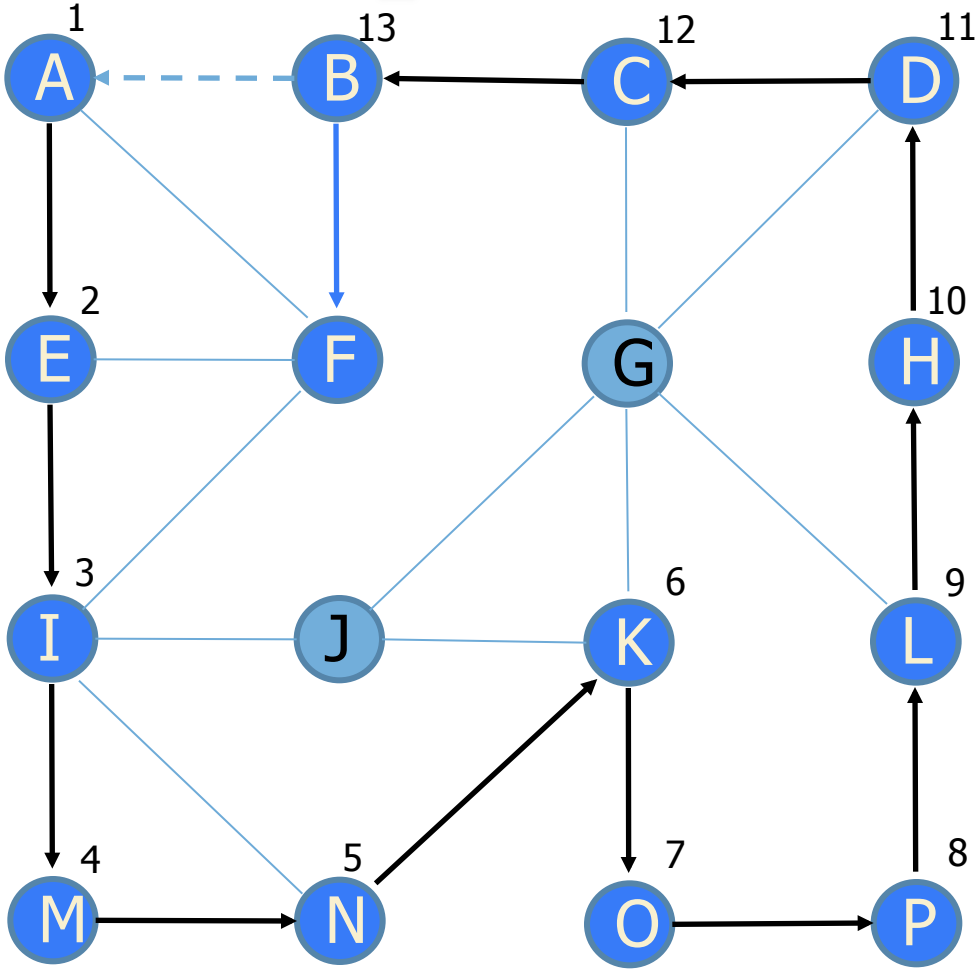
DFS - Depth First Search



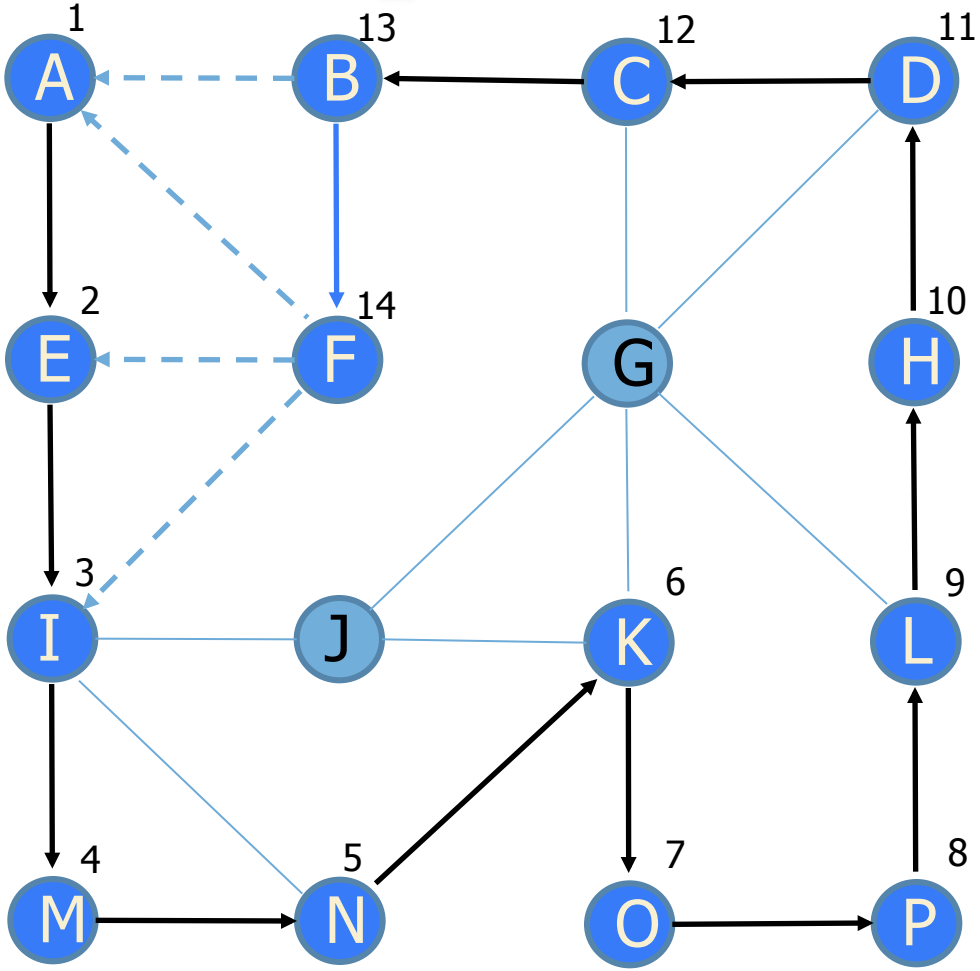
DFS - Depth First Search



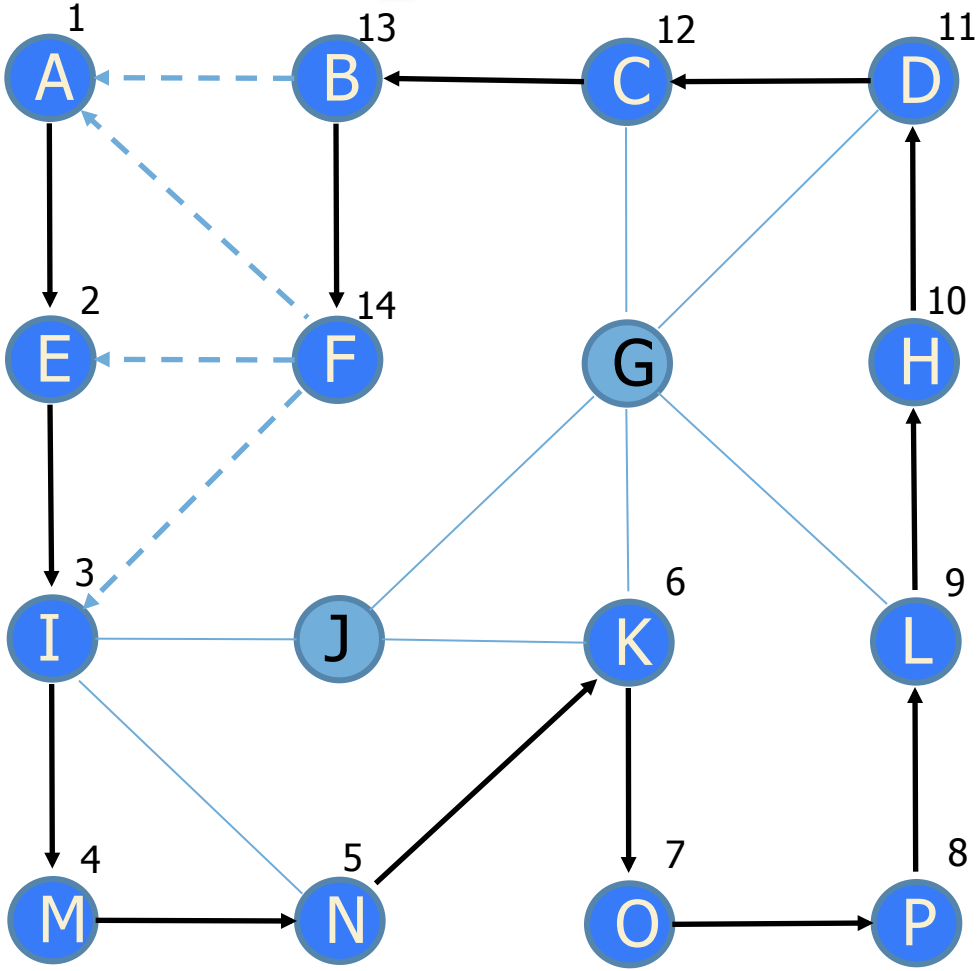
DFS - Depth First Search



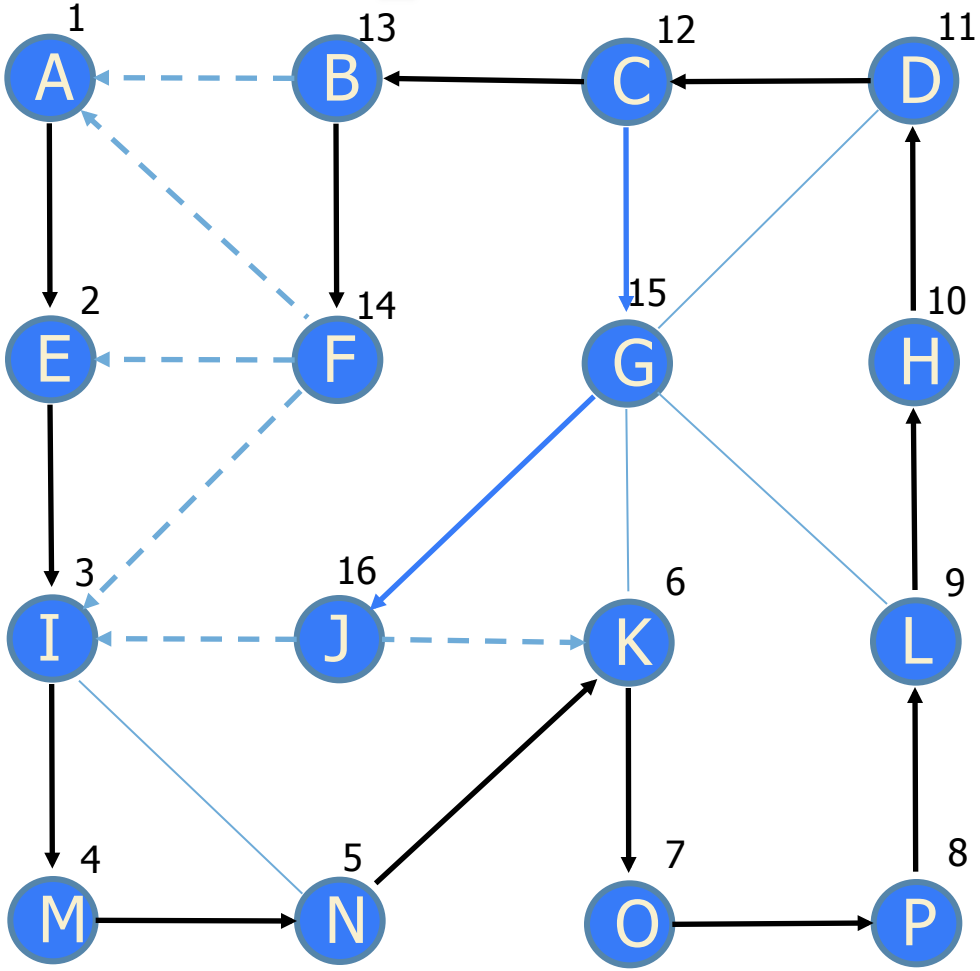
DFS - Depth First Search



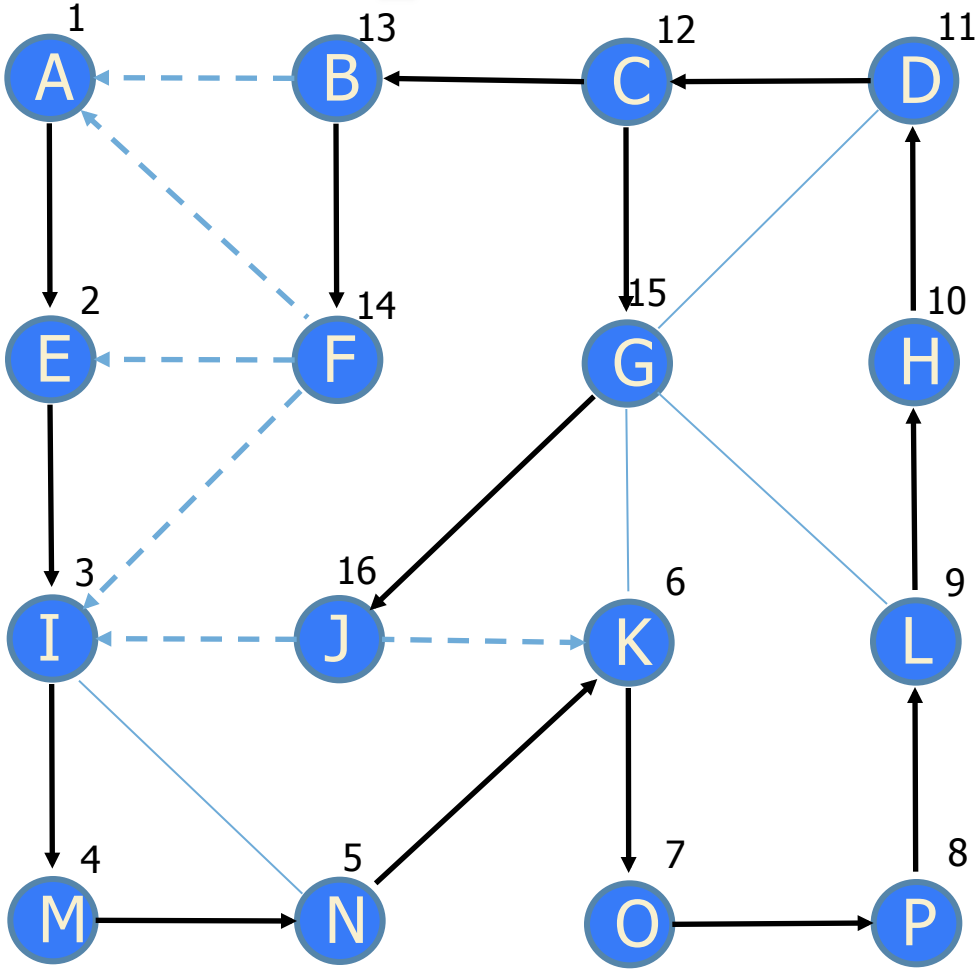
DFS - Depth First Search



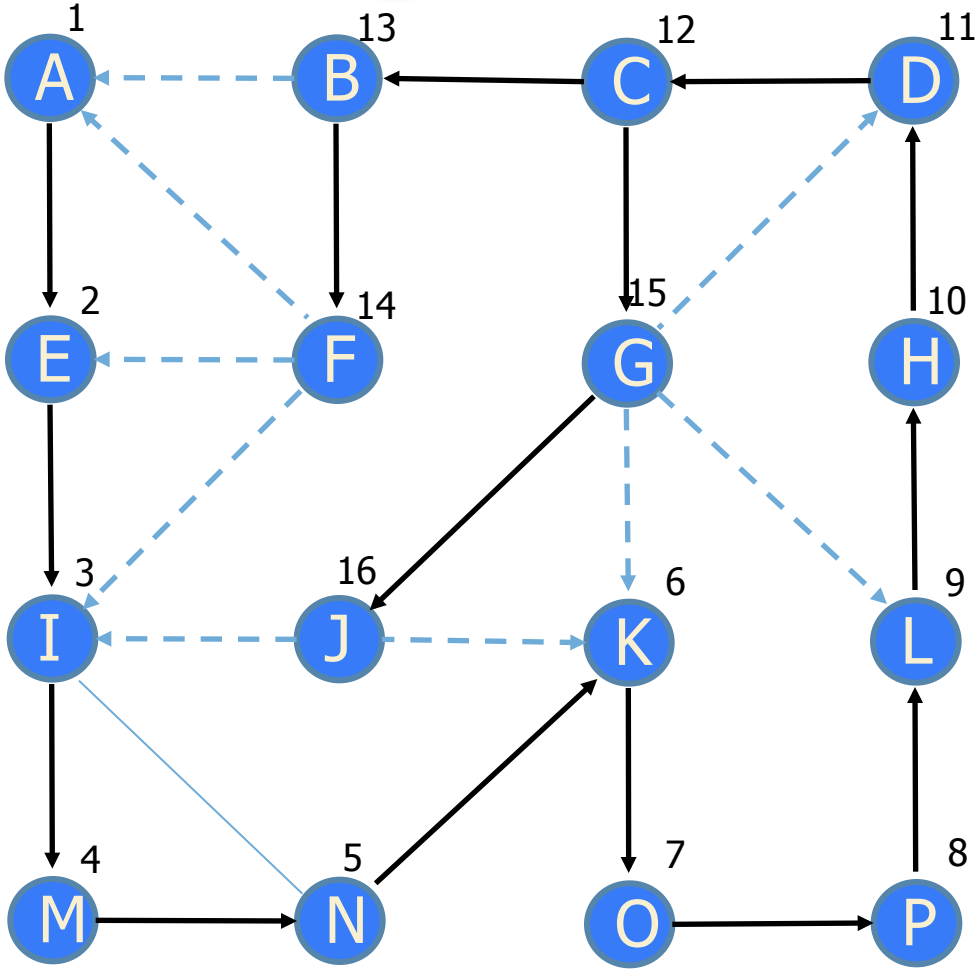
DFS - Depth First Search



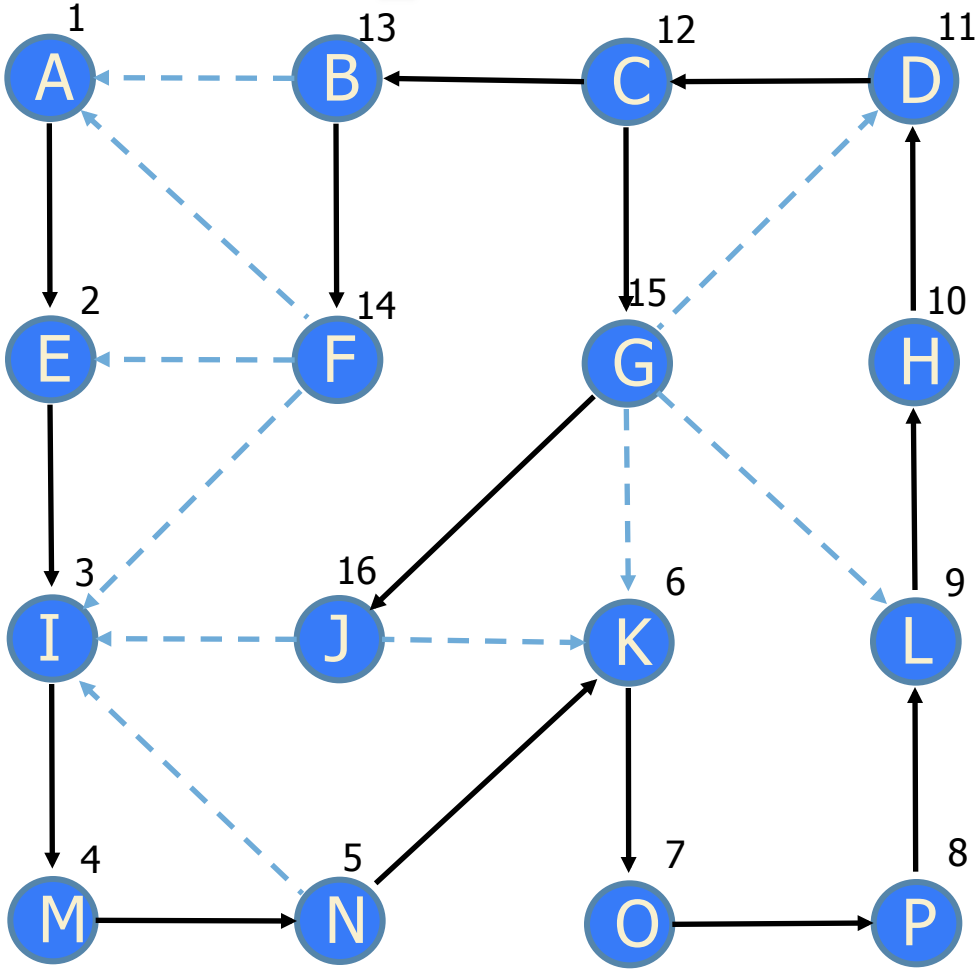
DFS - Depth First Search



DFS - Depth First Search



DFS - Depth First Search

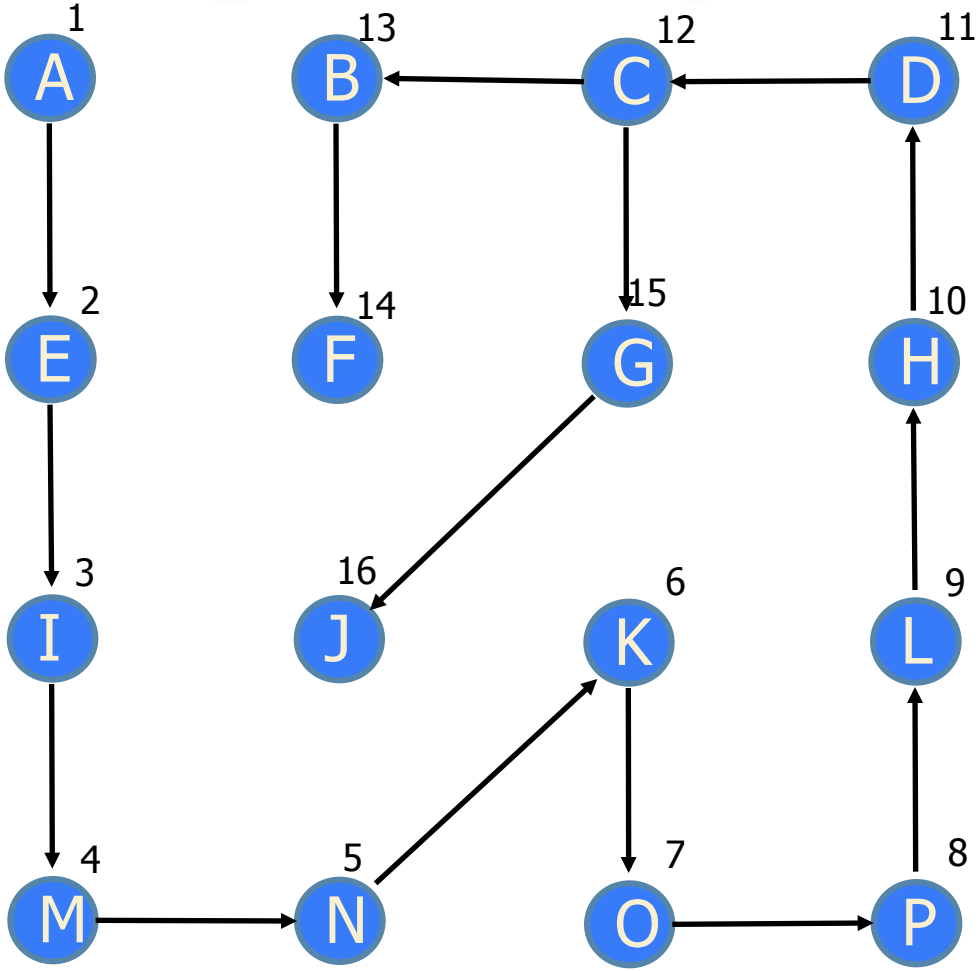


Running Time = $O(|E| + |V|)$

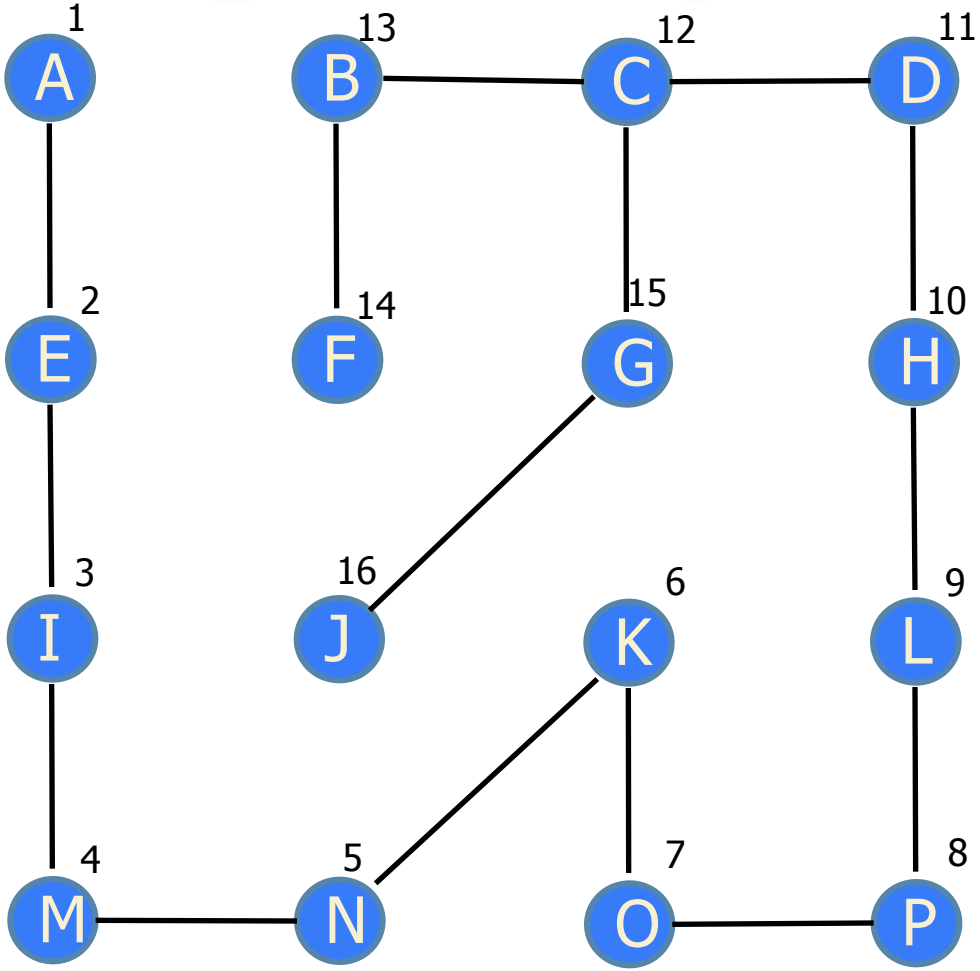
```
DFS(G)
    for each u ∈ G.V
        u.color = WHITE
    time = 0
    for each u ∈ G.V
        if u.color == WHITE
            DFS-VISIT(G, u)
```

```
DFS-VISIT(G, u)
  time = time + 1
  u.d = time
  u.color = GRAY                          // discover u
  for each v ∈ G.Adj[u]                   // explore (u, v)
      if v.color == WHITE
          DFS-VISIT(v)
  u.color = BLACK
  time = time + 1
  u.f = time                              // finish u
```

Spanning Tree



Spanning Tree



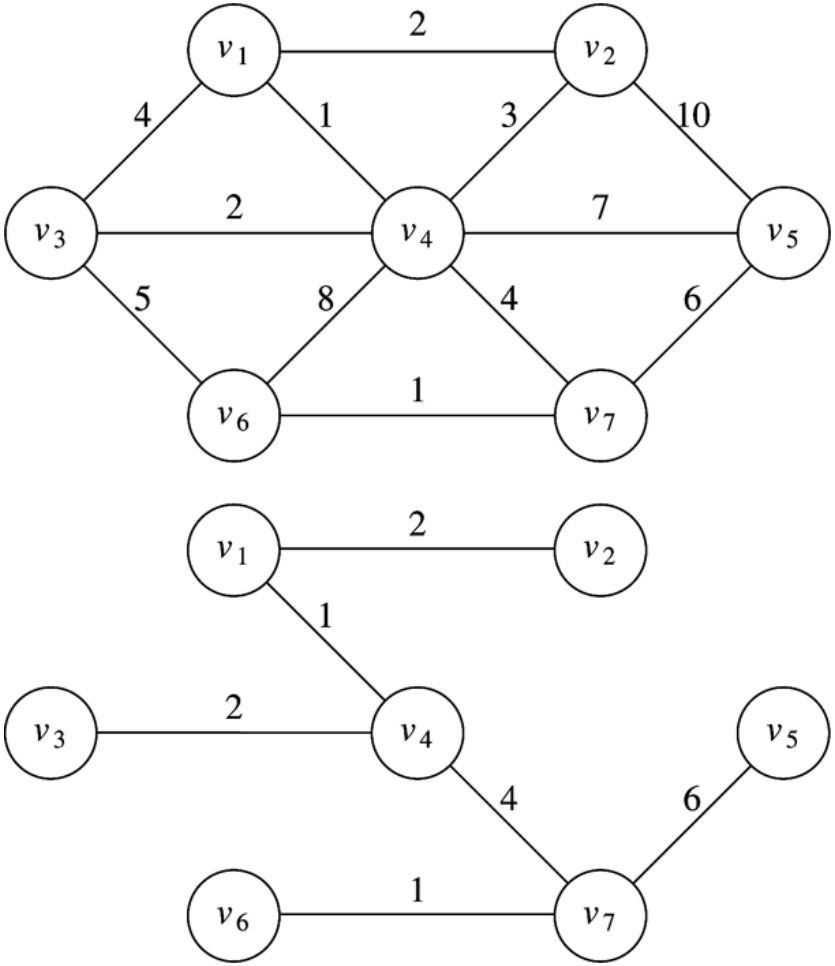
MST – Minimum Spanning Tree

- Spanning Tree – Given an undirected, connected graph G, a spanning tree T is a connected acyclic subgraph (tree) that contains all vertices of the graph.
- Minimum Spanning Tree – the spanning tree of the smallest weight, where the weight is the sum of the weights on all T's edges.
- MST Problem – Find the minimum spanning tree for a given weighted connected graph.
- Not always a unique MST.

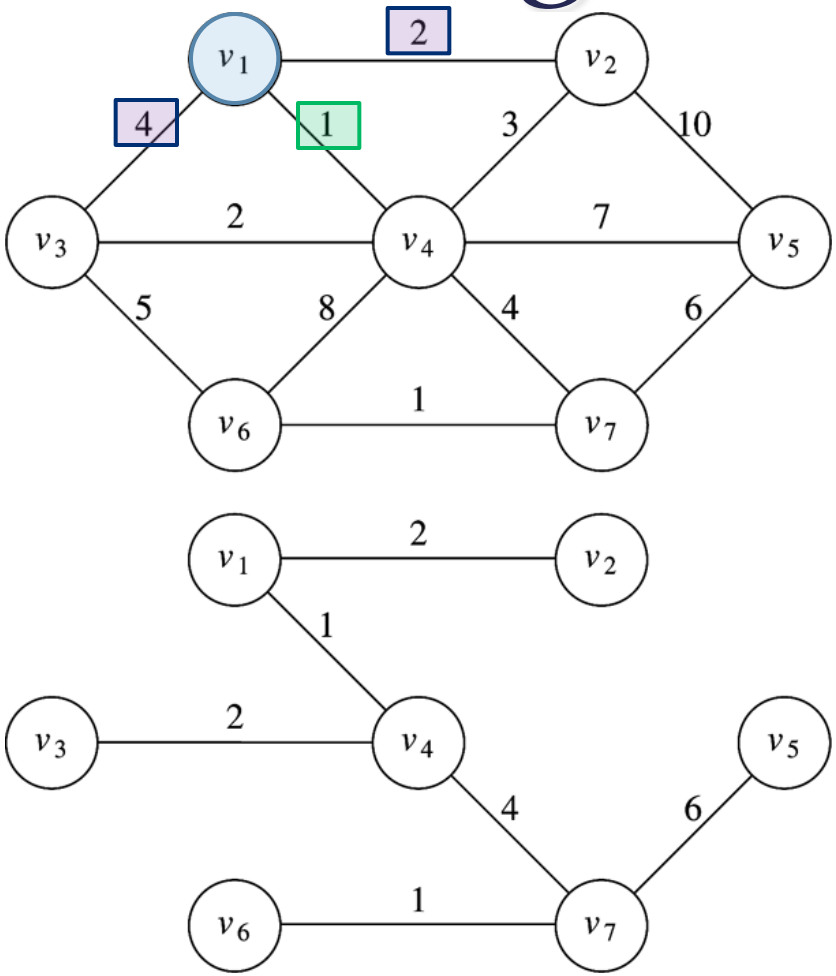
MST

- Useful for minimizing the amount of wiring needed for
  - Phone lines
  - Cable lines
- Used in wireless sensor networks
- Used in network routing algorithms

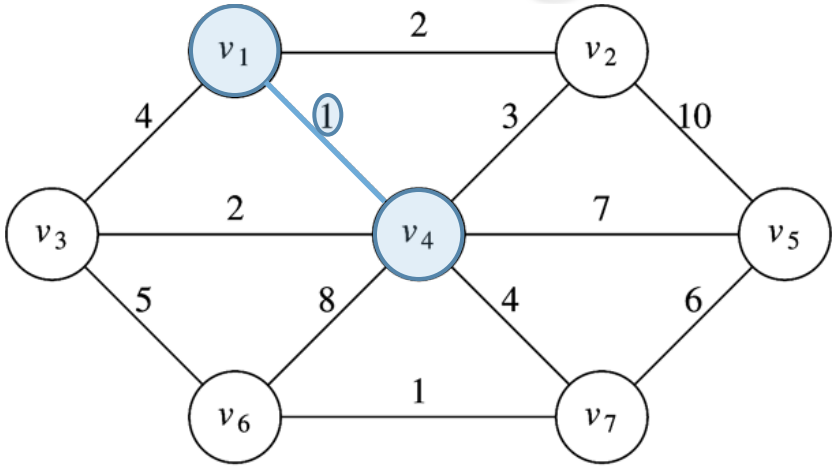
Prim's Algorithm

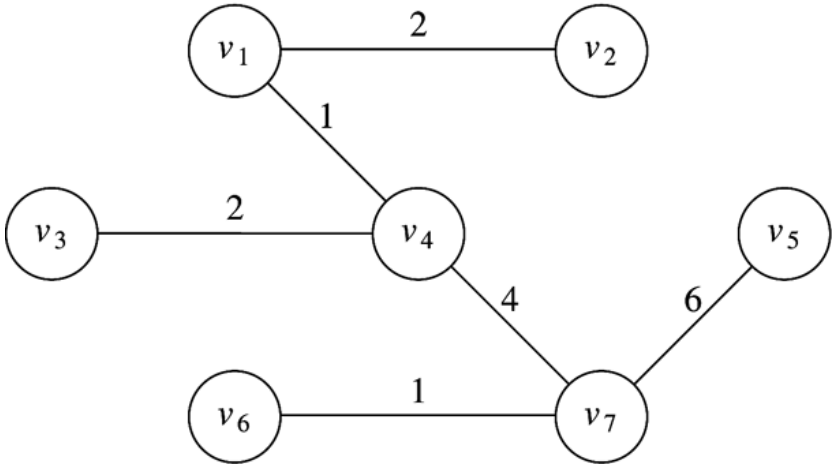


Prim's Algorithm

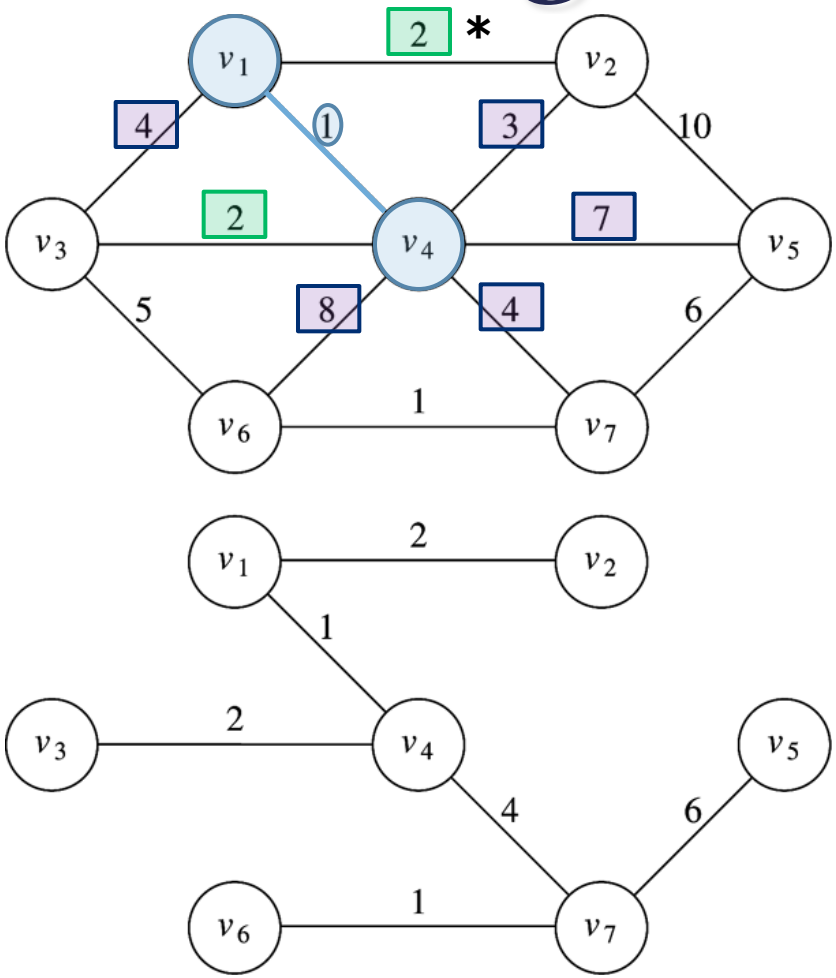


Prim's Algorithm

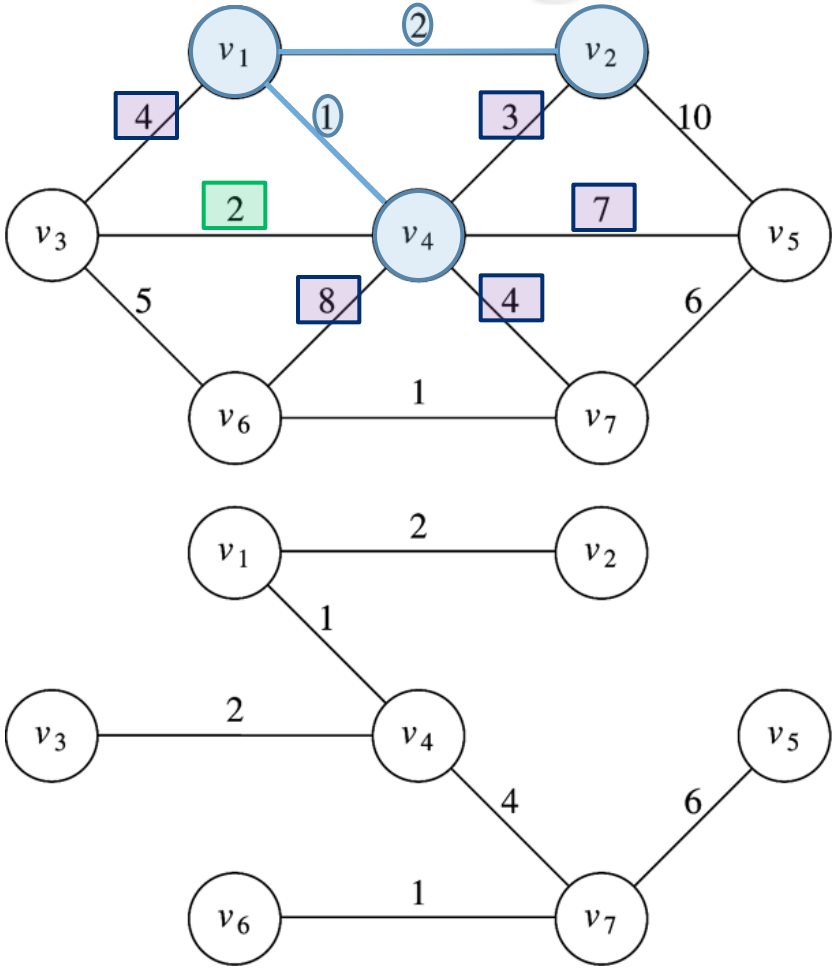




Prim's Algorithm



Prim's Algorithm



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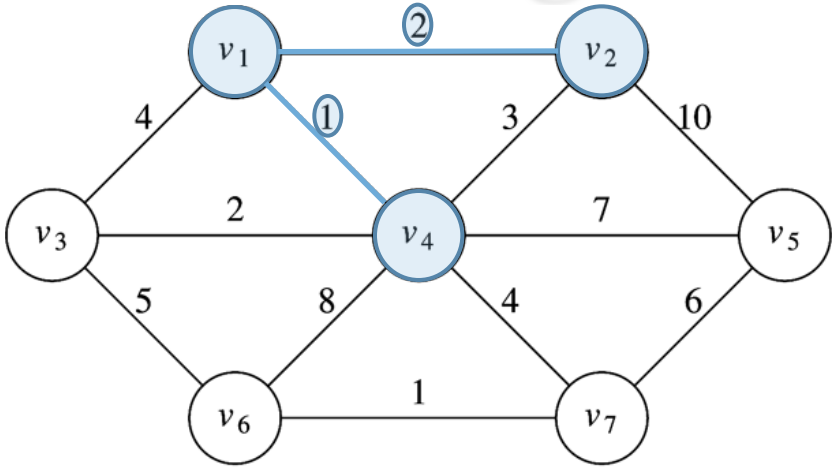
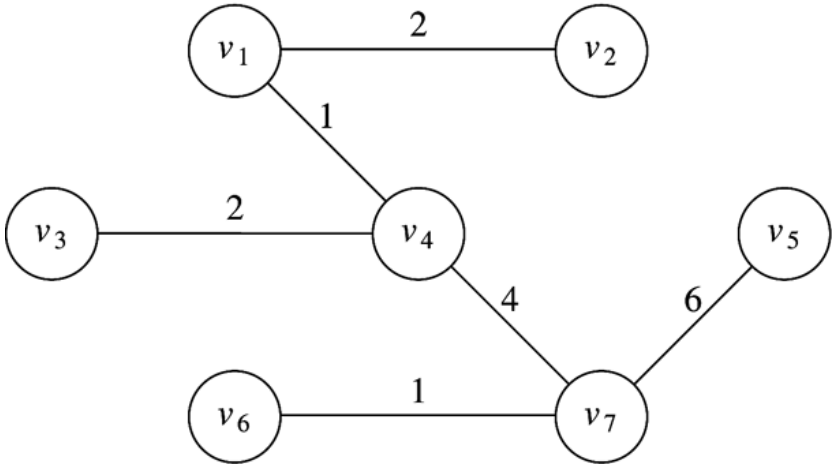
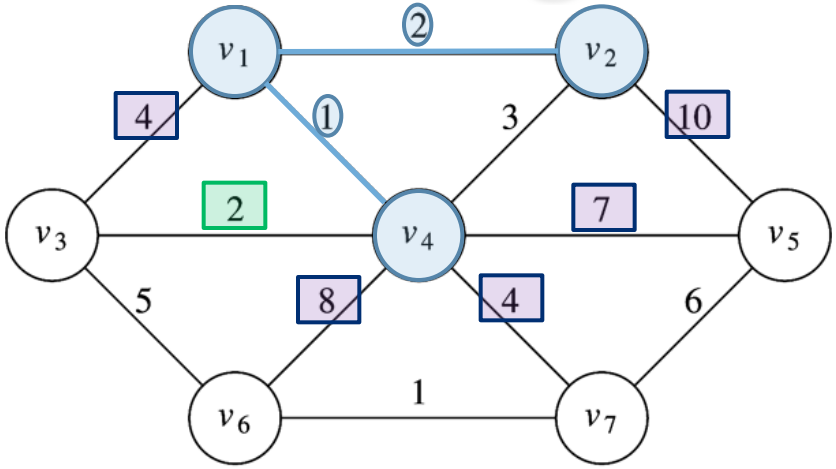
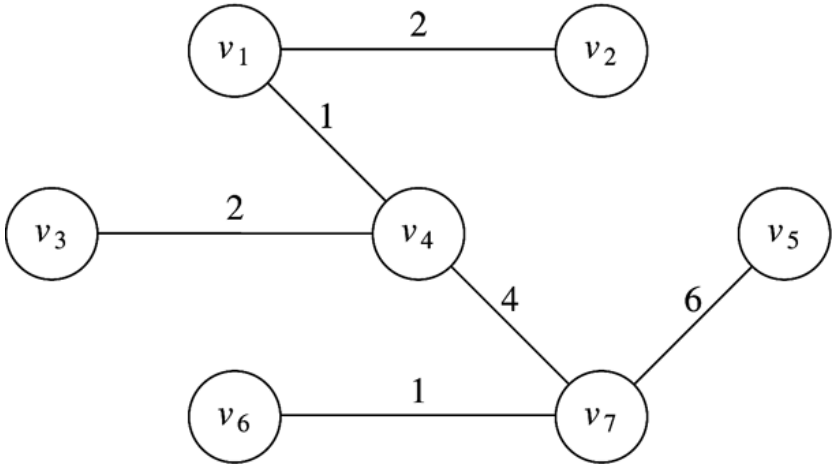


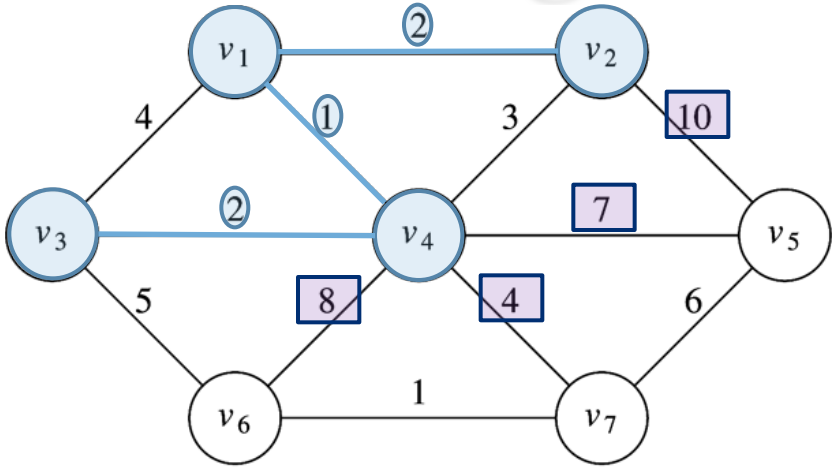


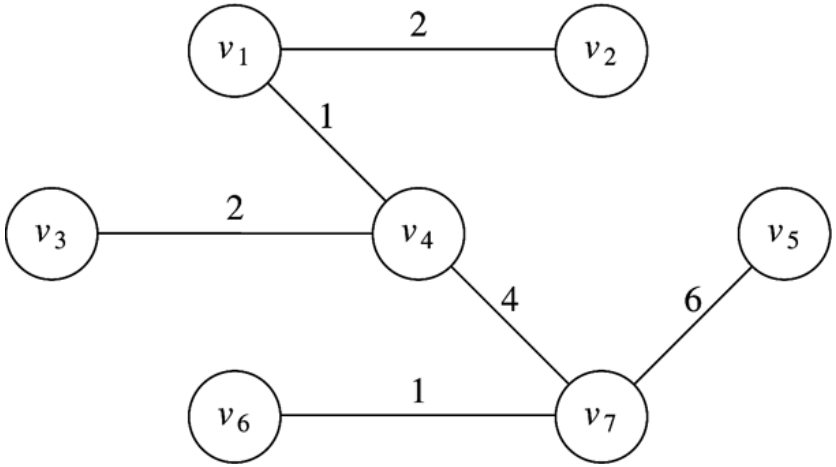
Prim's Algorithm



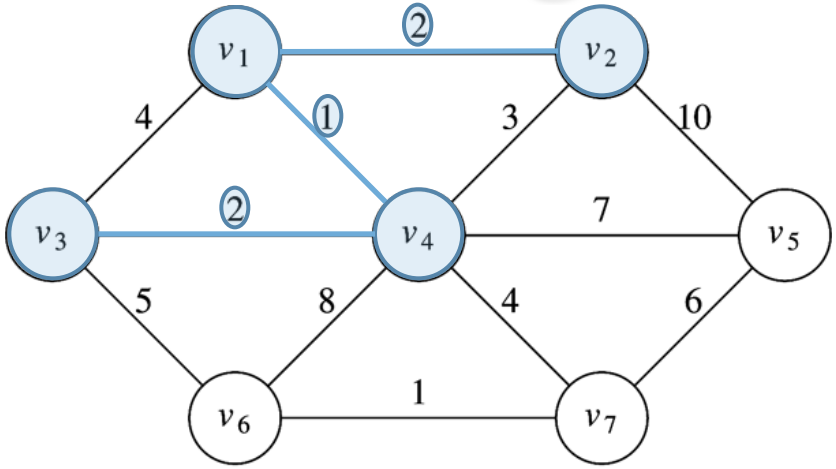


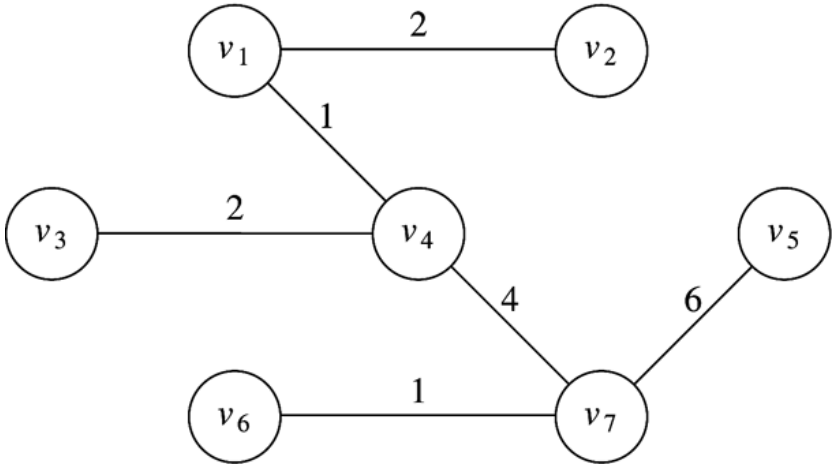
Prim's Algorithm





Prim's Algorithm





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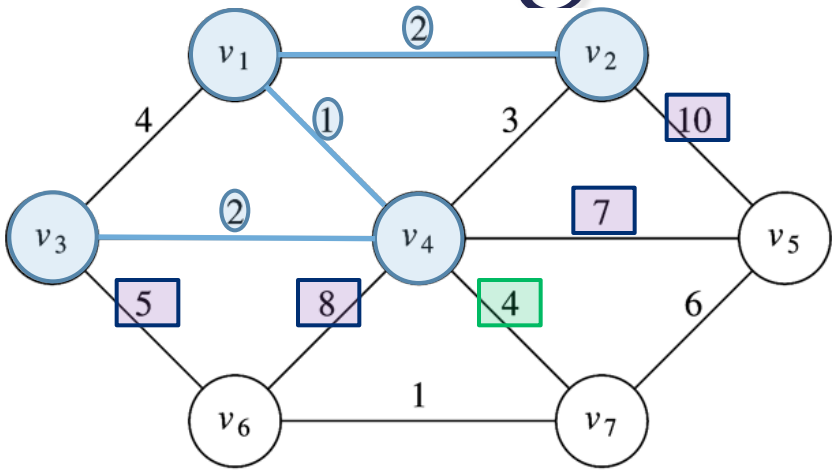

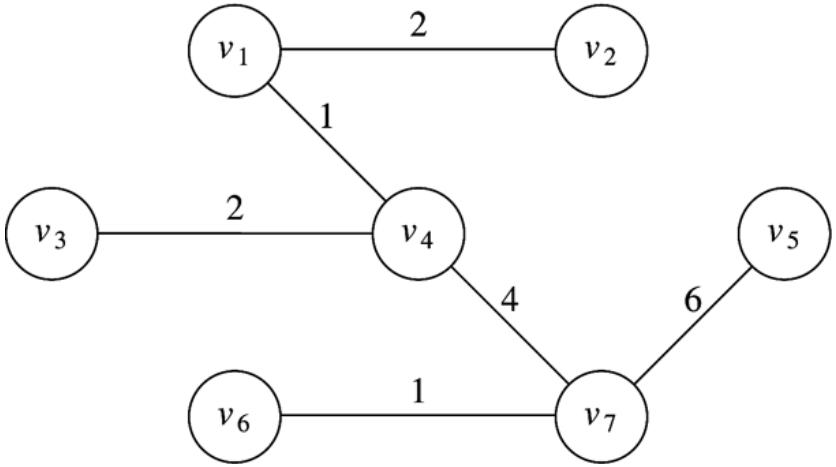

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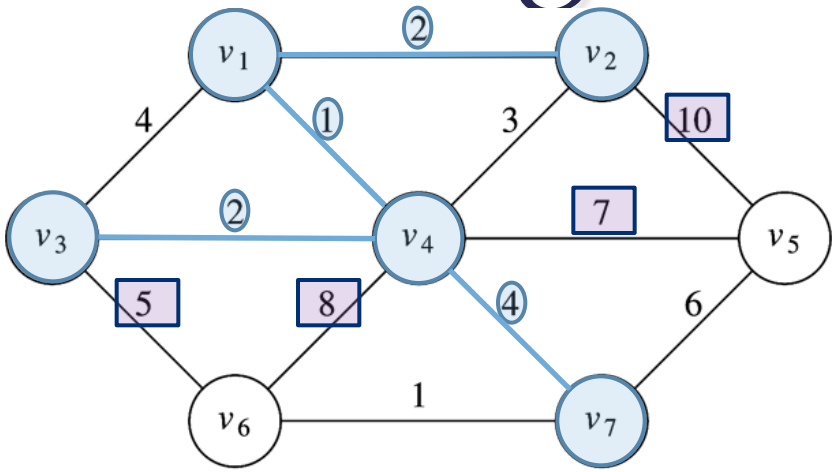
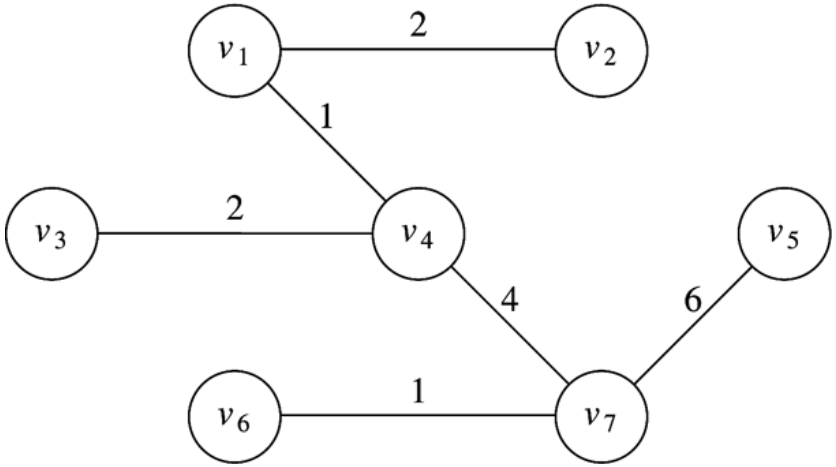
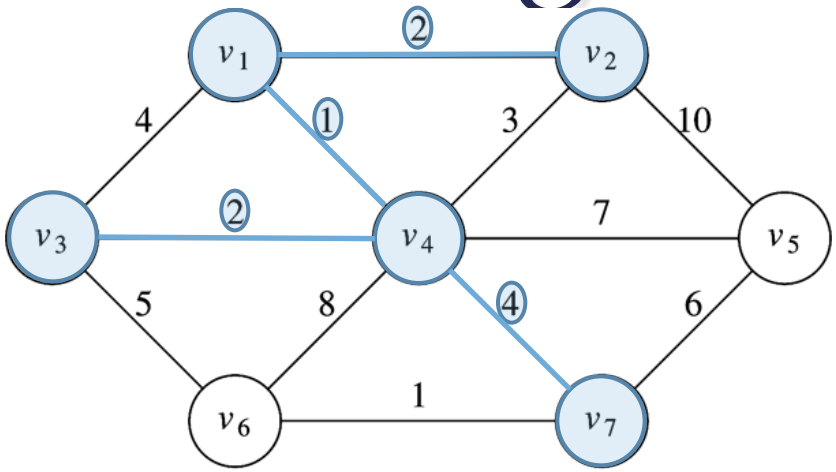
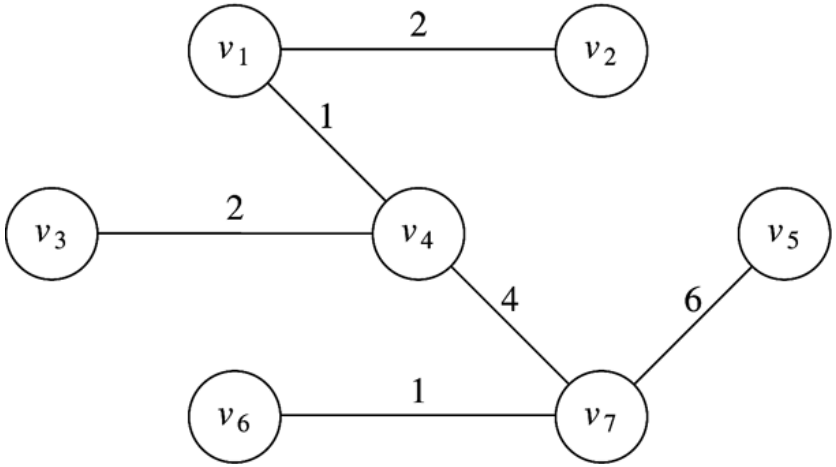




Prim's Algorithm





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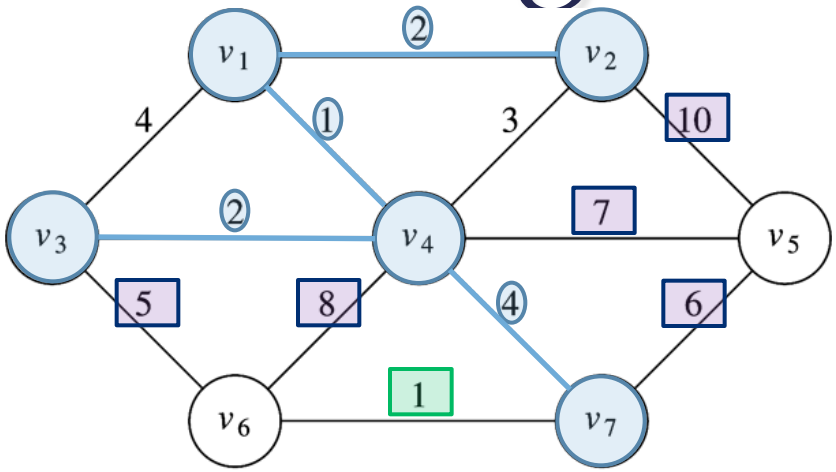

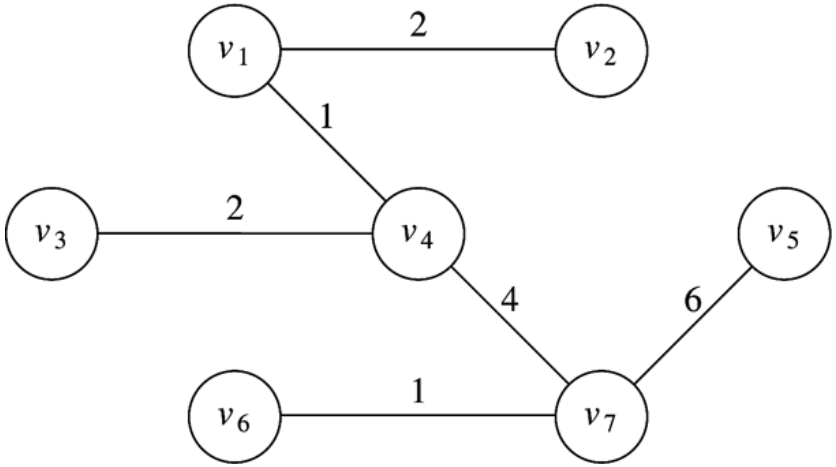

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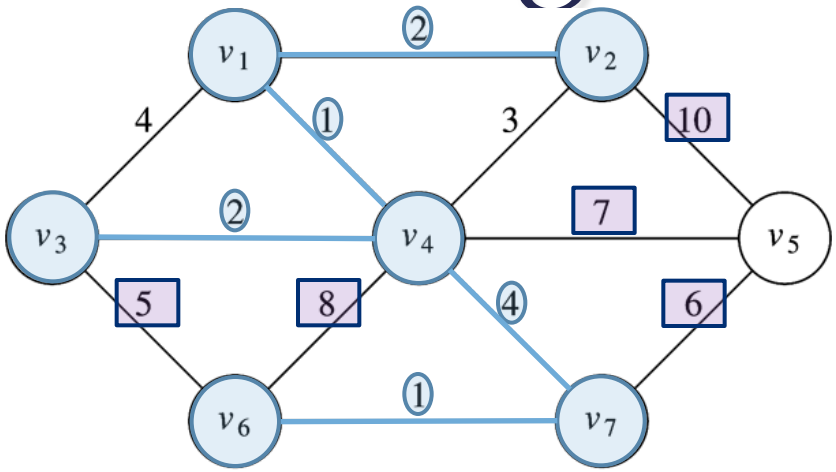
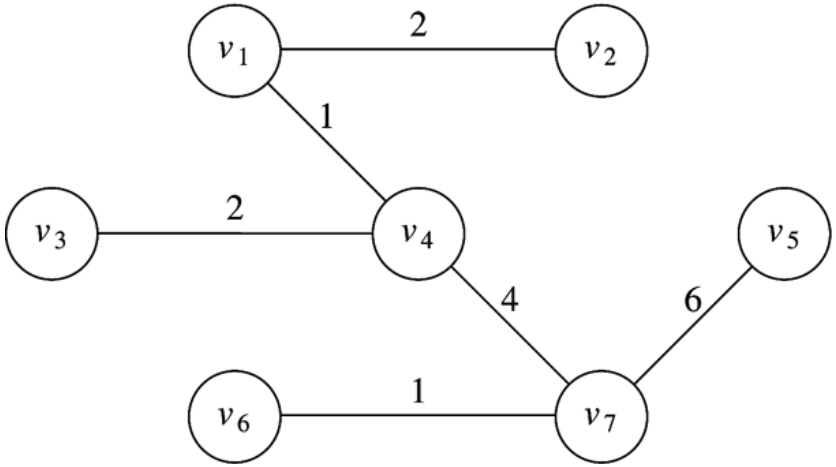
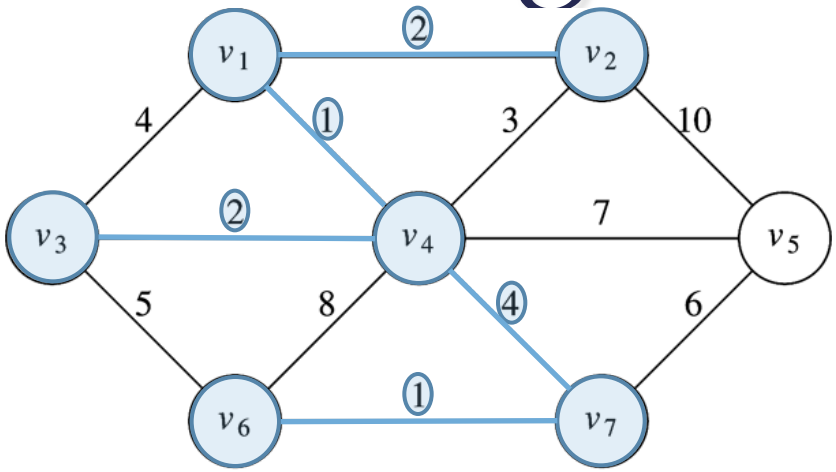
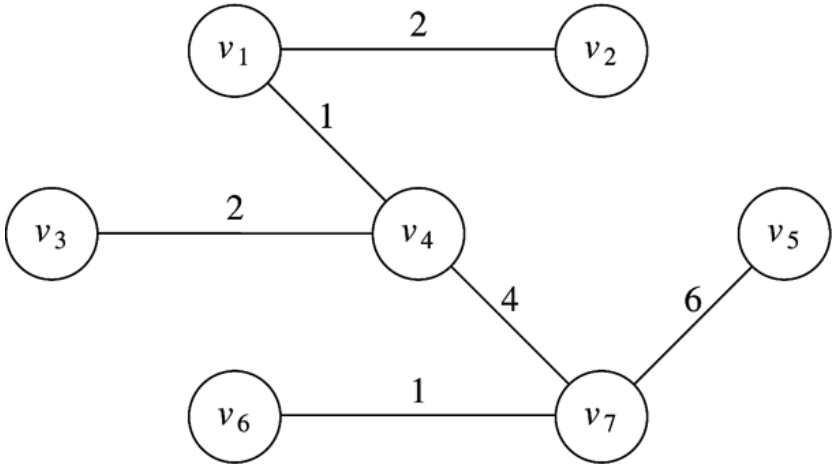




Prim's Algorithm





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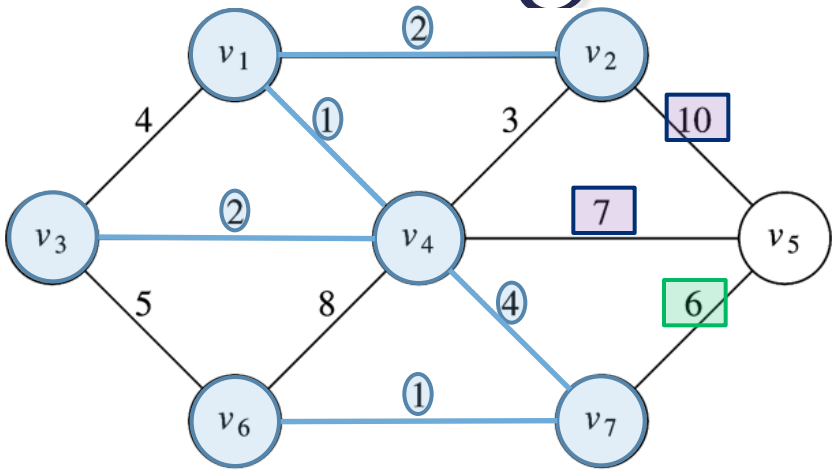

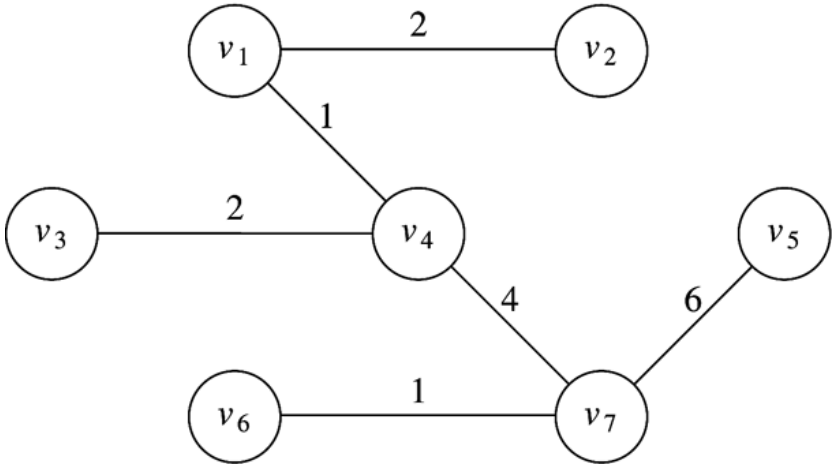

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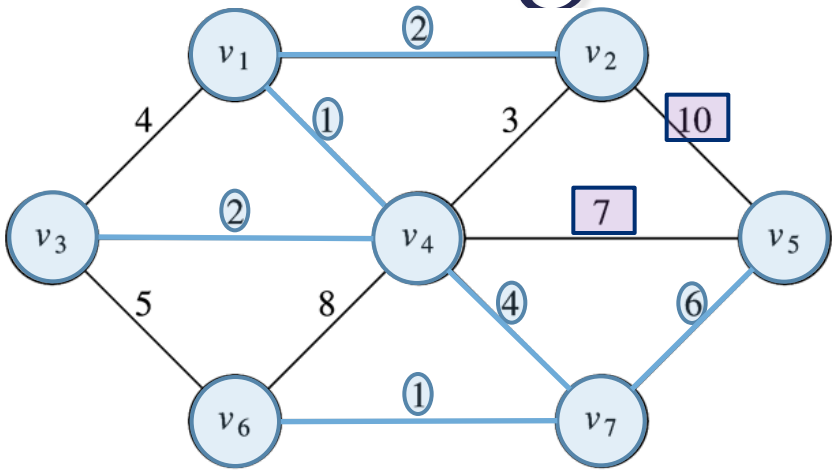

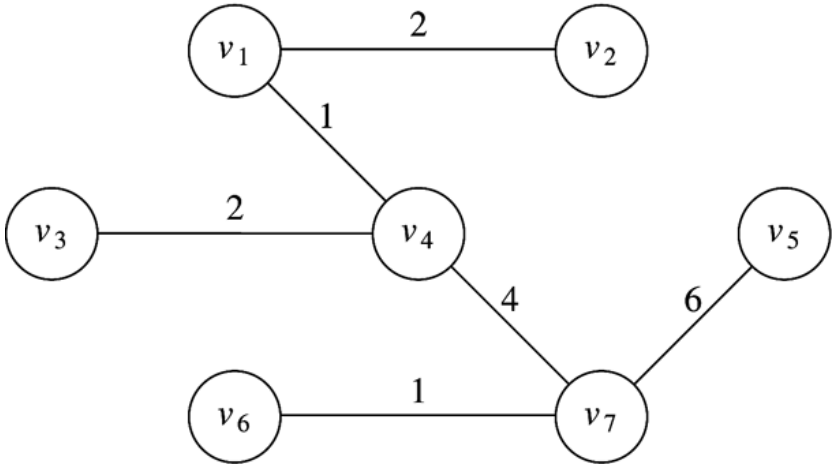

Prim's Algorithm