Sem-II - Calculus

Practical - I

Topic :- Limit and Continuity

Q.I] $\lim_{x \to a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \right]$

$\lim_{x \to a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \times \frac{\sqrt{a+2x} + \sqrt{3x}}{\sqrt{a+2x} + \sqrt{3x}} \times \frac{\sqrt{3a+x} + 2\sqrt{x}}{\sqrt{3a+x} + 2\sqrt{x}} \right]$

$\lim_{x \to a} \frac{(a+2x-3x)(\sqrt{3a+x} + 2\sqrt{x})}{(3a+x-4x)(\sqrt{a+2x} + \sqrt{3x})}$

$\frac{1}{3} \lim_{x \to a} \frac{(a-x)(\sqrt{3a+x} + 2\sqrt{x})}{(a-x)(\sqrt{a+2x} + \sqrt{3x})}$

$\frac{1}{3} \times \frac{\sqrt{3a+a} + 2\sqrt{a}}{\sqrt{a+2a} + \sqrt{3a}}$

$\frac{1}{3} \times \frac{\sqrt{4a} + 2\sqrt{a}}{\sqrt{3a} + \sqrt{3a}}$

$\frac{1}{3} \times \frac{2\sqrt{a} + 2\sqrt{a}}{2\sqrt{3a}}$

$\frac{1}{3} \times \frac{\cancel{2}4\sqrt{a}}{\cancel{2}\sqrt{3a}}$

$\frac{2}{3} \times \frac{1}{\sqrt{3}} = \frac{2}{3\sqrt{3}}$

2) $\lim_{y \to 0} \left[\frac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \right]$

$\lim_{y \to 0} \left[\frac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \times \frac{\sqrt{a+y} + \sqrt{a}}{\sqrt{a+y} + \sqrt{a}} \right]$

$\lim_{y \to 0} \frac{a + y - a}{y\sqrt{a+y}\,(\sqrt{a+y} + \sqrt{a})}$

$\lim_{y \to 0} \frac{y}{y\sqrt{a+y}\,(\sqrt{a+y} + \sqrt{a})}$

$= \frac{1}{\sqrt{a+0}\,(\sqrt{a+0} + \sqrt{a})}$

$= \frac{1}{\sqrt{a}\,(2\sqrt{a})}$

$= \frac{1}{2a}$

3] $\lim_{x \to \pi/6} \frac{\cos x - \sqrt{3}\sin x}{\pi - 6x}$

Taking $x - \frac{\pi}{6} = h$

$\therefore x = h + \pi/6$

where $h \to 0$

$$\lim_{h \to 0} \frac{\cos(h+\pi/6) - \sqrt{3}\sin(h+\pi/6)}{\pi - 6(h+\pi/6)}$$

Using $\cos(A+B) = \cos A \cos B - \sin A \cdot \sin B$

& $\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$

$$\lim_{h \to 0} \frac{\cos h \cdot \cos \pi/6 - \sin h \cdot \sin \pi/6 - \sqrt{3}(\sin h \cos \pi/6 + \cos h \sin \pi/6)}{\pi - \cancel{6}\frac{(6h+\pi)}{\cancel{6}}}$$

$$\lim_{h \to 0} \frac{\cos \cdot \frac{\sqrt{3}}{2} h - \sin \frac{h}{2} - \sqrt{3}\left(\sin \frac{\sqrt{3}}{2} h + \cos h \frac{1}{2}\right)}{-6h}$$

$$\lim_{h \to 0} \frac{-\sin \frac{4h}{2}}{-6h}$$

$$\lim_{h \to 0} \frac{\sin 4h}{12h}$$

$$\frac{1}{3} \lim_{h \to 0} \frac{\sin h}{h} = \frac{1}{3} \times 1$$

$$= \frac{1}{3}$$

4) $\lim_{x \to \infty} \dfrac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}}$

By Rationalising Numerator & Denominator

$$\lim_{x \to \infty} \left[\frac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}} \times \frac{\sqrt{x^2+5} + \sqrt{x^2-3}}{\sqrt{x^2+5} + \sqrt{x^2-3}} \times \frac{\sqrt{x^2+3} + \sqrt{x^2+1}}{\sqrt{x^2+3} + \sqrt{x^2+1}} \right]$$

$$\lim_{x \to \infty} \left[\frac{(x^2+5-x^2+3)(\sqrt{x^2+3} + \sqrt{x^2+1})}{(x^2+3-x^2-1)(\sqrt{x^2+5} + \sqrt{x^2-3})} \right]$$

$$\lim_{x \to \infty} \left[\frac{\cancel{8}^{4}}{\cancel{2}} \frac{\sqrt{x^2+3} + \sqrt{x^2+1}}{\sqrt{x^2+5} + \sqrt{x^2-3}} \right]$$

$$4 \lim_{x \to \infty} \frac{\sqrt{x^2\left(1+\frac{3}{x^2}\right)} + \sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{\sqrt{x^2+\left(\frac{5}{x^2}\right)} + \sqrt{x^2\left(1-\frac{3}{x^2}\right)}}$$

After Applying limit we get,

$= 4$

5. i] Examine the continuity :-

i) $f(x) = \dfrac{\sin 2x}{\sqrt{1-\cos 2x}}$ for $0 < x \leq \dfrac{\pi}{2}$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \frac{\sin \cancel{2} \frac{\pi}{\cancel{2}}}{\sqrt{1-\cos \cancel{2} \frac{\pi}{\cancel{2}}}}$$

$$f\left(\frac{\pi}{2}\right) = 0$$

f at $\pi/2$ is defined.

ii) $\lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{x \to \frac{\pi}{2}^+} \frac{\cos x}{\pi - 2x}$ 031

Put $x - \frac{\pi}{2} = h$

$\therefore x = \frac{\pi}{2} + h$

As $x \to 0 \therefore h \to 0$

$$\lim_{h \to 0} \frac{\cos(\frac{\pi}{2} + h)}{\pi - 2(\frac{\pi}{2} + h)}$$

$$\lim_{h \to 0} \frac{\cos h}{\pi - \cancel{2}\left(\frac{\pi + 2h}{\cancel{2}}\right)}$$

$$\lim_{h \to 0} \frac{\cos h}{-2h}$$

$$-\frac{1}{2} \lim_{h \to 0} \frac{\cos h}{h}$$

Applying the limit

$$-\frac{1}{2} \times 1$$

$$= -\frac{1}{2}$$

function is continuous at $-\frac{1}{2}$.

iii) $f(x) = \frac{x^2-9}{x-3}$ $0 < x < 3$

$= x+3$ $3 \le x < 6$

$= \frac{x^2-9}{x+3}$ $6 \le x < 9$

at $x=3$ & $x=6$.

$$f(x) = \frac{x^2-9}{x-3} = 0$$

f at $x=3$

$$\lim_{x\to3^+} f(x) = \lim_{x\to3^+} x+3$$

$$f(3) = x+3 = 6$$

f is defined at $x=3$

$$\lim_{x\to3^+} f(x) = \lim_{x\to3^+} (x+3) = 6$$

$$\lim_{x\to3^-} f(x) = \lim_{x\to3^-} \frac{x^2-9}{x-3}$$

$$= \frac{(x-3)(x+3)}{(x-3)}$$

$$= x+3$$

L.H.S = R.H.S

$$\lim_{x\to3^-} f(x) = \lim_{x\to3^+} f(x)$$

for $x=6$

$$f(6) = \frac{x^2-9}{x+3} = 3$$

$$\lim_{x\to6^+} \frac{x^2-9}{x+3} = \lim_{x\to6^+} \frac{(x-3)\cancel{(x+3)}}{\cancel{(x+3)}}$$

$$= 6-3$$

$$= 3$$

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$$\lim_{x \to 6^-} x+3 = 9$$

L·H S ≠ R·H·S

function is not continuous at 6

6· Find the value of k that function $f(x)$ is continuous at indicated point

i] $f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2}, & x < 0 \\ = k, & x = 0 \end{cases}$ at $x = 0$

∵ f is continuous at $x = 0$

∴ $\lim_{x \to 0} f(x) = f(0)$

$$\lim_{x \to 0} \frac{1-\cos 4x}{x^2} = k$$

$$2 \lim_{x \to 0} \frac{2\sin^2 2x}{x^2} = k$$

Applying limit

$2(2)^2 = k$

$k = 8$

ii] $f(x) = \begin{cases} (\sec^2 x)^{\cot^2 x}, & x \neq 0 \\ = k, & x = 0 \end{cases}$ at $x = 0$

$$\lim_{x \to 0} (\sec^2 x)^{\cot^2 x} = k$$

$$\lim_{x \to 0} (1+\tan^2 x)^{\cot^2 x} = k.$$

$$\lim_{x \to 0} (1+\tan^2 x)^{\frac{1}{\tan^2 x}} = k$$

Applying limit

$e = k$

iii) $f(x) = \dfrac{\sqrt{3} - \tan x}{\pi - 3x}$, $x \neq \dfrac{\pi}{3}$

$= k$, $x = \dfrac{\pi}{3}$ $\Bigg\}$ at $x = \dfrac{\pi}{3}$

$$\lim_{x \to \frac{\pi}{3}} \frac{\sqrt{3} - \tan x}{\pi - 3x}$$

Let,

$x - \pi/3 = h$

$x = \dfrac{\pi}{3} + h$

As $x \to 0$, $h \to 0$

$$\lim_{h \to 0} \frac{\sqrt{3} - \tan(\frac{\pi}{3} + h)}{\pi - 3\left(\frac{\pi}{3} + h\right)}$$

$$\lim_{h \to 0} \frac{\sqrt{3} - \tan(\pi/3) + \tan h}{1 - \tan \pi/3 . \tan h}$$

$$\lim_{h \to 0} \frac{\dfrac{\cancel{\sqrt{3} - \sqrt{3}} \times \sqrt{3}\tan h - \sqrt{3} - \tan h}{1 - \sqrt{2}\tan h}}{-3h}$$

$$\lim_{h \to 0} \frac{\dfrac{\sqrt{3} - 3\tan - \sqrt{3}\tan h}{1 - \sqrt{3}\tan h}}{-h}$$

$$\lim_{h\to 0} \frac{\frac{\sqrt{3} \ldots}{1-\sqrt{3}\tan h}}{-3h}$$

$$\lim_{h\to 0} = \frac{\frac{4\tan h}{1-\sqrt{3}\tan h}}{-3h}$$

$$\lim_{h\to 0} \frac{-4\tan h}{-3\tan(1-\sqrt{3}\tan h)}$$

$$\frac{4}{3} \lim_{h\to 0} \frac{\tan h}{h}$$

$$\lim_{h\to 0} \frac{1}{1-\sqrt{3}} \tan h$$

$$= \frac{4}{3}.$$

7. Discuss the continuity of the following function of these function have removable discontinuity. Redefine the function so as to make it continuous

i] $f(x) = \left.\begin{array}{ll} \frac{1-\cos 3x}{x\tan x} & x \neq 0 \\ = 9 & x=0 \end{array}\right\}$ at $x=0$

$\lim_{x\to 0} f(x) = \frac{1-\cos 3x}{x\tan x}$ $\lim_{x\to 0} \frac{9}{2} \neq a = f(0)$

f is not continuous at $x=0$

$$f(x) = \frac{1-\cos 3x}{x\tan x}$$

$$= \frac{2 \sin^2 3/2\, x}{x \tan x}$$

$$= \frac{\frac{2 \sin^2 3/2\, x}{x^2} \cdot x^2}{\frac{x \tan x}{x^2} \cdot x^2}$$

$$= 2 \lim_{x \to 0} \frac{(3/2)^2}{1}$$

$$= \frac{9}{2}$$

$$\lim_{x \to 0} f(x) = \frac{9}{2} = f(0)$$

f is not continuous at $x = 0$.

Redefine function,

$$f(x) = \frac{1 - \cos 3x}{x \tan x}, \quad x \neq 0$$

$$= \frac{9}{2}$$

Now, $\lim_{x \to 0} f(x) = f(0)$

$\therefore$ f has removable discontinuity at $x = 0$

ii) $$\left.\begin{array}{ll} f(x) = \dfrac{(e^{3x} - 1) \sin x}{x^2} & x \neq 0 \\ = \pi/6 & x = 0 \end{array}\right\} \text{at } x = 0$$

$$\lim_{x \to 0} \frac{e^{3x} - 1}{x} \lim_{x \to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{x}$$

$\lim_{x \to 0} 3 \cdot \frac{e^{3x}-1}{3x}$ $\lim_{x \to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{x}$

$3 \log e \frac{\pi}{\cancel{180}_{60}} = \frac{\pi}{60} = f(0)$

f is continuous at $x=0$.

8) $f(x) = \frac{e^{x^2} - \cos x}{x^2}$ $x=0$

is continuous at $x=0$

$\lim_{x \to 0} f(x) = f(0)$

$= \lim_{x \to 0} \frac{e^{x^2} - \cos x}{x^2} = f(0)$

$= \lim_{x \to 0} \frac{e^{x^2} - \cos x - 1 + 1}{x^2}$

$\lim_{x \to 0} \frac{(e^{x^2}-1) + (1-\cos x)}{x^2}$

$\lim_{x \to 0} \frac{e^{x^2}-1}{x^2} + \lim_{x \to 0} \frac{1-\cos x}{x^2}$

$\log e + \cancel{\lim_{x \to 0}} \frac{2\sin^2 x/2}{x^2}$

$\log e + 2 \lim_{x \to 0} \left(\frac{\sin x/2}{x}\right)^2$

Multiply with 2 Num & Denominator.

$= 1 + \cancel{2} \times \frac{1}{\cancel{4}2} = \frac{3}{2} \neq f(0)$

9) $f(x) = \dfrac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x} \quad x \neq \pi/2$

$f(0)$ is continuous at $x = \pi/2$

$$\lim_{x\to\pi/2} \frac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x} \times \frac{\sqrt{2}+\sqrt{1+\sin x}}{\sqrt{2}+\sqrt{1+\sin x}}$$

$$\lim_{x\to\pi/2} \frac{2 - 1 + \sin x}{\cos^2 x\,(\sqrt{2}+\sqrt{1+\sin x})}$$

$$= \lim_{x\to\pi/2} \frac{1+\sin x}{(1-\sin x)(1+\sin x)(\sqrt{2}+\sqrt{1+\sin x})}$$

$$= \lim_{x\to\pi/2} \frac{1}{(1-\sin x)(\sqrt{2}+\sqrt{1+\sin x})}$$

$$= \frac{1}{2(\sqrt{2}+\sqrt{2})}$$

$$= \frac{1}{2(2\sqrt{2})}$$

$$= \frac{1}{4\sqrt{2}}$$

$$f(\pi/2) = \frac{1}{4\sqrt{2}}$$

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2. Derivative

Q.1) Show that the following function defined from $\mathbb{R}$ to $\mathbb{R}$ are defferientiable

i) $\cot x$

$f(x) = \cot x$

$$f(a) = \lim_{x \to 0} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\cot x - \cot a}{x - a}$$

$$= \lim_{x \to 0} \frac{\frac{1}{\tan x} - \frac{1}{\tan a}}{x - a}$$

$$= \lim_{x \to 0} \frac{\tan a - \tan x}{(x - a)\tan x \cdot \tan x}$$

put $x - a = h$

$x = a + h$

as $x \to a$, $h \to 0$.

$$f(h) = \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{(\cancel{a}+h-\cancel{a})\tan(a+h)\tan a}$$

$$= \lim_{h \to 0} \frac{\tan a - \tan(a+h)}{h \times \tan(a+h)\tan a}$$

$$= \lim_{h \to 0} \frac{\tan a - \left[\frac{\tan a + \tan h}{1 - \tan a \cdot \tan h}\right]}{h\,[\tan(a+h)(\tan a)]}$$

$$= \lim_{h \to 0} \frac{\tan a - \tan^2 a \cdot \tan h - [\tan a + \tan h]}{h[\tan(a+h) \times \tan a] \times [1 - \tan a \cdot \tan h]}.$$

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$$= \lim_{h\to 0} \frac{\tan^2 a - 0 - [1+0]}{h[\tan a \times \tan a] \times [1-0]}$$

$$= \lim_{h\to 0} (-1) \times \frac{\tan^2 a + 1}{h(\tan^2 a)}$$

$$= \frac{-\sec^2 a}{\tan^2 a}$$

$$= -\cos^2 a$$

$\therefore Df(a) = -\cos^2 a$

$\therefore f$ is differentiable $\forall a \in R$.

(2) Cosec x

$f(x) = \operatorname{cosec} x$

$$Df(a) = \lim_{x\to 0} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x\to a} \frac{\operatorname{cosec} x - \operatorname{cosec} a}{x-a}$$

$$= \lim_{x\to a} \frac{\frac{1}{\sin x} - \frac{1}{\sin a}}{x-a}$$

$$= \lim_{x\to a} \frac{\sin a - \sin x}{\sin a \cdot \sin x (x-a)}$$

Let $x - a = h$

$x = a + h$

As $x \to a, h \to 0$

$$= \lim_{h \to 0} \frac{\sin a - \sin(a+h)}{\sin a \cdot \sin x\,(h)}$$

$$= \frac{\lim_{h \to 0} 2\cos\left(\frac{a+a+h}{2}\right) \cdot \sin\left(\frac{a-a-h}{2}\right)}{\lim_{h \to 0} \sin a \cdot \sin x\,(h)}$$

$$= \frac{\lim_{h \to 0} 2\cos\left(a+\frac{h}{2}\right) \sin\left(\frac{-h}{2}\right)}{\lim_{h \to 0} \sin a \cdot \sin(a+h)(h)}$$

$$= \lim_{h \to 0} \frac{2\cos\left(a+\frac{h}{2}\right) \cdot \sin\left(\frac{-h}{2}\right) \times \left(\frac{-1}{2}\right)}{\sin a \cdot \sin(a+h)\left(\frac{-h}{2}\right)}$$

$$= \frac{-\cos(a) \cdot (1)}{\sin^2 a}$$

$$= \frac{-\cos a}{\sin^2 a}$$

$$= -\cot a \cdot \operatorname{cosec} a$$

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(3) $\text{Sec } x$

$f(x) = \text{Sec } x$

$$Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\text{Sec } x - \text{Sec } a}{x - a}$$

$$= \lim_{x \to a} \frac{\frac{1}{\cos x} - \frac{1}{\cos a}}{x - a}$$

$$= \lim_{x \to a} \frac{\cos a - \cos x}{(\cos x)(\cos a)(x - a)}$$

Put $x = a + h$

As $x \to a$ $h \to 0$

$$= \lim_{h \to 0} \frac{\cos a - \cos(a+h)}{\cos(a+h) \cos a \times h}$$

$$= \lim_{h \to 0} \frac{2 \sin\left(\frac{a+a+h}{2}\right) \sin\left(\frac{a+h-a}{2}\right)}{\cos(a+h) \cdot \cos a \times h}$$

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$$= \lim_{h \to 0} \frac{x \sin\left(a + \frac{h}{2}\right) \sin\left(\frac{h}{2}\right)}{\cos(a+h) \cdot \cos a \cdot \frac{h}{2}} \times \frac{1}{x}$$

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$$= \frac{\sin(a+0) \quad (1)}{\cos(a+0) \cos a}$$

$$= \frac{\sin a}{\cos^2 a}$$

$$= \tan a \cdot \sec a.$$

Q.2) If $f(x) = 4x+1$, $x \leq 2$

$= x^2+5$, $x > 0$ at $x=2$, then find f is differentiable or not?

$\rightarrow$ LHD:

$$Df(2^-) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x+1 - (8+1)}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x+1-9}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x-8}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$= 4.$$

RHD :-

$$Df(2^+) = \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2^+} \frac{x^2+5-9}{x-2}$$

$$= \lim_{x\to 2^+} \frac{\cancel{2(x^2-2)}\ x^2-4}{x-2}$$

$$= \lim_{x\to 2^+} x+2$$

$$= 4$$

$\therefore$ R.H.D = L.H.D

f is differentiable at $x=2$.

Q.3) If $f(x) = 4x+7$, $x<3$

$= x^2+3x+1$, $x\geq 3$ at $x=3$, then

find f is differentiable or not?

$\rightarrow$ LHD:

$$DF(2^-) = \lim_{x\to 3^-} \frac{4x+7-(12+7)}{x-3}$$

$$= \lim_{x\to 3^-} \frac{4x-12}{x-3}$$

$$= \lim_{x\to 3^-} \frac{4(x-3)}{x-3} = 4.$$

$$Df(3^+) = \lim_{x \to 3^+} \frac{x^2+3x+1-(19)}{x-3}$$

$$= \lim_{x \to 3^+} \frac{x^2+3x-18}{x-3}$$

$$= \lim_{x \to 3^+} \frac{(x+6)(x-3)}{x-3}$$

$$= x+6$$

$$= 9$$

LHD $\neq$ RHD

$\therefore$ f is not differentiable at $x=3$.

Q4) If $f(x) = 8x-5$, $x \leq 2$

$= 3x^2-4x+7$, $x > 2$ at $x=2$, then find f is differentiable or not

→ LHD :

$$Df(2^-) = \lim_{x \to 2^-} \frac{8x-5-(11)}{x-2}$$

$$= \lim_{x \to 2^-} \frac{8x-16}{x-2}$$

$$= \lim_{x \to 2^-} \frac{8(x-2)}{x-2}$$

$$= 8.$$

R.H.D:-

$$Df(2^+) = \lim_{x \to 2^+} \frac{3x^2 - 4x + 7 - (11)}{x-2}$$

$$= \lim_{x \to 2^+} \frac{3x^2 - 4x - 4}{x-2}$$

$$= \lim_{x \to 2^+} \frac{3x^2 - 6x + 2x - 4}{x-2}$$

$$= \lim_{x \to 2^+} \frac{3x(x-2) + 2(x-2)}{x-2}$$

$$= \lim_{x \to 2^+} \frac{(3x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$= 3(2) + 2$$

$$= 8.$$

Practical No - 3

Topic : Application of Derivatives

Q.1) Find the intervals in which function is increasing or decreasing

1) $f(x) = x^3 - 5x - 11$
2) $f(x) = x^2 - 4x$
3) $f(x) = 2x^3 + x^2 - 20x + 4$
4) $f(x) = x^3 - 27x + 5$
5) $f(x) = 69 - 24x - 9x^2 + 2x^3$.

Q.2) Find the intervals in which function is concave upwards.

i) $y = 3x^2 - 2x^3$
ii) $y = x^4 - 6x^3 + 12x^2 + 5x + 7$
iii) $y = x^3 - 27x + 5$
iv) $y = 69 - 24x - 9x^2 + 2x^3$
v) $y = 2x^3 + x^2 - 20x + 4$

Sol^n:

Q.1) i) $f(x) = x^3 - 5x - 11$

$\therefore f'(x) = 3x^2 - 5$

$\therefore f$ is increasing iff $f(x) = 0$.

$3x^2 - 5 > 0$

$3(x^2 - 5/3) > 0$

$x \in (-\infty, -\sqrt{5/3}) \cup (\sqrt{5/3}, \infty)$ and f is decreasing

$f'(x) < 0$

$\therefore 3x^2 - 5 < 0$

$3(x^2 - 5/3) < 0$

$\therefore (x - \sqrt{5/3})(x + \sqrt{5/3}) < 0$

$-\sqrt{5/3} \quad + \sqrt{5/3}$

2) $f(x) = x^2 - 4x$

$f'(x) = 2x - 4$

$\therefore f(x)$ is increasing iff $f'(x) > 0$

$\therefore 2x - 4 > 0$

$\therefore 2(x-2) > 0$

$\therefore x - 2 > 0$

$x \in (2, \infty)$

and f is decreasing iff $f'(x) < 0$

$\therefore 2x - 4 < 0$

$\therefore 2(x-2) < 0$

$\therefore x - 2 < 0$

$x \in (-\infty, 2)$

3) $f(x) = 2x^3 + x^2 - 20x + 4$

$\therefore f'(x) = 6x^2 + 2x - 20$

$\therefore f$ is increasing iff $f'(x) > 0$

$\therefore 6x^2 - (3x^2 + x - 10) > 0$

$\therefore 2(3x^2 + x - 10) > 0$

$\therefore 3x^2 + x - 10 > 0$

$\therefore 3x^2 + 6x - 5x - 10 > 0$

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$\therefore 3x(x+2)-5(x+2)>0$

$\therefore (x+2)(3x-5)>0$

$+ \quad - \quad +$

$2 \qquad 5/3$

$x \in (-\infty,-2) \cup (5/3,\infty)$

and f is decreasing iff $f'(x)<0$

$\therefore 6x^2+2x-20<0$

$\therefore 2(3x^2+x-10)<0$

$\therefore 3x^2+x-10<0$

$\therefore 3x^2+6x-5x-10<0$

$\therefore 3x(x+2)-5(x+2)<0$

$\therefore (x+2)(3x-5)<0$

$+ \quad - \quad +$

$-2 \qquad 5/3$ $\qquad x \in (-2, 5/3)$

(4) $f(x) = x^3-27x+5$

$f'(x) = 3x^2-27$

$\therefore f$ is increasing iff $f'(x)>0$

$\therefore 3(x^2-9)>0$

$\therefore (x-3)(x+3)>0$

$+ \quad - \quad +$

$-3 \qquad 3$

$\therefore x \in (-\infty,-3) \cup (3,\infty)$

and f is decreasing iff $f'(x)<0$

$\therefore 3x^2-27<0$

$\therefore 3(x^2-9)<0$

$\therefore (x-3)(x+3)<0$

$+ \quad - \quad +$

$-3 \qquad 3$ $\qquad x \in (-3,3)$

043

5) $f(x) = 2x^3 - 9x^2 - 24x + 69$

$f'(x) = 6x^2 - 18x - 24$

$\therefore$ f is increasing iff $f'(x) > 0$

$\therefore 6x^2 - 18x - 24 > 0$

$\therefore 6(x^2 - 3x - 4) > 0$

$\therefore x^2 - 4x + x - 4 > 0$

$\therefore x(x-4) + 1(x-4) > 0$

$\therefore (x-4)(x+1) > 0$

$+ \quad - \quad +$

$-1 \quad 4$

$\therefore x \in (-\infty, -1) \cup (4, \infty)$

and f is decreasing iff $f'(x) < 0$

$\therefore 6x^2 - 18x - 24 < 0$

$\therefore 6(x^2 - 3x - 4) < 0$

$\therefore x^2 - 4x + x - 4 < 0$

$\therefore x(x-4) + 1(x-4) < 0$

$\therefore (x-4)(x+1) < 0$

$+ \quad - \quad +$

$-1 \quad 4$

$\therefore x \in (-1, 4)$

Q.2) Sol^n.

i) $y = 3x^2 - 2x^3$

$\therefore f(x) = 3x^2 - 2x^3$

$\therefore f'(x) = 6x - 6x^2$

$\therefore f''(x) = 6 - 12x$

f is concave upward if $f''(x) > 0$

$\therefore (6 - 12x) > 0$

$\therefore 12(6/12 - x) > 0$

$x - 1/2 > 0$

$x > 1/2$

$\therefore f''(x) > 0$

$\therefore x \in (1/2, \infty)$

2) $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

$f'(x) = 4x^3 - 18x^2 + 24x + 5$

$f''(x) = 12x^2 - 36x + 24$

f is concave upward if $f''(x) > 0$

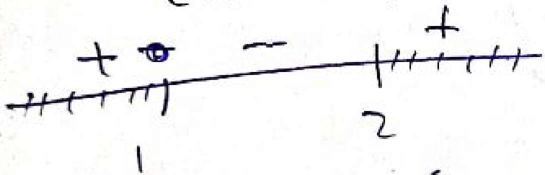
$\therefore 12x^2 - 36x + 24 > 0$

$\therefore 12(x^2 - 3x + 2) > 0$

$\therefore x^2 - 2x - x + 2 > 0$

$\therefore x(x-2) - 1(x-2) > 0$

$\therefore (x-2)(x-1) > 0$



$x \in (-\infty, 1) \cup (2, \infty)$

3) $y = x^3 - 27x + 5$

$f'(x) = 3x^2 - 27$

$f''(x) = 6x$

f is concave upward iff $f''(x) > 0$

$\therefore 6x > 0$

$\therefore x > 0$

$\therefore x \in (0, \infty)$

4) $y = 69 - 24x - 9x^2 + 2x^3$

$f(x) = 2x^3 - 9x^2 - 24x + 64$

$f'(x) = 6x^3 - 18x - 24$

$f''(x) = 12 - 18$

f is concave upward iff $f''(x) > 0$.

$\therefore 12x - 18 > 0$

$\therefore 12(x - 18/12) > 0$

$\therefore x - 3/2 > 0$

$\therefore x > 3/2$

$\therefore x \in (3/2, \infty)$

5) $y = 2x^3 + x^2 - 20x + 4$

$f(x) = 2x^3 + x^2 - 20x + 4$

$f'(x) = 6x^2 + 2x - 20$

$f''(x) = 12x + 2$

f is concave upward iff $f''(x) > 0$

$\therefore f''(x) > 0$

$\therefore 12x + 2 > 0$

$\therefore 12(\cancel{x} + 2/12) > 0$

$\therefore \cancel{x} + 1/6 > 0$

$\therefore x < -1/6$

$\therefore f''(x) > 0$

$\therefore$ There exist no interval.

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Practical-4

Topic: Application of Derivative and Newtons method

Q.1) Find maximum and minimum values of following functions

i) $f(x) = x^2 + \frac{16}{x^2}$.

ii) $f(x) = 3 - 5x^3 + 3x^5$

iii) $f(x) = x^3 - 3x^2 + 1$ in $[-\frac{1}{2}, 4]$

iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$ in $[-2, 3]$

Q.2) Find the real of following equation by Newton's method.

i) $f(x) = x^3 - 3x^2 - 55x + 9.5$ when $x_0 = 0$

ii) $f(x) = x^3 - 4x - 9$ in $[2, 3]$

iii) $f(x) = x^3 - 1.8x^2 - 10x + 12$ in $[1, 2]$.

FAQ

Sol^n:

Q.1) i) $f(x) = x^2 + \frac{16}{x^2}$

$f'(x) = 2x - \frac{32}{x^3}$

Now, Consider,

$f'(x) = 0$

$2x - \frac{32}{x^3} = 0$

$2x = \frac{32}{x^3}$

$x^4 = \frac{32}{2}$

$x^4 = 16$

$x = \pm 2$

$f''(x) = 2 + \frac{96}{x^4}$

$f''(2) = 2 + \frac{96}{2^4}$

$= 2 + \frac{96}{16}$

$= 2 + 6$

$= 8 > 0$

$\therefore$ f has minimum value at $x = 2$

$\therefore$ f has minimum value at $x=-2$

Function reaches minimum value at $x=2$ and $x=-2$

ii) $f(x) = 3 - 5x^3 + 3x^5$

$\therefore f'(x) = -15x^2 + 15x^4$.

Consider,

$f'(x) = 0$

$\therefore -15x^2 + 15x^4 = 0$

$15x^4 = 15x^2$

$x^2 = 1$

$x = \pm 1$

$\therefore f''(x) = -30x + 60x^3$

$\therefore f''(1) = -30 + 60$

$= 30 > 0$

$\therefore$ f has minimum value at $x=1$

$\therefore f(1) = 3 - 5(1)^3 + 3(1)^5$.

$f(-1) = 3 - 5(-1)^3 + 3(-1)^5$

$= 3 + 5 - 3$

$= 5$

$\therefore$ f has the maximum value 5 at $5 = -1$ and has the minimum value 1 at $x=1$

iii) $f(x) = x^3 - 3x^2 + 1$

$\therefore f'(x) = 3x^2 - 6x$

Consider,

$f'(x) = 0$

$\therefore 3x^2 - 6x = 0$

$3x(x-2) = 0$

$\therefore 3x = 0$ or $x - 2 = 0$

$\therefore x = 0$ or $x = 2$

$f''(x) = 6x - 6$

$\therefore f(0) = 6(0) - 6$

$= -6 < 0$

$\therefore f$ has maximum value at $x = 0$

$\therefore f(0) = (0)^3 - 3(0)^2 + 1 = 1$

$f''(2) = 6(2) - 6 = 12 - 6 = 6 > 0$

$\therefore f$ has maximum value at $x = 2$.

$f(2) = (2)^3 - 3(2)^3 + 1$

$= 8 - 3(4) + 1$

$= 9 - 12$

$= -3$

$\therefore f$ has maximum value 1 at $x = 0$ and f has minimum value -3 at $x = 2$

iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$

$\therefore f'(x) = 6x^2 - 6x - 12$

Consider,

$f'(x) = 0$

$\therefore 6x^2 - 6x - 12 = 0$

$\therefore 6(x^2 - x - 2) = 0$

$x^2 - x - 2 = 0$

$x(x+1) - 2(x+1) = 0$

$(x-2)(x+1) = 0$

$\therefore x = 2$ or $x = -1$

$\therefore f''(x) = 12x - 6$

$f''(2) = 12(2) - 6$

$= 24 - 6$

$= 18 > 0$

$\therefore f$ has minimum value at $x = 2$.

$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 1$

$= 2(8) - 3(4) - 24 + 1$

$= 16 - 12 - 24 + 1$

$= -19$

$f''(-1) = 12(-1) - 6$

$= -12 - 6$

$= -18 < 0$

$\therefore f$ has maximum value at $x = -1$.

$\therefore f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 1$

$= -2 - 3 + 12 + 1$

$= 8$

$\therefore f$ has maximum value 8 at $x = -1$ and f has minimum value -19 at $x = 2$

Q.2) i) $f(x) = x^3 - 3x^2 - 55x + 9.5$

$f'(x) = 3x^2 - 6x - 55$

By Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0 + \frac{9.5}{55}$$

$$\therefore x_1 = 0.1727$$

$$\therefore f(x_1) = (0.1727)^3 - 3(0.1727)^2 - 55(0.1727) + 9.5$$
$$= 0.0051 - 0.0895 - 9.4985 + 9.5$$
$$= -0.0829$$

$$f'(x) = 3(0.1727)^2 - 6(0.1727) - 55$$
$$= 0.895 - 1.0362 - 55$$
$$= -55.9467$$

$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 0.1727 - \frac{0.829}{55.9464}$ 048

$= 0.1712$

$f(x_2) = (0.1712)^3 - 3(0.1712)^3 - 55(0.1712) + 9.5$

$= 0.0050 - 0.0879 - 9.416 + 9.5$

$= 0.0011$

$f'(x_2) = 3(0.1712)^2 - 6(0.1712) - 55$

$= 0.0879 - 1.0272 - 55$

$= -55.9393$

$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$= 0.1712 + \frac{0.0011}{55.9393}$

$= 0.1712$

$\therefore$ The root of the eq^n is 0.1712.

ii) $f(x) = x^3 - 4x - 9$.

$f'(x) = 3x^2 - 9$

$f(2) = 2^3 - 4(2) - 9$

$= 8 - 8 - 9$

$= -9$

$f(3) = 3^3 - 4(3) - 9$

$= 27 - 12 - 9$

$= 6$

Let $x_0 = 3$ lie the initial approximation

By Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

n≥0

$$f_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 3 - \frac{6}{23}$$

$$= 2.7392$$

$$f(x_1) = (2.7392)^3 - 4(2.7392) - 9$$

$$= 20.5528 - 10.9568 - 9$$

$$= 0.596$$

$$f'(x_1) = 3(2.7392)^2 - 4$$

$$= 22.5096 - 4$$

$$= 18.5096$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.7392 - \frac{0.596}{18.5096}$$

$$= 2.7071$$

$$f(x^2) = (2.7071)^3 - 4(2.7071) - 9$$

$$= 19.8386 - 10.8284 - 9$$

$$= 0.0102$$

$$f'(x_2) = 3(2.7071)^2 - 4$$

$$= 21.9851 - 4$$

$$= 17.9851$$

$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$= 2.7071 - \frac{0.0102}{17.9851}$

$= 2.7071 - 0.0056$

$= 2.7015$

$f(x_3) = (2.7015)^3 - 4(2.7015) - 9$

$= 19.7158 - 10.806 - 9$

$= -0.0901$

$f'(x_3) = 3(2.7015)^2 - 4$

$= 21.8943 - 4$

$= 17.8943$

$x_4 = 2.7015 + \frac{0.0901}{17.8943}$

$= 2.7015 + 0.0050$

$= 2.7065.$

iii) $f(x) = x^3 - 1.8x^2 - 10x + 17$ $[1, 2]$

$f'(x) = 3x^2 - 3.6x - 10$

$f(1) = (1)^3 - 1.8(7)^2 - 10(1) + 7$

$= -1.8 - 10 + 17$

$= 6.2.$

$f(2) = (2)^3 - 1.8(2)^2 - 10(2) + 17$

$= 8 - 7.2 - 20 + 17$

$= -2.2$

Let $x_0 = 2$ lie initial approximation
By Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2 - \frac{2.2}{5.2}$$

$$= 2 - 0.4230$$

$$= 1.577$$

$$f(x_1) = (1.577)^3 - 1.8(1.577)^2 - 10(1.577) + 17$$

$$= 3.9219 - 4.4764 - 15.77 + 17$$

$$= 0.6755$$

$$f'(x_1) = 3(1.577)^2 - 3.6(1.577) - 10$$

$$= 7.4608 - 5.6772 - 10$$

$$= -8.2164$$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.577 + \frac{0.6755}{8.2164}$$

$$= 1.577 + 0.0822$$

$$= 1.6592$$

050

$$f(x_2) = (1.6592)^3 - 1.8(1.6592)^2 - 10(1.6592) + 17$$
$$= 4.5577 - 4.8553 - 16.592 + 17$$
$$= 0.0204$$

$$f'(x_2) = 3(1.6592)^2 - 3.6(1.6592) - 10$$
$$= 8.2588 - 5.97312 - 10$$
$$= -77143$$

$$x_3 = x_2 - f(x_2)/f'(x_2)$$
$$= 1.6592 + 0.0204/7.7143$$
$$= 1.6592 + 0.0026$$
$$= 1.6618$$

$$f(x_3) = (1.6618)^3 - 1.8(1.6618)^2 - 10(1.6618) + 17$$
$$= 4.5892 - 4.9708 - 16.618 + 17$$
$$= 0.0004$$

$$f'(x_3) = 3(1.6618)^2 - 3.6(1.6618) - 10$$
$$= 8.2847 - 5.9824 - 10$$
$$= -7.6977$$

$$x_4 = x_3 - f(x_3)/f'(x_3)$$
$$= 1.6187 + \frac{0.0004}{7.6977}$$

$$= 1.6618$$

20/01/2020

Practical-5

Topic - Integration

Q.1) Solve the following integration

1) $\frac{dx}{\sqrt{x^2+2x-3}}$

2) $\int (4e^{3x}+1)\,dx$

3) $\int (2x^2-3\sin x+5\sqrt{x})\,dx$

4) $\int \frac{x^3+3x+4}{\sqrt{x}}\,dx$

5) $\int t^7 \cdot \sin(2t^4)\,dt$

6) $\int \sqrt{x}\,(x^2-1)\,dx$

7) $\int 1/x^3 \cdot \sin(1/x^2)\,dx$

8) $\int \frac{\cos x}{\sqrt[3]{\sin^2 x}}\,dx$

9) $\int e^{\cos^2 x} \cdot \sin 2x\,dx$

10) $\int \left(\frac{x^2-2x}{x^3-3x^2+1}\right)dx$

Q.1) i) $I = \int \frac{dx}{\sqrt{x^2+2x-3}}$

$= \int \frac{dx}{\sqrt{(x+1)^2-(2)^2}}$

Comparing with $\int \frac{dx}{\sqrt{x^2-a^2}}$ $= x^2 = (x+1)^2$

$\therefore I = \log|x+\sqrt{x^2+a^2}| + c$

$= \log|x+1+\sqrt{(x+1)^2-(2)^2}| + c$

2) $I = \int (4e^{3x}+1)\,dx$

$= \int 4e^{3x}dx + \int 1\,dx$

$= \frac{4e^{3x}}{3} + x + C$

3) $I = \int (2x^2 - 3\sin x + 5\sqrt{x})\,dx$

$= 2\int x^2 dx - 3\int \sin x\,dx + 5\int \sqrt{x}\,dx$

$= \frac{2}{3}x^3 + 3\cos x + \frac{5x^2}{3}x^{3/2} + C$

$= \frac{2}{3}x^3 + 3\cos x + \frac{10}{3}x^{3/2} + C$

4) $\int \frac{x^3+3x+4}{\sqrt{x}}\,dx$

$I = \int \frac{x^3+3x+4}{\sqrt{x}}\,dx$

$= \int \frac{x^3+3x+4\,dx}{\sqrt{x}}$

$= \int \left(\frac{x^3}{x^{1/2}} + \frac{3x}{x^{1/2}} + \frac{4}{x^{1/2}}\right)dx$

$= \int x^{5/2}\,dx + 3\int x^{-1/2}\,dx + 4\int x^{-1/2}\,dx$

$= \frac{2}{7}x^{7/2} + 2x^{3/2} + 8\sqrt{x} + C$

5) $I = \int t^7 \cdot \sin(2t^4)\,dx$

let $t^4 = x$

$4t^3\,dt = dx$

$\int 4t^3 \cdot t^4 \sin(2t^4)\,dt$

$\therefore I = \frac{1}{4}\int x \cdot \sin(2x)\,dx$

$= \frac{1}{4}\left[x\int \sin 2x - \int\left[\int \sin 2x \cdot \frac{dx}{dx}\right]\right]$

$= \frac{1}{4}\left[-\frac{x\cos 2x}{2} + \frac{1}{2}\int \cos 2x \cdot 1\right]$

$= \frac{1}{4}\left[\frac{-x\cos 2x}{2} + \frac{1}{4}\sin 2x\right] + C$

$$= \frac{-1}{8} x \cdot \cos 2x + \frac{1}{16} \sin 2x + C$$

$$= \frac{-1}{8} t^4 \cos(2t^4) + \frac{1}{16} \sin(2t^4) + C$$

6) $\int \sqrt{x}\,(x^2-1)\,dx$

$$I = \int \sqrt{x}\,(x^2-1)\,dx$$

$$= \int (\sqrt{x} \cdot x^2 - \sqrt{x})\,dx$$

$$= \int (x^{5/2} - \sqrt{x})\,dx$$

$$= \frac{2}{7} x^{7/2} - \frac{2}{3} x^{3/2} + c$$

7) $I = \int \frac{1}{x^3} \sin\left(\frac{1}{x^2}\right) dx$

let $\frac{1}{x^2} = t$

$$x^{-2} = t$$

$$\frac{-2}{x^3}\,dx = dt$$

$$I = -\frac{1}{2} \int \frac{-2}{x^3} \cdot \sin\left(\frac{1}{x^2}\right) dx$$

$$= -\frac{1}{2} \int \sin t$$

$$= -\frac{1}{2}(-\cos t) + c$$

$$= \frac{1}{2} \cos t + C$$

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Resubstitution $t = 1/x^2$

$\therefore I = \frac{1}{2} \cos(1/x^2) + c$

$I = \frac{1}{2} \cos(1/x^2) + c$

8) $I = \int \frac{\cos x}{\sqrt[3]{\sin 2x}}$

let $\sin x = t$

$\therefore \cos x \, dx = dt$

$\therefore I = \int \frac{dt}{\sqrt[3]{t^2}}$

$= \int \frac{dt}{t^{2/3}}$

$= \int t^{-2/3} \, dt$

$= -3 t^{1/3} + c$

$= 3 (\sin x)^{1/3} + c$

$= 3 . \sqrt[3]{\sin x} + c$

9) $I = \int e^{\cos^2 x} \cdot \sin^2 x \, dx$

Let $\cos^2 x = t$

$-2\cos x \cdot \sin x \, dx = dt$

$-2\sin 2x \, dx = dt$

$\therefore I = \int -2\sin 2x \cdot e^{\cos^2 x} dx$

$= -\int e^t \, dt$

$= -e^t + C$

Resubstituting $t = \cos^2 x$.

$\therefore I = -e^{\cos^2 x} + C$

10) $I = \int \left(\frac{x^2 - 2x}{x^3 - 3x^2 + 1} \right) dx$

let $x^3 - 3x^2 + 1 = t$

$3(x^2 - 2x) \, dx = dt$

$(x^2 - 2x) \, dx = dt/3$

$\therefore I = \int 1/3t \, dt$

$= 1/3 \int dt/t$

$= \frac{1}{3} \log t + C$

$= \frac{1}{3} \log(x^3 - 3x^2 + 1) + C$

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Practical- 6

Topic - Application of integration & Numerical integration

Find the length of the following curve

) 2-t sin t $y = 1 - \cos t$ $[0, 2\pi]$

) for t belong to $[0, 2\pi]$

Ans: $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$x = t - \sin t$

$\frac{dx}{dt} = t - \cos t$

$y = 1 - \cos t$

$\frac{dy}{dt} = 0 - (-\sin t)$

$\frac{dy}{dt} = \sin t$

$L = \int_0^{2\pi} \sqrt{(t - \cos t + \sin t)^2 + (\sin t)^2}\, dt$

$= \int_0^{2\pi} \sqrt{1 - \cancel{2\cos t} \; 2\cos t + 1}\, dt$

$= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt$

$$= \int_0^{2\pi} 2 \left| \sin\frac{t}{2} \right| dt \qquad \sin^2\frac{t}{2} = \frac{1-\cos t}{2}$$

$$= \int_0^{2\pi} 2\sin\frac{t}{2}\, dt$$

$$= \left(-4\cos\left(\frac{t}{2}\right)\right)_0^{2\pi} = (-4\cos\pi) - (-4\cos 0)$$

$$= 4+4$$

$$= 8.$$

2) $y = \sqrt{4-x^2}$ $\quad x \in [-2, 2]$

sol: $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

$\frac{dy}{dx}$ $\quad = 2\int_0^2 \sqrt{1 + \left(\frac{-x}{\sqrt{4-x^2}}\right)^2}\, dx$

$= 2\int_0^2 \sqrt{1 + \frac{x^2}{4-x^2}}\, dx$

$= 4\int_0^2 \frac{1}{\sqrt{4-x^2}}\, dx$

$= 4\left(\sin^{-1}\left(\frac{x}{2}\right)\right)_0^2$

$= 2\pi$

3) $Y = x^{3/2}$ $\quad$ in $[0, 4)$

Sol: $f'(x) = \frac{3}{2} x^{1/2}$

$[f'(x)]^2 = \frac{9}{4} x$

$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$

$$= \int_0^7 \sqrt{1+\frac{9}{4}x}\,dx$$

put $u = 1+\frac{9}{4}x$, $du = \frac{9}{4}\,dx$

$$L = \int_1^{1+\frac{9}{4}x} \frac{4}{9}\sqrt{u}\,du = \left[\frac{4}{9}\cdot\frac{2}{3}\left(u^{3/2}\right)\right]_1^{1+\frac{9}{4}x}$$

$$= \frac{8}{27}\left[\left(1+\frac{9x}{4}\right)-1\right]$$

4) $x = 3\sin t$ $y = 3\cos t$

Sol^n: $\frac{dx}{dt} = 3\cos t$

$\frac{dy}{dt} = -3\sin t$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\,dt$$

$$= \int_0^{2\pi} \sqrt{(3\cos t)^2 + (-3\sin t)^2}\,dt$$

$$= \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\,dt$$

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$$= 2\int_0^{\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt$$

$$= \int_0^{2\pi} 3\sqrt{x}\; dt$$

$$= 3\int_0^{2\pi} x\, dt$$

$$= 3\,[x]_0^{2\pi}$$

$$= 3\,[2\pi - 0]$$

$$L = 6\pi \text{ units.}$$

5) $x = \frac{1}{6}y^3 + \frac{1}{2y}$ on $y = [1, 2]$

Sol^n: $\frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2}$

$$\frac{dx}{dy} = \frac{y^4 - 1}{2y^2}$$

$$L = \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_1^2 \sqrt{\frac{(y^4+1)^2}{(2y)^2}}\, dy$$

$$= \int_1^2 \frac{y^4 + 1}{2y^2}\, dy$$

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#20

$$= \frac{1}{2}\int_1^2 y^2\,dy + \frac{1}{2}\int_1^2 y^{-2}\,dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3} - \frac{y^{-1}}{1}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1\right]$$

$$= \frac{1}{2}\left[\frac{7}{3} - \frac{1}{2}\right]$$

$$= \frac{17}{12} \text{ units.}$$

2)

1) $\int_0^2 e^{x^2}\,dx$ with $n = 4$ —

Sol^n: $\int_0^2 e^{x^2}\,dx = 16.4526$

In each case the width of the subinterval be $\Delta x = \frac{2-0}{4} = \frac{1}{2}$

and so the sub interval will be

$[0, 0.5]$ $[0.5, 1]$. $[1, 1.5]$ $[1.5, 2]$

By Simpson rule,

$${}_{0}\int^{2} e^{x^2}\,dx = \frac{1/2}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)$$

$$\approx \frac{1/2}{3}\left(e^{0^2} + 4e^{(0.5)^2} + 2e^{(1)^2} + 4e^{(1.5)^2} + e^{(2)^2}\right)$$

$$\approx 17.3536$$

2) ${}_{1}\int^{4} x^2\,dx \quad n = 4$

$$\Delta x = \frac{4-0}{4} = 1$$

$${}_{a}\int^{b} f(x)\,dx = \frac{\Delta x}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right]$$

$$= \frac{1}{3}\left[y(0) + 4(1)^2 + 2(2)^2 + 4(3)^2 + 4^2\right]$$

$$= \frac{1}{3}\left[0^2 + 4(1)^2 + 2(2)^2 + 4(3)^2 + 4^2\right]$$

$$= \frac{64}{3}$$

3) $\int_0^{\pi/3} \sqrt{\sin x}\, dx \quad n = 6$

Sol: $\Delta x = \frac{b-a}{n}$

$$\Delta x = \frac{b-a}{n} = \frac{\pi/3 - 0}{6} = \frac{\pi}{18}$$

x	0	$\pi/8$	$2\pi/18$	$3\pi/18$	$4\pi/18$	$5\pi/18$
y	0	0.4167	0.584	0.707	0.801	0.87
	y_0	y_1	y_2	y_3	y_4	y_5

$$\int_0^{\pi/3} \sqrt{\sin x}\, dx \approx \frac{\Delta x}{3} \left(y_0 + 4(y_1 + y_3 + 1/8) + 2(y_2 + y_4) +\right.$$

$$= \frac{\pi/18}{3} \left(0 + 4(0.4167 + 0.707 + 0.875) + 0.93\right.$$

$$\approx 2(0.584 + 0.801) + 0.930$$

$$\approx 0.681$$

Practical - 7

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i) Solve the following differential equations

1) $x \frac{dy}{dx} + \frac{1}{x} y = \frac{e^x}{x}$

$p(x) = \frac{1}{x} \qquad Q(x) = \frac{e^x}{x}$

If $e^{\int 1/x \, dx}$

x

$y(IF) = \int Q(x)(IF)dx + c$

$= \int \frac{e^x}{x} \cdot x \cdot dx + c$

$= \int e^x dx + c$

$x \cdot y = e^x + c$

2) $e^x \frac{dy}{dx} + 2e^x y = 1$

Sol^n: $\frac{dy}{dx} + \frac{2e^x}{e^x} = \frac{1}{e^x}$

$\frac{dy}{dx} + 2y = \frac{1}{e^x}$

$\frac{dy}{dx} + 2y = e^{-x}$

$P(x) = 2 \qquad Q(x) = e^{-x}$

$\int P(x) = dx$

If $= e^{\int 2 dx}$

$= e^{2x}$

$Y(IF) = \int Q(x)(IF)dx + c$

$$= \int e^{x} dx + C$$

$$= y \cdot e^{2x}$$

$$= e^{x} + C$$

3) $x \dfrac{dy}{dx} = \dfrac{\cos x}{x} - 2y$

Sol^n:

$$x \frac{dy}{dx} = \frac{\cos x}{x} - 2y$$

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{\cos x}{x^2}$$

$$P(x) = 2(x) \quad Q(x) = \frac{\cos x}{x^2}$$

$$IF = e^{\int P(x)\, dx}$$

$$= e^{\int 2x\, dx}$$

$$Y(IF) = \int Q(x)\ (IF)\, dx + C$$

$$= \int \frac{\cos x}{x^2} - x^2\, dx + C$$

$$= \int \cos x + C$$

$$\therefore x^2 y = \sin x + C$$

4) $x \cdot \dfrac{dy}{dx} + 3y = \dfrac{\sin x}{x^2}$

Sol^n:

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{\sin x}{x^3}$$

$$P(x) = 3/x \qquad Q(x) = \sin x / x^3$$

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$P(x) = \int 3/x \, dx$

$= x^3$

$IF = e^{\int P(x) dx}$

$= e^x$

$Y(IF) = \int q(x) \, (IF) dx + C$

$= \int \frac{\sin x}{x^3} \cdot x^3 dx + C$

$= \int \sin x + C$

$x^3 y = -\cos x + C$

5) $e^{2x} \frac{dy}{dx} + 2e^{2x} y = 2x$

Sol:

$\frac{dy}{dx} + 2y = \frac{2x}{e^x x}$

$P(x) = 2$

$q(x) = \frac{2x}{e} = 2x e^{-2x}$

$(IF) = e^{\int P(x) dx}$

$= e^{\int 2 dx}$

$= e^{2x}$

$= e^{2x}$

$Y(IF) = \int q(x) \, (IF) dx + C$

$= \int 2x e^{-2x} e^{2x} + C$

$y e^{2x} = \int 2x + C = x^2 + C$

6.20

6) $Sec^2 x \cdot \tan y\, dx + Sec^2 y \tan x\, dy = 0$

Sol^n: $Sec^2 x \cdot \tan y\, dx = -Sec^2 y \tan x\, dy$

$$\frac{Sec^2 x}{\tan x}\, dx = -\frac{Sec^2 x}{\tan y}\, dy$$

$$\int \frac{Sec^2 x}{\tan x}\, dx = -\int \frac{Sec^2 x}{\tan y}\, dy$$

$$\int \frac{Sec^2 x}{\tan x}\, dx = -\int \frac{Sec^2 x}{\tan y}\, dy$$

$$\log|\tan x| = -\log|\tan y| + C$$

$$\log|\tan x + \tan y| = c$$

$$\tan x + \tan y = e^c$$

7) $\frac{dy}{dx} = Sin^2(x - y + 1)$

put $x - y + 1 = V$

$$x - y + 1 = V$$

$$1 - \frac{dy}{dx} = \frac{dv}{dx}$$

$$\cancel{1 - \frac{dv}{dx}} = \frac{dy}{dx}$$

$$\cancel{1} - \frac{dv}{dx} = Sin^2 v$$

$$\frac{dv}{dx} = 1 - Sin^2 v$$

$$\frac{dv}{dx} = \cos^2 x$$

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$$\frac{dv}{\cos^2 x} = dx$$

$$\int \sec^2 x \, dv = \int dx$$

$\tan v = x + C$ $\quad \tan(x - y + 1) = x + c$

8) $$\frac{dy}{dx} = \frac{2x + 3y - 1}{6x + 9y + 6}$$

put $2x + 3y = u$

$$2 + \frac{3dy}{dx} = \frac{du}{dx}$$

$$\frac{du}{dx} = \frac{1}{3}\left(\frac{dv}{x} - 2\right)$$

$$= \frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{1}{3}\left(\frac{v-1}{v+2}\right)$$

$$\frac{dv}{dx} = \frac{v-1}{v+2} + 2$$

$$\frac{dv}{dx} = \frac{v - 1 + 2v + u}{v + 2}$$

$$= \frac{3v + 3}{v + 2}$$

$$= \frac{3(v+1)}{(v+2)}$$

$$= \int \frac{v+2}{v+1} dv = 3dx$$

$$= \int \frac{v+1}{v} dx + \int \frac{1}{v+1} dv = \int 3dx.$$

Practical - 8

Topic - Euler's Method

Q.1) $\frac{dy}{dx} = y + e^x - 2$ $y(0) = 2$ $h = 0.5$ find $y(2) = 0$

Sol:

$f(x) = y + e^x - 2$ $x_0 = 0$ $y(0) = 2$ $h = 0.5$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	2	1	2.5
1	0.5	2.5	2.1487	3.5743
2	1	3.5743	4.2925	5.7205
3	1.5	5.7205	8.2021	9.8215
4	2	9.8215		

$\therefore Y(2) = 9.8215$

Q.2) $\frac{dy}{dx} = 1 + y^2$, $y(0) = 1$ $h = 0.2$ find $y(1) = ?$

$y_0 = 0$, $y_0 = 0$, $h = 0.2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	0	1	0.2
1	0.2	0.2	1.04	0.408
2	0.4	0.408	1.1664	0.6412
3	0.6	0.6412	1.4111	0.9234
4	0.8	0.9234	1.8526	1.2939
5	1	1.2939		

$Y(1) = 1.2939$

Q.3) $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$ $y(0)=1$, $h=0.2$ find $y(1)=?$

$x_0 = 0$ $y(0) = 1$ $h = 0.2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	1	0	1
1	0.2	1	0.4472	1.0894
2	0.4	1.0894	0.6059	1.2105
3	0.6	01.2105	0.7040	1.3513
4	0.8	1.3513	0.7696	1.5051
5	1	1.5051		

$y(1) = 1.5051$

Q.4) $dy = 3x^2 +$ $y(1)=2$ find $y(2)$ $h=0.5$
$h=0.25$

$y_0 = 2$ $x_0 = 1$ $h = 0.5$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	~~7.75~~ 4	4
1	1.5	4	7.75	7.875
2	2	7.875		

$y(2) = 7.875$

15) $y_0 = 2 \quad x_0 = 1$

2)

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	3
1	1.25	3	5.6875	4.4218
2	1.5	4.4218	59.6569	19.3360
3	1.75	19.3360	1122.6426	299.9960
4	2	299.9960		

$y(2) = 299.9960$

5) $\frac{dy}{dx} = \sqrt{xy} + 2 \quad y(1) = 1 \quad h = 0.2$

$x_0 = 1 \quad y_0 = 1 \quad h = 0.2$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	1	3	3.6
1	1.2	3.6		

$y(1.2) = 3.6$

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Practical - 9

580 Limits & Partial Order Derivate

Q.1) i) $\lim\limits_{(x,y)\to(-4,-1)} \dfrac{x^3-3y+y^2-1}{xy+5}$

At $(-4,-1)$, Denominator $\neq 0$

$\therefore$ By applying limit

$= \dfrac{(-4)^3-3(-1)+(-1)^2-1}{-4(-1)+5}$

$= \dfrac{-64+3+1-1}{4+5}$

$= \dfrac{-61}{9}$

ii) $\lim\limits_{(x,y)\to(2,0)} \dfrac{(y+1)(x^2+y^2-4x)}{x+3y}$

At $(2,0)$, Denominator $\neq 0$

$\therefore$ By applying limit,

$= \dfrac{(0+1)((2)^2+0-4(2))}{2+0}$

$= \dfrac{1(4+0-8)}{2}$

$= -\dfrac{4}{2}$

$= -2$

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iii) $\lim_{(x,y,z)\to(1,1,1)} \frac{x^2 - y^2 z^2}{x^2 - x^2 y^2}$

At $(1,1,1)$, Denominator $= 0$

$\therefore \lim_{(x,y,z)\to(1,1,1)} \frac{x^2 - y^2 z^2}{x^2 - x^2 y^2}$

$= \lim_{(x,y,z)\to(1,1,1)} \frac{(x - yz)(x + yz)}{x^2 (x - yz)}$

$= \lim_{(x,y,z)\to(1,1,1)} \frac{x + yz}{x^2}$

On Applying limit

$= \frac{1 + 1(1)}{(1)^2}$

$= 2$

Q2) i) $f(x,y) = xye^{x^2+y^2}$

$\therefore f_x = \frac{\partial}{\partial x}(f(x,y))$

$= \frac{\partial}{\partial x}(xye^{x^2+y^2})$

$= ye^{x^2+y^2}(2x)$

$\therefore \; f_x = 2xye^{x^2+y^2}$

$f_y = \frac{d}{dy}(f(x,y))$

$= \frac{d}{dy}(xye^{x^2+y^2})$

$= xe^{x^2+y^2}(2y)$

$\therefore \; f_y = 2y\,xe^{x^2+y^2}$

ii) $f(x,y) = e^x \cos y$

$f_x = \frac{d}{dx}(f(x,y))$

$= \frac{d}{dx}(e^x \cos y)$

$\therefore \; f_x = e^x \cos y$

$f_y = \frac{d}{dy}(f(x,y))$

$= \frac{d}{dy}(e^x \cos y)$

$f_y = -e^x \sin y$

(ii) $f(x,y) = x^3 y^2 - 3x^2 y + y^3 + 1$ 064

$f_x = \frac{d}{dx}(f(x,y))$

$= \frac{d}{dx}(x^3 y^2 - 3x^2 y + y^3 + 1)$

$f_x = 3x^2 y^2 - 6xy$

$f_y = \frac{d}{dy}(f(x,y))$

$= \frac{d}{dy}(x^3 y^2 - 3x^2 y + y^3 + 1)$

$f_y = 2x^3 y - 3x^2 + 3y^2$

Q.3) i) $f(x,y) = \frac{2x}{1+y^2}$

$f_x = \frac{d}{dx}\left(\frac{2x}{1+y^2}\right)$

$= \frac{1+y^2 \frac{d}{dx}(2x) - 2x \frac{d}{dx}(1+y^2)}{(1+y^2)^2}$

$= \frac{2+2y^2 - 0}{(1+y^2)^2}$

$$= \frac{2\cancel{(1+y^2)}}{\cancel{(1+y^2)}(1+y^2)}$$

$$= \frac{2}{1+y^2}$$

At $(0,0)$

$$= \frac{2}{1+0}$$

$$= 2$$

$$f_y = \frac{d}{dy}\left(\frac{2x}{1+y^2}\right)$$

$$= \frac{1+y^2 \frac{d}{dx}(2x) - 2x\frac{d}{dx}(1+y^2)}{(1+y^2)^2}$$

$$= \frac{1+y^2(0) - 2x(2y)}{(1+y^2)^2}$$

$$= \frac{-4xy}{(1+y^2)^2}$$

At $(0,0)$,

$$= \frac{-4(0)(0)}{(1+0)^2}$$

$$= 0$$

Q(4) i) $f(x,y) = \dfrac{y^2 - xy}{x^2}$

$$f_x = \frac{x^2 \frac{d}{dx}(y^2 - xy) - (y^2 - xy)\frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{x^2(-y) - (y^2 - xy)(2x)}{x^4}$$

$$= \frac{-x^2 y - 2x(y^2 - xy)}{x^4}$$

$$f_y = \frac{2y - x}{x^2}$$

$$f_{xx} = \frac{d}{dx}\left(\frac{-x^2 y - 2x(y^2 - xy)}{x^4}\right)$$

$$= \frac{x^4\left(\frac{d}{dx}(-x^2 y - 2xy^2 + 2x^2 y)\right) - (-x^2 y - 2xy^2 + 2x^2 y)\frac{d}{dx}(x^4)}{(x^4)^2}$$

$$= \frac{x^4(-2xy - 2y^2 + 4xy) - 4x^3(-x^2 y - 2xy^2 + 2x^2 y)}{x^6} \quad (1)$$

$$f_{yy} = \frac{d}{dy}\left(\frac{2y - x}{x^2}\right)$$

$$= \frac{2 - 0}{x^2} = \frac{2}{x^2}$$

$$f_{xy} = \frac{d}{dy}\left(\frac{-x^2 y - 2xy^2 + 2x^2 y}{x^4}\right) \quad (2)$$

$$= \frac{-x^2 - 4xy + 2x^2}{x^4} \quad (3)$$

$$f_{yx} = \frac{d}{dx}\left(\frac{2y - x}{x^2}\right)$$

$$= \frac{x^2 \frac{d}{dx}(2y-x) - (2y-x)\frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{-x^2 - 4xy + 2x^2}{x^4}$$

from (3) & (4)

$f_{xy} = f_{yx}$

ii) $f(x,y) = x^2 + 3x^2y^2 - \log(x^2+1)$

$f_x = \frac{d}{dx}(x^2+3x^2y - \log(x^2+1))$ $\quad f_y = \frac{d}{dx}(x^2+3y^2y^2 - \log(x^2$

$= 3x^2 + 6xy^2 - \frac{2x}{x^2+1}$ $\quad = 0 + 6x^2y - 0$

$= 6x^2y$

$$f_{xx} = 6x + 6y^2 - \left(\frac{x^2+1\frac{d(2x)}{dx} - 2x\frac{d(x^2+1)}{dx}}{(x^2+1)^2}\right)$$

$$= 6x + 6y^2 - \left(\frac{2(x^2+1) - 4x^2}{(x^2+1)^2}\right) \text{ ——— } 1$$

$$f_{yy} = \cancel{\frac{d}{dy}}(6x^2y)$$

$$= 6x^2 \text{ ——— } 2$$

$$f_{xy} = \frac{d}{dy}\left(3x^2 + 6xy^2 - \frac{2x}{x^2+1}\right)$$

$= 0 + 12xy - 0$

$= 12xy$ ——— 3

$f_{yx} = \frac{d}{dx}(6x^2 y)$

$= 12xy$

from (3) & (4)

$\therefore f_{xy} = f_{yx}$

ii) $f(x,y) = \sin(xy) + e^{x+y}$

$\Rightarrow f_x = y\cos(xy) + e^{x+y}(1)$

$= y\cos(xy) + e^{x+y}$

$f_y = x\cos(xy) + e^{x+y}(1)$

$= x\cos(xy) + e^{x+y}$

$\therefore f_{xx} = \frac{d}{dy}(x\cos(xy) + e^{x+y})$

$= -x\sin(xy)(x) + e^{x+y}(1)$

$= -x^2\sin(xy) + e^{x+y}$ —— (1)

$\therefore f_{yy} = \frac{d}{dy}(x\cos(xy) + e^{x+y})$

$= -x\sin(xy)(x) + e^{x+y}(1)$

$= -x^2\sin(xy) + e^{x+y}$ —— (2)

$f_{xy} = \frac{d}{dy}(x\cos(xy) + e^{x+y})\frac{d}{dy}(y\cos(xy) + e^{x+y})$

$= -y^2\sin(xy) + \cos(xy) + e^{x+y}$ —— (3)

$f_{yx} = \frac{d}{dx}(x\cos(xy) + e^{x+y})$

$= -x^2\sin(xy) + \cos(xy) + e^{x+y}$ —— (4)

$\therefore$ from (3) & (4)

$f_{xy} \neq f_{yx}$

iiai)

Q.5) i) $f(x,y) = \sqrt{x^2+y^2}$ at $(1,1)$

$\longrightarrow f(1,1) = \sqrt{(1)^2+(1)^2} = \sqrt{2}$

$fx = \frac{1}{2\sqrt{x^2+y^2}}(2x)$

$= \frac{x}{\sqrt{x^2+y^2}}$

fx at $(1,1) = \frac{1}{\sqrt{2}}$

$fy = \frac{1}{2\sqrt{x^2+y^2}}(2y)$

$= \frac{y}{\sqrt{x^2+y^2}}$

fy at $(1,1) = \frac{1}{\sqrt{2}}$

$\therefore L(x,y) = f(a,b) + fx(a,b)(x-a) + fy(a,b)(y-b)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x-1) + \frac{1}{\sqrt{2}}(y-1)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x-1+y-1)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x+y-2)$

$= \cancel{\sqrt{2}} + \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y - \frac{2}{\cancel{\sqrt{2}}}$

$= \frac{x+y}{\sqrt{2}}$

ii) $f(x,y) = 1 - x + y\sin x$ at $(\frac{\pi}{2}, 0)$

$f(\frac{\pi}{2}, 0) = 1 - \frac{\pi}{2} + 0 = 1 - \frac{\pi}{2}$

$fx = 0 - 1 + y\cos x$

fx at $(\frac{\pi}{2}, 0) = -1 + 0$

$= -1$

$fy = 0 - 0 + \sin x$

fy at $(\frac{\pi}{2}, 0) = \sin\frac{\pi}{2} = 1$

$L(x,y) = f(a,b) + fx(a,b)(x-a) + fy(a,b)(y-b)$

$= 1 - \frac{\pi}{2} + (-1)\left(x - \frac{\pi}{2}\right) + 1\,(y-0)$

$= 1 - \cancel{\frac{\pi}{2}} - x + \cancel{\frac{\pi}{2}} + y$

$= 1 - x + y$

iii) $f(x,y) = \log x + \log y$ at $(1,1)$

$f(1,1) = \log(1) + \log(1) = 0$

$f(x) = \frac{1}{x} + 0$ $\qquad fy = 0 + \frac{1}{y}$

fx at $(1,1) = 1$ $\qquad fy$ at $(1,1) = 1$

$L(x,y) = f(a,b) + fx(a,b)(x-a) + fy(a,b)(y-b)$

$= 0 + \cancel{1}(x-1) + 1(y-1)$

$= x - 1 + y - 1$

$= x + y - 2$

Practical - 10

Aim :- Directional Derivative of Given Vector

Q.1)

i) $f(x,y) = x + 2y - 3 \quad a = (1,-1) \quad u = 3i - j$

Here, $u = 3i - j$ is not a unit vector

$|\bar{u}| = \sqrt{(3)^2 + (-1)^2} = \sqrt{9+1} = \sqrt{10}$

Unit vector along $\frac{u}{|u|} = \frac{1}{\sqrt{10}}(3,-1)$.

$= \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$f(a+hu) = f(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{10}\right)$

$f(a) = f(1,-1) = (1) + 2(-1) - 3$

$= 1 - 2 - 3 = -4$

$f(a+hu) = f(1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

$= \left(1 + \frac{3}{\sqrt{10}}, -1 - \frac{(-h)}{\sqrt{10}}\right)$

$f(a+hu) = \left(1 + \frac{3}{\sqrt{10}}, -1 \frac{-h}{\sqrt{10}}\right) - 3$

$= 1 + \frac{3}{\sqrt{10}} - 2 - \frac{2h}{\sqrt{10}} - 3$

$f(a+hu) = -4 + \frac{h}{\sqrt{10}}$

$$D_u f(a) = \lim_{h \to 0} \frac{f(a+hu) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{-4 + h\sqrt{10} + 4}{h}$$

$$D_u f(a) = \frac{1}{\sqrt{10}}$$

) $f(x) = y^2 - 4x + 1$ $a = (3, 4)$ $u = i + 5j$

Here $u = i + 5j$ is not a unit vector

$|\bar{u}| = \sqrt{(1)^2 + (5)^2} = \sqrt{1+25} = \sqrt{26}$

Unit vector along u is $\frac{u}{|u|} = \frac{1}{\sqrt{26}} (1, 5)$

$$= \left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$$

$f(a) = f(3, 4) = (4)^2 - 4(3) + 1 = 5$

$f(a + hu) = f(3, 4) + h \left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$

$$= f\left(3 + \frac{h}{\sqrt{26}}, 4 + \frac{5h}{\sqrt{26}}\right).$$

$$f_{x,y}(a + hu) = \left(4 + \frac{5h}{\sqrt{26}}\right)^2 - 4\left(3 + \frac{h}{\sqrt{26}}\right) + 1$$

$$= 16 + \frac{25h^2}{26} + \frac{40h}{\sqrt{26}} - 12 - \frac{4h}{\sqrt{26}} + 1$$

$$= \frac{25h^2}{26} + \frac{40h}{\sqrt{20}} - \frac{4h}{\sqrt{20}} + 5$$

$$= \frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + 5$$

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$$D_u f(a) = \lim_{h \to 0} \frac{\frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + \cancel{8} - \cancel{8}}{h}$$

$$D_u f(a) = \frac{25h}{20} + \frac{36}{\sqrt{26}}$$

3) $2x + 3y \quad a = (1, 2) \quad u = 3i + 4j$

$f(x, y) = x^y + y^x = a = (1, 1)$

$f_x = y \cdot x^{y-1} + y^x \log y$

$f_y = x^y \log x + x y^{x-1}$

$Df(x, y) = (f_x, f_y)$

$= (y x^{y-1} + y^2 \log y, x^y \log x + x y^{x-1})$

$f(1,1) = (1 + 0, 1 + 0)$

$= (1, 1)$

$\Rightarrow$ Here $u = 3i + 4j$ is not a unit vector

$|\bar{u}| = \sqrt{(3)^2 + (4)^2} = \sqrt{25} = 5$

Unit vector along u is $\frac{u}{|u|} = \frac{5}{5}(3, 4)$

$= \left(\frac{3}{5}, \frac{4}{5}\right)$

$f(a) = f(3, 2) = 2(1) + 3(2) = 8$

$f(a + hu) = f(1,2) + h\left(\frac{3}{5}, \frac{4}{5}\right)$

$= f\left(1 + \frac{3h}{5}, 2 + \frac{4h}{5}\right)$

$f(a + hu) = 2\left(1 + \frac{3h}{5}\right) + 3\left(2 + \frac{4h}{5}\right)$

$$= 2 + \frac{6h}{5} + 6 + \frac{12h}{3}$$

$$= \frac{18h}{5} + 8$$

$$D_u f(a) = \lim_{h \to 0} \frac{\frac{18h}{5} + \cancel{8} - \cancel{8}}{k}$$

$$= \frac{18}{5}$$

Q.2) Find gradient vector for the following function.

1) $f(x, y) = (\tan^{-1} x) \cdot y^2 \qquad a = (1, -1)$

$$f_x = \frac{1}{1+x^2} \cdot y^2$$

$$f_y = 2y \cdot \tan^{-1} x$$

$$\Delta f(x, y) = (f_x, f_y).$$

$$= \left(\frac{y^2}{1+x^2}, 2y \tan^{-1} x\right)$$

$$f(1, -1) = \left(\frac{1}{2}, \tan^{-1}(1)(-2)\right)$$

$$= \left(\frac{1}{2}, \frac{\pi}{4}(-2)\right)$$

$$= \left(\frac{1}{2}, -\frac{\pi}{2}\right)$$

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iii) $f(x_4, z) = xy^2 - e^{x+y+z}$ $a = (1, -1, 0)$

$f_x = yz - e^{x+y+z}$

$f_y = xz - e^{x+y+z}$

$f_z = xy - e^{x+y+z}$

$\Delta f(x, y, z) = \cancel{xy} f_x, f_y, f_z$

$= y^2 - e^{x+y+z}, xz - e^{x+y+z}, xe_y - e^{x}$

$f(1, -1, 0) = ((-1)(0) - e^{(1+(-1)+0)}, (1)(0) - e^{1+(-1)+0}, (1)(-1) - e^{1+(-1)+0})$

$= 0 - e^0, 0 - e^0, -1 - e^0)$

$= (-1, -1, -2)$.

3)

1) $x^2 \cos y + e^{xy} = 2$ at $(1, 0)$

$d_x = \cos y - 2x + e^{xy} \cdot y$.

$d_y = x^2(-\sin y) + e^{xy} \cdot x$.

$(x_0, y_0) = (1, 0) = x_0 = 1$ $y_0 = 0$.

Eqⁿ of tangent.

$f_x(x - x_0) + f_y(y - y_0) = 0$

$f_x(x_0, y_0) = \cos 0^2(1) + e^0 0$

$= 1(2) + 0$

$= 2$

$f_y(x_0, y_0) = (1)^2(-\sin 0) + e^0 \cdot 1$

$= 0 + 1 \cdot 1$

$= 1$

$2(x-1) + 1(y-0) = 0$

$2x - 2 + y = 0$

$2x + y - 2 = 0$

It is the required eqn of tangent eqn of normal

$= ax + by + c = 0$

$\therefore bx + ay + d = 0$

$(1)1 + 2(y) + d = 0$ at $(1, 0)$

$1 + 2y + d = 0$

$1 + 2(0) + d = 0$

$d + 1 = 0$

$d = -1$

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ii) $x^2 + y^2 - 2x + 3y + 2 = 0$ at $(2, -2)$

$f_x = 2x + 0 - 2 + 0 + 0$

$= 2x - 2$

$f_y = 0 + 2y - 0 + 3 + 0$

$= 2y + 3$

$(x_0, y_0) = (2, -2)$

$\therefore x = 2 \quad , y_0 = -2$

$f_x(x_0, y_0) = 2(2) - 2 = 2$

$f_y(x_0, y_0) = 2(-2) + 3 = -1$

Eqn of tangent,

$f_x(x - x_0) + f_y(y - y_0) = 0$

$2(x-2) + (-1(y+2)) = 0$

$2x - 2 - y - 2 = 0$

$2x - y - 4 = 0$

It is the required eqn of tangent of Normal

$ax + by + c = 0$

$bx + ay + d = 0$

$\therefore -1(x) + 2(y) + d = 0$

$-x + 2y + d = 0$ at $(2, -2)$

$-2 + 2(-2) + d = 0$

$-2 - 4 + d = 0$

$-6 + d = 0$

$d = 6.$

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Q.4) i) $x^2 - 2\cancel{x}y^2 + 3y + xz = 7$ at $(2,1,0)$

$f_x = 2x - 0 + 0 + z$

$f_x = 2x + z.$

$f_y = 0 - 2z + 3 + 0$

$= 2z + 3$

$f_z = 0 - 2y + 0 + x$

$= x - 2y.$

$(x_0, y_0, z_0) = (2, 1, 0)$

$\therefore x_0 = 2, \; y_0 = 1, \; z_0 = 0$

$f_x(x_0, y_0, z_0) = 2(2) + 0 = 4.$

$f_y(x_0, y_0, z_0) = 2(0) + 3 = 3$

$f_z(x_0, y_0, z_0) = -2(1) + 2 = 0$

Equation of tangent

$f_x(x_0 - x_0) + f_y(y_0 - y_0) + f_z(z_0 - z_0) = 0$

$4(x-2) + 3(y-1) + 0(z-0) = 0$

$4x - 8 + 3y - 3 = 0$

$4x + 3y - 11 = 0$

This is required eq^n of tangent eq^n of normal at $(4, 3, -11)$

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$$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z}$$

$$\frac{x - 2}{4} = \frac{y - 1}{3} = \frac{z + 11}{0}$$

ii) $3xyz - x - y + z = -4$ at $(1, -1, 2)$

$3xyz - x - y + z + 4 = 0$

$f_x = 3yz - 1 - 0 + 0 + 0$
$= 3yz - 1$

$f_y = 3xz - 0 - 1 + 0 + 0$
$= 3xz - 1$

$f_z = 3xy - 0 - 0 + 1 + 0$
$= 3xy + 1$

$(x_0, y_0, z_0) = (1, -1, 2)$

$\therefore x_0 = 1, \ y_0 = -1, \ z_0 = 2$

$f_x(x_0, y_0, z_0) = 3(-1)(2) - 1 = -7$

$f_y(x_0, y_0, z_0) = 3(+1)(2) - 1 = 5$

$f_z(x_0, y_0, z_0) = 3(1)(-1) + 1 = -2$

Eq^n of tangent

$-7(x-1)+5(y+1)-2(z-2)=0$ 072

$-7x+7+5y+5-2z+4=0$

$-7x+5y-2z+16=0$

This is ~~req~~ eq^n of tangent

Eq^n of Normal at $(-7,5,-2)$

$$\frac{x-x_0}{fx}=\frac{y-y_0}{fy}=\frac{z-z_0}{fz}$$

$$\therefore \frac{x-1}{-7}=\frac{y+1}{1}=\frac{z-2}{-2}$$

5) 1) $f(x,y)=3x^2+y^2-3xy+6x-4y$

$fx=6x+0-3y+6-0$

$=6x-3y+6$

$fy=0+2y-3x+0-4$

$=2y-3x-4$

$fx=0$

$6x-3y+6=0$

$3(2x-y+2)=0$

$2x-y+2=0$

$2x-y=-2$ —— ①

$fy=0$

$2y-3x-4=0$

$2y-3x=4$ —— ②

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Multiply eq① with eq ②

$4x - 2y = -4$

$2y - 3x = 4$

$x = 0$

Subs value of x in eqⁿ ①.

$2(0) - y = -2$

$-y = -2$

$y = 2$

Critical points are $(0, 2)$

$r = f_{xx} = 6$

$t = f_{yy} = 2$

$s = f_{xy} = -3$

Here $r > 0$

$= rt - s^2$

$= 6(2) - (-3)^2$

$= 12 - 9$

$= 3 > 0$

$\therefore f$ has max at $(0, 2)$

$3x^2 + y^2 - 3xy + 6x - 4y$ at $(0, 2)$

$3(0)^2 + 2^2 - 3(0)(2) + 6(0) - 4(2)$

$0 + 4 + 0 + 0 - 8 = -4$

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ii) $f(x,y) = 2x^4 + 3x^2y - y^2$

$f_x = 8x^3 + 6xy$

$f_y = 3x^2 - 2y$

$f_x = 0$

$8x^3 + 6xy = 0$

$2x(4x^2 + 3y) = 0$

$4x^2 + 3y = 0$ — (1)

$f_y = 0$

$3x^2 - 2y = 0$ — (2)

Multiply eq (1) with 3 and eq^n (2) with 4.

$12x^2 + 9y = 0$

$-12x^2 - 8y = 0$

$y = 0$

Sub value of y in eq^n (1)

$4x^2 + 3(0) = 0$

$4x^2 = 0$

$x = 0$

Critical point is $(0,0)$

$r = f_{xx} = 24x^2 + 6x$

$t = f_{yy} = 0 - 2 = -2$

$s = f_{xy} = 6x - 0 = 6x = 6(0) = 0$

r at $(0,0)$

$= 24(0) + 6(0) = 0$

$\therefore r = 0$

$rt - s^2 = 0(-2) - (s)^2$

$= 0 - 0 = 0$

$r = 0$ & $rt - s^2 = 0$

Nothing to say

$f(x,y)$ at $(0,0)$

$2(0)^4 + 3(0)^2(0) - 0$

$= 0 + 0 - 0 = 0.$

iii) $f(x,y) = x^2 - y^2 + 2x + 8y - 70$

$f_x = 2x + 2$

$f_y = -2y + 8$

$f_x = 0$

$\therefore 2x + 2 = 0$

$x = -1$

$f_y = 0$

$-2y + 8 = 0$ 074

$y = \frac{-8}{-2} = 4$

Critical Point is $(-1, 4)$

$r = f_{xx} = 2$

$t = f_{yy} = -2$

$s = f_{xy} = 0$

$r > 0$

$rt - s^2 = 2(-2) - 0^2$

$= -4 - 0$

$= -4 \not< 0$

$f(x, y)$ at $(-1, 4)$

$(-1) - (4)^2 + 2(-1) + 8(4) - 70$

$= 1 + 16 - 2 + 32 - 70$

$= 11 + 30 - 70$

$= 37 - 70$

$= 33$

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