Multi-channel Communications Fall 2022

Lecture 7
“Optimal” Beamforming

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Introduction

- Previously we examined the use of multiple antennas to improve SNR through beam-forming
- However, we didn't examine the "optimal" set of weights, especially in the presence of multiple signals (i.e., interference)
- Today we examine several ways to determine the optimal weights
  - i.e., we examine several optimality criteria

System Model

- At the receiver we observe

$$\mathbf{r}(t) \quad = \quad \mathbf{a}_0 x(t) + \sum_{k=1}^{N_I} \mathbf{a}_k i_k(t) + \mathbf{n}(t)$$

- Channel vector **h** is often called array factor or manifold **a**
- Desired signal = $\mathbf{a}_0$
- Array response to interference = $\mathbf{a}_k$
- At each symbol period, we create the decision statistic (after matched filtering): $z = \mathbf{w}^{H}\mathbf{r}$
- What is the "optimal" choice of **w** ?

Optimum Array Processing

- How do we define "optimum"?
  - Maximum SNR
  - Eliminating interference (zero-forcing)
  - Maximum SINR
  - Minimum Mean Square Error
  - Maximum Likelihood
  - Minimum Noise Variance
- We will look at the first four today

Maximum SNR

- First let us consider the case where we only have noise in the channel

$$\mathbf{r}(t) \quad = \quad \mathbf{a}x(t) + \mathbf{n}(t)$$

- Noise is assumed spatially (and temporally) white

$$E\left\{\mathbf{n}(t)\mathbf{n}^H(t)\right\} = \sigma_n^2\mathbf{I}$$

- What weights maximize SNR?

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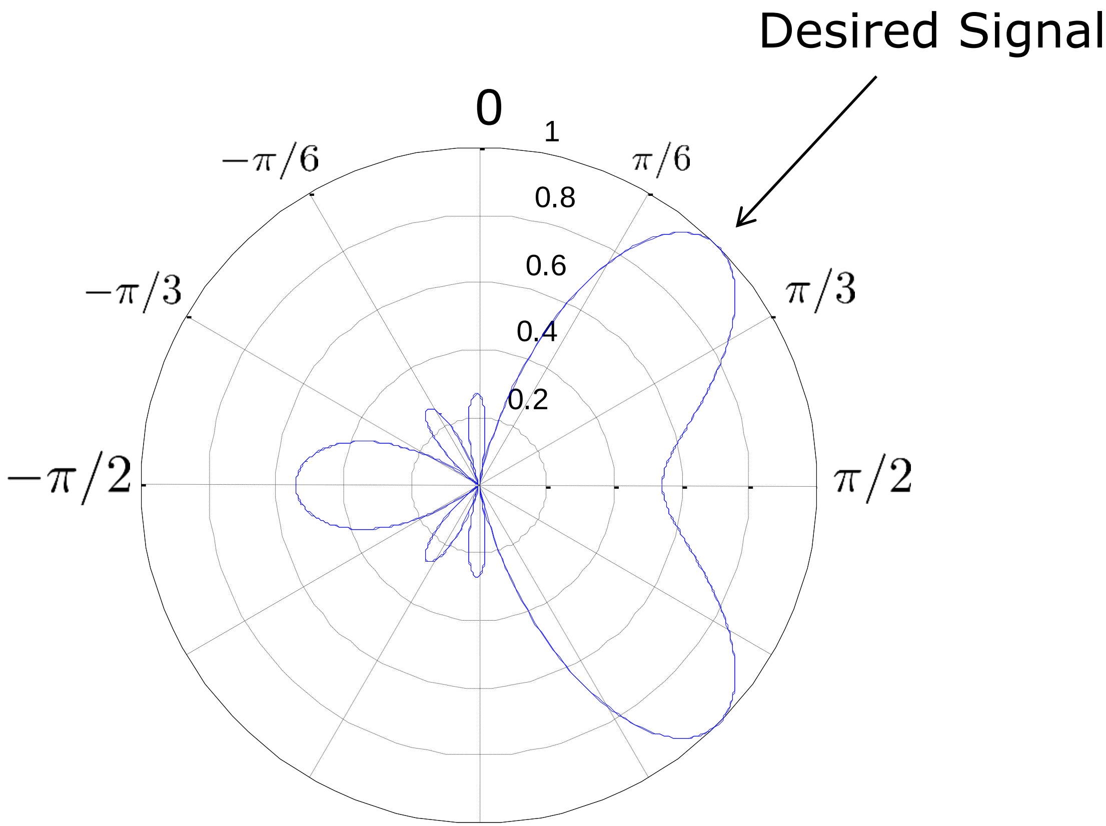
Interpretation

- The maximum SNR weight vector is simply the complex conjugate of the desired signal's channel vector (or array factor also called "array manifold" or "array response")
- This is equivalent to beamsteering to the angle-of-arrival when the array is closely spaced (i.e., when the received signal is a simple plane wave)
- This is also analogous to maximal ratio combining when the channels are uncorrelated (i.e., when we have a diversity array)

Maximum SNR - Example



- $M_r = 4$
- $d = \lambda/2$
- AOA $\theta = \pi/4$

Interference

- Now, let us consider the case where we also have interference in the channel

$$\mathbf{r}(t) = \mathbf{a}_0 x(t) + \sum_{k=1}^{N_I} \mathbf{a}_k i_k(t) + \mathbf{n}(t)$$

$$z = \mathbf{w}^H \mathbf{r}$$

- How do we completely eliminate the interference ? (termed zero-forcing beamformer)

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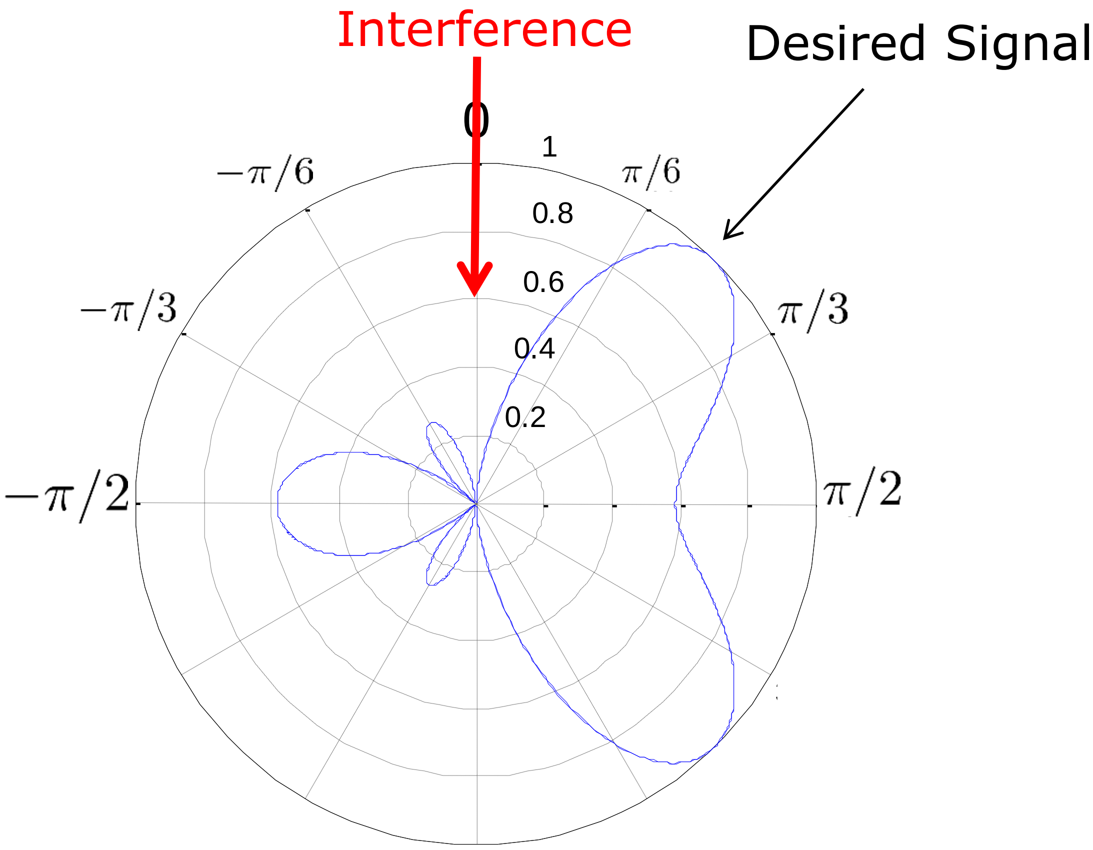
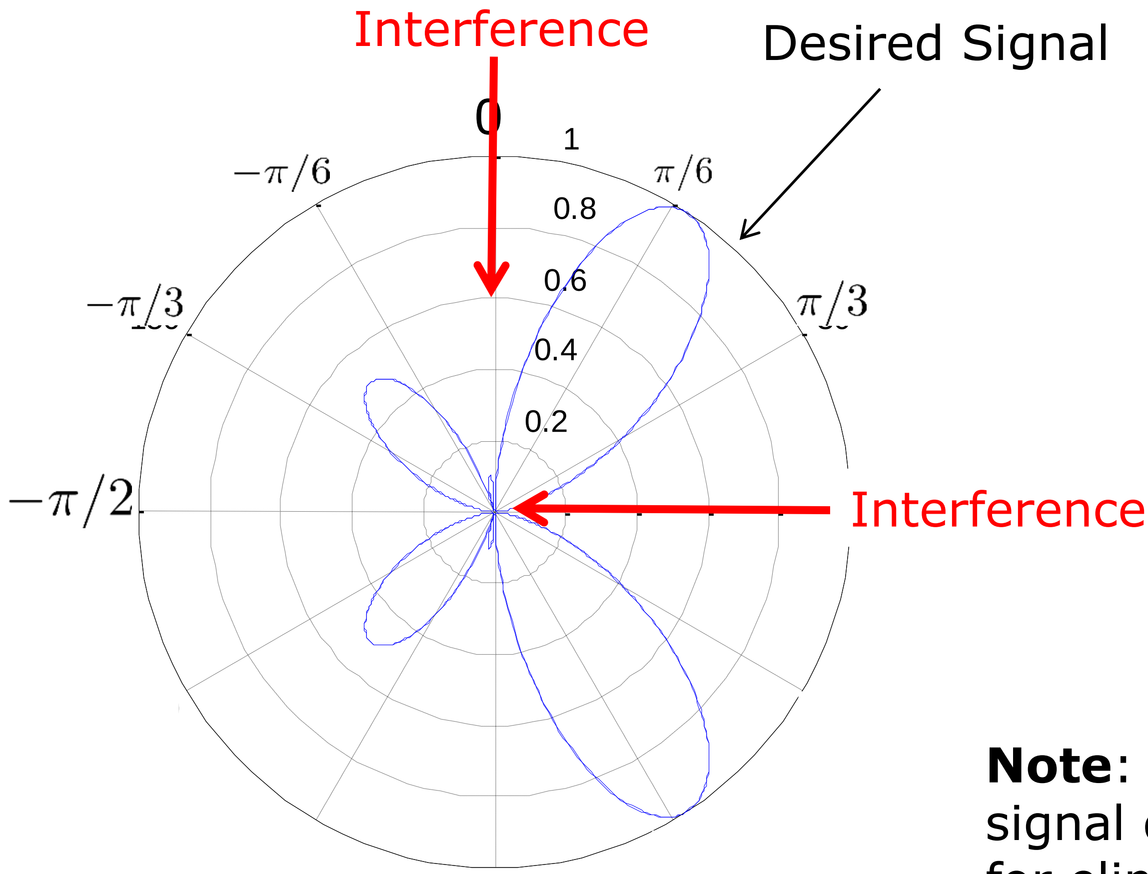
Zero-forcing Beamforming



- $M_r = 4$
- $d = \lambda/2$
- AOA = $\pi/4$

- Interferer from AOA = 0

Zero-forcing Beamforming

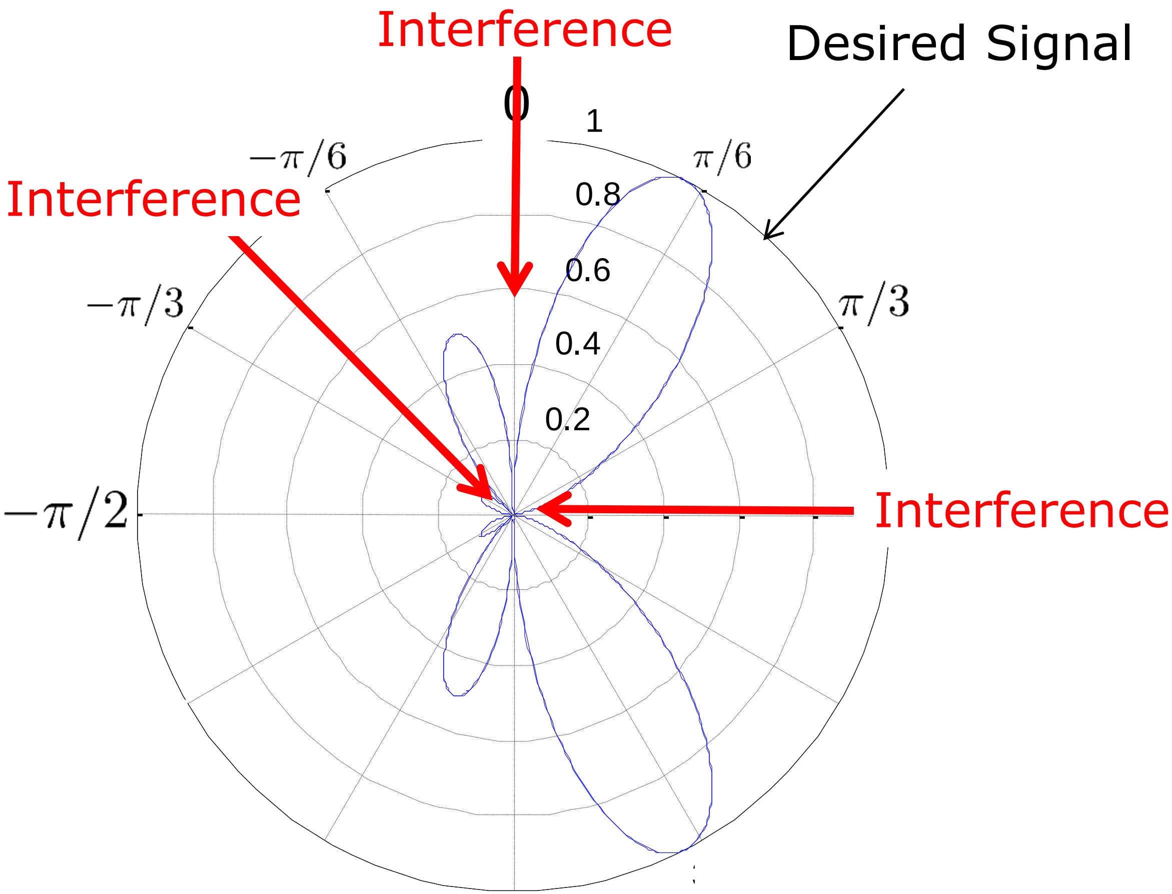


- $M_r = 4$
- d = $\lambda/2$
- AOA = $\pi/4$

- Interferers from AOAs = $[\pi/2, 0]$

Note: Gain in desired signal direction is sacrificed for eliminating interference

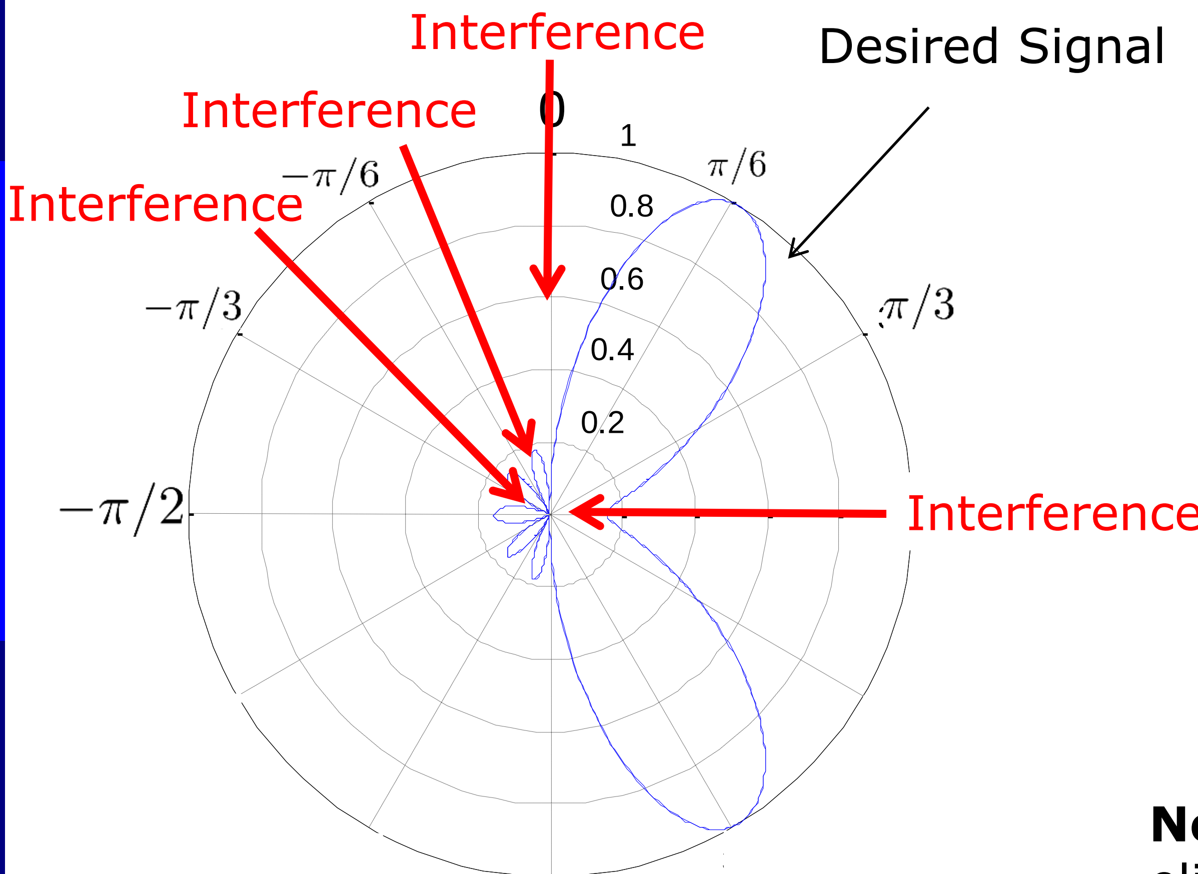
Zero-forcing Beamforming



- $M_r = 4$
- d = $\lambda/2$
- AOA = $\pi/4$

- Interferers from AOAs = [$\pi/2$, 0 -$\pi/4$]

Zero-forcing Beamforming



- $M_r = 4$
- d = $\lambda/2$
- DOA = $\pi/4$

- Interferers from AOAs = [$\pi/2$,0° -$\pi/4$,-32°]

Note: Array cannot eliminate (null) all four interferers. Insufficient degrees of freedom

Observations

- Zero-forcing can completely eliminate up to M_r-1 interferers
- Ignores noise and thus is not optimal in terms of SINR or square error
  - Note that the gain in direction of desired user is typically reduced in the effort to eliminate interference
- When interferer is near the desired signal, the SNR will be very low since the gain in the direction of the desired signal will be low

Maximum SINR

- Now, let us again consider the case where we have interference in the channel

$$\mathbf{r}(t) \quad = \quad \mathbf{a}_0 x(t) + \sum_{k=1}^{N_I} \mathbf{a}_k i_k(t) + \mathbf{n}(t)$$

- Noise is white

$$E\left\{\mathbf{n}^H(t)\mathbf{n}(t)\right\} = \sigma_n^2 \mathbf{I}$$

- What weights maximize SINR?

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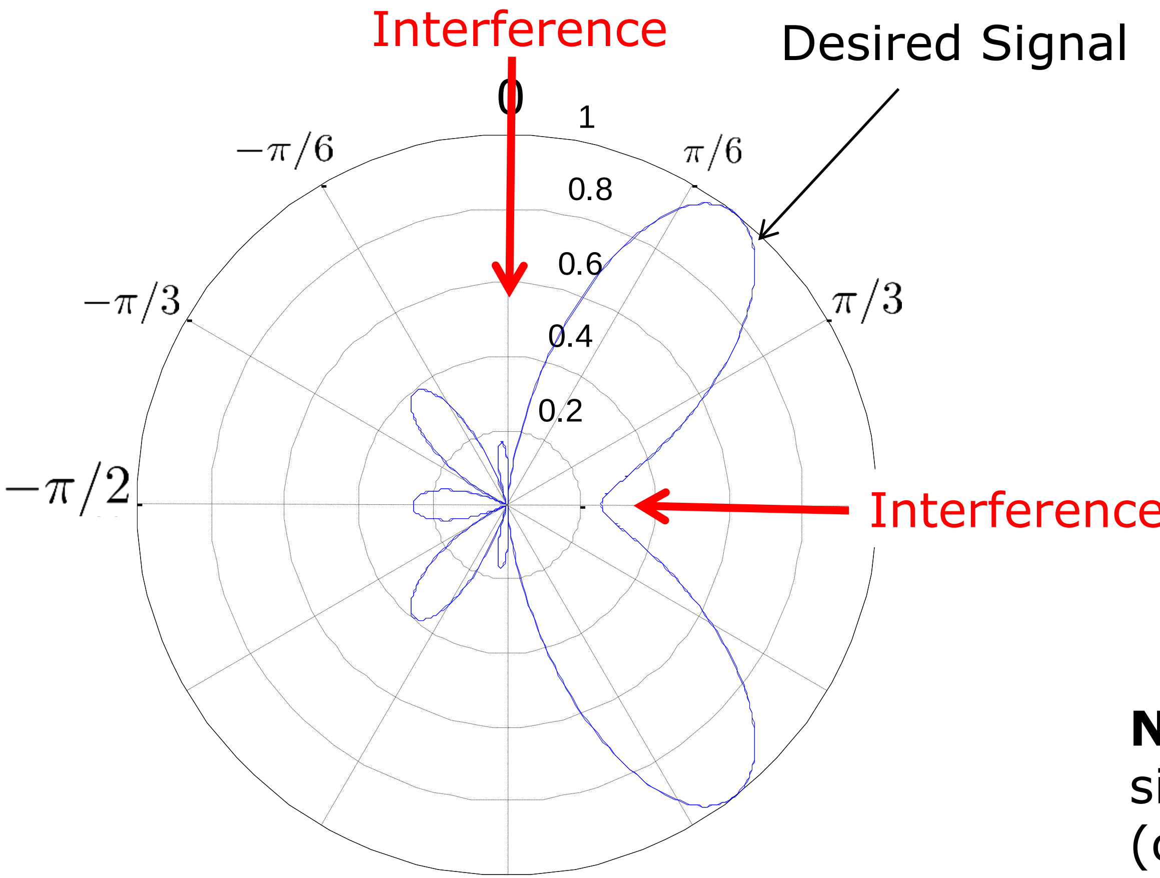
Max SINR Beamforming



- M_r = 4
- AOA =π/4
- Interferers from DOAs = [0 , π/2]
- INR = 3dB

Note: Gain in desired signal direction is increased (compared to zero-forcing) at the sacrifice of increased interference (balances noise and interference)

Minimum Mean Square Error BF

- Another means of balancing the performance versus noise and interference is to find the weights that minimize the square error

$$\mathbf{r}(t) = \mathbf{a}_0 x(t) + \sum_{k=1}^{N_I} \mathbf{a}_k i_k(t) + \mathbf{n}(t)$$

$$z = \mathbf{w}^H \mathbf{r}$$

- Want to find the weights such that

$$\mathbf{w}_{MMSE} = \min_{\mathbf{w}} \mathrm{E}\left\{|z - s|^2\right\} = \min_{\mathbf{w}} \mathrm{E}\left\{\left|\mathbf{w}^H \mathbf{r} - s\right|^2\right\}$$

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Adaptive Algorithms - LMS

- Up to this point we have considered the optimal weights for the beamformer, but not how to obtain them.
- One approach is simply to estimate the optimal weights using estimates of $\mathbf{R}_{rr}$ and $\mathbf{a}$
- Another approach is to use an *adaptive algorithm* to adaptively update the weights
  - Useful when the statistics are changing
- One such approach is based on the method of steepest descent known as the Least Mean Squares (LMS) [illegible]
  - Adaptively finds [illegible]ghts

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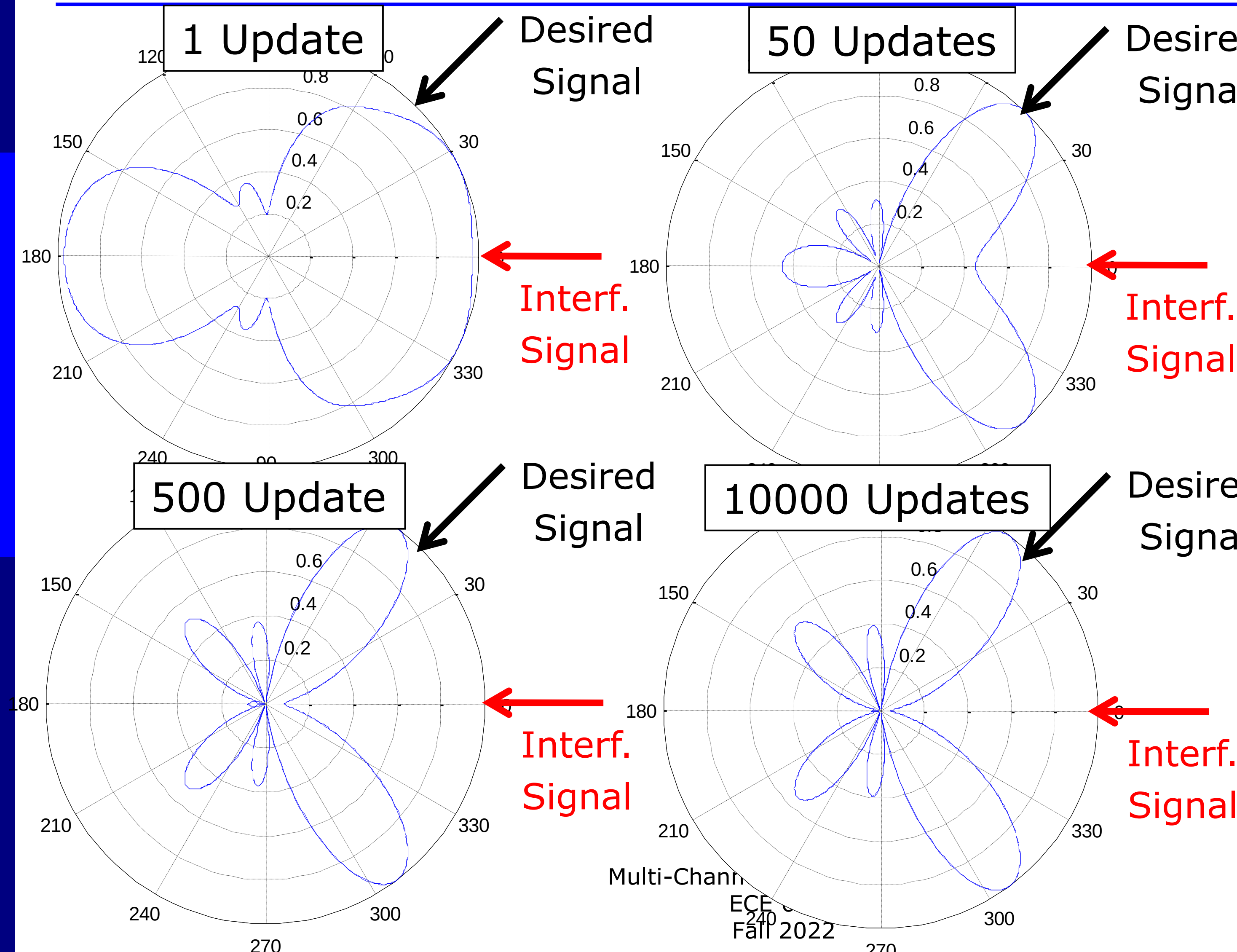
LMS - Example



SNR = 10dB
DOA = 45^{o}
Interference from 90^{o}
δ = -0.001

The RLS Algorithm

- The downside of the LMS algorithm is that it converges very slowly
- An alternative to LMS is to update weights that minimize the square error for a given set of samples
- This is referred to as Recursive Least Squares (RLS) which is more complex than LMS but converges faster

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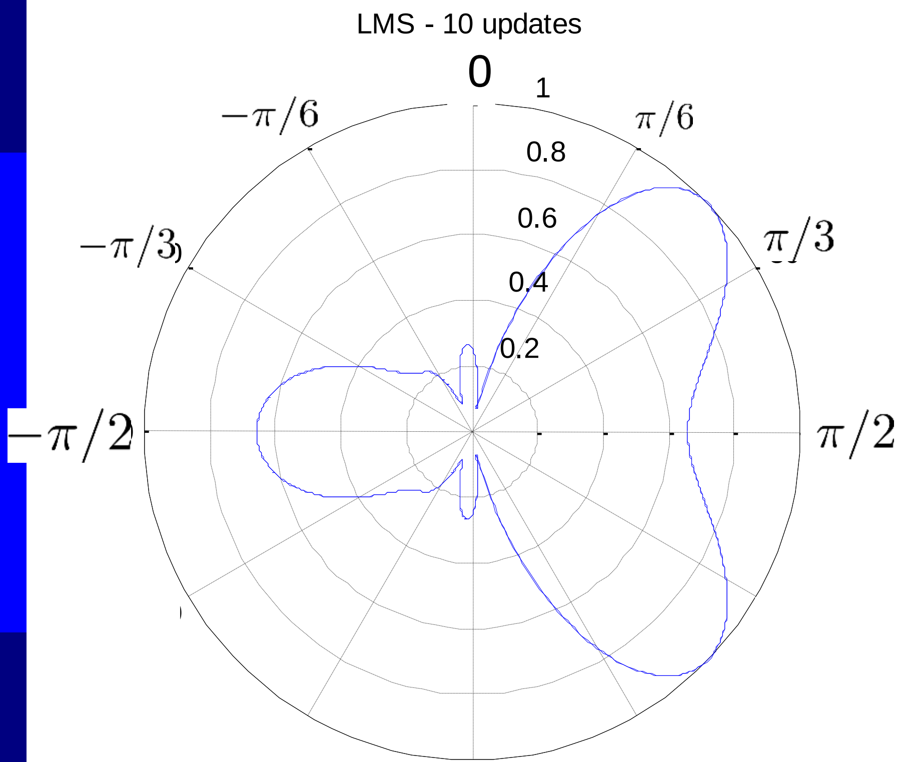
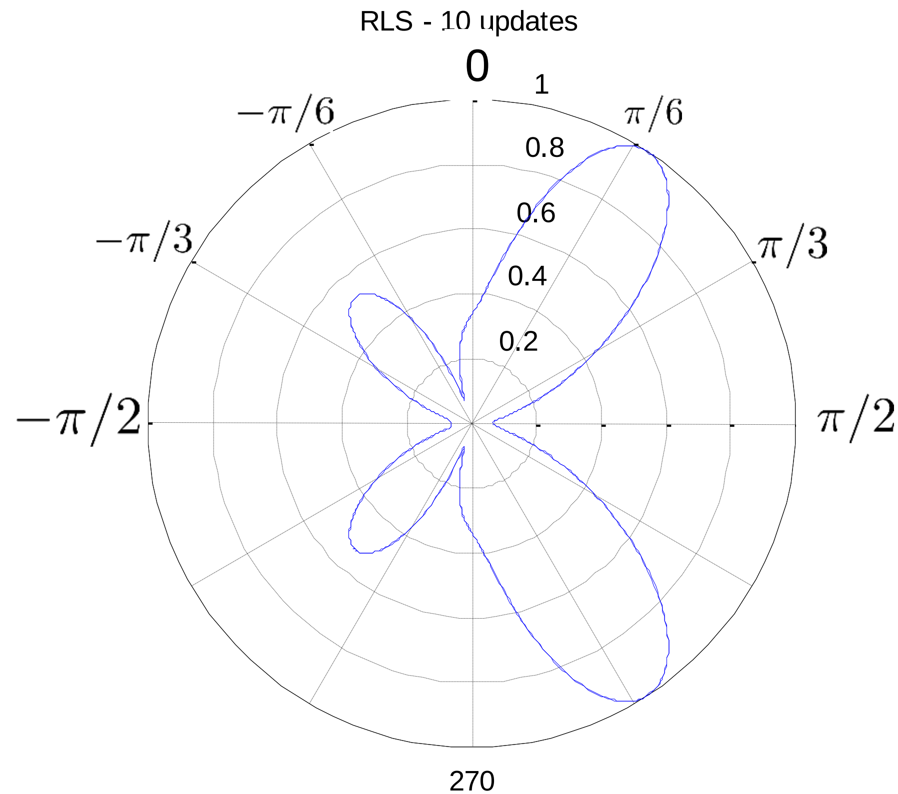
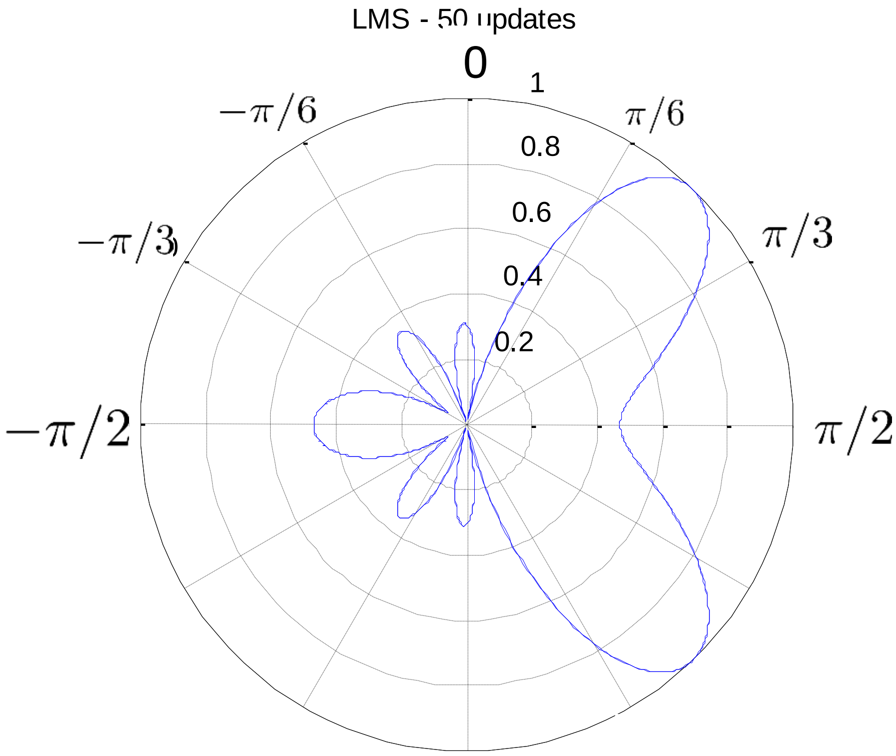
Comparison (10 updates)

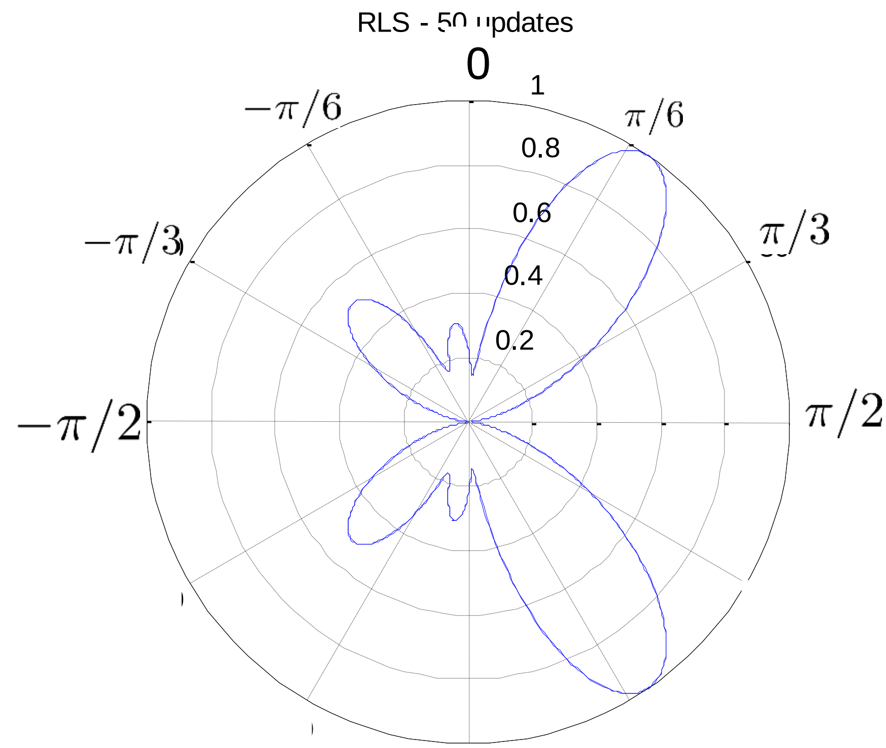




- $M_r = 4$; AOA = $\pi/4$ $\delta_s = -0.001$; $\chi = 0.999$
- Interferers from AOAs = $[0, \pi/2]$

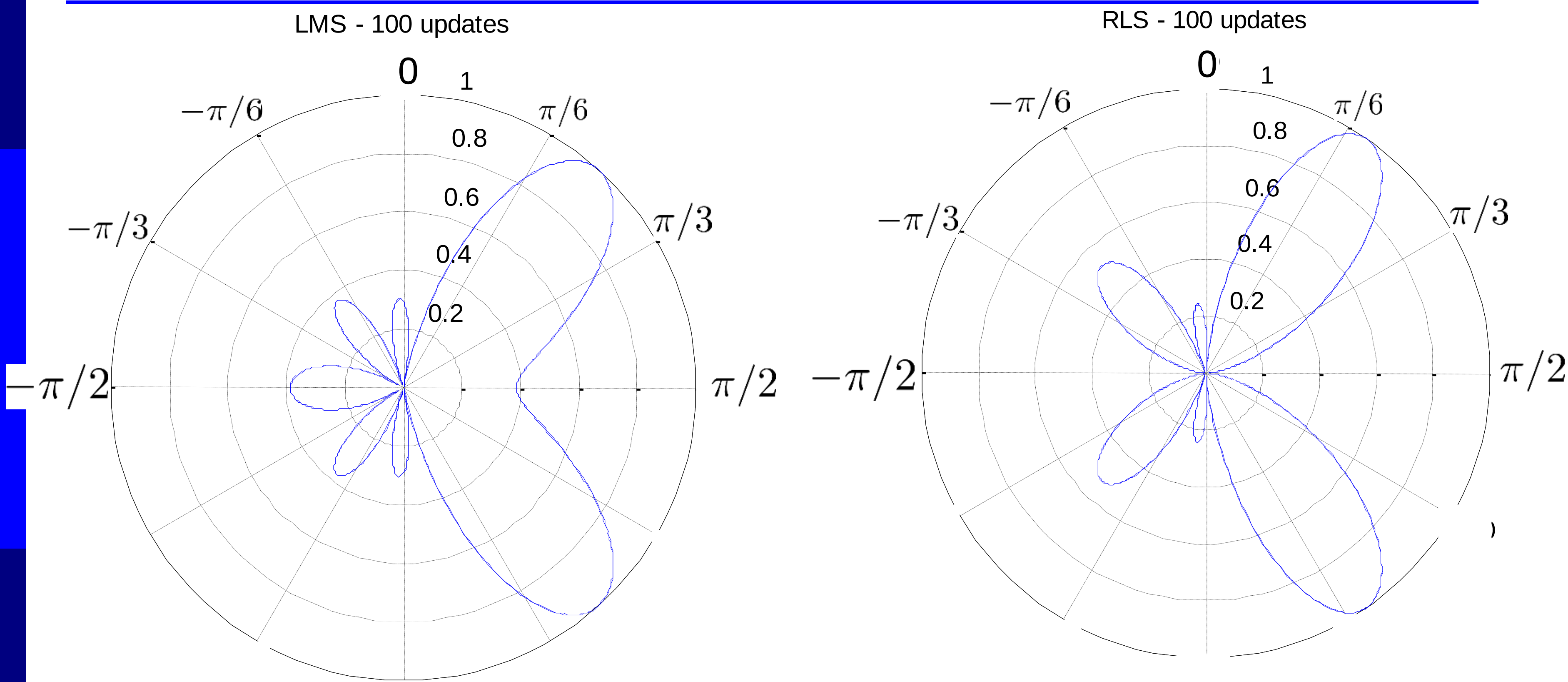
Comparison – cont. (50 updates)





- M_r = 4; AOA = 45^o; δ_s = -0.001; χ = 0.999
- Interferers from AOAs = [90^o ,0^o]

Comparison – cont. (100 updates)



- M_r = 4; AOA = $\pi/4$; δ_s = -0.001; χ = 0.999
- Interferers from AOAs = [$\pi/2$,0]

Conclusions

- Today we have examined several approaches to finding the “optimal” beamforming weights
- Although the we have done this within the context of classic beamforming (i.e., assuming plane wave propagation), the solutions actually are more general and also apply to any array including diversity arrays
  - Also apply to any array geometry
  - However, for diversity arrays there is no concept of a beampattern
- We have also looked at two well-known techniques for adapting the weights to achieve either the MMSE or LS weights