Multi-Channel Communications

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Part I

Preliminaries

Notation

The notation used in this manuscript is listed in the following tables. More specifically, Tables 1 and 2 describe the general notation used in this text. Tables 3 and 4 represent the English letter variables used in the text while Table 5 describe the Greek variables used in the manuscript.

Symbol	Meaning	Notes
$\tilde{x}(t)$	complex function	This notation will always indicate a complex function. However, at times we may have complex functions without this notation. Depends on context.
$\tilde{x}^*(t)$	complex conjugate	if $\tilde{x}(t) = a(t) + jb(t)$ then $\tilde{x}^*(t) = a(t) - jb(t)$
$E\{x(t)\}$ $E\{\mathbf{x}\}$	expectation of random process $x(t)$ expectation of random vector $\mathbf{x}$	If not clear from the context, the variable over which the expectation is taken will be made clear by using a subscript. e.g., $E_\lambda\{\mathbf{x}\}$ indicates expectation over λ
$\mathcal{F}\{x(t)\}$ $\mathcal{F}^{-1}\{X(f)\}$ $\mathcal{F}\{x[n]\}$ $\mathcal{F}^{-1}\{X[k]\}$	Fourier Transform of $x(t)$ Inverse Fourier Transform of $X(f)$ Discrete Transform of $x[n]$ Inverse Discrete Transform of $X[k]$	$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$ $x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$ $X[k] = \mathcal{F}\{x[n]\} = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$ $x[n] = \mathcal{F}^{-1}\{X[k]\} = \sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn}$ In matrix notation, $\mathbf{X} = \mathbf{D}_N\mathbf{x}$, where $\mathbf{D}_N$ defined below $\mathbf{x} = \mathbf{D}_N^H\mathbf{X}$ where $\mathbf{x}, \mathbf{X} \in \mathcal{C}^N$
$\mathbf{v}$	column vector	$\overline{\mathbf{v}}$ will also be used when bold isn't clear if size of vector isn't clear from context $\mathbf{v}^{a\times 1}$ will indicate the size
$[\mathbf{v}]_i$	ith element of vector	
$\mathbf{H}$	matrix	If size isn't clear from context, $\mathbf{H}^{m\times n}$ will be used to indicate an $m \times n$ matrix
$[\mathbf{H}]_{i,j}$	$\{i, j\}$th element of matrix $\mathbf{H}$	
$\Re^{m\times n}$	set of all $m \times n$ real matrices	
$\mathcal{C}^{m\times n}$	set of all $m \times n$ complex matrices	
$\mathbf{I}^{m\times m}$	$m \times m$ identity matrix	
$\mathbf{0}^{m\times n}$	$m \times n$ matrix of zeros	
$span(\mathcal{S})$	set of vectors spanned by $\mathcal{S}$	
$col(\mathbf{A})$	space spanned by columns of $\mathbf{A}$	column space of $\mathbf{A}$
$row(\mathbf{A})$	space spanned by rows of $\mathbf{A}$	row space of $\mathbf{A}$
$vec(\mathbf{A})$	stacked columns of matrix $\mathbf{A}$	converts an $m \times n$ matrix to an $mn \times 1$ vector
$\mathbf{u}^T$ $\mathbf{A}^T$	transpose of $\mathbf{u}$ transpose of $\mathbf{A}$	
$\lVert\mathbf{u}\rVert_p$	p-norm of $\mathbf{u}$	$\lVert\mathbf{u}\rVert$ is the 2-norm

Table 1: General Notation Used in the Class (Part 1 of 2)

$\mathbf{A}^*$ $\mathbf{A}^H$ $\mathbf{A}^{-H}$	element by element conjugate of $\mathbf{A}$ conjugate transpose of $\mathbf{A}$ conjugate transpose of $\mathbf{A^{-1}}$	 $\mathbf{A}^{-H} = (\mathbf{A}^{-1})^H$
$\mathbf{A}^\dagger$	pseudo-inverse of $\mathbf{A}$	$\mathbf{A}^\dagger = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T$
Symbol	Meaning	Notes
$\mathbf{A}^{1/2}$	matrix square root	
$trace(\mathbf{A})$	trace of $\mathbf{A}$	sum of diagonal elements
$R(\mathbf{A})$	rank of $\mathbf{A}$	
$row(\mathbf{A})$	row space of $\mathbf{A}$	
$col(\mathbf{A})$	column space of $\mathbf{A}$	
$dim(\mathbf{A})$	dimension of $\mathbf{A}$	number of vectors in any basis
$\lambda(\mathbf{A})$ $\sigma(\mathbf{A})$	eigenvalues of $\mathbf{A}$ singular values of $\mathbf{A}$	also called the spectrum of $\mathbf{A}$ λ_{min}, λ_{max} are min and max eigenvalues $\lambda_k(\mathbf{A})$ is the kth eigenvalue of $\mathbf{A}$ σ_{min}, σ_{max} are min and max singular values
$\|\mathbf{A}\|_F$	Frobenius norm of $\mathbf{A}$	
$\mathbf{A}^{-1}$	inverse of $\mathbf{A}$	
$\|\mathcal{V}\|$	cardinality of set $\mathcal{V}$	sets will generally use cal font
$\nabla f(\mathbf{x})$	gradient of function $f(\mathbf{x})$	argument is a vector $\mathbf{x}$; output is scalar

Table 2: General Notation Used in the Class (Part 2 of 2)

Symbol	Meaning	Notes
$\mathbf{a}(\theta)$ $\mathbf{A}(\bar{\theta})$	array factor Matrix of array factors	also array manifold vector $\mathbf{A}(\bar{\theta}) = [\mathbf{a}_1\mathbf{a}_2 \ldots \mathbf{a}_N]$
B	bandwidth	
C	capacity	
d D	distance Directivity	scalar
$\mathbf{D}_N$ $\mathbf{D}_{(LL)}$ $\mathbf{D}_{(NL)}$	DFT matrix, $[\mathbf{D}]_{m,n} = e^{-j2\pi(m-1)(n-1)/N}$ first L rows and columns of N-point DFT matrix ($L \times L$ matrix) first L columns DFT matrix ($N \times L$ matrix)	$\mathbf{D}_N^H$ is IDFT matrix $\mathbf{D}_N$ also known as Fourier matrix
f f_c f_d f_o f_{sa} Δf	frequency center (or carrier) frequency max Doppler frequency digital frequency offset sampling frequency sub-carrier spacing in OFDM	 $f_d = \frac{v}{\lambda}$ $f_o = f_{co}T_{eff}$ (f_{co} = cont. time) $f_{sa} = \frac{1}{T_{sa}}$
$f_X(x)$ $F_X(x)$	pdf of random variable X cdf of random variable X	
G_r	Receive Antenna Gain	scalar when in dB use $G_r(dB)$
G_t $G(\theta)$ $g_A(\theta)$	Transmit Antenna Gain Array Factor individual element pattern	scalar when in dB use $G_t(dB)$ gain pattern due to array gain pattern due to element
$h(\tau, t)$, $\mathbf{H}(\tau, t)$ $\tilde{h}(\tau, t)$ $\tilde{h}$ $H(f, t)$ $\mathbf{H}$ $\mathbf{H}_w$	channel impulse response time-varying frequency response channel matrix white channel matrix	if time-invariant, t is dropped if narrowband, τ is dropped can be scalar or vector or matrix if matrix, size is $M_r \times N_t$ complex baseband channel complex baseband channel sample $\tilde{h}$ and $\tilde{\alpha}$ are essentially the same $\mathcal{F}\{h(t, \tau)\}$ complex channel samples channel elements are iid complex GRV
$I(x; y)$	mutual information between x and y	
$J_o(\tau)$	zeroth-order Bessel function of the first kind	
L	diversity order	Number of branches in a diversity system (could be equal to M_r)
M_r	number of receive antennas	
n	path loss exponent	scalar

Table 3: Variables Used in the Class (Part 1 of 2)

Symbol	Meaning	Notes
n	noise (sample)	scalar or vector
$n(t)$	noise function	typically AWGN
$n_i(t)$	noise function on ith rx antenna	typically AWGN
N	number of components	typically multipath components
		Number of sub-carriers in OFDM
N_{cp}	samples in cyclic prefix	
N_s	number of signals in the environment	
N_t	number of transmit antennas	
P_r	Received Power	scalar
		when in dB use $P_r(dB)$
P_t	Transmit Power	scalar
		when in dB use $P_t(dB)$
$p(t)$	pulse shape	
$r(t)$	received signal with noise	r[k] is the sampled signal
$r_i(t)$	received signal with noise on ith antenna	
$\mathbf{r}(t)$	received signal on all antennas	if sampled, $\mathbf{r}$
$R(\tau)$	auto-correlation function of a random process	
$\mathbf{R}_{tx}$	transmit side correlation matrix	
	of the random channel vector	seen across the transmit antennas
$\mathbf{R}_{rx}$	receive side correlation matrix	
	of the random channel vector	seen across the receive antennas
$\mathbf{R}$	correlation matrix	$E\{vec(\mathbf{H})vec(\mathbf{H})^H\}$
R_s	symbol rate	
R_{ofdm}	OFDM symbol rate	
$s_{i,k}$	symbol transmitted from the ith antenna	
	during kth time slot	o
$S(f)$	Power Spectral Density of Random Process	Fourier Transform of $R(\tau)$
$S_A(\theta)$	Azimuthal Power Spectral Density	power per received angle
		$P_r = \int_{-\pi}^{\pi} S_A(\theta)G(\theta)d\theta$
		$G(\theta)$ is the receive antenna gain
$S_D(f)$	Doppler Spectrum	$\mathcal{F}\left\{\tilde{R}(\tau)\right\}$
t	time	
T_s	symbol duration	
T_{sa}	sample time/duration	$T_{sa} = \frac{1}{f_{sa}}$
T_o	OFDM symbol duration	$T_o = T_{eff} + T_{cp} + T_w$
T_{eff}	effective OFDM symbol duration	$T_{eff} = 1/\Delta f$
T_{cp}	cyclic prefix duration	
T_w	window duration	
$U[a,b]$	uniform distribution	range is [a,b]
v or $\mathbf{v}$	velocity	scalar or vector
w or $\mathbf{w}$	weights	scalar or vector
$x(t)$	transmit signal	
$x_i(t)$	transmit signal from ith antenna	
$\mathbf{x}(t)$	is transmit signal from multiple antennas	Will also use $\overline{x}(t)$ for this vector
z	decision statistic	after channel compensation

Table 4: Variables Used in the Class (Part 2 of 2)

Symbol	Meaning	Notes
α	magnitude of single channel	$\tilde{\alpha}$ is complex channel $\alpha e^{j\phi}$
		h will sometimes also be used for the single antenna channel
γ	SNR	
$\delta(t)$	impulse function	
Δ	distance	
ΔX	change in X	
ϵ	error	*e.g.* in the adaptation of antenna weights
η	decision statistic	before channel compensation ($z = \alpha^* \eta$)
ψ	general angle of an array	$\psi = \frac{2\pi d}{\lambda} \sin(\theta)$
λ	wavelength	scalar
λ	eigenvalue of a square matrix	context clarifies
μ	mean	
μ_i	ith moment about zero	
ν	discrete symbol time	represents discrete time
∇	gradient	$\hat{\nabla}$ is the estimated gradient
Ω_A	Solid Beam Angle	
ϕ	phase	
$\phi(\omega)$	Characteristic Function	$\mathrm{E}_\gamma \left\{ e^{j\omega\gamma} \right\}$
Φ	angle-of-departure	defined relative to array normal
$\Phi(s)$	Moment Generating Function	$\mathrm{E}_\gamma \left\{ e^{s\gamma} \right\}$
$\rho(d)$	spatial correlation	measured at distance d
θ	angle-of-arrival	defined relative to array normal
σ_n	noise variance	
σ_A	noise variance	
σ	singular value of a matrix	context clarifies
$\Sigma_{\mathbf{x}}$	covariance matrix of the vector $\mathbf{x}$	$\Sigma_{\mathbf{x}} = \mathrm{E}\left\{ \mathbf{x}\mathbf{x}^H \right\} - \mathrm{E}\left\{ \mathbf{x} \right\} \mathrm{E}\left\{ \mathbf{x} \right\}^H$
τ	delay	
ζ	element angle	defined for a circular array
ψ	array variable	$= \frac{2\pi d}{\lambda} \sin(\theta)$

Table 5: Greek Variables Used in the Class

Chapter 1

Matrix Theory: Basic Definitions

1.1 Introduction

Multi-channel communications by its nature involves the consideration of multiple channels simultaneously, whether they be in time, space or frequency. These channels may be separate at the time of transmission, but mixed upon reception. Or they may stay orthogonal throughout. In either case, vector and matrix notation is a convenient way to represent multiple channels both at the transmitter and receiver. Further, matrix theory is very convenient in describing the characteristics of these signals and the wireless channels that carry them. Thus, in this chapter we will provide a brief overview of vectors and matrices, with an emphasis on those concepts that are important to understanding MIMO and OFDM. For a more complete treatment of the topic of Linear Algebra and Matrix Analysis, please see [1]. In this section we will specifically discuss the following:

- Definitions
- Vector Spaces
- Functions of Matrices and Vectors
- Matrix Inversion
- Matrix Factorization (LU, LDU, Cholesky)
- Singular Value and Eigenvalue Decomposition

1.2 Basic Definitions

An n-element column vector is a group of n elements in vertical form:

$$\mathbf{v}^{n\times 1} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

An n-element row vector is a group of n elements in horizontal form:

$$\mathbf{v}^{1\times n} = \begin{bmatrix} v_1 & v_2 & v_3 & \cdots & v_n \end{bmatrix}$$

If not specified, assume that any lower case bold letter is a column vector and any upper case bold letter is a matrix.

A matrix $\mathbf{V}^{m\times n}$ is defined as an $m \times n$ array of mn elements:

$$\mathbf{V}^{m\times n} = \begin{bmatrix} v_{11} & v_{12} & v_{13} & \cdots & v_{1n} \\ v_{21} & v_{22} & v_{23} & \cdots & v_{2n} \\ v_{31} & v_{32} & v_{33} & \cdots & v_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{m1} & v_{m2} & v_{m3} & \cdots & v_{mn} \end{bmatrix}$$

$\Re^{m\times n}$ represents the set of all $m \times n$ real matrices.

$\mathcal{C}^{m\times n}$ is the set of all complex matrices of dimension $m \times n$. Note that $\mathcal{C} \in \{\Re^{m\times n} + j\Re^{m\times n}\}$.

Example 1.2.1. Consider modulation symbols s_i transmitted from four different transmitters using the same pulse shape and received at four different receivers over the same frequency band. If there is independent noise generated at each receiver, how can we use matrix notation to represent the matched filter outputs assuming independent channels h_{ij} between the ith receiver and jth transmitter? How would we represent the received signal if the transmit and receive frequency bands were mutually orthogonal?

Solution: We can write the transmitted symbols as a vector $\mathbf{s} = [s_1, s_2, s_3, s_4]^T$. The channel between four transmitters and five receivers can be written as a 4×4 matrix with h_{ij} as the (i, j)th element:

$$\mathbf{H} = \begin{bmatrix} h_{1,1} & h_{1,2} & h_{1,3} & h_{1,4} \\ h_{2,1} & h_{2,2} & h_{2,3} & h_{2,4} \\ h_{3,1} & h_{3,2} & h_{3,3} & h_{3,4} \\ h_{4,1} & h_{4,2} & h_{4,3} & h_{4,4} \end{bmatrix} \tag{1.1}$$

The matched filter outputs and the noise samples can both be written as 5×1 vectors $\tilde{\mathbf{r}}$ and $\mathbf{n}$ respectively. That gives the following representation:

$$\mathbf{r} = \mathbf{H}\mathbf{s} + \mathbf{n} \tag{1.2}$$

If the frequency bands are orthogonal, the equation would be the same but we would have a different definition for the channel matrix $\mathbf{H}$. Specifically, it would be diagonal:

$$\mathbf{H} = \begin{bmatrix} h_{1,1} & 0 & 0 & 0 \\ 0 & h_{2,2} & 0 & 0 \\ 0 & 0 & h_{3,3} & 0 \\ 0 & 0 & 0 & h_{4,4} \end{bmatrix} \tag{1.3}$$

1.3 Vector Measures

Two column vectors $\mathbf{u}$ and $\mathbf{v}$ (with equal length) are said to be **orthonormal** if

$$\mathbf{u}^T\mathbf{v} = 0$$

and

$$\mathbf{u}^T\mathbf{u} = \mathbf{v}^T\mathbf{v} = 1$$

The p-norm of a vector $\mathbf{v}$ is defined as

$$\|\mathbf{v}\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{1/p}$$

Example 1.3.1. Consider the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$. Are they orthonormal? How about $\mathbf{u}_2 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \end{bmatrix}$? What is the 2-norm of $\mathbf{v}_1$?

Solution: It is straightforward to show that $\mathbf{u}_1^T\mathbf{v}_1 = 0$. Thus, the two vectors are orthogonal. However, $||\mathbf{u}||_2 = 2$ and $||\mathbf{v}||_2 = 2$. Thus, neither vector has unit norm and the two vectors are NOT orthonormal.

On the other hand $\mathbf{u}_2^T\mathbf{v}_2 = 0$, $||\mathbf{u}_1||_2 = 1$ and $||\mathbf{v}_1||_2 = 1$. Thus $\mathbf{u}_2$ and $\mathbf{v}_2$ are orthonormal.

1.3.1 Vector Spaces

A set of vectors $\mathcal{V}$ is defined as a vector space over $\mathcal{F}$ when basic vector addition and scalar multiplication properties hold. The most fundamental are the closure properties:

- $\mathbf{x} + \mathbf{y} \in \mathcal{V}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$
- $\alpha\mathbf{x} \in \mathcal{V}$ for all $\alpha \in \mathcal{F}$ and $\mathbf{x} \in \mathcal{V}$.

Simple example: the set $\Re^n$ of real vectors of length n is a vector space over $\Re$.

A **subspace** is defined as a non-empty set of vectors taken from a vector space that also forms a separate vector space.

1.3.2 Linear Independence

A set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots \mathbf{v}_n\}$ is said to be **linearly independent** when the only set of coefficients α_i that satisfy the equation

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + ...\alpha_n\mathbf{v}_n = \mathbf{0}$$

is $\alpha_1 = \alpha_2 = \alpha_3 = \ldots \alpha_n = 0$.

Alternatively, a set of vectors is linearly independent if none of them can be written as a linear combination of finitely many other vectors in the collection.

A set of vectors which is not linearly independent is called **linearly dependent**.

1.3.3 Span

A set of vectors $\mathcal{S}$ is said to **span** the vector space $\mathcal{V}$ if every vector in $\mathcal{V}$ can be generated from a linear combination of the vectors in $\mathcal{S}$.

We denote this relationship by $\mathcal{V} = \mathit{span}(\mathcal{S})$ or $\mathcal{S}$ spans $\mathcal{V}$.

If a set of vectors $\mathcal{S}$ is a spanning set and is also **linearly independent** it is said to be a **basis** for the vector space $\mathcal{V}$.

Example 1.3.2. The unit vectors $\left\{ \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ span $\Re^3$.

That is, we can construct any vector in $\Re^3$ using a weighted sum of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$.

NOTE: These vectors also form a basis for $\Re^3$.

1.4 Matrices

1.4.1 Matrix Addition

Consider two matrices **A** and **B** which are $m \times n$ matrices. The sum of two matrices is defined as the sum of the individual elements:

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} & \cdots & A_{1n}+B_{1n} \\ A_{21}+B_{21} & A_{22}+B_{22} & \cdots & A_{2n}+B_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A_{m1}+B_{m1} & A_{m2}+B_{m2} & \cdots & A_{mn}+B_{mn} \end{bmatrix}_{m \times n}$$

1.4.2 Scalar Multiplication

When multiplying a matrix **A** by a scalar α the result is that every element in the matrix is multiplied by that scalar. In other words:

$$\alpha\mathbf{A} = \begin{bmatrix} \alpha A_{11} & \alpha A_{12} & \cdots & \alpha A_{1n} \\ \alpha A_{21} & \alpha A_{22} & \cdots & \alpha A_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha A_{m1} & \alpha A_{m2} & \cdots & \alpha A_{mn} \end{bmatrix}$$

1.4.3 Matrix Multiplication

The multiplication of two matrices requires that the number of columns of the first matrix is the same as the number of rows of the second matrix. Given the $m \times n$ matrix **A** and the $n \times k$ matrix **B**, the multiplication of the two matrices is defined as

$$\mathbf{AB} = \begin{bmatrix} \sum_{i=1}^{n} A_{1i}B_{i1} & \sum_{i=1}^{n} A_{1i}B_{i2} & \cdots & \sum_{i=1}^{n} A_{1i}B_{ik} \\ \sum_{i=1}^{n} A_{2i}B_{i1} & \sum_{i=1}^{n} A_{2i}B_{i2} & \cdots & \sum_{i=1}^{n} A_{2i}B_{ik} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{i=1}^{n} A_{mi}B_{i1} & \sum_{i=1}^{n} A_{mi}B_{i2} & \cdots & \sum_{i=1}^{n} A_{mi}B_{ik} \end{bmatrix}_{m \times k}$$

Note that in general, $\mathbf{AB} \neq \mathbf{BA}$. In fact, both products can only be formed when the two matrices are square with the same dimensions.

However, if **A**, **B** and **AB** are all symmetric and $\mathbf{A}, \mathbf{B}$,both are of the same dimensions, then $\mathbf{AB} = \mathbf{BA}$. Alternatively, if $\mathbf{A}, \mathbf{B}$ are of the same dimensions and diagonal, then $\mathbf{AB} = \mathbf{BA}$.

1.4.4 Matrix Transpose

The transpose of a matrix is the matrix obtained by interchanging the rows and columns of that matrix.

The transpose of the $m \times n$ matrix $\mathbf{A}$ is a $n \times m$ matrix denoted as $\mathbf{A}^T$. That is:

$$\mathbf{A}^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{m1} \\ A_{12} & A_{22} & \cdots & \alpha A_{m2} \\ \vdots & \vdots & \cdots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{mn} \end{bmatrix}_{n \times m}$$

From this definition, it should be clear that $\left(\mathbf{A}^T\right)^T = \mathbf{A}$.

1.4.5 Matrix Conjugate

The conjugate of a matrix is the element-by-element conjugate. We will denote the conjugate of the matrix $\mathbf{A}$ as $\mathbf{A}^*$:

$$\mathbf{A}^* = \begin{bmatrix} A^*_{11} & A^*_{12} & \cdots & A^*_{1n} \\ A^*_{21} & A^*_{22} & \cdots & A^*_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A^*_{m1} & A^*_{m2} & \cdots & A^*_{mn} \end{bmatrix}_{m \times n}$$

1.4.6 Matrix Hermitian

The conjugate transpose $\mathbf{A}^H$ (sometimes called the **Hermitian** of $\mathbf{A}$) is then:

$$\mathbf{A}^H = \begin{bmatrix} A^*_{11} & A^*_{21} & \cdots & A^*_{m1} \\ A^*_{12} & A^*_{22} & \cdots & \alpha A^*_{m2} \\ \vdots & \vdots & \cdots & \vdots \\ A^*_{1n} & A^*_{2n} & \cdots & A^*_{mn} \end{bmatrix}_{n \times m}$$

1.4.7 Matrix Definitions

A **symmetric matrix** is one for which $\mathbf{A} = \mathbf{A}^T$. A **Hermitian matrix** is one for which $\mathbf{A} = \mathbf{A}^H$. It is straightforward to show that

$$\begin{aligned} (\mathbf{AB})^T &= \mathbf{B}^T\mathbf{A}^T \\ (\mathbf{AB})^H &= \mathbf{B}^H\mathbf{A}^H \end{aligned}$$

Because of these facts, $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$ are symmetric matrices:

$$\left(\mathbf{A}^T\mathbf{A}\right)^T = \mathbf{A}^T(\mathbf{A}^T)^T = \mathbf{A}^T\mathbf{A}$$

Similarly, for a complex matrix $\mathbf{A}$, $\mathbf{A}^H\mathbf{A}$ is a Hermitian matrix.

Important Matrix Measures

The **trace** of an $n \times n$ matrix is the sum of its diagonal elements. That is

$$trace\,(\mathbf{A}) = A_{11} + A_{22} + ...A_{nn} = \sum_{i=1}^{n} A_{ii}$$

The **Frobenius norm** of a matrix is defined as

$$|\mathbf{A}|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m} |A_{ij}|^2}$$

Later we will also discuss the **rank** and **determinant** of a matrix.

The Matrix Inverse

For a square matrix $\mathbf{A}$ of size $n \times n$, the matrix $\mathbf{B}$ that satisfies the conditions

$$\mathbf{AB} = \mathbf{I}_n$$

and

$$\mathbf{BA} = \mathbf{I}_n$$

where $\mathbf{I}_n$ is an $n \times n$ identity matrix is called the **inverse** of $\mathbf{A}$, denoted $\mathbf{B} = \mathbf{A}^{-1}$.
For a matrix to be invertible it must be square, but not all square matrices are invertible.

A matrix that is not invertible is called a **singular matrix**. If it is invertible, it is **nonsingular**.

Properties of the Matrix Inverse

For two square nonsingular matrices $\mathbf{A}$ and $\mathbf{B}$ of the same dimensions:

- $\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$
- $\mathbf{AB}$ is also nonsingular
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- $\left(\mathbf{A}^{-1}\right)^{T} = \left(\mathbf{A}^{T}\right)^{-1}$

The Matrix Inversion Lemma

A very handy identity is the matrix inversion lemma (also known as the Woodberry matrix identity) which states

$$(\mathbf{A} + \mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\left(\mathbf{C}^{-1} + \mathbf{DA}^{-1}\mathbf{B}\right)^{-1}\mathbf{DA}^{-1} \tag{1.4}$$

Orthogonal Matrices

An orthogonal matrix is defined to be a real matrix $\mathbf{P}_{n\times n}$ whose columns (or rows) constitute an orthonormal basis for $\mathcal{R}^n$. Note that the following properties hold:

- $\mathbf{P}$ has orthonormal columns
- $\mathbf{P}$ has orthonormal rows
- $\mathbf{P}^T\mathbf{P} = \mathbf{P}\mathbf{P}^T = \mathbf{I}$
- $\mathbf{P}^{-1} = \mathbf{P}^T$
- $\|\mathbf{Px}\|_2 = \|\mathbf{x}\|_2$ for every $\mathbf{x} \in \mathcal{R}^n$

Unitary Matrices

A unitary matrix is defined to be a complex matrix $\mathbf{U}_{n\times n}$ whose columns (or rows) constitute an orthonormal basis for $\mathcal{C}^n$. Note that the following properties hold:

- $\mathbf{U}$ has orthonormal columns
- $\mathbf{U}$ has orthonormal rows
- $\mathbf{U}^H\mathbf{U} = \mathbf{U}\mathbf{U}^H = \mathbf{I}$
- $\mathbf{U}^{-1} = \mathbf{U}^H$
- $\|\mathbf{Ux}\|_2 = \|\mathbf{x}\|_2$ for every $\mathbf{x} \in \mathcal{C}^n$

Example 1.4.1. Consider the matrix $\mathbf{A} = \begin{bmatrix} 10 & 2 & 9 \\ -1 & 4 & 12 \\ 20 & 25 & -10 \end{bmatrix}$. What is $trace(\mathbf{A})$? What is $|\mathbf{A}|_F$? Is $\mathbf{A}$ orthogonal?

Solution: We can easily find that $trace(\mathbf{A}) = 4$ and $|\mathbf{A}|_F$ =38.4. Further, it is easy to see that $\mathbf{A}^T\mathbf{A} \neq \mathbf{I}$, thus $\mathbf{A}$ is not orthogonal.

Example 1.4.2. Consider the matrix $\mathbf{B} = \begin{bmatrix} -0.5 & -0.5 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0.5 & 0.5 & \sqrt{2}/2 \end{bmatrix}$.

What is $trace(\mathbf{B})$? What is $|\mathbf{B}|_F$? Is $\mathbf{B}$ orthogonal? What is the inverse of $\mathbf{B}$?

Solution: Calculating the trace we find that $trace(\mathbf{B}) = -0.5$ and $|\mathbf{B}|_F = 1.73$. Further, it is easy to show that $\mathbf{B}^T\mathbf{B} = \mathbf{I}$ and $\mathbf{B}\mathbf{B}^T = \mathbf{I}$. Also, since $\mathbf{B}$ is orthogonal,

$$\mathbf{B}^{-1} = \mathbf{B}^T = \begin{bmatrix} -0.5 & \sqrt{2}/2 & 0.5 \\ -0.5 & -\sqrt{2}/2 & 0.5 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix} \tag{1.5}$$

Null Space

The **null space** of an $m \times n$ matrix $\mathbf{A}$ is the set X of $n \times 1$ vectors $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$. The *left-hand null space* of $\mathbf{A}$ is the set of $m \times 1$ vectors y such that $\mathbf{A^T y} = \mathbf{0}$. In other words, the left-hand null space of $\mathbf{A}$ is the null space of $\mathbf{A}^T$.

Example 1.4.3. Consider the matrix $\mathbf{A} = \begin{bmatrix} 10 & 2 & 9 \\ -1 & 4 & 12 \\ 20 & 25 & -10 \end{bmatrix}$. Is the vector $\mathbf{v} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ in the null space of $\mathbf{A}$?

Solution: Taking $\mathbf{Av}$ we get

$$\mathbf{A} = \begin{bmatrix} 10 & 2 & 9 \\ -1 & 4 & 12 \\ 20 & 25 & -10 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 17 \\ 7 \\ -15 \end{bmatrix} \tag{1.6}$$

Thus, $\mathbf{v}$ is not in the null space of $\mathbf{A}$.

1.5 Matrix Factorization

1.5.1 The LU Factorization

If $\mathbf{A}$ is an $n \times n$ matrix that is non-singular and all leading principal submatrices $\mathbf{A}_k$ are nonsingular, $\mathbf{A}$ can be factored into the product $\mathbf{LU}$ where $\mathbf{L}$ is lower triangular (all zeros above the diagonal) and $\mathbf{U}$ is upper triangular (all zeros below the diagonal). That is

$$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} L_{11} & 0 & 0 & \cdots & 0 \\ L_{21} & L_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{n1} & L_{n2} & L_{n3} & \cdots & L_{nn} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} & \cdots & U_{1n} \\ 0 & U_{22} & U_{23} & \cdots & U_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & U_{nn} \end{bmatrix} \tag{1.7}$$

Note that $L_{ii} = 1\ \forall i$ and $U_{ii} \neq 0\ \forall i$.

1.5.2 The LDU Factorization

The **LU** factorization can be converted to a factorization with ones on the diagonals of both the lower and upper triangular matrices by factoring **U**:

$$\mathbf{U} = \begin{bmatrix} U_{11} & 0 & 0 & \cdots & 0 \\ 0 & U_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & U_{nn} \end{bmatrix} \begin{bmatrix} 1 & U_{12}/U_{11} & U_{13}/U_{11} & \cdots & U_{1n}/U_{11} \\ 0 & 1 & U_{23}/U_{22} & \cdots & U_{2n}/U_{22} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Thus, a matrix that can be factored in an **LU** factorization can also be factored as $\mathbf{A} = \mathbf{LDU}$ where **L** and **U** are upper and lower triangular matrices respectively with 1's on their diagonals and **D** is a diagonal matrix.

1.5.3 Cholesky Factorization

If an $n \times n$ matrix **A** that has an **LDU** factorization is symmetric, the **LDU** factorization is $\mathbf{A} = \mathbf{LDL}^T$. Such a matrix is called **positive definite** if all of the diagonal values of **D** are greater than zero.

A positive definite matrix can also be factored as $\mathbf{A} = \mathbf{R}^T\mathbf{R}$ where **R** is upper triangular with all positive diagonal entries. This factorization is known as the **Cholesky factorization** of **A**. Note that $\mathbf{R} = \mathbf{D}^{1/2}\mathbf{L}^T$.

Example 1.5.1. Consider the matrix $\mathbf{A} = \begin{bmatrix} 10 & 2 & 9 \\ -1 & 4 & 12 \\ 20 & 25 & -10 \end{bmatrix}$. Calculate an LU factorization of **A**. Determine the LDU factorization of **A**. **Solution:** Using Matlab, we can find the LU factorization:

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -0.1 & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix} \tag{1.8}$$

$$\mathbf{U} = \begin{bmatrix} 10 & 2 & 9 \\ 0 & 4.2 & 12.9 \\ 0 & 0 & -92.5 \end{bmatrix} \tag{1.9}$$

Now, the LDU factorization is similar, but we factor a diagonal out of $\mathbf{U}$.

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ -0.1 & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix} \tag{1.10}$$

$$\mathbf{U} = \begin{bmatrix} 1 & 0.2 & 0.9 \\ 0 & 1 & 3.1 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.11}$$

$$\mathbf{D} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 4.2 & 0 \\ 0 & 0 & -92.5 \end{bmatrix} \tag{1.12}$$

1.5.4 Rank of a Matrix

An important measure of a matrix is its **rank**.

There are many ways to define rank, but a basic definition is as follows: The rank of a matrix $\mathbf{A}$, denoted $R(\mathbf{A})$ the number of linearly independent rows of the matrix or the number of linearly independent columns of the matrix (note that they are the same).

Alternatively, the rank is the number of non-zero rows in the matrix after it has been transformed to row-echelon form.

Rank is also equal to the number of non-zero singular values (to be discussed shortly) of $\mathbf{A}$.

Notes on $R(\mathbf{A})$

- The rank of $\mathbf{A}$ is the dimension of the column space of $\mathbf{A}$ (which is the same as the dimension of the row space of $\mathbf{A}$).

- If $\mathbf{A}$ is a square matrix of size $n \times n$ and $R(\mathbf{A}) = n$, then we say that $\mathbf{A}$ has **full rank**.

- If the rank of an $m \times n$ matrix $\mathbf{A}$ is greater than or equal to n, the null space is empty.

- For a full-rank square matrix, the null space is empty.

- If $\mathbf{A}$ is an $m \times n$ matrix, then $dim(col(\mathbf{A})) + dim(nul(\mathbf{A})) = n$ where $dim(\mathcal{S})$ is the dimension (i..e, the number of vectors in any basis) of the space $\mathcal{S}$.

- If $\mathbf{A}$ is an $m \times n$ matrix, then $row(\mathbf{A}) = col(\mathbf{A}^T)$ and $col(\mathbf{A}) = row(\mathbf{A}^T)$.

- Note that for an $m \times n$ matrix $\mathbf{A}$,

$$R(\mathbf{A}) = R(\mathbf{A}^T) = R(\mathbf{A}\mathbf{A}^T) = R(\mathbf{A}^T\mathbf{A})$$

- For a square $n \times n$ matrix $\mathbf{A}$, it is non-singular only if the rank of $\mathbf{A}$ is n. In other words $R(\mathbf{A}) = n$ if $\mathbf{A}$ is non-singular.
- A diagonal matrix $\mathbf{A}$ which has non-zero values on the diagonal is full rank.
- It can be shown that $R(\mathbf{A}+\mathbf{B}) \leq R(\mathbf{A}) + R(\mathbf{B})$.
- Based on the preceding two properties adding a diagonal matrix to any matrix will make it full rank.

1.5.5 Row and Column Spaces

Rank can also be defined in terms of the row or column space of $\mathbf{A}$.

The **row space** of an $m \times n$ matrix, $\mathbf{A}$, denoted by $row(\mathbf{A})$ is a vector space which is the set of all linear combinations of the row vectors of $\mathbf{A}$.

The **column space** of an $m \times n$ matrix, $\mathbf{A}$, denoted by $col(\mathbf{A})$ is the set of all linear combinations of the column vectors of $\mathbf{A}$.

If $\mathbf{A}$ is an $m \times n$ real matrix, then $col(\mathbf{A})$ is a subspace of $\Re^m$ and $row(\mathbf{A})$ is a subspace of $\Re^n$.

More specifically, $col(\mathbf{A})$ is the span of the columns of $\mathbf{A}$ and $row(\mathbf{A})$ is the span of the rows of $\mathbf{A}$.

1.5.6 Singular Value Decomposition

For each $m \times n$ matrix $\mathbf{A}$, there are orthogonal matrices $\mathbf{U}^{m\times m}$ and $\mathbf{V}^{n\times n}$ and a diagonal matrix $\mathbf{D}^{r\times r} = diag\,(\sigma_1\sigma_2\ldots\sigma_r)$ such that

$$\mathbf{A} = \mathbf{U}\begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^T$$

The values $\sigma_1, \sigma_2, \ldots, \sigma_r$ are called the non-zero singular values of $\mathbf{A}$. When $r < p = \min\{m, n\}$, $\mathbf{A}$ also has $p - r$ zero singular values.

This factorization is known as the **singular value decomposition** of $\mathbf{A}$ and the columns in $\mathbf{U}$ and $\mathbf{V}$ are called the left and right singular vectors of $\mathbf{A}$.

Note that r is the *rank* of $\mathbf{A}$. Further, we will define the maximum singular value as σ_{max} and the minimum singular value as σ_{min}.

Note that $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices.

If $\mathbf{A}$ is $n \times n$ and nonsingular, we can write the SVD of $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T$$

and the SVD of the inverse of $\mathbf{A}$ can be written

$$\mathbf{A}^{-1} = \mathbf{V}\mathbf{D}^{-1}\mathbf{U}^T$$

The SVD provides valuable information about the matrix $\mathbf{A}$

- The unit vectors of $\mathbf{U}$ (i.e., the left singular vectors) that correspond to the non-zero singular values span the range of $\mathbf{A}$.

- The singular vectors corresponding to the zero singular values span the null space of $\mathbf{A}$.

Singular values define the matrix 2-norm. For an $m \times n$ matrix $\mathbf{A}$:

$$\|\mathbf{A}\|_2 = \max_{\|\mathbf{x}\|_2 = 1} \|\mathbf{A}\mathbf{x}\|_2$$

To interpret this, we say that for any $n \times 1$ vector $\mathbf{x}$ with a unit norm, the matrix norm of $\mathbf{A}$ is the maximum norm possible after transforming $\mathbf{x}$ by $\mathbf{A}$. Further it can be shown that

$$\max_{\|\mathbf{x}\|_2 = 1} \|\mathbf{A}\mathbf{x}\|_2 = \sigma_{max}.$$

Thus, $\|\mathbf{A}\|_2 = \sigma_{max}$

The *condition number* of a matrix κ is the ratio of the largest to smallest singular value of a matrix. That is

$$\kappa = \frac{\max_i \sigma_i}{\min_i \sigma_i}$$

1.5.7 The Pseudo-Inverse

The pseudo inverse (sometimes called the Moore-Penrose psuedoinverse) of an $m \times n$ matrix $\mathbf{A}$ with rank is $R(\mathbf{A}) = min(m, n)$ is defined as

$$\mathbf{A}^{\dagger} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T$$

If $R(\mathbf{A}) < min(m, n)$, then the pseudo inverse can be defined through the SVD of $\mathbf{A}$. Specifically, if $\mathbf{A} = \mathbf{U}\begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^T$ is the SVD of $\mathbf{A}$, then the pseudo inverse is defined as

$$\mathbf{A}^{\dagger} = \mathbf{V}\begin{pmatrix} \mathbf{D}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^T$$

Thus, when considering the psuedoinverse of $\mathbf{A}$, the singular values of the inverse are equal to $\frac{1}{\sigma_i}$ where σ_i are the non-zero singular values of $\mathbf{A}$.

Additionally, the right singular vectors of $\mathbf{A}^{\dagger}$ are the left singular vectors of $\mathbf{A}$ and vice-versa.

Example 1.5.2. Consider the matrix $\mathbf{A} = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 6 & 4 \\ 1 & 1 & 0 \end{bmatrix}$. Determine a SVD for $\mathbf{A}$. What does the SVD tell us?

Solution: The SVD is written as $\mathbf{UDV}^T = \mathbf{A}$. Using Matlab we can find one possible solution:

$$\begin{aligned} \mathbf{U} &= \begin{bmatrix} -0.54 & 0.75 & -0.37 \\ -0.83 & -0.55 & 0.09 \\ -0.13 & 0.36 & 0.92 \end{bmatrix} \\ \mathbf{D} &= \begin{bmatrix} 8.9 & 0 & 0 \\ 0 & 2.1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \mathbf{V} &= \begin{bmatrix} -0.38 & 0.72 & 0.58 \\ -0.82 & 0.03 & -0.58 \\ -0.43 & -0.69 & 0.58 \end{bmatrix} \end{aligned}$$

We can see that the rank of $\mathbf{A}$ is two and it is a singular matrix. There are only two modes, one much stronger than the other.

Example 1.5.3. Consider the matrix $\mathbf{B} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Determine a SVD for $\mathbf{B}$. What does the SVD tell us? Provide an SVD for $\mathbf{B}^{-1}$.

Solution: The SVD is written as $\mathbf{UDV}^T = \mathbf{B}$. Using Matlab we can find one possible solution:

$$
\begin{aligned}
\mathbf{U} &= \begin{bmatrix} -0.45 & 0.82 & -0.35 \\ -0.87 & -0.5 & -0.04 \\ -0.21 & 0.28 & 0.94 \end{bmatrix} \\
\mathbf{D} &= \begin{bmatrix} 7.3 & 0 & 0 \\ 0 & 2.1 & 0 \\ 0 & 0 & 0.58 \end{bmatrix} \\
\mathbf{V} &= \begin{bmatrix} -0.45 & 0.82 & -0.35 \\ -0.87 & -0.50 & -0.04 \\ -0.21 & 0.28 & 0.94 \end{bmatrix}
\end{aligned}
$$

We can see that the rank of $\mathbf{B}$ is three (i.e., it is non-singular). There are three modes, although the third is very weak. Further, we can find $\mathbf{B}^{-1} = \mathbf{VD}^{-1}\mathbf{U}^T$.

1.5.8 Eigenvalues and Eigenvectors

For an $n \times n$ matrix $\mathbf{A}$, the scalars λ and $n \times 1$ vectors $\mathbf{x} \neq \mathbf{0}$ satisfying

$$\mathbf{Ax} = \lambda\mathbf{x}$$

are called the **eigenvalues** and **eigenvectors** of $\mathbf{A}$. The set of distinct eigenvalues $\lambda(\mathbf{A})$ is known as the **spectrum** of $\mathbf{A}$.

Some additional notes on eigenvalues and eigenvalues:

* The absolute value of the maximum eigenvalue is also known as the **spectral radius** of $\mathbf{A}$.
* The eigenvectors corresponding to distinct eigenvalues is a linearly independent set.
* If no eigenvalue of $\mathbf{A}$ is repeated, then $\mathbf{A}$ is diagonalizable.
* Symmetric and Hermitian matrices have real eigenvalues.

1.5.9 Eigenvalues vs Singular Values

For $\mathbf{A} \in \mathcal{C}^{m\times n}$ with $R(\mathbf{A}) = r$:

* The nonzero eigen values of $\mathbf{A}^H\mathbf{A}$ and $\mathbf{AA}^H$ are equal and positive.
* The nonzero singular values of $\mathbf{A}$ are equal to the positive square roots of the non-zero eigenvalues of $\mathbf{A}^H\mathbf{A}$

1.5.10 Eigen Decomposition

A square $n \times n$ matrix $\mathbf{A}$ that has n linearly independent eigenvectors can be factorized into

$$\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^{-1} \tag{1.13}$$

where the i column of $\mathbf{Q}$ corresponds to the ith eigenvector of $\mathbf{A}$ and Λ is diagonal with the ith element of the diagonal equal to the ith eigenvalue of $\mathbf{A}$. If $\mathbf{A}$ is also symmetric, then

$$\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^T$$

Positive Definite Matrices

If $\lambda_i > 0 \forall i$, then we can set $\mathbf{B} = \Lambda^{1/2}\mathbf{Q}^T$ and

$$\mathbf{A} = \mathbf{B}^T\mathbf{B} \tag{1.14}$$

Such a matrix when $\mathbf{B}$ is non-singular is termed **positive definite**. In other words, consider the eigenpair $(\lambda, \mathbf{x})$:

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{1.15}$$

Multiplying both sides by $\mathbf{x}^T$ and solving for λ gives

$$\begin{aligned} \mathbf{x}^T\mathbf{A}\mathbf{x} &= \mathbf{x}^T\lambda\mathbf{x} \\ \lambda &= \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \end{aligned}$$

Continuing,

$$\begin{aligned} \lambda &= \frac{\mathbf{x}^T\mathbf{B}^T\mathbf{B}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \\ &= \frac{\|\mathbf{B}\mathbf{x}\|_2^2}{\|\mathbf{x}\|_2^2} \end{aligned}$$

If $\mathbf{B}$ is non-singular all eigenvalues will be positive and thus we say that $\mathbf{A}$ is **positive definite**.

However, if $\mathbf{B}$ is singular, all of the eigenvalues will be non-negative and we say that $\mathbf{A}$ is **positive semi-definite**.

Notes on Eigendecomposition

The matrix $\mathbf{B}$ in the previous decomposition is not unique.

However, there is only one upper triangular matrix with positive diagonals $\mathbf{R}$ such that $\mathbf{A} = \mathbf{R}^T\mathbf{R}$. This is known as the **Cholesky Factorization** of $\mathbf{A}$ as discussed above.

For a square matrix, the inverse of a matrix can be found from its eigendecomposition. Specifically,

$$\mathbf{A}^{-1} = \mathbf{Q}\Lambda^{-1}\mathbf{Q}^{-1}$$

Formal Definition of Positive Definiteness

If a matrix $\mathbf{A}$ is Hermitian, it is **positive definite** if for any vector $\mathbf{x} \neq \mathbf{0}$

$$\mathbf{x}^H \mathbf{A} \mathbf{x} > 0$$

$\mathbf{A}$ is **semipositive definite** if for any vector $\mathbf{x}$

$$\mathbf{x}^H \mathbf{A} \mathbf{x} \geq 0$$

$\mathbf{A}$ is **negative definite** if for any vector $\mathbf{x} \neq \mathbf{0}$

$$\mathbf{x}^H \mathbf{A} \mathbf{x} < 0$$

Note that $\mathbf{A}$ is positive definite if and only if all of its eigenvalues are positive.

Hermitian Matrices

If the $n \times n$ matrix $\mathbf{A}$ is Hermitian, its eigenvalues are real and we can decompose $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{U} \Lambda \mathbf{U}^H$$

where

$$\mathbf{U} = [\mathbf{u}_1 \mathbf{u}_2 \ldots \mathbf{u}_n]$$

are the orthogonal eigenvectors. This means that $\mathbf{U}$ is unitary. Further, the diagonal matrix Λ is made up of the eigenvalues of $\mathbf{A}$. Note that since $\mathbf{U}$ is unitary, we can also write:

$$\mathbf{U}^H \mathbf{A} \mathbf{U} = \Lambda$$

Further,

$$\mathbf{A} \mathbf{U} = \mathbf{U} \Lambda$$

which means that for any eigenvector $\mathbf{u}_i$ we can write

$$\mathbf{A} \mathbf{u}_i = \lambda_i \mathbf{u}_i \tag{1.16}$$

Interpreting Eigenvalues

The eigenvectors are somewhat arbitrary. Any orthonormal basis will suffice. Once the eigenvectors are chosen, the eigenvalues tell us the real information about the matrix.

To see this consider a symmetric $n \times n$ matrix $\mathbf{A}$. The eigendecomposition can be written as

$$\mathbf{A} = \mathbf{Q} \Lambda \mathbf{Q}^T$$

Now consider the matrix applied to an arbitrary vector $\mathbf{x}$:

$$\begin{aligned} \mathbf{y} &= \mathbf{A} \mathbf{x} \\ &= \mathbf{Q} \Lambda \mathbf{Q}^T \mathbf{x} \end{aligned}$$

Let us consider the multiplication steps one at a time. The product $\mathbf{z}_1 = \mathbf{Q}^T\mathbf{x}$ can be written as

$$\mathbf{z}_1 = \begin{bmatrix} \mathbf{q}_1^T\mathbf{x} \\ \mathbf{q}_2^T\mathbf{x} \\ \vdots \\ \mathbf{q}_n^T\mathbf{x} \end{bmatrix}$$

Thus, the ith element of $\mathbf{z}_1$ is the inner product of the input vector x with the ith eigenvector. We can think of this as correlation (how much of each eigenvector is in $\mathbf{x}$).

Next, we have

$$\begin{aligned} \mathbf{z}_2 &= \Lambda\mathbf{Q}^T\mathbf{x} \\ &= \lambda\mathbf{z}_1 \\ &= \begin{bmatrix} \lambda_1\mathbf{q}_1^T\mathbf{x} \\ \lambda_2\mathbf{q}_2^T\mathbf{x} \\ \vdots \\ \lambda_n\mathbf{q}_n^T\mathbf{x} \end{bmatrix} \end{aligned} \tag{1.17}$$

Thus, the next step is to weight each inner product with the appropriate eigenvalue. (We are weighting each correlation measure by their relative importance as determined by the eigenvalues.

Finally,

$$\begin{aligned} \mathbf{y} &= \mathbf{Q}\mathbf{z}_2 \\ &= \mathbf{Q}\begin{bmatrix} \lambda_1\mathbf{q}_1^T\mathbf{x} \\ \lambda_2\mathbf{q}_2^T\mathbf{x} \\ \vdots \\ \lambda_n\mathbf{q}_n^T\mathbf{x} \end{bmatrix} \\ &= \sum_{i=1}^{n}\lambda_i(\mathbf{q}_i^T\mathbf{x})\mathbf{q}_i \end{aligned}$$

Thus, we see that the output is the weighted sum of the eigenvectors. The weighting is the product of the correlation of the input with each eigenvector and the eigenvalue of interest. Thus, we can think of the matrix multiplication process as a decomposition, followed by a weighting, followed by a synthesis.

Example 1.5.4. What does the previous analysis tell us about the eigenvalues of an $n \times n$ identity matrix?

Example 1.5.5. Consider the matrix $\mathbf{B} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Determine an eigen decomposition for $\mathbf{B}$. What does the decomposition tell us? Compare to the SVD of this matrix. What does this suggest about the SVD and eigen decomposition of symmetric matrices?

Solution: The eigen decomposition is written as $\mathbf{Q\Lambda Q}^T = \mathbf{B}$. Using Matlab we can find one possible solution:

$$\begin{aligned} \mathbf{Q} &= \begin{bmatrix} -0.35 & 0.82 & 0.45 \\ -0.04 & -0.5 & 0.87 \\ 0.94 & 0.28 & 0.21 \end{bmatrix} \\ \mathbf{\Lambda} &= \begin{bmatrix} 0.58 & 0 & 0 \\ 0 & 2.1 & 0 \\ 0 & 0 & 7.3 \end{bmatrix} \end{aligned}$$

We can see that the eigenvalues are equal to the singular values. Further, we see from the SVD that $\mathbf{U} = \mathbf{V}$. Also, we can see that the eigen decomposition and the SVD are the same fo r$\mathbf{B}$. For symmetric matrices we can conclude that the two decompositions are the same.

Example 1.5.6. Consider the matrix $\mathbf{C} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 6 & -1 \\ -1 & -1 & -1 \end{bmatrix}$. Determine an eigen decomposition for $\mathbf{C}$. What does the decomposition tell us? Compare to the SVD of this matrix. What does this suggest about the SVD and eigen decomposition of symmetric matrices?

Solution: The eigen decomposition is written as $\mathbf{Q\Lambda Q}^T = \mathbf{B}$. Using Matlab we can find one possible solution:

$$\begin{aligned} \mathbf{Q} &= \begin{bmatrix} -0.35 & 0.82 & 0.45 \\ -0.04 & -0.5 & 0.87 \\ 0.94 & 0.28 & 0.21 \end{bmatrix} \\ \mathbf{\Lambda} &= \begin{bmatrix} 0.58 & 0 & 0 \\ 0 & 2.1 & 0 \\ 0 & 0 & 7.3 \end{bmatrix} \end{aligned}$$

We can see that the eigenvalues are equal to the singular values. Further, we see from the SVD that $\mathbf{U} = \mathbf{V}$. Also, we can see that the eigen decomposition and the SVD are the same fo r$\mathbf{B}$. For symmetric matrices we can conclude that the two decompositions are the same.

1.5.11 Range Space

The **range space** of an $m \times n$ matrix $\mathbf{A}$ is the set of $m \times 1$ vectors Y generated as $\mathbf{y} = \mathbf{A}\mathbf{x}$ when $\mathbf{x} \in \Re^n$.

The range space is denoted as $col(\mathbf{A})$ and is a subspace of $\Re^m$.

$col(\mathbf{A})$ is the space spanned by the columns of $\mathbf{A}$, also called the column space of $\mathbf{A}$.

$col(\mathbf{A}^T)$ is the space spanned by the rows of $\mathbf{A}$, also called the row space of $\mathbf{A}$, denoted as $row(\mathbf{A})$.

1.5.12 Using the Matrix Inverse

A system of n linear equations in n unknowns can be written as a single matrix equation as

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

where $\mathbf{x}$ is an n-dimensional vector of the n unknowns, $\mathbf{A}$ is the $n \times n$ matrix of coefficients, and $\mathbf{b}$ is an $n \times 1$ vector.

If the matrix $\mathbf{A}$ is nonsingular, the solution is given by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Note: In general, computing the inverse is a computationally intensive operation, and thus solving systems of equations are typically done in other ways.

1.5.13 DFT Matrix

The DFT matrix (also called the Fourier matrix), denoted $\mathbf{D}_N$, is an $N \times N$ matrix where the elements are defined as $[\mathbf{D}]_{m,n} = e^{-j2\pi(m-1)(n-1)/N}$. Note that multiplying an N-element column vector $\mathbf{x}$ by $\mathbf{D}_N$ creates the DFT of $\mathbf{x}$. That is the DFT of a vector $\mathbf{x}$ is defined as

$$X[k] = \mathcal{F}\left\{x[n]\right\} = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} \tag{1.18}$$

This can be written as

$$\mathbf{X} = \mathbf{D}_N\mathbf{x} \tag{1.19}$$

where $\mathbf{x}, \mathbf{X} \in \mathcal{C}^N$. Similarly, $\mathbf{D}_N^H$ is the IDFT matrix since $\mathbf{x} = \mathbf{D}_N^H\mathbf{X}$.

$\mathbf{D}_N^H$ is IDFT matrix

1.5.14 Circulant Matrices

A *circulant* matrix is a matrix entirely defined by any of its rows, as all other rows are simply circular shifts of that first row with offsets defined by the row indices. Alternatively a circulant matrix is described by any of its columns with all other columns simply circular shifts of the first column. Note that $\mathbf{D}_N$ is a circular matrix.

Note that if an $N \times N$ matrix $\mathbf{A}$ is circulant, the following important properties hold:

- ∗ The eigenvectors of $\mathbf{A}$ equal the columns of the DFT matrix $\mathbf{D}_N$.

- ∗ The eigenvalues of $\mathbf{A}$ equal the entries of $\mathbf{D}_N\mathbf{a}$ where $\mathbf{a}$ is any column of $\mathbf{A}$. In other words, the eigenvalues are the DFT of any of the columns.

1.5.15 The Kronecker Product

The Kronecker product is analogous to the outer product in vector multiplication. Specifically, if we have an $N \times M$ matrix $\mathbf{A}$ and a $K \times T$ $\mathbf{A}$ the Kronecker product is an $NK \times MT$ matrix $\mathbf{C}$:

$$\mathbf{C} = \mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} [\mathbf{A}]_{1,1}\mathbf{B} & \cdots & [\mathbf{A}]_{1,M}\mathbf{B} \\ \vdots & \ddots & \vdots \\ \mathbf{A}]_{N,1}\mathbf{B} & \cdots & [\mathbf{A}]_{N,M}\mathbf{B} \end{bmatrix} \tag{1.20}$$

Note that the Kronecker product as the following properties:

- ∗ $(\mathbf{A} \otimes \mathbf{B})^* = \mathbf{A}^* \otimes \mathbf{B}^*$

- ∗ $(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$

Part II

Multi-Input Multi-Output Transmission and Reception

Chapter 2

The MIMO Wireless Channel

2.1 Introduction

In this chapter, we describe the various effects of the wireless channel, with an emphasis on the *spatial* channel, since that is of primary importance in MIMO systems. We start by describing the three primary propagation effects: (1) the impact of distance, (2) the impact of motion, and (3) the impact of multipath. The first effect is primarily a large-scale effect (i.e., we notice its impact over large distances - say greater than 10m), while the second and third effects are small-scale effects (their effects are noticed over small distances - say smaller than a few meters). In a later chapter we will revisit the wireless channel with a focus on frequency selectivity as it is of primary importance when studying OFDM. For more complete treatments of the wireless channel, including the MIMO channel, please see [3], Chapter 3, [4], Chapter 3, as well as classic discussions of the SISO channel in [5, 6, 8].

2.2 Basic Propagation Effects

2.2.1 Impact of Distance

Consider a sinusoidal transmit signal at frequency f in a single dimension measured by a perfect antenna at a distance d from the transmitter. The measured electric field (ignoring polarization) is

$$\begin{aligned} E(f,t,d) &= \frac{\cos\left(2\pi \mathrm{f}(\mathrm{t}-\mathrm{d}/\mathrm{c})\right)}{d} \\ &= \frac{1}{d}\cos\left(2\pi \mathrm{ft}-\phi(\mathrm{d})\right) \end{aligned} \tag{2.1}$$

where $\phi(d) = \frac{2\pi f d}{c}$ and c is the speed of light.

From this equation, we can see that the sinusoidal field is attenuated and phase shifted depending on the distance d. The phase shift is of marginal consequence, but the attenuation due to distance is a major effect. This inverse relationship to distance results in a power loss relative to d^2. Specifically, it can be shown that the received signal power P_r in a free-space environment (i.e., in the absence of objects) can be written as

$$P_r = P_t G_t G_r \left(\frac{\lambda}{4\pi d}\right)^2 \tag{2.2}$$

where P_t is the transmit power, G_t and G_r are the transmit and receive antenna gains along the direction of the signal propagation, and $\lambda = c/f$ is the wavelength of transmission. This is known as the Friis Transmission Formula. From this equation we can see that the received signal power decreases with d^2. Additionally, for constant antenna gains, the received power also decreases with frequency.

The above received power formula applies only to free-space. However, it has been found empirically that in more complicated environments, the large scale path loss increases with d^n where n is the termed the path loss exponent and is generally $2 < n < 5$. In other words,

$$P_r = P_o \left(\frac{d_o}{d}\right)^n \tag{2.3}$$

where P_o is a reference power received at reference distance d_o. The reference distance is generally smaller than the distance being characterized d.

Path loss is a *large scale* effect in that it doesn't change significantly over small distances (say one meter). Changes in the received signal power over smaller distances (or time) is due to the impact of multipath and Doppler shift. These *small scale* effects will be discussed next.

2.2.2 Movement and Doppler Shift

Again consider a sinusoidal transmit signal, but with a mobile receiver[1]. Now if the receive antenna (starting from distance d_o) moves with velocity v away from the transmit antenna, the measured field is

$$\begin{aligned} E(f,t,d) &= \frac{\cos\left(2\pi \mathrm{f}\left(\mathrm{t} - \frac{\mathrm{d_o+vt}}{\mathrm{c}}\right)\right)}{d_o + vt} \\ &= \frac{1}{d_o + vt}\cos\left(2\pi(\mathrm{f}-\mathrm{f_d})\mathrm{t} - \phi_\mathrm{o}\right) \end{aligned} \tag{2.4}$$

where $f_d = \frac{fv}{c} = \frac{v}{\lambda}$ is the *Doppler Shift* and $\phi_o = \frac{2\pi f d_o}{c}$. For small changes in distance,

$$E(f,t,d) \approx \frac{1}{d_o}\cos\left(2\pi(\mathrm{f}-\mathrm{f_d})\mathrm{t} - \phi_\mathrm{o}\right) \tag{2.5}$$

And the primary impact that we observe due to movement is a Doppler shift. Note that in general, the shift in frequency due to movement is $f_i = f_d \cos(\theta)$ where $f_d = \frac{v}{\lambda}$ is the maximum Doppler shift and θ is the angle of arrival of the signal relative to the velocity. In the case above, $\theta = 180^o$ resulting in the largest negative shift of f_d. Had the receiver been moving *towards* the transmitter ($\theta = 0$), the frequency shift would have been positive f_d.

[1]Although we focus on a mobile receiver, a mobile transmitter and static receiver would cause the same effect.

2.2.3 Multipath

The third major effect due to the wireless channel is multipath. That is, because the transmitter and receiver are not located in free space, the receiver will observe multiple versions of the signal which have been reflected, scattered or diffracted by objects in the environment. The multiple copies of the signal will arrive at different relative delays and relative strengths. The difference in delay will cause two primary effects. If the delay is large relative to the modulated symbol time, the received signal will experience *inter-symbol interference or ISI.* We will discuss this effect in more detail later. Secondly, due to the fact that the carrier frequency is typically large (hundreds of MHz to tens of GHz), even small delay differences will cause substantial phase differences between the two sinusoids. The phase differences can (and will) cause destructive interference.

To see this, consider the situation depicted in Figure 2.1. In this example there is a perfect reflector located $d_r = 10$m from the transmitter which is transmitting a continuous tone signal at frequency f. We examine the magnitude of the received E-field at various distances d between the transmitter and receiver. The received E-field can be written as (note the negative sign due to the perfect reflector):

$$E(f,t,d) \quad = \quad \frac{\cos\left(2\pi \mathrm{f}\left(\mathrm{t}-\frac{\mathrm{d}}{\mathrm{c}}\right)\right)}{d} - \frac{\cos\left(2\pi \mathrm{f}\left(\mathrm{t}-\frac{2\mathrm{d_r}-\mathrm{d}}{\mathrm{c}}\right)\right)}{2d_r - d}$$

Thus, at the receiver we have the sum of two sinusoids with the same frequency, f, but different gains and phases. The phase difference is

$$\begin{aligned} \Delta\phi &= \frac{2\pi f(2d_r - d)}{c} + \pi - \frac{2\pi f d}{c} \\ &= \frac{4\pi f(d_r - d)}{c} + \pi \end{aligned} \tag{2.6}$$

Thus:

* When $\Delta\phi = 2k\pi$ for integer k, the two sinusoids will add *constructively* and the signal due to two paths is larger than without multipath.

* When $\Delta\phi = k\pi$ for odd integer k, the two signals add *destructively.*

* Thus a pattern of constructive/destructive interference is created that gradually attenuates due to longer distances. Specifically, the peak received magnitude will decay with $\frac{1}{d}$.

To make this more concreate, consider an example where $t = 0.01$, $f = 2GHz$, $c = 2.98 \times 10^8$ and $d_r = 10$m. The magnitude of the E-field is plotted in Figure 2.2. We can see that there are repeating patterns of constructive and destructive interference. Note that because the reflection is immediately behind the receiver, changing the distance dm causes a $2d/\lambda$ change in the phase difference. As a result, we see the pattern repeat at multiples of $\lambda/2 = 0.075$m in this case. Additionally, the overall received signal amplitude is decaying at $\frac{1}{d}$ as expected. (This corresponds to a power decay of $\frac{1}{d^2}$).

Of course, if there is multipath and receiver movement, the phase relationship of the two paths will vary with time due to the time varying phase of each multipath component (i.e., their respective frequency shifts).

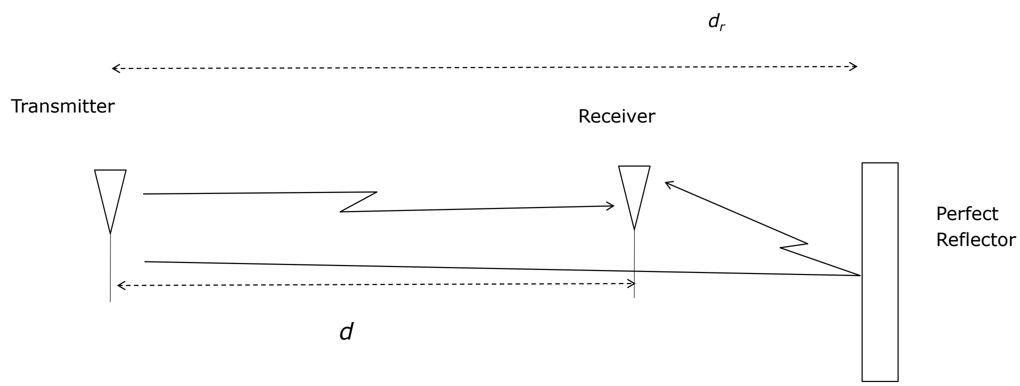

Figure 2.1: Example of Multipath with a Perfect Reflector

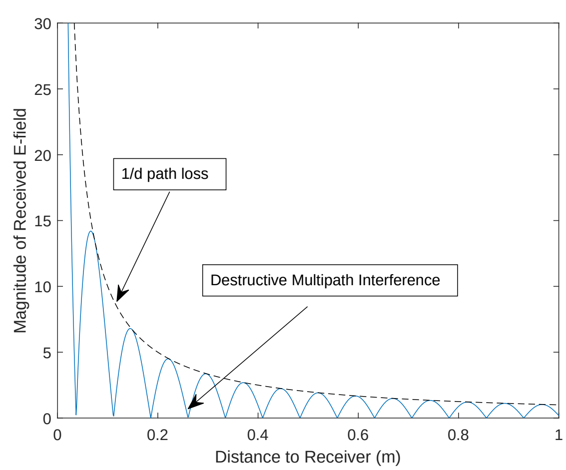

Figure 2.2: Example of Multipath Interference with a Perfect Reflector. Note that the electric field strength (plotted for a fixed time versus distance) decays with distance and experiences destructive multipath interference at multiples of half the wavelength.

2.2.4 Small and Large Scale Effects

The first propagation effect discussed (attenuation vs distance) is typically termed a *large scale effect* whereas the second and third propagation effects are termed *small scale effects*. The reason for this is that the attenuation over distance doesn't change appreciably over small relative distance changes, whereas the small scale effects do. For example consider Figure 2.3. The figure shows the magnitude of the received E-field for the example in Figure 2.2 when the receiver is moving towards the reflector at $v = 5$m/s and $d_r = 150m$. The overall attenuation due to distance is clearly seen in left hand plot which shows the received signal magnitude over 50m. However, the impact of multipath and Doppler is not visible. On the right, when the magnitude is viewed over a short distance (less than a meter) the attenuation due to distance is not discernible, but the multipath/Doppler effects clearly are. Thus, we refer to the effects which are discernible over large distances as large scale effects, and those which are discernible over short distances as small scale effects.

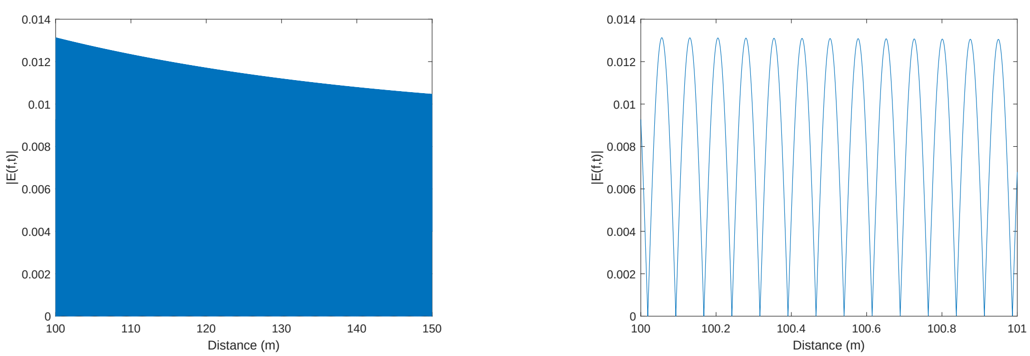

Figure 2.3: Large vs Small Scale Propagation Effects: The magnitude of the received E-field at a given distance when there is a perfect reflector placed at 150m from the transmitter and the receiver is traveling away from the transmitter (and towards the reflector) at 5m/s.

2.3 The General Wireless Channel in Complex Baseband

Thus, there are two dominant small-scale effects in the wireless channel: multipath and Doppler. In general, the received signal is a superposition of scaled and time-delayed versions of the transmit signal where the scaling, delay and number of multipath components is time-varying. This general scenario is shown in Figure 2.4. Together, multipath and Doppler result in the phenomenon known as *fading*, time-varying signal strength due to the changing phase relationships between multipath components.

At bandpass we can write the transmit and receive signals (ignoring noise):

$$x(t) = \Re\left\{\tilde{x}(t)e^{j2\pi f_c t}\right\} \tag{2.7}$$

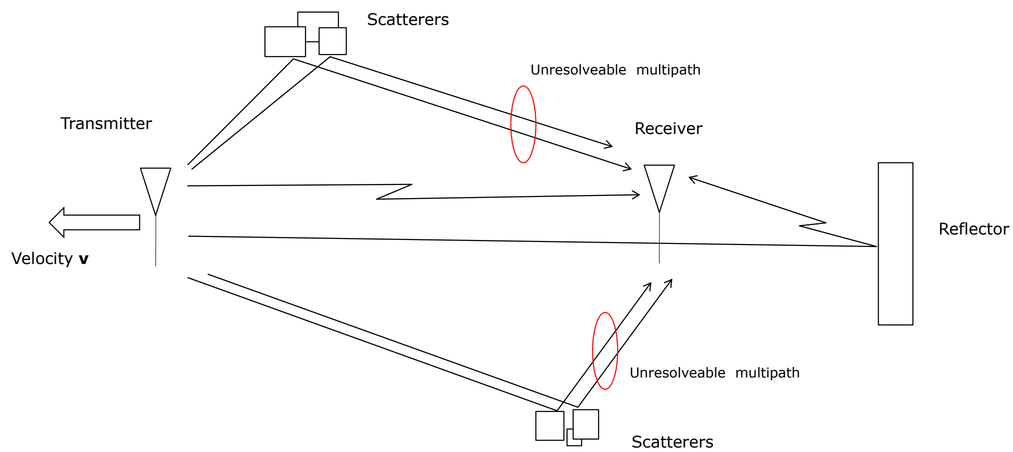

Figure 2.4: Depiction of the General Wireless Channel

$$r(t) = \Re\left\{\sum_{i=1}^{N(t)} \alpha_i(t)\tilde{x}(t-\tau_i(t))e^{j(2\pi f_c(t-\tau_i(t))+\phi_i)}\right\} \tag{2.8}$$

Let $\phi_i(t) = 2\pi f_c \tau_i(t) + \phi_i$. Then,

$$r(t) = \Re\left\{\left(\sum_{i=1}^{N(t)} \alpha_i(t)e^{j\phi_i(t)}\tilde{x}(t-\tau_i(t))\right)e^{j2\pi f_c t}\right\} \tag{2.9}$$

Assuming a linear, time-varying channel, from linear system theory we can write represent the channel by its time-varying impulse response. Thus, the complex baseband received signal $\tilde{r}(t)$ can be written as the convolution of the complex baseband of the input and channel impulse responses. Thus, the received signal we can written as:

$$r(t) = \Re\left\{\left[\underbrace{\int_{-\infty}^{\infty}\tilde{h}(\tau,t)\tilde{x}(t-\tau)d\tau}_{\tilde{r}(t)}\right]e^{j2\pi f_c t}\right\} \tag{2.10}$$

Which means that

$$\tilde{h}(\tau,t) = \sum_{i=1}^{N(t)} \tilde{\alpha}_i(t)\delta(\tau-\tau_i(t)) \tag{2.11}$$

is the time-varying impulse response of the channel in complex baseband and $\tilde{\alpha}_i(t) = \alpha_i(t)e^{j\phi_i(t)}$

is the complex channel gain of the ith multipath component. Further, the time-varying transfer function of the channel is

$$\tilde{H}(f,t) = \int_{-\infty}^{\infty} \tilde{h}(\tau,t)e^{-j2\pi f\tau}d\tau \tag{2.12}$$

2.4 Random Channels

In general it is impossible to know the channel $\tilde{h}(\tau,t)$ in advance or with certainty. Thus, when characterizing system performance we rely on statistical characterizations of the channel. There are three aspects of this statistical characterization which dictate link performance: delay spread, Doppler spread, and angle spread. Delay spread is a measure of the range of delays τ_i. If the delay spread is large (larger than the signal's modulated symbol duration T_s) inter-symbol interference occurs. In the frequency domain, we observe *frequency selective* fading since the channel transfer function is not flat over the signal bandwidth. In other words, the signal experiences different gains at different frequencies. This distortion is particularly harmful to signal detection. On the other hand if $\tau_i(t) << T_s$, $\forall i,t$, the fading experienced by the signal is termed *flat fading* and ISI is absent.

Doppler spread represents the range of Doppler shifts f_i experienced by the various multipath components. Clearly, from our previous discussion $-f_d < f_i < f_d$. Thus, the mobile velocity and transmit frequency dictate the Doppler spread. Doppler spread results in time-domain selectivity. If $f_d \propto \frac{1}{T_s}$ we term the fading *fast* fading. On the other hand, if $f_d << \frac{1}{T_s}$, the fading is termed *slow* fading.

Angle spread is a measure of the range of angles-of-arrival of the various multipath components. Clarke's model (which we will discuss next) assumes that the angle spread is 2π. Angle spread dictates spatial fading which we will discuss later in this chapter. In general, the larger the angle spread, the more spatially selective the channel is.

2.4.1 Spatial Selectivity and Correlation

Define the received signal power as a function of angle-of-arrival θ as the power spectrum $S_A(\theta)$. Then, assuming that all multipath components have equal gain, we can write the complex baseband received signal (without noise) in terms of the transmit signal $x(t)$ as

$$y(t) = \int_{-\pi}^{\pi} \sqrt{S_A(\theta)G(\theta)}x(t - D(\theta)/c)d\theta \tag{2.13}$$

where $G(\theta)$ is the normalized receive antenna gain and $D(\theta)$ is the delay due to the angle-of-arrival θ and c is the speed of light. Assuming the transmit signal is a sinusoid, at complex baseband the signal is a constant $x(t - D(\theta)/c) = xe^{-j\phi(\theta)}$. Further, assuming an omni-directional antenna with normalized gain pattern $G(\theta) = 1$

$$y(t) = y = h_o x \tag{2.14}$$

where h_o is the channel:

$$h_o = \int_{-\pi}^{\pi} \sqrt{S_A(\theta)}e^{-j\phi(\theta)}d\theta \tag{2.15}$$

Now, if we consider another point in space, d meters away, we find the channel to be

$$h_1 = \int_{-\pi}^{\pi} \sqrt{S_A(\theta)} e^{-j[\phi(\theta)+2\pi d/\lambda \sin(\theta)]} d\theta \tag{2.16}$$

where θ is defined relative to a line normal to the line connecting the two points[2] The correlation of the channels between those two points can be found as

$$\rho(d) = \mathrm{E}\left\{\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \sqrt{S_A(\theta_1)} e^{-j\phi(\theta_1)} \sqrt{S_A(\theta_2)} e^{j[\phi(\theta_2)+2\pi d/\lambda \sin(\theta_2)]} d\theta_1 d\theta_2\right\} \tag{2.17}$$

where the expectation is over $\phi(\theta)$. Making the reasonable assumption that $\phi(\theta)$ is uniformly distributed over $[-\pi, \pi)$, the expectation is zero unless $\theta_1 = \theta_2$., thus

$$\rho(d) = \int_{-\pi}^{\pi} S_A(\theta_1) e^{j2\pi d/\lambda \sin(\theta)} d\theta \tag{2.18}$$

Thus, we can see that the power azimuth spectrum directly controls the spatial correlation. We will show later that the larger the spread of the power azimuth spectrum, the faster the signal decorrelates in space.

2.4.2 Temporal Selectivity and Correlation

This result can also be used to determine the temporal correlation. Specifically, if we assume that the receiver is moving at velocity v, after τ seconds the receiver has moved $v\tau$ meters. Thus, using a simple substitution of variables we can calculate the temporal correlation $\tilde{R}(\tau)$ as

$$\begin{aligned} \tilde{R}(\tau) &= \int_{-\pi}^{\pi} S_A(\theta_1) e^{j2\pi \frac{v}{/\lambda} \tau \sin(\theta)} d\theta & (2.19) \\ &= \int_{-\pi}^{\pi} S_A(\theta_1) e^{j2\pi f_d \tau \sin(\theta)} d\theta & (2.20) \end{aligned}$$

We see that the temporal correlation at time delay τ depends directly on the maximum Doppler spread $f_d = \frac{v}{\lambda}$. Later we will see that, as one would expect, larger f_d leads to faster decorrelation in time.

2.4.3 Frequency Selectivity and Correlation

The power delay profile $\phi_h(\tau)$ is defined as the expected power of the channel impulse response as a function of delay τ:

$$\begin{aligned} \phi_h(\tau) &= \mathrm{E}\left\{\tilde{h}(\tau, t)\tilde{h}^*(\tau, t)\right\} & (2.21) \\ &= \mathrm{E}\left\{|\tilde{h}(\tau, t)|^2\right\} & (2.22) \end{aligned}$$

[2]It should be noted that this definition of the direction of arrival is somewhat different than the definition we used for determining the Doppler shift. In that case, the AOA was defined relative to the velocity.

Using this definition, we can write the frequency correlation (assuming wide sense stationarity) as

$$\begin{aligned} R_h(\Delta f) &= \mathrm{E}\left\{\tilde{H}(f,t)\tilde{H}^*(f+\Delta f,t)\right\} && (2.23)\\ &= \mathrm{E}\left\{\int_{-\infty}^{\infty}\tilde{h}(\tau,t)e^{-j2\pi f\tau}d\tau\int_{-\infty}^{\infty}\tilde{h}^*(\tau,t)e^{j2\pi(f+\Delta f)\tau}d\tau\right\} && (2.24)\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{E}\left\{h(\tau_1,t)h^*(\tau_2,t)\right\}e^{j2\pi[(f+\Delta f)\tau_2-f\tau_1]}d\tau_1 d\tau_2 && (2.25)\end{aligned}$$

Now, assuming uncorrelated scattering,

$$\mathrm{E}\left\{h(\tau_1,t)h^*(\tau_2,t)\right\}=0 \qquad \tau_1\neq\tau_2 \tag{2.26}$$

Thus,

$$\begin{aligned} R_h(\Delta f) &= \int_{-\infty}^{\infty}\mathrm{E}\left\{|h(\tau,t)|^2\right\}e^{j2\pi\Delta f\tau}d\tau && (2.27)\\ &= \int_{-\infty}^{\infty}\phi_h(\tau)e^{j2\pi\Delta f\tau}d\tau && (2.28)\end{aligned}$$

2.5 Narrowband Fading

In the early days of mobile communication, the signal bandwidth was fairly small. Thus, a common assumption in channel modeling was that the channel bandwidth was significantly larger than the signal bandwidth (i.e., fading was flat). As data rates increased, this assumption was no longer valid and the channel was observed to be frequency selective. Many developed channel models made the narrowband assumption. While, clearly that is not valid for modern communication systems, the advent of OFDM made the narrowband channel models relevant again. Specifically, since OFDM transmits a large number of narrowband signals, each narrowband signal will typically experience flat fading, and the narrowband assumption becomes valid. For the remainder of Part II, we will make the narrowband assumption in order to focus on the spatial effects of the channel which dominate MIMO link performance. We will however, revisit this narrowband assumption when we examine OFDM.

2.6 Clarke's Model for SISO Channels

Let us assume that the transmit signal is $x(t)=\cos\left(2\pi \mathrm{f_c t}\right)$. The received signal is comprised of a large number of multipath components arriving from different directions θ_i with respect to the velocity vector $\mathbf{v}$ and thus each has a different Doppler shift:

$$f_i=\frac{v}{\lambda}\cos\left(\theta_\mathrm{i}\right) \tag{2.29}$$

Clarke's Model [6, 8] makes a *Narrowband Assumption* that $\delta(\tau-\tau_i)=\delta(\tau-\tau_o)\forall i$ (see Figure 2.5). In other words, the difference in time-of-arrival of all multipath components is very small relative to the symbol duration and thus unresolvable. This is equivalent to the channel having a bandwidth that is much greater than the signal bandwidth (i.e., we assume that the signal is narrow relative to the channel).

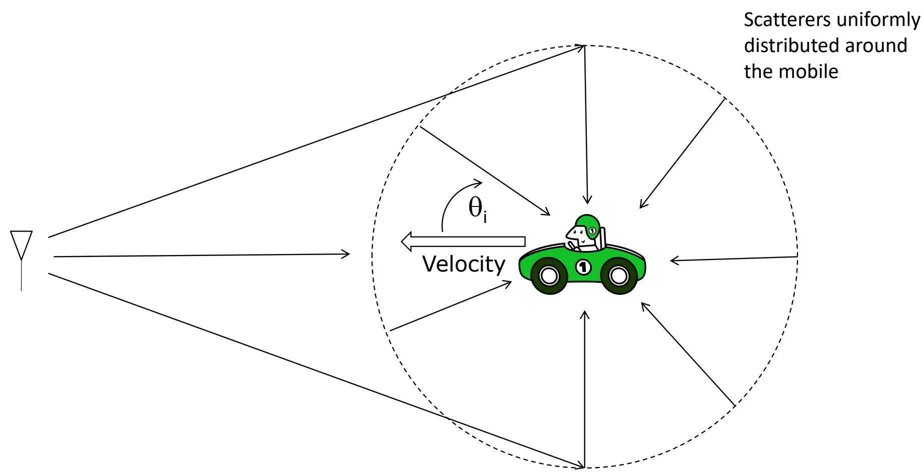

Figure 2.5: Scenario Assumed in Jakes/Clarke's Model. Scatterers causing multipath at the mobile receiver are uniformly distributed in angle around the mobile. θ_i is the angle-of-arrival of ith multipath component relative to the mobile velocity.

With a sinusoidal transmit signal (constant at baseband), we have

$$r(t) = \sqrt{2P_{avg}} \sum_{i=1}^{N} c_i \cos\left(2\pi \mathrm{f_c t} + 2\pi \mathrm{f_i t} + \phi_\mathrm{i}\right) \tag{2.30}$$

where the P_{avg} is the average received power, $\sum_i c_i^2 = 1$ and ϕ_i is assumed to be random and uniformly distributed over $[0, 2\pi)$ or deterministic and equally spaced over $[0, 2\pi)$. We can re-write this as

$$r(t) = \sqrt{P_{avg}} \sum_{i=1}^{N} \left[c_i \cos\left(2\pi \mathrm{f_i t} + \phi_\mathrm{i}\right)\cos\left(2\pi \mathrm{f_c t}\right) - \mathrm{c_i} \sin\left(2\pi \mathrm{f_i t} + \phi_\mathrm{i}\right)\sin\left(2\pi \mathrm{f_c t}\right)\right] \tag{2.31}$$

Thus, in complex baseband

$$\begin{aligned}
\tilde{r}(t) &= \sqrt{P_{avg}} \sum_{i=1}^{N} \left[c_i \cos\left(2\pi \mathrm{f_i t} + \phi_\mathrm{i}\right) + \mathrm{jc_i} \sin\left(2\pi \mathrm{f_i t} + \phi_\mathrm{i}\right)\right] \\
&= \sqrt{P_{avg}} \sum_{i=1}^{N} c_i e^{j(2\pi f_i t + \phi_i)} \\
&= \sqrt{P_{avg}} \sum_{i=1}^{N} c_i e^{j(2\pi f_d \cos(\theta_\mathrm{i})\mathrm{t} + \phi_\mathrm{i})}
\end{aligned} \tag{2.32}$$

where $f_d = \frac{v}{\lambda}$ is the maximum Doppler shift. Further, in this model we typically set $c_i = \frac{1}{\sqrt{N}}$.

From the above development we can see that $\tilde{h}(\tau, t) \approx \tilde{\gamma}(t)\delta(\tau)$ and thus $\tilde{r}(t) \approx \tilde{\gamma}(t)\tilde{x}(t)$.

2.6.1 Rayleigh and Rician Fading

From the Central Limit Thereom, as $N \to \infty$ the real ($\tilde{r}_I(t)$) and imaginary ($\tilde{r}_Q(t)$) portions of $\tilde{r}(t)$ tend towards Gaussian random processes. Further, since ϕ_i is uniformly distributed, they will be independent Gaussian random processes with equal power $P_{avg}/2$. In this case, the magnitude of the complex baseband channel $\alpha(t)$

$$\alpha(t) = \sqrt{\tilde{r}_I^2(t) + \tilde{r}_Q^2(t)} \tag{2.33}$$

will be *Rayleigh Distributed* and we have what is called the *Rayleigh Channel* or *Rayleigh Fading*. Specifically, if we let

$$\sigma^2 = \frac{P_{avg}}{2} = \mathrm{E}\left\{\tilde{r}_I^2(t)\right\} = \mathrm{E}\left\{\tilde{r}_Q^2(t)\right\} \tag{2.34}$$

we have the distribution of the envelope:

$$f_\alpha(x) = \begin{cases} \frac{x}{\sigma^2} exp\left(-\frac{x^2}{2\sigma^2}\right) & 0 \leq x \leq \infty \\ 0 & else \end{cases} \tag{2.35}$$

Note that we can write the impulse response for this model as

$$\tilde{h}(\tau, t) = \left\{\frac{1}{\sqrt{N}} \sum_{i=1}^{N} e^{-j(2\pi f_i t + \phi_i)}\right\} \delta(\tau) \tag{2.36}$$

which is a narrowband model (no frequency selectivity) but is also time-varying.

For a Rician channel, there is a direct LOS component. Specifically, if $\tilde{h}^{Ra}(t)$ is a unit power Rayleigh fading channel, then a Rician fading channel can be written as

$$\tilde{h}^{Ri}(t) = \sqrt{\frac{K}{K+1}} + \sqrt{\frac{1}{K+1}}\tilde{h}^{Ra}(t) \tag{2.37}$$

where $\tilde{h}^{Ri}(t)$ is a unit power Rician fading channel where the K-factor is the ratio of the power of the direct LOS component and the diffuse (Rayleigh) component. Note that $K \to \infty$ results in an AWGN channel and $K \to 0$ results in a Rayleigh channel. An example plot of Rayleigh and Ricean distributions is given in Figure 2.6 for a Ricean K-factor of $K = 10$. We can see that the probability of low amplitudes (close to zero) is much smaller for the Ricean channel due to the strong LOS component.

2.6.2 Temporal Correlation in Clarke's Model

We typically define the autocorrelation function of the received signal (assuming that the signal is wide sense stationary) as

$$R(\tau) = \mathrm{E}\left\{r(t)r(t+\tau)\right\} \tag{2.38}$$

or in complex baseband

$$\tilde{R}(\tau) = \mathrm{E}\left\{\tilde{r}(t)\tilde{r}^*(t+\tau)\right\} \tag{2.39}$$

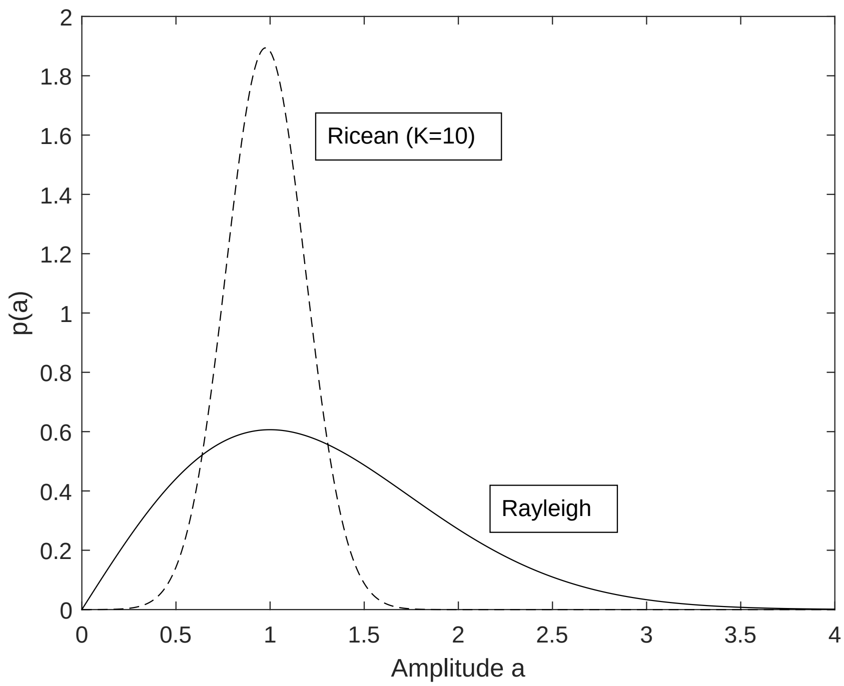

Figure 2.6: Rayleigh and Ricean (K=10) Probabilty Density Functions for Unit Average Energy.

where for Clarke's model, the expectation is over c_i, θ_i and ϕ_i. Assuming a sinusoidal transmit signal (to highlight the impact of the channel) we have

$$\begin{aligned}\tilde{R}(\tau) &= \mathrm{E}\left\{\tilde{r}(t)\tilde{r}(t+\tau)\right\} \\ &= \mathrm{E}\left\{\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N} e^{-j(2\pi f_i t+\phi_i)}\right)\left(\frac{1}{\sqrt{N}}\sum_{k=1}^{N} e^{j(2\pi f_k (t+\tau)+\phi_k)}\right)\right\} \\ &= \mathrm{E}\left\{\frac{1}{N}\sum_{i=1}^{N}\sum_{k=1}^{N} e^{-j(2\pi f_d \cos(\theta_\mathrm{i})\mathrm{t}-2\pi \mathrm{f_d} \cos(\theta_\mathrm{k})\mathrm{t}+\phi_\mathrm{i}-\phi_\mathrm{k})}\right\} \qquad (2.40)\end{aligned}$$

Taking the expectation over uniform distribution of θ_i, θ_k, ϕ_i and ϕ_k, we find that the expectation is zero for $i \neq k$. Thus, we have

$$\begin{aligned}\tilde{R}(\tau) &= \mathrm{E}\left\{\frac{1}{N}\sum_{i=1}^{N} e^{-j2\pi f_d \cos(\theta_\mathrm{i})\tau}\right\} \\ &= \frac{1}{N}\sum_{i=1}^{N}\int_0^{2\pi}\frac{1}{2\pi}e^{-j2\pi f_d \cos(\theta_\mathrm{i})\tau}d\theta_i \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-j2\pi f_d \cos(\theta)\tau}d\theta \\ &= J_o\left(2\pi f_d \tau\right) \qquad (2.41)\end{aligned}$$

where J_o is the zeroth order modified bessel function of the first kind. Note that all N integrals are the same since θ_i are i.i.d.

Alternatively, we can use the expression in () to determine temporal correlation using $S_A(\theta) = \frac{1}{2\pi}$:

$$\begin{aligned}\tilde{R}(\tau) &= \int_{-\pi}^{\pi}\frac{1}{2\pi}e^{j2\pi f_d \tau \sin(\theta)}d\theta \qquad (2.42) \\ &= J_o\left(2\pi f_d \tau\right) \qquad (2.43)\end{aligned}$$

This function is plotted versus τf_d in Figure 2.7. If the maximum Doppler frequency f_d increases, the temporal correlation will change faster. Alternatively, for a lower maximum Doppler frequency, the temporal correlation is higher for a specific delay.

2.6.3 Doppler Power Spectrum for Clarke's Model

The total power at the mobile can be written as the integral of the azimuthal power spectrum $S_A(\theta)$:

$$P_r = \int_{-\pi}^{\pi} S_A(\theta)G(\theta)d\theta \qquad (2.44)$$

where $G(\theta)$ is the receive antenna gain. We would like to translate this to Doppler Power Spectrum $S_D(f)$. Now $f = f_d \cos(\theta)$. Thus, two angles contribute to f, $\pm\theta$. What we want is

$$P_r = \int_{-f_d}^{f_d} S_D(f)df \qquad (2.45)$$

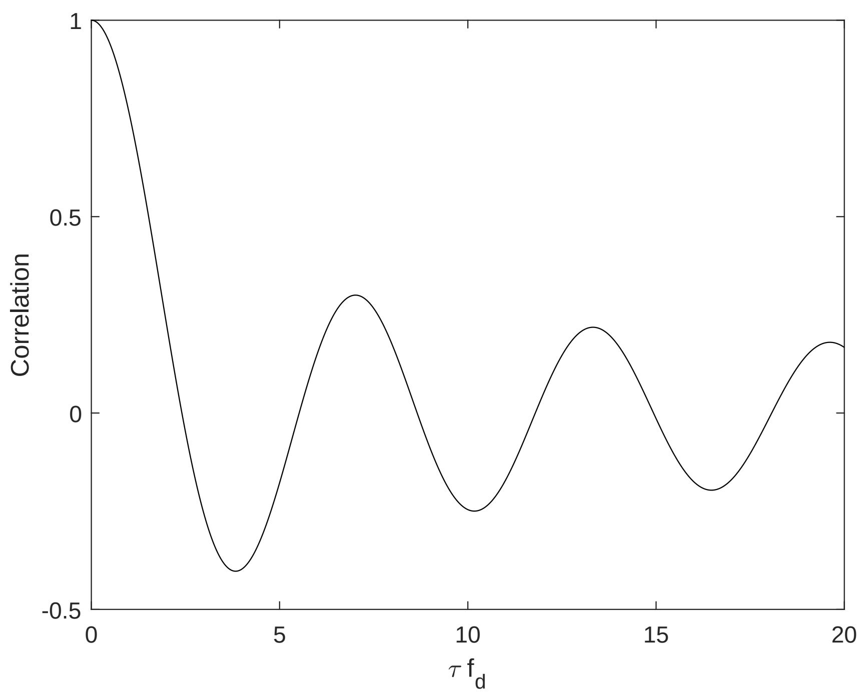

Figure 2.7: Temporal Correlation in Jakes/Clarkes Model. Note that the plot is normalized by the maximum Doppler frequency f_d.

Now, to convert from θ to f:

$$\theta = \cos^{-1}\left(\frac{\mathrm{f}}{\mathrm{f_d}}\right) \tag{2.46}$$

$$d\theta = -\frac{1}{f_d}\frac{1}{\sqrt{1-(f/f_d)^2}} \tag{2.47}$$

Thus,

$$\begin{aligned} P_r = & \int_{-f_d}^{f_d}\left\{S_A\left(\left|\cos^{-1}\left(\frac{\mathrm{f}}{\mathrm{f_d}}\right)\right|\right)G\left(\left|\cos^{-1}\left(\frac{\mathrm{f}}{\mathrm{f_d}}\right)^{-1}\right|\right)+\right. \\ & \left.S_A\left(-\left|\cos^{-1}\left(\frac{\mathrm{f}}{\mathrm{f_d}}\right)\right|\right)G\left(-\left|\cos^{-1}\left(\frac{\mathrm{f}}{\mathrm{f_d}}\right)\right|\right)\right\}\frac{-df}{f_d\sqrt{1-(f/f_d)^2}} \end{aligned} \tag{2.48}$$

Thus,

$$\begin{aligned} S_D(f) = & \frac{1}{f_d\sqrt{1-(f/f_d)^2}}S_A\left(\left|\cos^{-1}\left(\frac{\mathrm{f}}{\mathrm{f_d}}\right)\right|\right)G\left(\left|\cos^{-1}\left(\frac{\mathrm{f}}{\mathrm{f_d}}\right)^{-1}\right|\right)+\ldots \\ & S_A\left(-\left|\cos^{-1}\left(\frac{\mathrm{f}}{\mathrm{f_d}}\right)\right|\right)G\left(-\left|\cos^{-1}\left(\frac{\mathrm{f}}{\mathrm{f_d}}\right)\right|\right) \end{aligned} \tag{2.49}$$

For a uniform distribution in angle and a dipole antenna:

$$\begin{aligned} S_A(\theta) &= \frac{1}{2\pi} \\ G(\theta) &= \frac{3}{2} \end{aligned}$$

we have

$$S_D(f) = \begin{cases} \frac{3}{2\pi f_d\sqrt{1-(f/f_d)^2}} & |f| \le f_d \\ 0 & else \end{cases} \tag{2.50}$$

Finally, it can be shown that there is a close relationship between the Doppler spectrum and the correlation function. Specifically,

$$S_D(f) = \mathcal{F}\left\{\tilde{R}(\tau)\right\} \tag{2.51}$$

An example for $f_d = 100$Hz is plotted in Figure 2.8.

2.7 General Matrix Channel

Up to this point, we have considered a SISO channel (i.e., one transmit antenna and one receive antenna). Now, we will expand our discussion to a MIMO channel. Consider a set of N_t signals sent from N_t antennas written in vector complex baseband form as

$$\tilde{\mathbf{x}}(t) = \begin{bmatrix} \sum_{i=-\infty}^{\infty} s_{1,i}p(t-iT) \\ \sum_{i=-\infty}^{\infty} s_{2,i}p(t-iT) \\ \cdots \\ \sum_{i=-\infty}^{\infty} s_{N_t,i}p(t-iT) \end{bmatrix} \tag{2.52}$$

where $p(t)$ is the pulse shape used, $s_{j,i}$ is the ith symbol in time sent from the jth antenna. Let the channel between any transmit/receive pair be a slow, flat fading channel so that the received

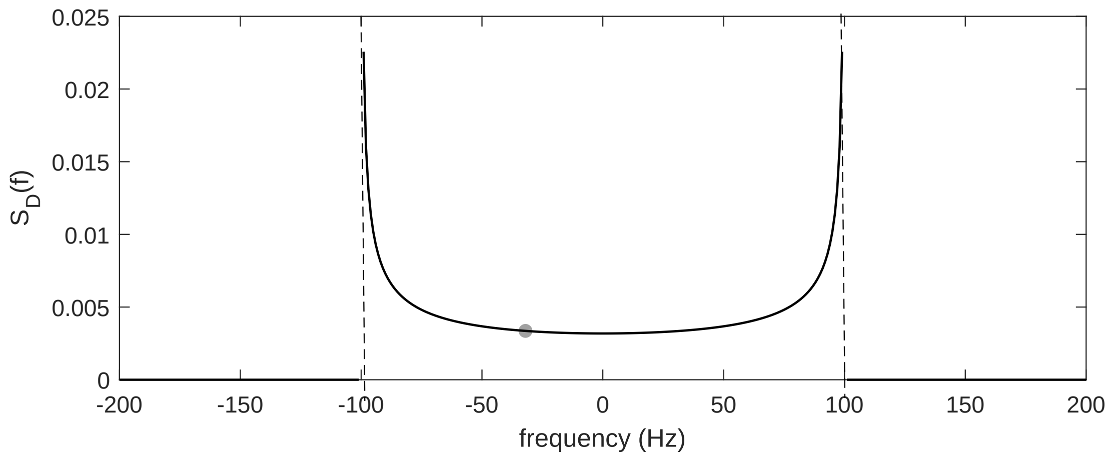

Figure 2.8: Doppler Spectrum in Jakes/Clarkes Model. Note that the plot assumes that the maximum Doppler frequency is f_d=100Hz.

signal on the ith receive antenna is

$$\tilde{r}_i(t) = \sum_{k=1}^{N_t} \int_{-\infty}^{\infty} \tilde{h}_{i,k}(\tau, t) x_k(t-\tau) d\tau + \tilde{n}_i(t) \tag{2.53}$$

$$= \sum_{k=1}^{N_t} \sum_{m=-\infty}^{\infty} \tilde{h}_{i,k}[mT] s_{k,i} p(t - mT) + \tilde{n}_i(t) \tag{2.54}$$

where $h_{i,k}(\tau, t)$ is the channel impulse response between the kth transmit antenna and the ith receive antenna. Our assumption that the channel is flat means that the transmit signal is not distorted and experiences a single time-varying gain. The assumption that the channel is slow means that the channel remains constant over a symbol. Thus, for any particular symbol, the received symbol on the i antenna from kth transmitter can be written as $h_{i,k}[mT]s_{k,m}p(t - mT)$. $\tilde{n}_i(t)$ is the noise process observed on the ith receive antenna (generally assumed to be white Gaussian noise).

The received signal during the mth time slot is passed through a matched filter and sampled:

$$r_i[m] = \int_{(m-1)T/2}^{mT/2} \tilde{r}_i(t) p(t - mT) dt \tag{2.55}$$

$$= \sum_{k=1}^{N_t} s_{k,m} \tilde{h}_{i,k}[m] + \tilde{n}_i[m] \tag{2.56}$$

where $x[n] = x(nT)$. Dropping the dependence on time,

$$\tilde{r}_i = \sum_{k=1}^{N_t} s_k \tilde{h}_{i,k} + \tilde{n}_i \tag{2.57}$$

$$= \tilde{\mathbf{h}}_i^T \mathbf{s} + \tilde{n}_i \tag{2.58}$$

where $\mathbf{s}$ is a vector of symbols transmitted from N_t antennas and $\tilde{\mathbf{h}}_i$ is the complex channel coefficients seen at the ith receiver from each of the N_t antennas. If there are M_r receive antennas, we can write the vector of received signals (after matched filtering and sampling) as:

$$\tilde{\mathbf{r}} = \tilde{\mathbf{H}}\mathbf{s} + \tilde{\mathbf{n}} \tag{2.59}$$

where $\tilde{\mathbf{r}}$ is the receive signal vector, $\mathbf{s}$ is the vector of transmit symbols, $\tilde{\mathbf{n}}$ is a vector of complex AWGN noise samples with $\mathrm{E}\left\{\tilde{\mathbf{n}}\right\} = \mathbf{0}^{M_r \times 1}$ and $\mathrm{E}\left\{\tilde{\mathbf{n}}\tilde{\mathbf{n}}^H\right\} = \sigma^2\mathbf{I}$, and $\tilde{\mathbf{H}}$ is the $M_r \times N_t$ channel matrix. $\tilde{\mathbf{H}}_{i,j}$ is the complex channel coefficient from the jth transmit antenna to the ith receive antenna.

If all transmit/receive channels are independent Rayleigh faded channels $\tilde{\mathbf{H}}$ is a matrix of independent complex Gaussian random variables, i.e., $\tilde{\mathbf{H}} = \tilde{\mathbf{H}}_w$.

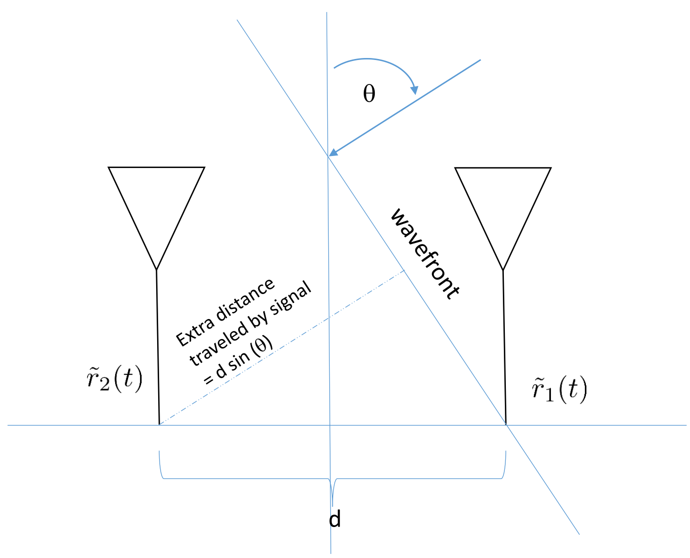

Figure 2.9: Wavefront impinging on two antennas (plane-wave assumption). Note that θ is defined here to be relative to the array normal.

2.8 Spatial Correlation

Consider a single multipath component received at two antennas $\tilde{r}_1(t)$ and $\tilde{r}_2(t)$ separated by distance d. The correlation between the two received signals can be written as

$$\rho(d) \quad = \quad \mathrm{E}\left\{\tilde{r}_1(t)\tilde{r}_2^*(t)\right\} \tag{2.60}$$

For a single multipath component arriving from angle θ (note that we have temporarily defined θ to be relative to the array normal as shown in Figure 2.9) we have

$$\tilde{r}_2(t) = \tilde{r}_1(t)e^{-j2\pi\frac{d}{\lambda}\sin(\theta)} \tag{2.61}$$

Thus, we have

$$\rho(d) = \mathrm{E}\left\{\tilde{r}_1(t)\tilde{r}_1^*(t)e^{j2\pi\frac{d}{\lambda}\sin(\theta)}\right\} \tag{2.62}$$

Clearly, the overall correlation will depend on the angular distribution of the arriving multipath. Assuming that the received signal is wide-sense stationary and the angle of arrival is uniformly distributed on $[0, 2\pi)$:

$$\begin{aligned}\rho(d) &= P_r\mathrm{E}\left\{e^{j2\pi\frac{d}{\lambda}\sin(\theta)}\right\} \\ &= \int_{-\pi}^{\pi}\frac{1}{2\pi}e^{j2\pi\frac{d}{\lambda}\sin(\phi)}d\phi \\ &= J_o\left(\frac{2\pi d}{\lambda}\right) \end{aligned} \tag{2.63}$$

NOTE: The imaginary part is zero. We could have also arrived at this result by using $S_A(\theta) = \frac{1}{2\pi}$ in (2.18).

Now, consider a Gaussian distribution for $S_A(\theta)$ with central angle θ_o

$$\begin{aligned}\rho(d) &= \int_{-\pi}^{\pi}S_A(\theta)e^{j2\pi d/\lambda \sin(\theta)}d\theta \\ &= \int_{-\pi}^{\pi}e^{j2\pi\frac{d}{\lambda}\sin(\theta)}\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(\theta-\theta_o)^2}{2\sigma^2}\right)d\theta\end{aligned}$$

Let $z = \frac{\theta-\theta_o}{\sigma}$ and $dz = \frac{d\theta}{\sigma}$.

$$\begin{aligned}\rho(d) &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}e^{j2\pi\frac{d}{\lambda}\sin(\sigma z+\theta_o)}\exp\left(-\frac{z^2}{2}\right)dz \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}e^{j2\pi\frac{d}{\lambda}(\sin(\sigma z)\cos(\theta_o)+\cos(\sigma z)\sin(\theta_o))}\exp\left(-\frac{z^2}{2}\right)dz\end{aligned}$$

Assuming relatively small angle spread σz will be small where $\exp -z^2/2$ is significant. Thus, we can make the approximation

$$\begin{aligned}\sin(\sigma \mathrm{z}) &\approx \sigma z \\ \cos(\sigma \mathrm{z}) &\approx 1\end{aligned}$$

Thus,

$$\begin{aligned}\rho(d) &\approx \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}e^{j2\pi\frac{d}{\lambda}(\sigma z\cos(\theta_o)+\sin(\theta_o))}\exp\left(-\frac{z^2}{2}\right)dz \\ &= \frac{e^{j2\pi\frac{d}{\lambda}\sin(\theta_o)}}{\sqrt{2\pi}}\int_{-\pi}^{\pi}e^{j2\pi\frac{d}{\lambda}\sigma z\cos(\theta_o)}\exp\left(-\frac{z^2}{2}\right)dz \qquad (2.64) \\ &= e^{j2\pi\frac{d}{\lambda}\sin(\theta_o)}e^{-\left(\frac{\left(2\pi\frac{d}{\lambda}\sigma\cos(\theta_o)\right)^2}{2}\right)} \qquad (2.65)\end{aligned}$$

As an example, consider $\sigma = 5^o$ with a central AOA of zero. (In the equation above, σ is defined in radians. For this example $\sigma = 0.0873$ radians.) The spatial correlation is clearly much higher (for the same separation distance) for the Gaussian distribution. In general, we can show an inverse relationship between angular spread (some measure of the width of the angle power distribution) and and spatial correlation. That is, for a given separation distance, a smaller angular spread results in higher correlation.

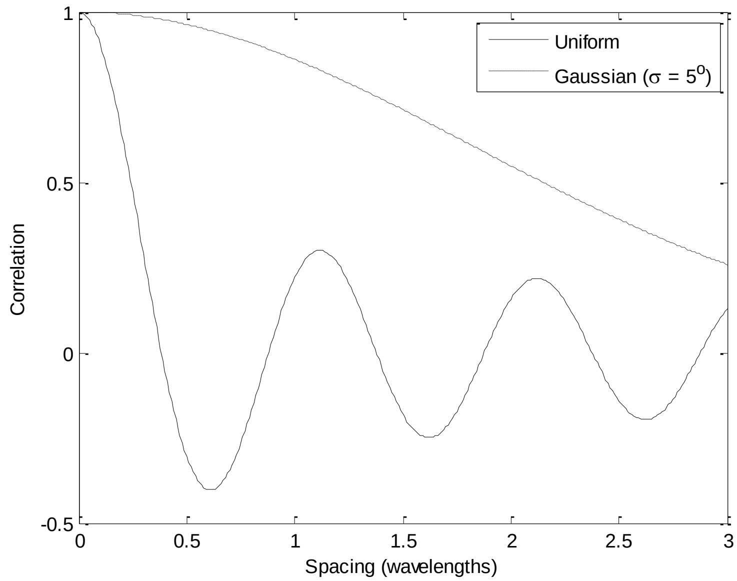

Figure 2.10: Spatial Correlation for Uniform Distribution and Gaussian Distribution ($\sigma = 5^o$)

2.9 Matrix (Spatial) Version of Clarke's Model

Clarke's narrowband SISO model discussed earlier can be extended to a MIMO channel. Consider the scenario depicted in Figure 2.11. If we consider a sinusoidal transmit signal and examine one multipath component from the first transmit antenna to the first receive antenna:

$$\tilde{r}_{11}(t) = e^{j(2\pi f_i t + \phi_i)} \tag{2.66}$$

where $f_i = \frac{v}{\lambda}\cos(\theta_{\rm i})$ assuming that the array baseline is perpendicular to the velocity vector and that the receiver is mobile (not the transmitter). Note that if the velocity were due to the transmitter movement, the angle-of-arrival θ_i should be replaced by the angle of departure at the transmitter, Φ_i. If we have a uniformly spaced linear array, the received signal at the other antennas can be written:

$$\begin{aligned} \tilde{r}_{21}(t) &= e^{j(2\pi f_i t + \phi_i - 2\pi \frac{d}{\lambda}\sin(\theta_{\rm i}))} \\ \tilde{r}_{31}(t) &= e^{j(2\pi f_i t + \phi_i - 2\pi \frac{2d}{\lambda}\sin(\theta_{\rm i}))} \\ &\vdots \\ \tilde{r}_{M_r1}(t) &= e^{j(2\pi f_i t + \phi_i - 2\pi \frac{(M_r-1)d}{\lambda}\sin(\theta_{\rm i}))} \end{aligned} \tag{2.67}$$

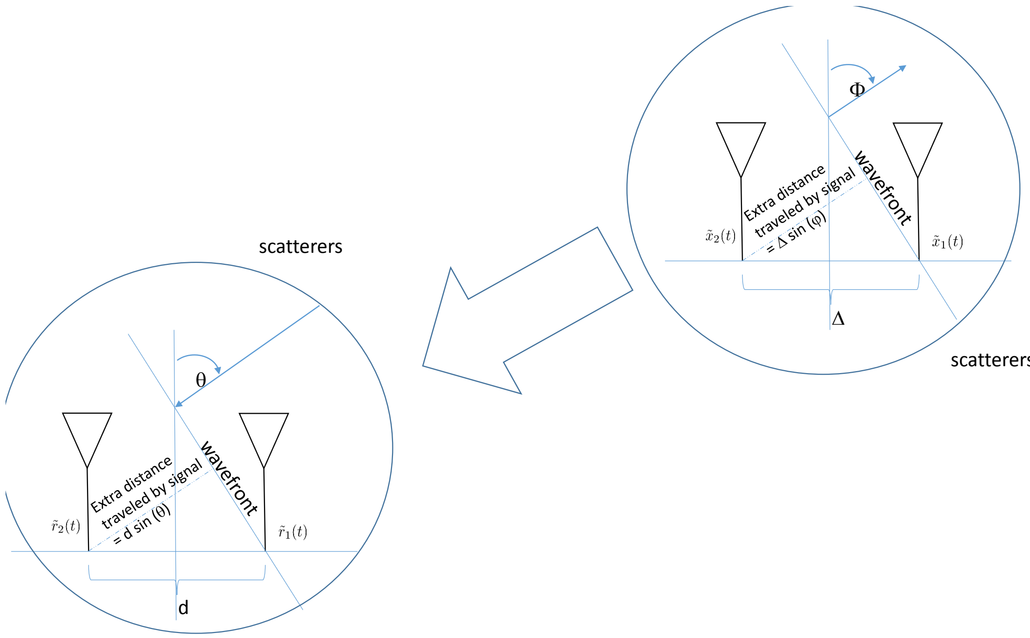

Figure 2.11: Assumed Geometry for Spatial Jakes Model

Similarly, we can write the signal from the multiple transmit antennas (assuming a uniform linear array with spacing Δ) and a departure angle Φ_i seen at first receive antenna:

$$\begin{aligned}
\tilde{r}_{12}(t) &= e^{j(2\pi f_i t + \phi_i - 2\pi \frac{\Delta}{\lambda}\sin(\Phi_\mathrm{i}))} \\
\tilde{r}_{13}(t) &= e^{j(2\pi f_i t + \phi_i - 2\pi \frac{2\Delta}{\lambda}\sin(\Phi_\mathrm{i}))} \\
&\vdots \\
\tilde{r}_{1N_t}(t) &= e^{j(2\pi f_i t + \phi_i - 2\pi \frac{(N_t-1)\Delta}{\lambda}\sin(\Phi_\mathrm{i}))}
\end{aligned} \tag{2.68}$$

In general,

$$x_{ab}(t) = e^{j\left(2\pi f_i t + \theta_i - \frac{2\pi}{\lambda}(a-1)d\sin(\phi_\mathrm{i}) - \frac{2\pi}{\lambda}(\mathrm{b}-1)\Delta\sin(\Phi_\mathrm{i})\right)} \tag{2.69}$$

If we define the $M_r \times 1$ vector

$$\mathbf{d} = \begin{bmatrix} 0 \\ d \\ 2d \\ \vdots \\ (M_r-1)d \end{bmatrix} \tag{2.70}$$

and $N_t \times 1$ vector

$$\overline{\boldsymbol{\Delta}} = \begin{bmatrix} 0 \\ \Delta \\ 2\Delta \\ \vdots \\ (N_t-1)\Delta \end{bmatrix} \tag{2.71}$$

we can write one component in the matrix channel as

$$\mathbf{H}_i(t) = \underbrace{e^{j(2\pi f_i t + \phi_i)}}_{\text{scalar,time}-\text{varyingcomponent}} \underbrace{e^{-j\left(\frac{2\pi}{\lambda}\mathbf{d}\sin(\theta_\mathrm{i})\right)}}_{\text{rxantennas}} \underbrace{e^{-j\left(\frac{2\pi}{\lambda}\overline{\boldsymbol{\Delta}}^T\sin(\Phi_\mathrm{i})\right)}}_{\text{txantennas}} \tag{2.72}$$

(Note that for a length N column vector $\mathbf{x}$, we define $e^{\mathbf{x}} = \begin{bmatrix} e^{x_1} \\ e^{x_1} \\ e^{x_1} \\ \vdots \\ e^{x_N} \end{bmatrix}$.) Finally, the narrowband Clarkes $M_r \times N_t$ Matrix channel can be written as

$$\begin{aligned} \mathbf{H}(t) &= \sqrt{\frac{1}{N}}\sum_{i=1}^{N}\mathbf{H}_i(t) \\ &= \sqrt{\frac{1}{N}}\sum_{i=1}^{N} e^{j(2\pi f_i t + \phi_i)} e^{-j\left(\frac{2\pi}{\lambda}\mathbf{d}\sin(\theta_\mathrm{i})\right)} e^{-j\left(\frac{2\pi}{\lambda}\overline{\boldsymbol{\Delta}}^T\sin(\Phi_\mathrm{i})\right)} \end{aligned} \tag{2.73}$$

where $f_i = \frac{v}{\lambda}\cos(\theta_\mathrm{i})$ (assuming that the receiver is moving. The distribution of θ_i and Φ_i depend on many factors. Typical assumptions include uniform (mobile), Gaussian, Laplacian (base station). The angle spread of the distribution determines the spatial correlation.

2.10 Matrix Correlation

Define matrix channel $\mathbf{H} = \begin{bmatrix} h_{11} & \cdots & \cdots & h_{1N_t} \\ h_{21} & \cdots & \cdots & h_{2N_t} \\ \vdots & & & \vdots \\ h_{M_r 1} & \cdots & \cdots & h_{M_r N_t} \end{bmatrix}$. We define the stacked vector

$$\mathbf{h}_{st} = vec\{\mathbf{H}\} = \begin{bmatrix} h_{11} \\ h_{21} \\ \vdots \\ h_{M_r 1} \\ h_{12} \\ h_{22} \\ \vdots \\ h_{M_r 2} \\ \vdots \\ h_{M_r N_t} \end{bmatrix} \tag{2.74}$$

The correlation matrix is defined as

$$\mathbf{R} = \mathrm{E}\left\{\mathbf{h}_{st}\mathbf{h}_{st}^H\right\} \tag{2.75}$$

2.10.1 Kronecker Model

The Kronecker model assumes that

$$\mathbf{R} = \mathbf{R}_{tx} \otimes \mathbf{R}_{rx} \tag{2.76}$$

where $\mathbf{R}_{tx}$ is the correlation across transmit antennas (termed *transmit correlation*), $\mathbf{R}_{rx}$ is the correlation across receive antennas (termed *receive correlation*), and $\otimes$ is the Kronecker product.

2.10.2 Example

Define a 2×2 channel matrix with $\mathrm{E}\left\{|h_{i,j}|^2\right\} = 1$

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \tag{2.77}$$

Determine the full correlation matrix $\mathbf{R}$ if the correlation between receive antennas is ρ and the correlation between transmit antennas is λ with and without the Kronecker assumption.

First, we can write

$$\mathbf{R}_{rx} = \mathrm{E}\left\{ \begin{bmatrix} h_{11} \\ h_{21} \end{bmatrix} \begin{bmatrix} h_{11}^* & h_{21}^* \end{bmatrix} \right\} = \begin{bmatrix} 1 & \rho \\ \rho^* & 1 \end{bmatrix} \tag{2.78}$$

$$\mathbf{R}_{rx} = \mathrm{E}\left\{ \begin{bmatrix} h_{12} \\ h_{22} \end{bmatrix} \begin{bmatrix} h_{12}^* & h_{22}^* \end{bmatrix} \right\} = \begin{bmatrix} 1 & \rho \\ \rho^* & 1 \end{bmatrix} \tag{2.79}$$

$$\mathbf{R}_{tx} = \mathrm{E}\left\{ \begin{bmatrix} h_{11} \\ h_{12} \end{bmatrix} \begin{bmatrix} h_{11}^* & h_{12}^* \end{bmatrix} \right\} = \begin{bmatrix} 1 & \lambda \\ \lambda^* & 1 \end{bmatrix} \tag{2.80}$$

$$\mathbf{R}_{tx} = \mathrm{E}\left\{ \begin{bmatrix} h_{21} \\ h_{22} \end{bmatrix} \begin{bmatrix} h_{21}^* & h_{22}^* \end{bmatrix} \right\} = \begin{bmatrix} 1 & \lambda \\ \lambda^* & 1 \end{bmatrix} \tag{2.81}$$

However, this doesn't completely define the channel correlation. We can see this by looking at $\mathbf{R}$:

$$\mathbf{R} = \mathrm{E}\left\{ \begin{bmatrix} h_{11} \\ h_{21} \\ h_{12} \\ h_{22} \end{bmatrix} \begin{bmatrix} h_{11}^* & h_{21}^* & h_{12}^* & h_{22}^* \end{bmatrix} \right\} = \begin{bmatrix} 1 & \rho & \lambda & ? \\ \rho^* & 1 & ? & \lambda \\ \lambda^* & ? & 1 & \rho \\ ? & \lambda^* & \rho^* & 1 \end{bmatrix} \tag{2.82}$$

Thus, without the Kronecker assumption, we cannot complete the correlation matrix. The Kronecker Model thus assumes that

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_{rx} & \lambda\mathbf{R}_{rx} \\ \lambda^*\mathbf{R}_{rx} & \mathbf{R}_{rx} \end{bmatrix} \tag{2.83}$$

$$= \begin{bmatrix} 1 & \rho & \lambda & \lambda\rho \\ \rho^* & 1 & \lambda\rho^* & \lambda \\ \lambda^* & \lambda^*\rho & 1 & \rho \\ \lambda^*\rho^* & \lambda^* & \rho^* & 1 \end{bmatrix} \tag{2.84}$$

Thus, with the Kronecker assumption, we can complete the correlation matrix.

2.11 Conclusion

Overcoming the wireless channel is the dominant challenge in wireless communications. Understanding multipath fading is a key to understanding the benefits (and thus prevalence) of MIMO-OFDM communications. In this chapter we have examined the key aspects of the MIMO wireless channel including the temporal, spatial and frequency statistics of the channel. We have primarily focused on the first two aspects, but will return to the frequency statistics later in this manuscript.