Multi-Channel Communications

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Part I

Preliminaries

Notation

The notation used in this manuscript is listed in the following tables. More specifically, Tables 1 and 2 describe the general notation used in this text. Tables 3 and 4 represent the English letter variables used in the text while Table 5 describe the Greek variables used in the manuscript.

Symbol	Meaning	Notes
$\tilde{x}(t)$	complex function	This notation will always indicate a complex function. However, at times we may have complex functions without this notation. Depends on context.
$\tilde{x}^*(t)$	complex conjugate	if $\tilde{x}(t) = a(t) + jb(t)$ then $\tilde{x}^*(t) = a(t) - jb(t)$
$E\{x(t)\}$ $E\{\mathbf{x}\}$	expectation of random process $x(t)$ expectation of random vector $\mathbf{x}$	If not clear from the context, the variable over which the expectation is taken will be made clear by using a subscript. e.g., $E_\lambda\{\mathbf{x}\}$ indicates expectation over λ
$\mathcal{F}\{x(t)\}$ $\mathcal{F}^{-1}\{X(f)\}$ $\mathcal{F}\{x[n]\}$ $\mathcal{F}^{-1}\{X[k]\}$	Fourier Transform of $x(t)$ Inverse Fourier Transform of $X(f)$ Discrete Transform of $x[n]$ Inverse Discrete Transform of $X[k]$	$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$ $x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$ $X[k] = \mathcal{F}\{x[n]\} = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn}$ $x[n] = \mathcal{F}^{-1}\{X[k]\} = \sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi}{N}kn}$ In matrix notation, $\mathbf{X} = \mathbf{D}_N\mathbf{x}$, where $\mathbf{D}_N$ defined below $\mathbf{x} = \mathbf{D}_N^H\mathbf{X}$ where $\mathbf{x}, \mathbf{X} \in \mathcal{C}^N$
$\mathbf{v}$	column vector	$\overline{\mathbf{v}}$ will also be used when bold isn't clear if size of vector isn't clear from context $\mathbf{v}^{a\times 1}$ will indicate the size
$[\mathbf{v}]_i$	ith element of vector	
$\mathbf{H}$	matrix	If size isn't clear from context, $\mathbf{H}^{m\times n}$ will be used to indicate an $m \times n$ matrix
$[\mathbf{H}]_{i,j}$	$\{i,j\}$th element of matrix $\mathbf{H}$	
$\Re^{m\times n}$	set of all $m \times n$ real matrices	
$\mathcal{C}^{m\times n}$	set of all $m \times n$ complex matrices	
$\mathbf{I}^{m\times m}$	$m \times m$ identity matrix	
$\mathbf{0}^{m\times n}$	$m \times n$ matrix of zeros	
$span(\mathcal{S})$	set of vectors spanned by $\mathcal{S}$	
$col(\mathbf{A})$	space spanned by columns of $\mathbf{A}$	column space of $\mathbf{A}$
$row(\mathbf{A})$	space spanned by rows of $\mathbf{A}$	row space of $\mathbf{A}$
$vec(\mathbf{A})$	stacked columns of matrix $\mathbf{A}$	converts an $m \times n$ matrix to an $mn \times 1$ vector
$\mathbf{u}^T$ $\mathbf{A}^T$	transpose of $\mathbf{u}$ transpose of $\mathbf{A}$	
$\|\mathbf{u}\|_p$	p-norm of $\mathbf{u}$	$\|\mathbf{u}\|$ is the 2-norm

Table 1: General Notation Used in the Class (Part 1 of 2)

Symbol	Meaning	Notes
$\mathbf{A}^*$	element by element conjugate of $\mathbf{A}$	
$\mathbf{A}^H$	conjugate transpose of $\mathbf{A}$	
$\mathbf{A}^{-H}$	conjugate transpose of $\mathbf{A}^{-1}$	$\mathbf{A}^{-H} = (\mathbf{A}^{-1})^H$
$\mathbf{A}^\dagger$	pseudo-inverse of $\mathbf{A}$	$\mathbf{A}^\dagger = \left(\mathbf{A}^T \mathbf{A}\right)^{-1} \mathbf{A}^T$
Symbol	Meaning	Notes
$\mathbf{A}^{1/2}$	matrix square root	
$trace(\mathbf{A})$	trace of $\mathbf{A}$	sum of diagonal elements
$R(\mathbf{A})$	rank of $\mathbf{A}$	
$row(\mathbf{A})$	row space of $\mathbf{A}$	
$col(\mathbf{A})$	column space of $\mathbf{A}$	
$dim(\mathbf{A})$	dimension of $\mathbf{A}$	number of vectors in any basis
$\lambda(\mathbf{A})$	eigenvalues of $\mathbf{A}$	also called the spectrum of $\mathbf{A}$
		λ_{min}, λ_{max} are min and max eigenvalues
		$\lambda_k(\mathbf{A})$ is the kth eigenvalue of $\mathbf{A}$
$\sigma(\mathbf{A})$	singular values of $\mathbf{A}$	σ_{min}, σ_{max} are min and max singular values
$\|\mathbf{A}\|_F$	Frobenius norm of $\mathbf{A}$	
$\mathbf{A}^{-1}$	inverse of $\mathbf{A}$	
$|\mathcal{V}|$	cardinality of set $\mathcal{V}$	sets will generally use cal font
$\nabla f(\mathbf{x})$	gradient of function $f(\mathbf{x})$	argument is a vector $\mathbf{x}$; output is scalar

Table 2: General Notation Used in the Class (Part 2 of 2)

Symbol	Meaning	Notes
$\mathbf{a}(\theta)$	array factor	also array manifold vector
$\mathbf{A}(\bar{\theta})$	Matrix of array factors	$\mathbf{A}(\bar{\theta}) = [\mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_N]$
B	bandwidth	
C	capacity	
d	distance	scalar
D	Directivity	
$\mathbf{D}_N$	DFT matrix, $[\mathbf{D}]_{m,n} = e^{-j2\pi(m-1)(n-1)/N}$	$\mathbf{D}_N^H$ is IDFT matrix
$\mathbf{D}_{(LL)}$	first L rows and columns of N-point DFT matrix ($L \times L$ matrix)	$\mathbf{D}_N$ also known as Fourier matrix
$\mathbf{D}_{(NL)}$	first L columns DFT matrix ($N \times L$ matrix)	
f	frequency	
f_c	center (or carrier) frequency	
f_d	max Doppler frequency	$f_d = \frac{v}{\lambda}$
f_o	digital frequency offset	$f_o = f_{co} T_{eff}$ (f_{co} = cont. time)
f_{sa}	sampling frequency	$f_{sa} = \frac{1}{T_{sa}}$
Δf	sub-carrier spacing in OFDM	
$f_X(x)$	pdf of random variable X	
$F_X(x)$	cdf of random variable X	
G_r	Receive Antenna Gain	scalar when in dB use $G_r(dB)$
G_t	Transmit Antenna Gain	scalar when in dB use $G_t(dB)$
$G(\theta)$	Array Factor	gain pattern due to array
$g_A(\theta)$	individual element pattern	gain pattern due to element
$h(\tau,t)$, $\mathbf{H}(\tau,t)$	channel impulse response	if time-invariant, t is dropped if narrowband, τ is dropped can be scalar or vector or matrix if matrix, size is $M_r \times N_t$
$\tilde{h}(\tau,t)$		complex baseband channel
$\tilde{h}$		complex baseband channel sample $\tilde{h}$ and $\tilde{\alpha}$ are essentially the same
$H(f,t)$	time-varying frequency response	$\mathcal{F}\{h(t,\tau)\}$
$\mathbf{H}$	channel matrix	complex channel samples
$\mathbf{H}_w$	white channel matrix	channel elements are iid complex GRV
$H(X)$	entropy of random variable X	context differentiates from channel functions
$I(x;y)$	mutual information between x and y	
$J_o(\tau)$	zeroth-order Bessel function of the first kind	
L	diversity order	Number of branches in a diversity system (could be equal to M_r)
M_r	number of receive antennas	

Table 3: Variables Used in the Class (Part 1 of 2)

Symbol	Meaning	Notes
n	path loss exponent	scalar
n	noise (sample)	scalar or vector
$n(t)$	noise function	typically AWGN
$n_i(t)$	noise function on ith rx antenna	typically AWGN
N	number of components	typically multipath components
		Number of sub-carriers in OFDM
N_{cp}	samples in cyclic prefix	
N_s	number of signals in the environment	
N_t	number of transmit antennas	
P_r	Received Power	scalar
		when in dB use $P_r(dB)$
P_t	Transmit Power	scalar
		when in dB use $P_t(dB)$
$p(t)$	pulse shape	
$r(t)$	received signal with noise	r[k] is the sampled signal
$r_i(t)$	received signal with noise on ith antenna	
$\mathbf{r}(t)$	received signal on all antennas	if sampled, $\mathbf{r}$
$R(\tau)$	auto-correlation function of a random process	
$\mathbf{R}_{tx}$	transmit side correlation matrix	
	of the random channel vector	seen across the transmit antennas
$\mathbf{R}_{rx}$	receive side correlation matrix	
	of the random channel vector	seen across the receive antennas
$\mathbf{R}$	correlation matrix	$E\{vec(\mathbf{H})vec(\mathbf{H})^H\}$
R_s	symbol rate	
R_{ofdm}	OFDM symbol rate	
$s_{i,k}$	symbol transmitted from the ith antenna	
	during kth time slot	o
$S(f)$	Power Spectral Density of Random Process	Fourier Transform of $R(\tau)$
$S_A(\theta)$	Azimuthal Power Spectral Density	power per received angle
		$P_r = \int_{-\pi}^{\pi} S_A(\theta)G(\theta)d\theta$
		$G(\theta)$ is the receive antenna gain
$S_D(f)$	Doppler Spectrum	$\mathcal{F}\left\{\tilde{R}(\tau)\right\}$
t	time	
T_s	symbol duration	
T_{sa}	sample time/duration	$T_{sa} = \frac{1}{f_{sa}}$
T_o	OFDM symbol duration	$T_o = T_{eff} + T_{cp} + T_w$
T_{eff}	effective OFDM symbol duration	$T_{eff} = 1/\Delta f$
T_{cp}	cyclic prefix duration	
T_w	window duration	
$U[a,b]$	uniform distribution	range is [a,b]
v or $\mathbf{v}$	velocity	scalar or vector
w or $\mathbf{w}$	weights	scalar or vector
$x(t)$	transmit signal	
$x_i(t)$	transmit signal from ith antenna	
$\mathbf{x}(t)$	is transmit signal from multiple antennas	Will also use $\overline{x}(t)$ for this vector
z	decision statistic	after channel compensation

Table 4: Variables Used in the Class (Part 2 of 2)

Symbol	Meaning	Notes
α	magnitude of single channel	$\tilde{\alpha}$ is complex channel $\alpha e^{j\phi}$
		h will sometimes also be used for the single antenna channel
γ	SNR	
$\delta(t)$	impulse function	
Δ	distance	
ΔX	change in X	
ϵ	error	*e.g.* in the adaptation of antenna weights
η	decision statistic	before channel compensation ($z = \alpha^* \eta$)
ψ	general angle of an array	$\psi = \frac{2\pi d}{\lambda} \sin(\theta)$
λ	wavelength	scalar
λ	eigenvalue of a square matrix	context clarifies
μ	mean	
μ_i	ith moment about zero	
ν	discrete symbol time	represents discrete time
∇	gradient	$\hat{\nabla}$ is the estimated gradient
Ω_A	Solid Beam Angle	
ϕ	phase	
$\phi(\omega)$	Characteristic Function	$\mathrm{E}_\gamma\left\{e^{j\omega\gamma}\right\}$
Φ	angle-of-departure	defined relative to array normal
$\Phi(s)$	Moment Generating Function	$\mathrm{E}_\gamma\left\{e^{s\gamma}\right\}$
$\rho(d)$	spatial correlation	measured at distance d
θ	angle-of-arrival	defined relative to array normal
σ_n	noise standard deviation	
σ_A	AOA standard deviation	
σ	singular value of a matrix	context clarifies
$\Sigma_{\mathbf{x}}$	covariance matrix of the vector $\mathbf{x}$	$\Sigma_{\mathbf{x}} = \mathrm{E}\left\{\mathbf{x}\mathbf{x}^H\right\} - \mathrm{E}\left\{\mathbf{x}\right\}\mathrm{E}\left\{\mathbf{x}\right\}^H$
τ	delay	
ζ	element angle	defined for a circular array
ψ	array variable	$= \frac{2\pi d}{\lambda} \sin(\theta)$

Table 5: Greek Variables Used in the Class

Chapter 1

Matrix Theory: Basic Definitions

1.1   Introduction

Multi-channel communications by its nature involves the consideration of multiple channels simultaneously, whether they be in time, space or frequency. These channels may be separate at the time of transmission, but mixed upon reception. Or they may stay orthogonal throughout. In either case, vector and matrix notation is a convenient way to represent multiple channels both at the transmitter and receiver. Further, matrix theory is very convenient in describing the characteristics of these signals and the wireless channels that carry them. Thus, in this chapter we will provide a brief overview of vectors and matrices, with an emphasis on those concepts that are important to understanding MIMO and OFDM. For a more complete treatment of the topic of Linear Algebra and Matrix Analysis, please see [1]. In this section we will specifically discuss the following:

- Definitions

- Vector Spaces

- Functions of Matrices and Vectors

- Matrix Inversion

- Matrix Factorization (LU, LDU, Cholesky)

- Singular Value and Eigenvalue Decomposition

1.2   Basic Definitions

An n-element column vector is a group of n elements in vertical form:

$$\mathbf{v}^{n \times 1} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

An n-element row vector is a group of n elements in horizontal form:

$$\mathbf{v}^{1 \times n} = \begin{bmatrix} v_1 & v_2 & v_3 & \cdots & v_n \end{bmatrix}$$

If not specified, assume that any lower case bold letter is a column vector and any upper case bold letter is a matrix.

A matrix $\mathbf{V}^{m \times n}$ is defined as an $m \times n$ array of mn elements:

$$\mathbf{V}^{m \times n} = \begin{bmatrix} v_{11} & v_{12} & v_{13} & \cdots & v_{1n} \\ v_{21} & v_{22} & v_{23} & \cdots & v_{2n} \\ v_{31} & v_{32} & v_{33} & \cdots & v_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{m1} & v_{m2} & v_{m3} & \cdots & v_{mn} \end{bmatrix}$$

$\Re^{m \times n}$ represents the set of all $m \times n$ real matrices.

$\mathcal{C}^{m \times n}$ is the set of all complex matrices of dimension $m \times n$. Note that $\mathcal{C} \in \{\Re^{m \times n} + j\Re^{m \times n}\}$.

Example 1.2.1. Consider modulation symbols s_i transmitted from four different transmitters using the same pulse shape and received at four different receivers over the same frequency band. If there is independent noise generated at each receiver, how can we use matrix notation to represent the matched filter outputs assuming independent channels h_{ij} between the ith receiver and jth transmitter? How would we represent the received signal if the transmit and receive frequency bands were mutually orthogonal?

Solution: We can write the transmitted symbols as a vector $\mathbf{s} = [s_1, s_2, s_3, s_4]^T$. The channel between four transmitters and five receivers can be written as a 4×4 matrix with h_{ij} as the (i, j)th element:

$$\mathbf{H} = \begin{bmatrix} h_{1,1} & h_{1,2} & h_{1,3} & h_{1,4} \\ h_{2,1} & h_{2,2} & h_{2,3} & h_{2,4} \\ h_{3,1} & h_{3,2} & h_{3,3} & h_{3,4} \\ h_{4,1} & h_{4,2} & h_{4,3} & h_{4,4} \end{bmatrix} \tag{1.1}$$

The matched filter outputs and the noise samples can both be written as 5×1 vectors $\tilde{\mathbf{r}}$ and $\mathbf{n}$ respectively. That gives the following representation:

$$\mathbf{r} = \mathbf{H}\mathbf{s} + \mathbf{n} \tag{1.2}$$

If the frequency bands are orthogonal, the equation would be the same but we would have a different definition for the channel matrix $\mathbf{H}$. Specifically, it would be diagonal:

$$\mathbf{H} = \begin{bmatrix} h_{1,1} & 0 & 0 & 0 \\ 0 & h_{2,2} & 0 & 0 \\ 0 & 0 & h_{3,3} & 0 \\ 0 & 0 & 0 & h_{4,4} \end{bmatrix} \tag{1.3}$$

1.3 Vector Measures

Two column vectors $\mathbf{u}$ and $\mathbf{v}$ (with equal length) are said to be **orthonormal** if

$$\mathbf{u}^T \mathbf{v} = 0$$

and

$$\mathbf{u}^T \mathbf{u} = \mathbf{v}^T \mathbf{v} = 1$$

The p-norm of a vector $\mathbf{v}$ is defined as

$$\|\mathbf{v}\|_p = \left(\sum_{i=1}^{n} |v_i|^p \right)^{1/p}$$

Example 1.3.1. Consider the vectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$. Are they orthonormal? How about

$\mathbf{u}_2 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0.5 \\ -0.5 \\ 0.5 \\ -0.5 \end{bmatrix}$? What is the 2-norm of $\mathbf{v}_1$?

Solution: It is straightforward to show that $\mathbf{u}_1^T \mathbf{v}_1 = 0$. Thus, the two vectors are orthogonal. However, $\|\mathbf{u}\|_2 = 2$ and $\|\mathbf{v}\|_2 = 2$. Thus, neither vector has unit norm and the two vectors are NOT orthonormal.

On the other hand $\mathbf{u}_2^T \mathbf{v}_2 = 0$, $\|\mathbf{u}_1\|_2 = 1$ and $\|\mathbf{v}_1\|_2 = 1$. Thus $\mathbf{u}_2$ and $\mathbf{v}_2$ are orthonormal.

1.3.1 Vector Spaces

A set of vectors $\mathcal{V}$ is defined as a vector space over $\mathcal{F}$ when basic vector addition and scalar multiplication properties hold. The most fundamental are the closure properties:

- $\mathbf{x} + \mathbf{y} \in \mathcal{V}$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$

- $\alpha\mathbf{x} \in \mathcal{V}$ for all $\alpha \in \mathcal{F}$ and $\mathbf{x} \in \mathcal{V}$.

Simple example: the set $\Re^n$ of real vectors of length n is a vector space over $\Re$.

A **subspace** is defined as a non-empty set of vectors taken from a vector space that also forms a separate vector space.

1.3.2   Linear Independence

A set of vectors $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots \mathbf{v}_n\}$ is said to be **linearly independent** when the only set of coefficients α_i that satisfy the equation

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \ldots\alpha_n\mathbf{v}_n = \mathbf{0}$$

is $\alpha_1 = \alpha_2 = \alpha_3 = \ldots \alpha_n = 0$.

Alternatively, a set of vectors is linearly independent if none of them can be written as a linear combination of finitely many other vectors in the collection.

A set of vectors which is not linearly independent is called **linearly dependent**.

1.3.3   Span

A set of vectors $\mathcal{S}$ is said to **span** the vector space $\mathcal{V}$ if every vector in $\mathcal{V}$ can be generated from a linear combination of the vectors in $\mathcal{S}$.

We denote this relationship by $\mathcal{V} = span(\mathcal{S})$ or $\mathcal{S}$ spans $\mathcal{V}$.

If a set of vectors $\mathcal{S}$ is a spanning set and is also **linearly independent** it is said to be a **basis** for the vector space $\mathcal{V}$.

Example 1.3.2. The unit vectors $\left\{ \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ span $\Re^3$.

That is, we can construct any vector in $\Re^3$ using a weighted sum of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$.

NOTE: These vectors also form a basis for $\Re^3$.

1.4   Matrices

1.4.1   Matrix Addition

Consider two matrices $\mathbf{A}$ and $\mathbf{B}$ which are $m \times n$ matrices. The sum of two matrices is defined as the sum of the individual elements:

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} & \cdots & A_{1n} + B_{1n} \\ A_{21} + B_{21} & A_{22} + B_{22} & \cdots & A_{2n} + B_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A_{m1} + B_{m1} & A_{m2} + B_{m2} & \cdots & A_{mn} + B_{mn} \end{bmatrix}_{m \times n}$$

1.4.2   Scalar Multiplication

When multiplying a matrix $\mathbf{A}$ by a scalar α the result is that every element in the matrix is multiplied by that scalar. In other words:

$$\alpha \mathbf{A} = \begin{bmatrix} \alpha A_{11} & \alpha A_{12} & \cdots & \alpha A_{1n} \\ \alpha A_{21} & \alpha A_{22} & \cdots & \alpha A_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha A_{m1} & \alpha A_{m2} & \cdots & \alpha A_{mn} \end{bmatrix}$$

1.4.3   Matrix Multiplication

The multiplication of two matrices requires that the number of columns of the first matrix is the same as the number of rows of the second matrix. Given the $m \times n$ matrix $\mathbf{A}$ and the $n \times k$ matrix $\mathbf{B}$, the multiplication of the two matrices is defined as

$$\mathbf{AB} = \begin{bmatrix} \sum_{i=1}^{n} A_{1i}B_{i1} & \sum_{i=1}^{n} A_{1i}B_{i2} & \cdots & \sum_{i=1}^{n} A_{1i}B_{ik} \\ \sum_{i=1}^{n} A_{2i}B_{i1} & \sum_{i=1}^{n} A_{2i}B_{i2} & \cdots & \sum_{i=1}^{n} A_{2i}B_{ik} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{i=1}^{n} A_{mi}B_{i1} & \sum_{i=1}^{n} A_{mi}B_{i2} & \cdots & \sum_{i=1}^{n} A_{mi}B_{ik} \end{bmatrix}_{m \times k}$$

Note that in general, $\mathbf{AB} \neq \mathbf{BA}$. In fact, both products can only be formed when the two matrices are square with the same dimensions.

However, if $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{AB}$ are all symmetric and $\mathbf{A}, \mathbf{B}$,both are of the same dimensions, then $\mathbf{AB} = \mathbf{BA}$. Alternatively, if $\mathbf{A}, \mathbf{B}$ are of the same dimensions and diagonal, then $\mathbf{AB} = \mathbf{BA}$.

1.4.4   Matrix Transpose

The transpose of a matrix is the matrix obtained by interchanging the rows and columns of that matrix.

The transpose of the $m \times n$ matrix $\mathbf{A}$ is a $n \times m$ matrix denoted as $\mathbf{A}^T$. That is:

$$\mathbf{A}^T = \left[\begin{array}{cccc} A_{11} & A_{21} & \cdots & A_{m1} \\ A_{12} & A_{22} & \cdots & \alpha A_{m2} \\ \vdots & \vdots & \cdots & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{mn} \end{array} \right]_{n \times m}$$

From this definition, it should be clear that $\left(\mathbf{A}^T \right)^T = \mathbf{A}$.

1.4.5   Matrix Conjugate

The conjugate of a matrix is the element-by-element conjugate. We will denote the conjugate of the matrix $\mathbf{A}$ as $\mathbf{A}^*$:

$$\mathbf{A}^* = \left[\begin{array}{cccc} A_{11}^* & A_{12}^* & \cdots & A_{1n}^* \\ A_{21}^* & A_{22}^* & \cdots & A_{2n}^* \\ \vdots & \vdots & \cdots & \vdots \\ A_{m1}^* & A_{m2}^* & \cdots & A_{mn}^* \end{array} \right]_{m \times n}$$

1.4.6   Matrix Hermitian

The conjugate transpose $\mathbf{A}^H$ (sometimes called the **Hermitian** of $\mathbf{A}$) is then:

$$\mathbf{A}^H = \left[\begin{array}{cccc} A_{11}^* & A_{21}^* & \cdots & A_{m1}^* \\ A_{12}^* & A_{22}^* & \cdots & \alpha A_{m2}^* \\ \vdots & \vdots & \cdots & \vdots \\ A_{1n}^* & A_{2n}^* & \cdots & A_{mn}^* \end{array} \right]_{n \times m}$$

1.4.7   Matrix Definitions

A **symmetric matrix** is one for which $\mathbf{A} = \mathbf{A}^T$. A **Hermitian matrix** is one for which $\mathbf{A} = \mathbf{A}^H$. It is straightforward to show that

$$\begin{aligned} (\mathbf{AB})^T &= \mathbf{B}^T \mathbf{A}^T \\ (\mathbf{AB})^H &= \mathbf{B}^H \mathbf{A}^H \end{aligned}$$

Because of these facts, $\mathbf{A}^T \mathbf{A}$ and $\mathbf{A} \mathbf{A}^T$ are symmetric matrices:

$$\left(\mathbf{A}^T \mathbf{A} \right)^T = \mathbf{A}^T (\mathbf{A}^T)^T = \mathbf{A}^T \mathbf{A}$$

Similarly, for a complex matrix $\mathbf{A}$, $\mathbf{A}^H \mathbf{A}$ is a Hermitian matrix.

Important Matrix Measures

The **trace** of an $n \times n$ matrix is the sum of its diagonal elements. That is

$$trace\left(\mathbf{A}\right) = A_{11} + A_{22} + ...A_{nn} = \sum_{i=1}^{n} A_{ii}$$

The **Frobenius norm** of a matrix is defined as

$$|\mathbf{A}|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m} |A_{ij}|^2}$$

Later we will also discuss the **rank** and **determinant** of a matrix.

The Matrix Inverse

For a square matrix $\mathbf{A}$ of size $n \times n$, the matrix $\mathbf{B}$ that satisfies the conditions

$$\mathbf{AB} = \mathbf{I}_n$$

and

$$\mathbf{BA} = \mathbf{I}_n$$

where $\mathbf{I}_n$ is an $n \times n$ identity matrix is called the **inverse** of $\mathbf{A}$, denoted $\mathbf{B} = \mathbf{A}^{-1}$.

For a matrix to be invertible it must be square, but not all square matrices are invertible.

A matrix that is not invertible is called a **singular matrix**. If it is invertible, it is **nonsingular**.

Properties of the Matrix Inverse

For two square nonsingular matrices $\mathbf{A}$ and $\mathbf{B}$ of the same dimensions:

- $\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$

- $\mathbf{AB}$ is also nonsingular

- $\left(\mathbf{AB}\right)^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

- $\left(\mathbf{A}^{-1}\right)^T = \left(\mathbf{A}^T\right)^{-1}$

The Matrix Inversion Lemma

A very handy identity is the matrix inversion lemma (also known as the Woodberry matrix identity) which states

$$\left(\mathbf{A} + \mathbf{BCD}\right)^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}\left(\mathbf{C}^{-1} + \mathbf{DA}^{-1}\mathbf{B}\right)^{-1}\mathbf{DA}^{-1} \tag{1.4}$$

Orthogonal Matrices

An orthogonal matrix is defined to be a real matrix $\mathbf{P}_{n \times n}$ whose columns (or rows) constitute an orthonormal basis for $\mathcal{R}^n$. Note that the following properties hold:

- **P** has orthonormal columns

- **P** has orthonormal rows

- $\mathbf{P}^T\mathbf{P} = \mathbf{P}\mathbf{P}^T = \mathbf{I}$

- $\mathbf{P}^{-1} = \mathbf{P}^T$

- $\|\mathbf{P}\mathbf{x}\|_2 = \|\mathbf{x}\|_2$ for every $\mathbf{x} \in \mathcal{R}^n$

Unitary Matrices

A unitary matrix is defined to be a complex matrix $\mathbf{U}_{n \times n}$ whose columns (or rows) constitute an orthonormal basis for $\mathcal{C}^n$. Note that the following properties hold:

- **U** has orthonormal columns

- **U** has orthonormal rows

- $\mathbf{U}^H\mathbf{U} = \mathbf{U}\mathbf{U}^H = \mathbf{I}$

- $\mathbf{U}^{-1} = \mathbf{U}^H$

- $\|\mathbf{U}\mathbf{x}\|_2 = \|\mathbf{x}\|_2$ for every $\mathbf{x} \in \mathcal{C}^n$

Example 1.4.1. Consider the matrix $\mathbf{A} = \begin{bmatrix} 10 & 2 & 9 \\ -1 & 4 & 12 \\ 20 & 25 & -10 \end{bmatrix}$. What is $trace(\mathbf{A})$? What is $|\mathbf{A}|_F$? Is $\mathbf{A}$ orthogonal?

Solution: We can easily find that $trace(\mathbf{A}) = 4$ and $|\mathbf{A}|_F = 38.4$. Further, it is easy to see that $\mathbf{A}^T\mathbf{A} \neq \mathbf{I}$, thus $\mathbf{A}$ is not orthogonal.

Example 1.4.2. Consider the matrix $\mathbf{B} = \begin{bmatrix} -0.5 & -0.5 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0.5 & 0.5 & \sqrt{2}/2 \end{bmatrix}$.

What is $trace(\mathbf{B})$? What is $|\mathbf{B}|_F$? Is $\mathbf{B}$ orthogonal? What is the inverse of $\mathbf{B}$?

Solution: Calculating the trace we find that $trace(\mathbf{B}) = -0.5$ and $|\mathbf{B}|_F = 1.73$. Further, it is easy to show

that $\mathbf{B}^T\mathbf{B} = \mathbf{I}$ and $\mathbf{BB}^T = \mathbf{I}$. Also, since $\mathbf{B}$ is orthogonal,

$$\mathbf{B}^{-1} = \mathbf{B}^T = \begin{bmatrix} -0.5 & \sqrt{2}/2 & 0.5 \\ -0.5 & -\sqrt{2}/2 & 0.5 \\ \sqrt{2}/2 & 0 & \sqrt{2}/2 \end{bmatrix} \tag{1.5}$$

Null Space

The **null space** of an $m \times n$ matrix $\mathbf{A}$ is the set X of $n \times 1$ vectors $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$. The *left-hand null space* of $\mathbf{A}$ is the set of $m \times 1$ vectors y such that $\mathbf{A^T y} = \mathbf{0}$. In other words, the left-hand null space of $\mathbf{A}$ is the null space of $\mathbf{A}^T$.

Example 1.4.3. Consider the matrix $\mathbf{A} = \begin{bmatrix} 10 & 2 & 9 \\ -1 & 4 & 12 \\ 20 & 25 & -10 \end{bmatrix}$. Is the vector $\mathbf{v} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ in the null space of $\mathbf{A}$?

Solution: Taking $\mathbf{Av}$ we get

$$\mathbf{A} = \begin{bmatrix} 10 & 2 & 9 \\ -1 & 4 & 12 \\ 20 & 25 & -10 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 17 \\ 7 \\ -15 \end{bmatrix} \tag{1.6}$$

Thus, $\mathbf{v}$ is not in the null space of $\mathbf{A}$.

1.5 Matrix Factorization

1.5.1 The LU Factorization

If $\mathbf{A}$ is an $n \times n$ matrix that is non-singular and all leading principal submatrices $\mathbf{A}_k$ are nonsingular, $\mathbf{A}$ can be factored into the product $\mathbf{LU}$ where $\mathbf{L}$ is lower triangular (all zeros above the diagonal) and $\mathbf{U}$ is upper triangular (all zeros below the diagonal). That is

$$\mathbf{A} = \mathbf{LU} = \begin{bmatrix} L_{11} & 0 & 0 & \cdots & 0 \\ L_{21} & L_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{n1} & L_{n2} & L_{n3} & \cdots & L_{nn} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} & \cdots & U_{1n} \\ 0 & U_{22} & U_{23} & \cdots & U_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & U_{nn} \end{bmatrix} \tag{1.7}$$

Note that $L_{ii} = 1 \ \forall i$ and $U_{ii} \neq 0 \ \forall i$.

1.5.2   The LDU Factorization

The **LU** factorization can be converted to a factorization with ones on the diagonals of both the lower and upper triangular matrices by factoring **U**:

$$
\mathbf{U} = \begin{bmatrix} U_{11} & 0 & 0 & \cdots & 0 \\ 0 & U_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & U_{nn} \end{bmatrix} \begin{bmatrix} 1 & U_{12}/U_{11} & U_{13}/U_{11} & \cdots & U_{1n}/U_{11} \\ 0 & 1 & U_{23}/U_{22} & \cdots & U_{2n}/U_{22} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}
$$

Thus, a matrix that can be factored in an **LU** factorization can also be factored as $\mathbf{A} = \mathbf{LDU}$ where **L** and **U** are upper and lower triangular matrices respectively with 1's on their diagonals and **D** is a diagonal matrix.

1.5.3   Cholesky Factorization

If an $n \times n$ matrix **A** that has an **LDU** factorization is symmetric, the **LDU** factorization is $\mathbf{A} = \mathbf{LDL}^T$. Such a matrix is called **positive definite** if all of the diagonal values of **D** are greater than zero.

A positive definite matrix can also be factored as $\mathbf{A} = \mathbf{R}^T\mathbf{R}$ where **R** is upper triangular with all positive diagonal entries. This factorization is known as the **Cholesky factorization** of **A**. Note that $\mathbf{R} = \mathbf{D}^{1/2}\mathbf{L}^T$.

Example 1.5.1. Consider the matrix $\mathbf{A} = \begin{bmatrix} 10 & 2 & 9 \\ -1 & 4 & 12 \\ 20 & 25 & -10 \end{bmatrix}$. Calculate an LU factorization of **A**. Determine the LDU factorization of **A**. **Solution:** Using Matlab, we can find the LU factorization:

$$
\mathbf{L} \quad = \quad \begin{bmatrix} 1 & 0 & 0 \\ -0.1 & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix} \tag{1.8}
$$

$$
\mathbf{U} \quad = \quad \begin{bmatrix} 10 & 2 & 9 \\ 0 & 4.2 & 12.9 \\ 0 & 0 & -92.5 \end{bmatrix} \tag{1.9}
$$

Now, the LDU factorization is similar, but we factor a diagonal out of **U**.

$$\mathbf{L} \;=\; \begin{bmatrix} 1 & 0 & 0 \\ -0.1 & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix} \tag{1.10}$$

$$\mathbf{U} \;=\; \begin{bmatrix} 1 & 0.2 & 0.9 \\ 0 & 1 & 3.1 \\ 0 & 0 & 1 \end{bmatrix} \tag{1.11}$$

$$\mathbf{D} \;=\; \begin{bmatrix} 10 & 0 & 0 \\ 0 & 4.2 & 0 \\ 0 & 0 & -92.5 \end{bmatrix} \tag{1.12}$$

1.5.4 Rank of a Matrix

An important measure of a matrix is its **rank**.

There are many ways to define rank, but a basic definition is as follows: The rank of a matrix **A**, denoted $R(\mathbf{A})$ the number of linearly independent rows of the matrix or the number of linearly independent columns of the matrix (note that they are the same).

Alternatively, the rank is the number of non-zero rows in the matrix after it has been transformed to row-echelon form.

Rank is also equal to the number of non-zero singular values (to be discussed shortly) of **A**.

Notes on $R(\mathbf{A})$

- The rank of **A** is the dimension of the column space of **A** (which is the same as the dimension of the row space of **A**).

- If **A** is a square matrix of size $n \times n$ and $R(\mathbf{A}) = n$, then we say that **A** has **full rank**.

- If the rank of an $m \times n$ matrix **A** is greater than or equal to n, the null space is empty.

- For a full-rank square matrix, the null space is empty.

- If **A** is an $m \times n$ matrix, then $dim(col(\mathbf{A})) + dim(nul(\mathbf{A})) = n$ where $dim(\mathcal{S})$ is the dimension (i..e, the number of vectors in any basis) of the space $\mathcal{S}$.

- If **A** is an $m \times n$ matrix, then $row(\mathbf{A}) = col(\mathbf{A}^T)$ and $col(\mathbf{A}) = row(\mathbf{A}^T)$.

- Note that for an $m \times n$ matrix **A**,

$$R(\mathbf{A}) = R(\mathbf{A}^T) = R(\mathbf{A}\mathbf{A}^T) = R(\mathbf{A}^T\mathbf{A})$$

- For a square $n \times n$ matrix $\mathbf{A}$, it is non-singular only if the rank of $\mathbf{A}$ is n. In other words $R(\mathbf{A}) = n$ if $\mathbf{A}$ is non-singular.

- A diagonal matrix $\mathbf{A}$ which has non-zero values on the diagonal is full rank.

- It can be shown that $R(\mathbf{A} + \mathbf{B}) \leq R(\mathbf{A}) + R(\mathbf{B})$.

- Based on the preceding two properties adding a diagonal matrix to any matrix will make it full rank.

1.5.5   Row and Column Spaces

Rank can also be defined in terms of the row or column space of $\mathbf{A}$.

The **row space** of an $m \times n$ matrix, $\mathbf{A}$, denoted by $row(\mathbf{A})$ is a vector space which is the set of all linear combinations of the row vectors of $\mathbf{A}$.

The **column space** of an $m \times n$ matrix, $\mathbf{A}$, denoted by $col(\mathbf{A})$ is the set of all linear combinations of the column vectors of $\mathbf{A}$.

If $\mathbf{A}$ is an $m \times n$ real matrix, then $col(\mathbf{A})$ is a subspace of $\Re^m$ and $row(\mathbf{A})$ is a subspace of $\Re^n$.

More specifically, $col(\mathbf{A})$ is the span of the columns of $\mathbf{A}$ and $row(\mathbf{A})$ is the span of the rows of $\mathbf{A}$.

1.5.6   Singular Value Decomposition

For each $m \times n$ matrix $\mathbf{A}$, there are orthogonal matrices $\mathbf{U}^{m \times m}$ and $\mathbf{V}^{n \times n}$ and a diagonal matrix $\mathbf{D}^{r \times r} = diag\left(\sigma_1 \sigma_2 \ldots \sigma_r\right)$ such that

$$\mathbf{A} = \mathbf{U} \begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{V}^T$$

The values $\sigma_1, \sigma_2, \ldots, \sigma_r$ are called the non-zero singular values of $\mathbf{A}$. When $r < p = \min\{m, n\}$, $\mathbf{A}$ also has $p - r$ zero singular values.

This factorization is known as the **singular value decomposition** of $\mathbf{A}$ and the columns in $\mathbf{U}$ and $\mathbf{V}$ are called the left and right singular vectors of $\mathbf{A}$.

Note that r is the *rank* of $\mathbf{A}$. Further, we will define the maximum singular value as σ_{max} and the minimum singular value as σ_{min}.

Note that $\mathbf{U}$ and $\mathbf{V}$ are orthogonal matrices.

If $\mathbf{A}$ is $n \times n$ and nonsingular, we can write the SVD of $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{V}^T$$

and the SVD of the inverse of $\mathbf{A}$ can be written

$$\mathbf{A}^{-1} = \mathbf{V}\mathbf{D}^{-1}\mathbf{U}^T$$

The SVD provides valuable information about the matrix $\mathbf{A}$

- The unit vectors of $\mathbf{U}$ (i.e., the left singular vectors) that correspond to the non-zero singular values span the range of $\mathbf{A}$.

- The singular vectors corresponding to the zero singular values span the null space of $\mathbf{A}$.

  Singular values define the matrix 2-norm. For an $m \times n$ matrix $\mathbf{A}$:

  $$\|\mathbf{A}\|_2 = \max_{\|\mathbf{x}\|_2=1} \|\mathbf{A}\mathbf{x}\|_2$$

  To interpret this, we say that for any $n \times 1$ vector $\mathbf{x}$ with a unit norm, the matrix norm of $\mathbf{A}$ is the maximum norm possible after transforming $\mathbf{x}$ by $\mathbf{A}$. Further it can be shown that

  $$\max_{\|\mathbf{x}\|_2=1} \|\mathbf{A}\mathbf{x}\|_2 = \sigma_{max}.$$

Thus, $\|\mathbf{A}\|_2 = \sigma_{max}$

The *condition number* of a matrix κ is the ratio of the largest to smallest singular value of a matrix. That is

$$\kappa = \frac{\max_i \sigma_i}{\min_i \sigma_i}$$

1.5.7   The Pseudo-Inverse

The pseudo inverse (sometimes called the Moore-Penrose psuedoinverse) of an $m \times n$ matrix $\mathbf{A}$ with rank is $R(\mathbf{A}) = min(m, n)$ is defined as

$$\mathbf{A}^\dagger = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T$$

If $R(\mathbf{A}) < min(m, n)$, then the pseudo inverse can be defined through the SVD of $\mathbf{A}$. Specifically, if $\mathbf{A} = \mathbf{U}\begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^T$ is the SVD of $\mathbf{A}$, then the pseudo inverse is defined as

$$\mathbf{A}^\dagger = \mathbf{V}\begin{pmatrix} \mathbf{D}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^T$$

Thus, when considering the psuedoinverse of $\mathbf{A}$, the singular values of the inverse are equal to $\frac{1}{\sigma_i}$ where σ_i are the non-zero singular values of $\mathbf{A}$.

Additionally, the right singular vectors of $\mathbf{A}^{\dagger}$ are the left singular vectors of $\mathbf{A}$ and vice-versa.

Example 1.5.2. Consider the matrix $\mathbf{A} = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 6 & 4 \\ 1 & 1 & 0 \end{bmatrix}$. Determine a SVD for $\mathbf{A}$. What does the SVD tell us?

Solution: The SVD is written as $\mathbf{UDV}^{T} = \mathbf{A}$. Using Matlab we can find one possible solution:

$$\mathbf{U} = \begin{bmatrix} -0.54 & 0.75 & -0.37 \\ -0.83 & -0.55 & 0.09 \\ -0.13 & 0.36 & 0.92 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 8.9 & 0 & 0 \\ 0 & 2.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} -0.38 & 0.72 & 0.58 \\ -0.82 & 0.03 & -0.58 \\ -0.43 & -0.69 & 0.58 \end{bmatrix}$$

We can see that the rank of $\mathbf{A}$ is two and it is a singular matrix. There are only two modes, one much stronger than the other.

Example 1.5.3. Consider the matrix $\mathbf{B} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Determine a SVD for $\mathbf{B}$. What does the SVD tell us? Provide an SVD for $\mathbf{B}^{-1}$.

Solution: The SVD is written as $\mathbf{UDV}^T = \mathbf{B}$. Using Matlab we can find one possible solution:

$$\mathbf{U} = \begin{bmatrix} -0.45 & 0.82 & -0.35 \\ -0.87 & -0.5 & -0.04 \\ -0.21 & 0.28 & 0.94 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 7.3 & 0 & 0 \\ 0 & 2.1 & 0 \\ 0 & 0 & 0.58 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} -0.45 & 0.82 & -0.35 \\ -0.87 & -0.50 & -0.04 \\ -0.21 & 0.28 & 0.94 \end{bmatrix}$$

We can see that the rank of $\mathbf{B}$ is three (i.e., it is non-singular). There are three modes, although the third is very weak. Further, we can find $\mathbf{B}^{-1} = \mathbf{VD}^{-1}\mathbf{U}^T$.

1.5.8 Eigenvalues and Eigenvectors

For an $n \times n$ matrix $\mathbf{A}$, the scalars λ and $n \times 1$ vectors $\mathbf{x} \neq \mathbf{0}$ satisfying

$$\mathbf{Ax} = \lambda\mathbf{x}$$

are called the **eigenvalues** and **eigenvectors** of $\mathbf{A}$. The set of distinct eigenvalues $\lambda(\mathbf{A})$ is known as the **spectrum** of $\mathbf{A}$.

Some additional notes on eigenvalues and eigenvalues:

* The absolute value of the maximum eigenvalue is also known as the **spectral radius** of $\mathbf{A}$.

* The eigenvectors corresponding to distinct eigenvalues is a linearly independent set.

* If no eigenvalue of $\mathbf{A}$ is repeated, then $\mathbf{A}$ is diagonalizable.

* Symmetric and Hermitian matrices have real eigenvalues.

1.5.9 Eigenvalues vs Singular Values

For $\mathbf{A} \in \mathcal{C}^{m \times n}$ with $R(\mathbf{A}) = r$:

* The nonzero eigen values of $\mathbf{A}^H\mathbf{A}$ and $\mathbf{AA}^H$ are equal and positive.

* The nonzero singular values of $\mathbf{A}$ are equal to the positive square roots of the non-zero eigenvalues of $\mathbf{A}^H\mathbf{A}$

1.5.10   Eigen Decomposition

A square $n \times n$ matrix $\mathbf{A}$ that has n linearly independent eigenvectors can be factorized into

$$\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^{-1} \tag{1.13}$$

where the i column of $\mathbf{Q}$ corresponds to the ith eigenvector of $\mathbf{A}$ and Λ is diagonal with the ith element of the diagonal equal to the ith eigenvalue of $\mathbf{A}$. If $\mathbf{A}$ is also symmetric, then

$$\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^{T}$$

Positive Definite Matrices

If $\lambda_i > 0 \forall i$, then we can set $\mathbf{B} = \Lambda^{1/2}\mathbf{Q}^{T}$ and

$$\mathbf{A} = \mathbf{B}^{T}\mathbf{B} \tag{1.14}$$

Such a matrix when $\mathbf{B}$ is non-singular is termed **positive definite**. In other words, consider the eigenpair $(\lambda, \mathbf{x})$:

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{1.15}$$

Multiplying both sides by $\mathbf{x}^{T}$ and solving for λ gives

$$\begin{aligned} \mathbf{x}^{T}\mathbf{A}\mathbf{x} &= \mathbf{x}^{T}\lambda\mathbf{x} \\ \lambda &= \frac{\mathbf{x}^{T}\mathbf{A}\mathbf{x}}{\mathbf{x}^{T}\mathbf{x}} \end{aligned}$$

Continuing,

$$\begin{aligned} \lambda &= \frac{\mathbf{x}^{T}\mathbf{B}^{T}\mathbf{B}\mathbf{x}}{\mathbf{x}^{T}\mathbf{x}} \\ &= \frac{\|\mathbf{B}\mathbf{x}\|_2^2}{\|\mathbf{x}\|_2^2} \end{aligned}$$

If $\mathbf{B}$ is non-singular all eigenvalues will be positive and thus we say that $\mathbf{A}$ is **positive definite**.

However, if $\mathbf{B}$ is singular, all of the eigenvalues will be non-negative and we say that $\mathbf{A}$ is **positive semi-definite**.

Notes on Eigendecomposition

The matrix $\mathbf{B}$ in the previous decomposition is not unique.

However, there is only one upper triangular matrix with positive diagonals $\mathbf{R}$ such that $\mathbf{A} = \mathbf{R}^{T}\mathbf{R}$. This is known as the **Cholesky Factorization** of $\mathbf{A}$ as discussed above.

For a square matrix, the inverse of a matrix can be found from its eigendecomposition. Specifically,

$$\mathbf{A}^{-1} = \mathbf{Q}\Lambda^{-1}\mathbf{Q}^{-1}$$

Formal Definition of Positive Definiteness

If a matrix $\mathbf{A}$ is Hermitian, it is **positive definite** if for any vector $\mathbf{x} \neq \mathbf{0}$

$$\mathbf{x}^H \mathbf{A} \mathbf{x} > 0$$

$\mathbf{A}$ is **semipositive definite** if for any vector $\mathbf{x}$

$$\mathbf{x}^H \mathbf{A} \mathbf{x} \geq 0$$

$\mathbf{A}$ is **negative definite** if for any vector $\mathbf{x} \neq \mathbf{0}$

$$\mathbf{x}^H \mathbf{A} \mathbf{x} < 0$$

Note that $\mathbf{A}$ is positive definite if and only if all of its eigenvalues are positive.

Hermitian Matrices

If the $n \times n$ matrix $\mathbf{A}$ is Hermitian, its eigenvalues are real and we can decompose $\mathbf{A}$ as

$$\mathbf{A} = \mathbf{U}\Lambda\mathbf{U}^H$$

where

$$\mathbf{U} = [\mathbf{u}_1\mathbf{u}_2 \dots \mathbf{u}_n]$$

are the orthogonal eigenvectors. This means that $\mathbf{U}$ is unitary. Further, the diagonal matrix Λ is made up of the eigenvalues of $\mathbf{A}$. Note that since $\mathbf{U}$ is unitary, we can also write:

$$\mathbf{U}^H \mathbf{A} \mathbf{U} = \Lambda$$

Further,

$$\mathbf{A}\mathbf{U} = \mathbf{U}\Lambda$$

which means that for any eigenvector $\mathbf{u}_i$ we can write

$$\mathbf{A}\mathbf{u}_i = \lambda_i \mathbf{u}_i \tag{1.16}$$

Interpreting Eigenvalues

The eigenvectors are somewhat arbitrary. Any orthonormal basis will suffice. Once the eigenvectors are chosen, the eigenvalues tell us the real information about the matrix.

To see this consider a symmetric $n \times n$ matrix $\mathbf{A}$. The eigendecomposition can be written as

$$\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^T$$

Now consider the matrix applied to an arbitrary vector $\mathbf{x}$:

$$\begin{aligned} \mathbf{y} &= \mathbf{A}\mathbf{x} \\ &= \mathbf{Q}\Lambda\mathbf{Q}^T\mathbf{x} \end{aligned}$$

Let us consider the multiplication steps one at a time. The product $\mathbf{z}_1 = \mathbf{Q}^T\mathbf{x}$ can be written as

$$\mathbf{z}_1 = \begin{bmatrix} \mathbf{q}_1^T\mathbf{x} \\ \mathbf{q}_2^T\mathbf{x} \\ \vdots \\ \mathbf{q}_n^T\mathbf{x} \end{bmatrix}$$

Thus, the ith element of $\mathbf{z}_1$ is the inner product of the input vector x with the ith eigenvector. We can think of this as correlation (how much of each eigenvector is in $\mathbf{x}$).

Next, we have

$$
\begin{aligned}
\mathbf{z}_2 &= \Lambda\mathbf{Q}^T\mathbf{x} \\
&= \lambda\mathbf{z}_1 \\
&= \begin{bmatrix} \lambda_1\mathbf{q}_1^T\mathbf{x} \\ \lambda_2\mathbf{q}_2^T\mathbf{x} \\ \vdots \\ \lambda_n\mathbf{q}_n^T\mathbf{x} \end{bmatrix}
\end{aligned}
\tag{1.17}
$$

Thus, the next step is to weight each inner product with the appropriate eigenvalue. (We are weighting each correlation measure by their relative importance as determined by the eigenvalues.

Finally,

$$
\begin{aligned}
\mathbf{y} &= \mathbf{Q}\mathbf{z}_2 \\
&= \mathbf{Q}\begin{bmatrix} \lambda_1\mathbf{q}_1^T\mathbf{x} \\ \lambda_2\mathbf{q}_2^T\mathbf{x} \\ \vdots \\ \lambda_n\mathbf{q}_n^T\mathbf{x} \end{bmatrix} \\
&= \sum_{i=1}^{n} \lambda_i(\mathbf{q}_i^T\mathbf{x})\mathbf{q}_i
\end{aligned}
$$

Thus, we see that the output is the weighted sum of the eigenvectors. The weighting is the product of the correlation of the input with each eigenvector and the eigenvalue of interest. Thus, we can think of the matrix multiplication process as a decomposition, followed by a weighting, followed by a synthesis.

Example 1.5.4. What does the previous analysis tell us about the eigenvalues of an $n \times n$ identity matrix?

Example 1.5.5. Consider the matrix $\mathbf{B} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 6 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Determine an eigen decomposition for

B. What does the decomposition tell us? Compare to the SVD of this matrix. What does this suggest about the SVD and eigen decomposition of symmetric matrices?

Solution: The eigen decomposition is written as $\mathbf{Q\Lambda Q}^T = \mathbf{B}$. Using Matlab we can find one possible solution:

$$\mathbf{Q} = \begin{bmatrix} -0.35 & 0.82 & 0.45 \\ -0.04 & -0.5 & 0.87 \\ 0.94 & 0.28 & 0.21 \end{bmatrix}$$

$$\mathbf{\Lambda} = \begin{bmatrix} 0.58 & 0 & 0 \\ 0 & 2.1 & 0 \\ 0 & 0 & 7.3 \end{bmatrix}$$

We can see that the eigenvalues are equal to the singular values. Further, we see from the SVD that $\mathbf{U} = \mathbf{V}$. Also, we can see that the eigen decomposition and the SVD are the same fo r**B**. For symmetric matrices we can conclude that the two decompositions are the same.

Example 1.5.6. Consider the matrix $\mathbf{C} = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 6 & -1 \\ -1 & -1 & -1 \end{bmatrix}$. Determine an eigen decomposition

for **C**. What does the decomposition tell us? Compare to the SVD of this matrix. What does this suggest about the SVD and eigen decomposition of symmetric matrices?

Solution: The eigen decomposition is written as $\mathbf{Q\Lambda Q}^T = \mathbf{B}$. Using Matlab we can find one possible solution:

$$\mathbf{Q} = \begin{bmatrix} -0.35 & 0.82 & 0.45 \\ -0.04 & -0.5 & 0.87 \\ 0.94 & 0.28 & 0.21 \end{bmatrix}$$

$$\mathbf{\Lambda} = \begin{bmatrix} 0.58 & 0 & 0 \\ 0 & 2.1 & 0 \\ 0 & 0 & 7.3 \end{bmatrix}$$

We can see that the eigenvalues are equal to the singular values. Further, we see from the SVD that $\mathbf{U} = \mathbf{V}$. Also, we can see that the eigen decomposition and the SVD are the same fo r**B**. For symmetric matrices we can conclude that the two decompositions are the same.

1.5.11   Range Space

The **range space** of an $m \times n$ matrix $\mathbf{A}$ is the set of $m \times 1$ vectors Y generated as $\mathbf{y} = \mathbf{A}\mathbf{x}$ when $\mathbf{x} \in \Re^n$.

The range space is denoted as $col(\mathbf{A})$ and is a subspace of $\Re^m$.

$col(\mathbf{A})$ is the space spanned by the columns of $\mathbf{A}$, also called the column space of $\mathbf{A}$.

$col(\mathbf{A}^T)$ is the space spanned by the rows of $\mathbf{A}$, also called the row space of $\mathbf{A}$, denoted as $row(\mathbf{A})$.

1.5.12   Using the Matrix Inverse

A system of n linear equations in n unknowns can be written as a single matrix equation as

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

where $\mathbf{x}$ is an n-dimensional vector of the n unknowns, $\mathbf{A}$ is the $n \times n$ matrix of coefficients, and $\mathbf{b}$ is an $n \times 1$ vector.

If the matrix $\mathbf{A}$ is nonsingular, the solution is given by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Note: In general, computing the inverse is a computationally intensive operation, and thus solving systems of equations are typically done in other ways.

1.5.13   DFT Matrix

The DFT matrix (also called the Fourier matrix), denoted $\mathbf{D}_N$, is an $N \times N$ matrix where the elements are defined as $[\mathbf{D}]_{m,n} = e^{-j2\pi(m-1)(n-1)/N}$. Note that multiplying an N-element column vector $\mathbf{x}$ by $\mathbf{D}_N$ creates the DFT of $\mathbf{x}$. That is the DFT of a vector $\mathbf{x}$ is defined as

$$X[k] = \mathcal{F}\{x[n]\} = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi}{N}kn} \tag{1.18}$$

This can be written as

$$\mathbf{X} = \mathbf{D}_N\mathbf{x} \tag{1.19}$$

where $\mathbf{x}, \mathbf{X} \in \mathcal{C}^N$. Similarly, $\mathbf{D}_N^H$ is the IDFT matrix since $\mathbf{x} = \mathbf{D}_N^H\mathbf{X}$.

$\mathbf{D}_N^H$ is IDFT matrix

1.5.14   Circulant Matrices

A *circulant* matrix is a matrix entirely defined by any of its rows, as all other rows are simply circular shifts of that first row with offsets defined by the row indices. Alternatively a circulant matrix is described by any of its columns with all other columns simply circular shifts of the first column. Note that $\mathbf{D}_N$ is a circular matrix.

Note that if an $N \times N$ matrix $\mathbf{A}$ is circulant, the following important properties hold:

* The eigenvectors of $\mathbf{A}$ equal the columns of the DFT matrix $\mathbf{D}_N$.

* The eigenvalues of $\mathbf{A}$ equal the entries of $\mathbf{D}_N\mathbf{a}$ where $\mathbf{a}$ is any column of $\mathbf{A}$. In other words, the eigenvalues are the DFT of any of the columns.

1.5.15 The Kronecker Product

The Kronecker product is analogous to the outer product in vector multiplication. Specifically, if we have an $N \times M$ matrix $\mathbf{A}$ and a $K \times T$ $\mathbf{A}$ the Kronecker product is an $NK \times MT$ matrix $\mathbf{C}$:

$$\mathbf{C} = \mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} [\mathbf{A}]_{1,1}\mathbf{B} & \cdots & [\mathbf{A}]_{1,M}\mathbf{B} \\ \vdots & \ddots & \vdots \\ \mathbf{A}]_{N,1}\mathbf{B} & \cdots & [\mathbf{A}]_{N,M}\mathbf{B} \end{bmatrix} \tag{1.20}$$

Note that the Kronecker product as the following properties:

* $(\mathbf{A} \otimes \mathbf{B})^* = \mathbf{A}^* \otimes \mathbf{B}^*$

* $(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1}$

Part II

Multi-Input Multi-Output Transmission and Reception

Chapter 2

The MIMO Wireless Channel

2.1 Introduction

In this chapter, we describe the various effects of the wireless channel, with an emphasis on the *spatial* channel, since that is of primary importance in MIMO systems. We start by describing the three primary propagation effects: (1) the impact of distance, (2) the impact of motion, and (3) the impact of multipath. The first effect is primarily a large-scale effect (i.e., we notice its impact over large distances - say greater than 10m), while the second and third effects are small-scale effects (their effects are noticed over small distances - say smaller than a few meters). In a later chapter we will revisit the wireless channel with a focus on frequency selectivity as it is of primary importance when studying OFDM. For more complete treatments of the wireless channel, including the MIMO channel, please see [4], Chapter 3, [5], Chapter 3, as well as classic discussions of the SISO channel in [6, 7, 9].

2.2 Basic Propagation Effects

2.2.1 Impact of Distance

Consider a sinusoidal transmit signal at frequency f in a single dimension measured by a perfect antenna at a distance d from the transmitter. The measured electric field (ignoring polarization) is

$$
\begin{aligned}
E(f,t,d) &= \frac{\cos\left(2\pi \mathrm{f}(\mathrm{t} - \mathrm{d/c})\right)}{d} \\
&= \frac{1}{d}\cos\left(2\pi \mathrm{ft} - \phi(\mathrm{d})\right)
\end{aligned}
\tag{2.1}
$$

where $\phi(d) = \frac{2\pi f d}{c}$ and c is the speed of light.

From this equation, we can see that the sinusoidal field is attenuated and phase shifted depending on the distance d. The phase shift is of marginal consequence, but the attenuation due to distance is a major effect. This inverse relationship to distance results in a power loss relative to d^2. Specifically, it can be shown that the received signal power P_r in a free-space environment (i.e.,

in the absence of objects) can be written as

$$P_r = P_t G_t G_r \left(\frac{\lambda}{4\pi d} \right)^2 \tag{2.2}$$

where P_t is the transmit power, G_t and G_r are the transmit and receive antenna gains along the direction of the signal propagation, and $\lambda = c/f$ is the wavelength of transmission. This is known as the Friis Transmission Formula. From this equation we can see that the received signal power decreases with d^2. Additionally, for constant antenna gains, the received power also decreases with frequency.

The above received power formula applies only to free-space. However, it has been found empirically that in more complicated environments, the large scale path loss increases with d^n where n is the termed the path loss exponent and is generally $2 < n < 5$. In other words,

$$P_r = P_o \left(\frac{d_o}{d} \right)^n \tag{2.3}$$

where P_o is a reference power received at reference distance d_o. The reference distance is generally smaller than the distance being characterized d.

Path loss is a *large scale* effect in that it doesn't change significantly over small distances (say one meter). Changes in the received signal power over smaller distances (or time) is due to the impact of multipath and Doppler shift. These *small scale* effects will be discussed next.

2.2.2   Movement and Doppler Shift

Again consider a sinusoidal transmit signal, but with a mobile receiver[1]. Now if the receive antenna (starting from distance d_o) moves with velocity v away from the transmit antenna, the measured field is

$$
\begin{aligned}
E(f,t,d) &= \frac{\cos\left(2\pi f\left(t - \frac{d_o + vt}{c}\right)\right)}{d_o + vt} \\
&= \frac{1}{d_o + vt}\cos\left(2\pi(f - f_d)t - \phi_o\right) \tag{2.4}
\end{aligned}
$$

where $f_d = \frac{fv}{c} = \frac{v}{\lambda}$ is the *Doppler Shift* and $\phi_o = \frac{2\pi f d_o}{c}$. For small changes in distance,

$$E(f,t,d) \approx \frac{1}{d_o}\cos\left(2\pi(f - f_d)t - \phi_o\right) \tag{2.5}$$

And the primary impact that we observe due to movement is a Doppler shift. Note that in general, the shift in frequency due to movement is $f_i = f_d \cos(\theta)$ where $f_d = \frac{v}{\lambda}$ is the maximum Doppler shift and θ is the angle of arrival of the signal relative to the velocity. In the case above, $\theta = 180^o$ resulting in the largest negative shift of f_d. Had the receiver been moving *towards* the transmitter ($\theta = 0$), the frequency shift would have been positive f_d.

---

[1] Although we focus on a mobile receiver, a mobile transmitter and static receiver would cause the same effect.

2.2.3 Multipath

The third major effect due to the wireless channel is multipath. That is, because the transmitter and receiver are not located in free space, the receiver will observe multiple versions of the signal which have been reflected, scattered or diffracted by objects in the environment. The multiple copies of the signal will arrive at different relative delays and relative strengths. The difference in delay will cause two primary effects. If the delay is large relative to the modulated symbol time, the received signal will experience *inter-symbol interference or ISI*. We will discuss this effect in more detail later. Secondly, due to the fact that the carrier frequency is typically large (hundreds of MHz to tens of GHz), even small delay differences will cause substantial phase differences between the two sinusoids. The phase differences can (and will) cause destructive interference.

To see this, consider the situation depicted in Figure 2.1. In this example there is a perfect reflector located $d_r = 10$m from the transmitter which is transmitting a continuous tone signal at frequency f. We examine the magnitude of the received E-field at various distances d between the transmitter and receiver. The received E-field can be written as (note the negative sign due to the perfect reflector):

$$E(f,t,d) = \frac{\cos\left(2\pi f\left(t - \frac{d}{c}\right)\right)}{d} - \frac{\cos\left(2\pi f\left(t - \frac{2d_r - d}{c}\right)\right)}{2d_r - d}$$

Thus, at the receiver we have the sum of two sinusoids with the same frequency, f, but different gains and phases. The phase difference is

$$\begin{aligned} \Delta\phi &= \frac{2\pi f(2d_r - d)}{c} + \pi - \frac{2\pi f d}{c} \\ &= \frac{4\pi f(d_r - d)}{c} + \pi \end{aligned} \tag{2.6}$$

Thus:

* When $\Delta\phi = 2k\pi$ for integer k, the two sinusoids will add *constructively* and the signal due to two paths is larger than without multipath.

* When $\Delta\phi = k\pi$ for odd integer k, the two signals add *destructively*.

* Thus a pattern of constructive/destructive interference is created that gradually attenuates due to longer distances. Specifically, the peak received magnitude will decay with $\frac{1}{d}$.

To make this more concrete, consider an example where $t = 0.01$, $f = 2GHz$, $c = 2.98 \times 10^8$ and $d_r = 10$m. The magnitude of the E-field is plotted in Figure 2.2. We can see that there are repeating patterns of constructive and destructive interference. Note that because the reflection is immediately behind the receiver, changing the distance dm causes a $2d/\lambda$ change in the phase difference. As a result, we see the pattern repeat at multiples of $\lambda/2 = 0.075$m in this case. Additionally, the overall received signal amplitude is decaying at $\frac{1}{d}$ as expected. (This corresponds to a power decay of $\frac{1}{d^2}$).

Of course, if there is multipath and receiver movement, the phase relationship of the two paths will vary with time due to the time varying phase of each multipath component (i.e., their respective frequency shifts).
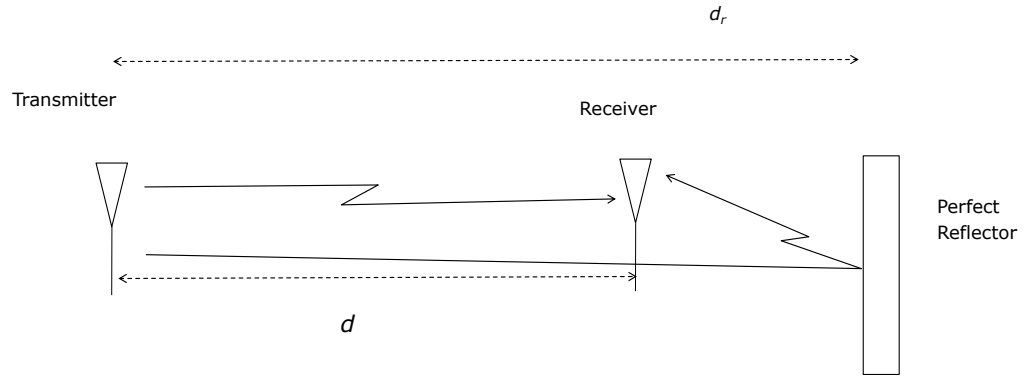
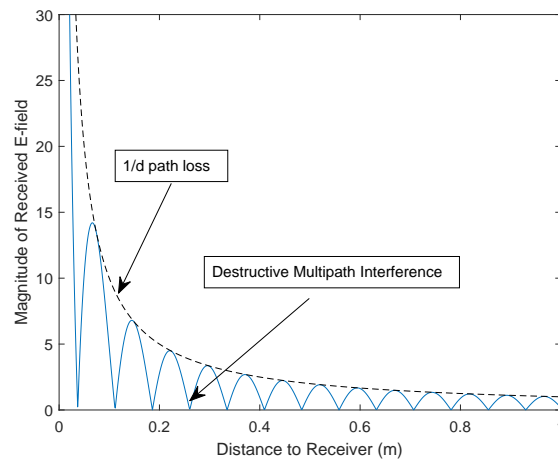Figure 2.1: Example of Multipath with a Perfect Reflector

Figure 2.2: Example of Multipath Interference with a Perfect Reflector. Note that the electric field strength (plotted for a fixed time versus distance) decays with distance and experiences destructive multipath interference at multiples of half the wavelength.

2.2.4 Small and Large Scale Effects

The first propagation effect discussed (attenuation vs distance) is typically termed a *large scale effect* whereas the second and third propagation effects are termed *small scale effects*. The reason for this is that the attenuation over distance doesn't change appreciably over small relative distance changes, whereas the small scale effects do. For example consider Figure 2.3. The figure shows the magnitude of the received E-field for the example in Figure 2.2 when the receiver is moving towards the reflector at $v = 5$m/s and $d_r = 150m$. The overall attenuation due to distance is clearly seen in left hand plot which shows the received signal magnitude over 50m. However, the impact of multipath and Doppler is not visible. On the right, when the magnitude is viewed over a short distance (less than a meter) the attenuation due to distance is not discernible, but the multipath/Doppler effects clearly are. Thus, we refer to the effects which are discernible over large distances as large scale effects, and those which are discernible over short distances as small scale effects.
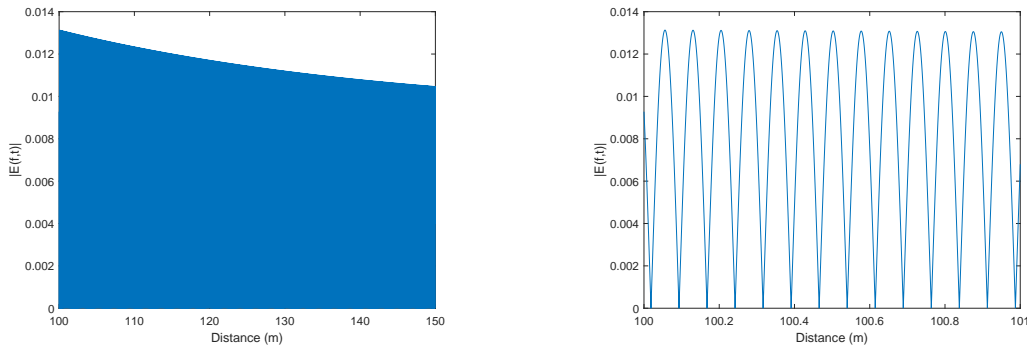
Figure 2.3: Large vs Small Scale Propagation Effects: The magnitude of the received E-field at a given distance when there is a perfect reflector placed at 150m from the transmitter and the receiver is traveling away from the transmitter (and towards the reflector) at 5m/s.

2.3 The General Wireless Channel in Complex Baseband

Thus, there are two dominant small-scale effects in the wireless channel: multipath and Doppler. In general, the received signal is a superposition of scaled and time-delayed versions of the transmit signal where the scaling, delay and number of multipath components is time-varying. This general scenario is shown in Figure 2.4. Together, multipath and Doppler result in the phenomenon known as *fading*, time-varying signal strength due to the changing phase relationships between multipath components.

At bandpass we can write the transmit and receive signals (ignoring noise):

$$x(t) = \Re\left\{\tilde{x}(t)e^{j2\pi f_c t}\right\} \tag{2.7}$$
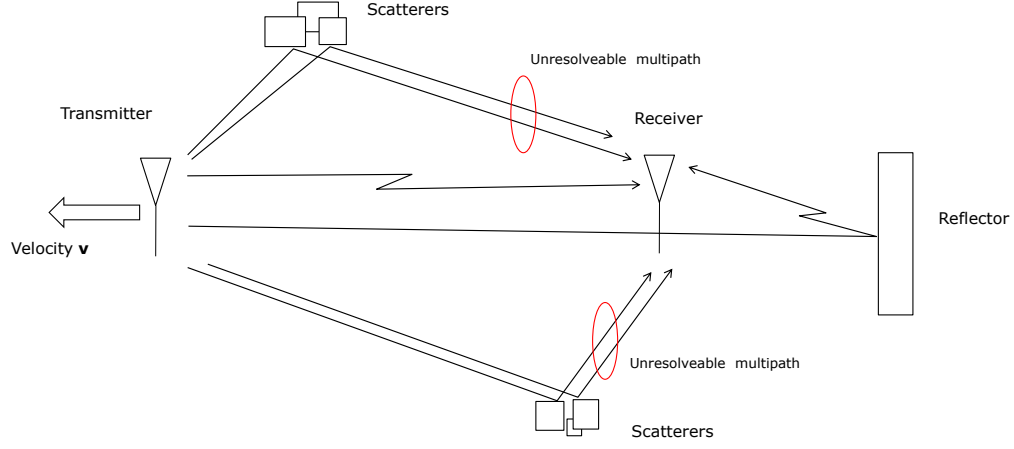
Figure 2.4: Depiction of the General Wireless Channel

$$r(t) = \Re \left\{ \sum_{i=1}^{N(t)} \alpha_i(t) \tilde{x}(t - \tau_i(t)) e^{j(2\pi f_c(t - \tau_i(t)) + \phi_i)} \right\} \tag{2.8}$$

Let $\phi_i(t) = 2\pi f_c \tau_i(t) + \phi_i$. Then,

$$r(t) = \Re \left\{ \left(\sum_{i=1}^{N(t)} \alpha_i(t) e^{j\phi_i(t)} \tilde{x}(t - \tau_i(t)) \right) e^{j2\pi f_c t} \right\} \tag{2.9}$$

Assuming a linear, time-varying channel, from linear system theory we can write represent the channel by its time-varying impulse response. Thus, the complex baseband received signal $\tilde{r}(t)$ can be written as the convolution of the complex baseband of the input and channel impulse responses. Thus, the received signal we can written as:

$$r(t) = \Re \left\{ \left[\underbrace{\left[\int_{-\infty}^{\infty} \tilde{h}(\tau, t) \tilde{x}(t - \tau) d\tau \right]}_{\tilde{r}(t)} \right] e^{j2\pi f_c t} \right\} \tag{2.10}$$

Which means that

$$\tilde{h}(\tau, t) = \sum_{i=1}^{N(t)} \tilde{\alpha}_i(t) \delta(\tau - \tau_i(t)) \tag{2.11}$$

is the time-varying impulse response of the channel in complex baseband and $\tilde{\alpha}_i(t) = \alpha_i(t) e^{j\phi_i(t)}$

is the complex channel gain of the ith multipath component. Further, the time-varying transfer function of the channel is

$$\tilde{H}(f,t) = \int_{-\infty}^{\infty} \tilde{h}(\tau,t)e^{-j2\pi f\tau}d\tau \tag{2.12}$$

2.4   Random Channels

In general it is impossible to know the channel $\tilde{h}(\tau,t)$ in advance or with certainty. Thus, when characterizing system performance we rely on statistical characterizations of the channel. There are three aspects of this statistical characterization which dictate link performance: delay spread, Doppler spread, and angle spread. Delay spread is a measure of the range of delays τ_i. If the delay spread is large (larger than the signal's modulated symbol duration T_s) inter-symbol interference occurs. In the frequency domain, we observe *frequency selective* fading since the channel transfer function is not flat over the signal bandwidth. In other words, the signal experiences different gains at different frequencies. This distortion is particularly harmful to signal detection. On the other hand if $\tau_i(t) << T_s$, $\forall i,t$, the fading experienced by the signal is termed *flat fading* and ISI is absent.

Doppler spread represents the range of Doppler shifts f_i experienced by the various multipath components. Clearly, from our previous discussion $-f_d < f_i < f_d$. Thus, the mobile velocity and transmit frequency dictate the Doppler spread. Doppler spread results in time-domain selectivity. If $f_d \propto \frac{1}{T_s}$ we term the fading *fast* fading. On the other hand, if $f_d << \frac{1}{T_s}$, the fading is termed *slow* fading.

Angle spread is a measure of the range of angles-of-arrival of the various multipath components. Clarke's model (which we will discuss next) assumes that the angle spread is 2π. Angle spread dictates spatial fading which we will discuss later in this chapter. In general, the larger the angle spread, the more spatially selective the channel is.

2.4.1   Spatial Selectivity and Correlation

Define the received signal power as a function of angle-of-arrival θ as the power spectrum $S_A(\theta)$. Then, assuming that all multipath components have equal gain, we can write the complex baseband received signal (without noise) in terms of the transmit signal $x(t)$ as

$$y(t) = \int_{-\pi}^{\pi} \sqrt{S_A(\theta)G(\theta)}x(t - D(\theta)/c)d\theta \tag{2.13}$$

where $G(\theta)$ is the normalized receive antenna gain and $D(\theta)$ is the delay due to the angle-of-arrival θ and c is the speed of light. Assuming the transmit signal is a sinusoid, at complex baseband the signal is a constant $x(t - D(\theta)/c) = xe^{-j\phi(\theta)}$. Further, assuming an omni-directional antenna with normalized gain pattern $G(\theta) = 1$

$$y(t) = y = h_o x \tag{2.14}$$

where h_o is the channel:

$$h_o = \int_{-\pi}^{\pi} \sqrt{S_A(\theta)}e^{-j\phi(\theta)}d\theta \tag{2.15}$$

Now, if we consider another point in space, d meters away, we find the channel to be

$$h_1 = \int_{-\pi}^{\pi} \sqrt{S_A(\theta)} e^{-j[\phi(\theta)+2\pi d/\lambda \sin(\theta)]} d\theta \tag{2.16}$$

where θ is defined relative to a line normal to the line connecting the two points[2] The correlation of the channels between those two points can be found as

$$\rho(d) = \mathrm{E} \left\{ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sqrt{S_A(\theta_1)} e^{-j\phi(\theta_1)} \sqrt{S_A(\theta_2)} e^{j[\phi(\theta_2)+2\pi d/\lambda \sin(\theta_2)]} d\theta_1 d\theta_2 \right\} \tag{2.17}$$

where the expectation is over $\phi(\theta)$. Making the reasonable assumption that $\phi(\theta)$ is uniformly distributed over $[-\pi, \pi)$, the expectation is zero unless $\theta_1 = \theta_2$., thus

$$\rho(d) = \int_{-\pi}^{\pi} S_A(\theta_1) e^{j2\pi d/\lambda \sin(\theta)} d\theta \tag{2.18}$$

Thus, we can see that the power azimuth spectrum directly controls the spatial correlation. We will show later that the larger the spread of the power azimuth spectrum, the faster the signal decorrelates in space.

2.4.2   Temporal Selectivity and Correlation

This result can also be used to determine the temporal correlation. Specifically, if we assume that the receiver is moving at velocity v, after τ seconds the receiver has moved $v\tau$ meters. Thus, using a simple substitution of variables we can calculate the temporal correlation $\tilde{R}(\tau)$ as

$$\tilde{R}(\tau) = \int_{-\pi}^{\pi} S_A(\theta) e^{j2\pi \frac{v}{\lambda} \tau \sin(\theta)} d\theta \tag{2.19}$$

$$= \int_{-\pi}^{\pi} S_A(\theta) e^{j2\pi f_d \tau \sin(\theta)} d\theta \tag{2.20}$$

We see that the temporal correlation at time delay τ depends directly on the maximum Doppler spread $f_d = \frac{v}{\lambda}$. Later we will see that, as one would expect, larger f_d leads to faster decorrelation in time.

2.4.3   Frequency Selectivity and Correlation

The power delay profile $\phi_h(\tau)$ is defined as the expected power of the channel impulse response as a function of delay τ:

$$\phi_h(\tau) = \mathrm{E} \left\{ \tilde{h}(\tau, t) \tilde{h}^*(\tau, t) \right\} \tag{2.21}$$

$$= \mathrm{E} \left\{ |\tilde{h}(\tau, t)|^2 \right\} \tag{2.22}$$

---

[2]It should be noted that this definition of the direction of arrival is somewhat different than the definition we used for determining the Doppler shift. In that case, the AOA was defined relative to the velocity.

Using this definition, we can write the frequency correlation (assuming wide sense stationarity) as

$$R_h(\Delta f) \quad = \quad \mathrm{E}\left\{ \tilde{H}(f,t)\tilde{H}^*(f+\Delta f,t) \right\} \tag{2.23}$$

$$= \quad \mathrm{E}\left\{ \int_{-\infty}^{\infty} \tilde{h}(\tau,t)e^{-j2\pi f\tau}d\tau \int_{-\infty}^{\infty} \tilde{h}^*(\tau,t)e^{j2\pi(f+\Delta f)\tau}d\tau \right\} \tag{2.24}$$

$$= \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{E}\left\{ h(\tau_1,t)h^*(\tau_2,t) \right\} e^{j2\pi[(f+\Delta f)\tau_2 - f\tau_1]}d\tau_1 d\tau_2 \tag{2.25}$$

Now, assuming uncorrelated scattering,

$$\mathrm{E}\left\{ h(\tau_1,t)h^*(\tau_2,t) \right\} = 0 \qquad \tau_1 \neq \tau_2 \tag{2.26}$$

Thus,

$$R_h(\Delta f) \quad = \quad \int_{-\infty}^{\infty} \mathrm{E}\left\{ |h(\tau,t)|^2 \right\} e^{j2\pi\Delta f\tau}d\tau \tag{2.27}$$

$$= \quad \int_{-\infty}^{\infty} \phi_h(\tau)e^{j2\pi\Delta f\tau}d\tau \tag{2.28}$$

2.5   Narrowband Fading

In the early days of mobile communication, the signal bandwidth was fairly small. Thus, a common assumption in channel modeling was that the channel bandwidth was significantly larger than the signal bandwidth (i.e., fading was flat). As data rates increased, this assumption was no longer valid and the channel was observed to be frequency selective. Many developed channel models made the narrowband assumption. While, clearly that is not valid for modern communication systems, the advent of OFDM made the narrowband channel models relevant again. Specifically, since OFDM transmits a large number of narrowband signals, each narrowband signal will typically experience flat fading, and the narrowband assumption becomes valid. For the remainder of Part II, we will make the narrowband assumption in order to focus on the spatial effects of the channel which dominate MIMO link performance. We will however, revisit this narrowband assumption when we examine OFDM.

2.6   Clarke's Model for SISO Channels

Let us assume that the transmit signal is $x(t) = \cos(2\pi f_c t)$. The received signal is comprised of a large number of multipath components arriving from different directions θ_i with respect to the velocity vector $\mathbf{v}$ and thus each has a different Doppler shift:

$$f_i = \frac{v}{\lambda}\cos(\theta_i) \tag{2.29}$$

Clarke's Model [7, 9] makes a *Narrowband Assumption* that $\delta(\tau - \tau_i) = \delta(\tau - \tau_o)\forall i$ (see Figure 2.5). In other words, the difference in time-of-arrival of all multipath components is very small relative to the symbol duration and thus unresolvable. This is equivalent to the channel having a bandwidth that is much greater than the signal bandwidth (i.e., we assume that the signal is narrow relative to the channel).
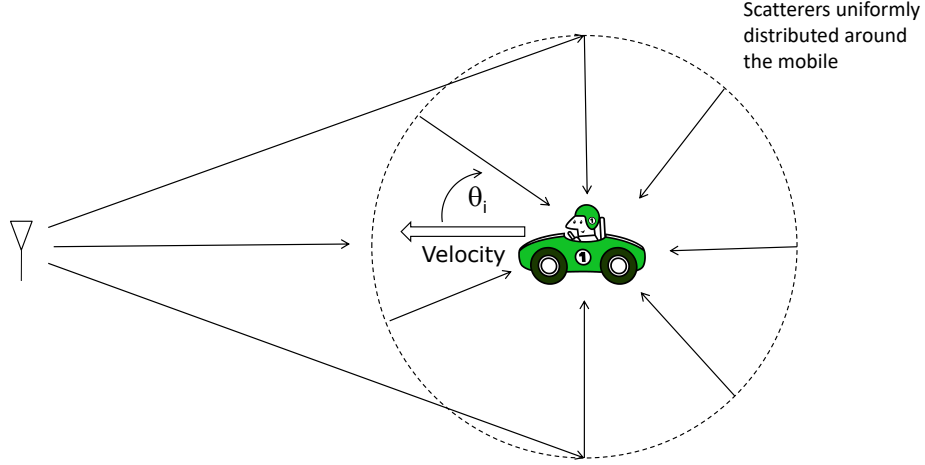
Figure 2.5: Scenario Assumed in Jakes/Clarke's Model. Scatterers causing multipath at the mobile receiver are uniformly distributed in angle around the mobile. θ_i is the angle-of-arrival of ith multipath component relative to the mobile velocity.

With a sinusoidal transmit signal (constant at baseband), we have

$$r(t) = \sqrt{2P_{avg}} \sum_{i=1}^{N} c_i \cos\left(2\pi f_c t + 2\pi f_i t + \phi_i\right) \tag{2.30}$$

where the P_{avg} is the average received power, $\sum_i c_i^2 = 1$ and ϕ_i is assumed to be random and uniformly distributed over $[0, 2\pi)$ or deterministic and equally spaced over $[0, 2\pi)$. We can re-write this as

$$r(t) = \sqrt{P_{avg}} \sum_{i=1}^{N} \left[c_i \cos\left(2\pi f_i t + \phi_i\right) \cos\left(2\pi f_c t\right) - c_i \sin\left(2\pi f_i t + \phi_i\right) \sin\left(2\pi f_c t\right)\right] \tag{2.31}$$

Thus, in complex baseband

$$\begin{aligned}
\tilde{r}(t) &= \sqrt{P_{avg}} \sum_{i=1}^{N} \left[c_i \cos\left(2\pi f_i t + \phi_i\right) + j c_i \sin\left(2\pi f_i t + \phi_i\right)\right] \\
&= \sqrt{P_{avg}} \sum_{i=1}^{N} c_i e^{j(2\pi f_i t + \phi_i)} \\
&= \sqrt{P_{avg}} \sum_{i=1}^{N} c_i e^{j(2\pi f_d \cos(\theta_i) t + \phi_i)}
\end{aligned}$$

$$\tag{2.32}$$

where $f_d = \frac{v}{\lambda}$ is the maximum Doppler shift. Further, in this model we typically set $c_i = \frac{1}{\sqrt{N}}$.

From the above development we can see that $\tilde{h}(\tau, t) \approx \tilde{\gamma}(t)\delta(\tau)$ and thus $\tilde{r}(t) \approx \tilde{\gamma}(t)\tilde{x}(t)$.

2.6.1 Rayleigh and Rician Fading

From the Central Limit Thereom, as $N \to \infty$ the real ($\tilde{r}_I(t)$) and imaginary ($\tilde{r}_Q(t)$) portions of $\tilde{r}(t)$ tend towards Gaussian random processes. Further, since ϕ_i is uniformly distributed, they will be independent Gaussian random processes with equal power $P_{avg}/2$. In this case, the magnitude of the complex baseband channel $\alpha(t)$

$$\alpha(t) = \sqrt{\tilde{r}_I^2(t) + \tilde{r}_Q^2(t)} \tag{2.33}$$

will be *Rayleigh Distributed* and we have what is called the *Rayleigh Channel* or *Rayleigh Fading*. Specifically, if we let

$$\sigma^2 = \frac{P_{avg}}{2} = \text{E}\left\{\tilde{r}_I^2(t)\right\} = \text{E}\left\{\tilde{r}_Q^2(t)\right\} \tag{2.34}$$

we have the distribution of the envelope:

$$f_\alpha(x) = \begin{cases} \frac{x}{\sigma^2} exp\left(-\frac{x^2}{2\sigma^2}\right) & 0 \leq x \leq \infty \\ \\ 0 & else \end{cases} \tag{2.35}$$

Note that we can write the impulse response for this model as

$$\tilde{h}(\tau, t) = \left\{\frac{1}{\sqrt{N}}\sum_{i=1}^{N} e^{-j(2\pi f_i t + \phi_i)}\right\}\delta(\tau) \tag{2.36}$$

which is a narrowband model (no frequency selectivity) but is also time-varying.

For a Rician channel, there is a direct LOS component. Specifically, if $\tilde{h}^{Ra}(t)$ is a unit power Rayleigh fading channel, then a Rician fading channel can be written as

$$\tilde{h}^{Ri}(t) = \sqrt{\frac{K}{K+1}} + \sqrt{\frac{1}{K+1}}\tilde{h}^{Ra}(t) \tag{2.37}$$

where $\tilde{h}^{Ri}(t)$ is a unit power Rician fading channel where the K-factor is the ratio of the power of the direct LOS component and the diffuse (Rayleigh) component. Note that $K \to \infty$ results in an AWGN channel and $K \to 0$ results in a Rayleigh channel. An example plot of Rayleigh and Ricean distributions is given in Figure 2.6 for a Ricean K-factor of $K = 10$. We can see that the probability of low amplitudes (close to zero) is much smaller for the Ricean channel due to the strong LOS component.

2.6.2 Temporal Correlation in Clarke's Model

We typically define the autocorrelation function of the received signal (assuming that the signal is wide sense stationary) as

$$R(\tau) = \text{E}\left\{r(t)r(t+\tau)\right\} \tag{2.38}$$

or in complex baseband

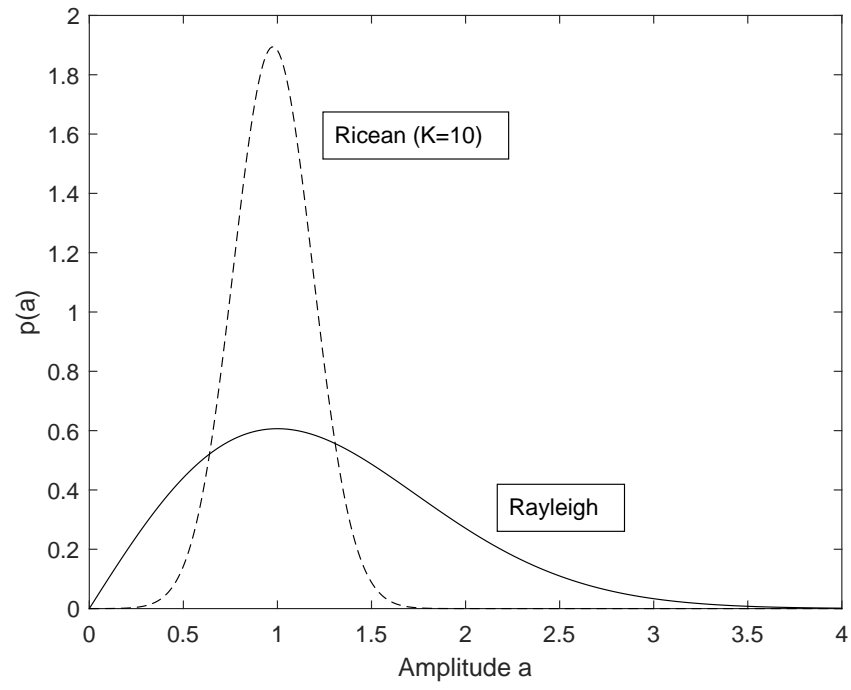$$\tilde{R}(\tau) = \text{E}\left\{\tilde{r}(t)\tilde{r}^*(t+\tau)\right\} \tag{2.39}$$

Figure 2.6: Rayleigh and Ricean (K=10) Probabilty Density Functions for Unit Average Energy.

where for Clarke's model, the expectation is over c_i, θ_i and ϕ_i. Assuming a sinusoidal transmit signal (to highlight the impact of the channel) we have

$$
\begin{aligned}
\tilde{R}(\tau) &= \mathrm{E}\left\{\tilde{r}(t)\tilde{r}(t+\tau)\right\} \\
&= \mathrm{E}\left\{\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}e^{-j(2\pi f_i t+\phi_i)}\right)\left(\frac{1}{\sqrt{N}}\sum_{k=1}^{N}e^{j(2\pi f_k(t+\tau)+\phi_k)}\right)\right\} \\
&= \mathrm{E}\left\{\frac{1}{N}\sum_{i=1}^{N}\sum_{k=1}^{N}e^{-j(2\pi f_d\cos(\theta_i)\mathrm{t}-2\pi f_d\cos(\theta_k)\mathrm{t}+\phi_i-\phi_k)}\right\}
\end{aligned}
\tag{2.40}
$$

Taking the expectation over uniform distribution of θ_i, θ_k, ϕ_i and ϕ_k, we find that the expectation is zero for $i \neq k$. Thus, we have

$$
\begin{aligned}
\tilde{R}(\tau) &= \mathrm{E}\left\{\frac{1}{N}\sum_{i=1}^{N}e^{-j2\pi f_d\cos(\theta_i)\tau}\right\} \\
&= \frac{1}{N}\sum_{i=1}^{N}\int_{0}^{2\pi}\frac{1}{2\pi}e^{-j2\pi f_d\cos(\theta_i)\tau}d\theta_i \\
&= \frac{1}{2\pi}\int_{0}^{2\pi}e^{-j2\pi f_d\cos(\theta)\tau}d\theta \\
&= J_o\left(2\pi f_d\tau\right)
\end{aligned}
\tag{2.41}
$$

where J_o is the zeroth order modified bessel function of the first kind. Note that all N integrals are the same since θ_i are i.i.d.

Alternatively, we can use the expression in (2.20) to determine temporal correlation using $S_A(\theta) = \frac{1}{2\pi}$:

$$
\begin{aligned}
\tilde{R}(\tau) &= \int_{-\pi}^{\pi}\frac{1}{2\pi}e^{j2\pi f_d\tau\sin(\theta)}d\theta \tag{2.42} \\
&= J_o\left(2\pi f_d\tau\right) \tag{2.43}
\end{aligned}
$$

This function is plotted versus τf_d in Figure 2.7. If the maximum Doppler frequency f_d increases, the temporal correlation will change faster. Alternatively, for a lower maximum Doppler frequency, the temporal correlation is higher for a specific delay.

2.6.3 Doppler Power Spectrum for Clarke's Model

The total power at the mobile can be written as the integral of the azimuthal power spectrum $S_A(\theta)$:

$$
P_r = \int_{-\pi}^{\pi}S_A(\theta)G(\theta)d\theta
\tag{2.44}
$$

where $G(\theta)$ is the receive antenna gain. We would like to translate this to Doppler Power Spectrum $S_D(f)$. Now $f = f_d\cos(\theta)$. Thus, two angles contribute to f, $\pm\theta$. What we want is
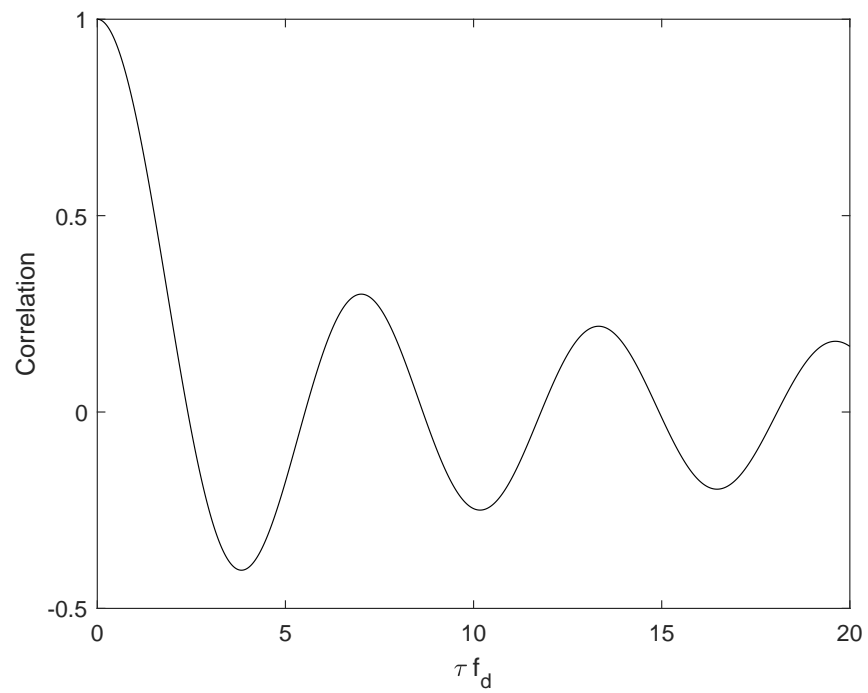
$$
P_r = \int_{-f_d}^{f_d}S_D(f)df
\tag{2.45}
$$

Figure 2.7: Temporal Correlation in Jakes/Clarkes Model. Note that the plot is normalized by the maximum Doppler frequency f_d.

Now, to convert from θ to f:

$$\theta \;=\; \cos^{-1}\left(\frac{\text{f}}{\text{f}_{\text{d}}}\right) \tag{2.46}$$

$$d\theta \;=\; -\frac{1}{f_d}\frac{1}{\sqrt{1-(f/f_d)^2}} \tag{2.47}$$

Thus,

$$P_r \;=\; \int_{-f_d}^{f_d}\left\{S_A\left(\left|\cos^{-1}\left(\frac{\text{f}}{\text{f}_{\text{d}}}\right)\right|\right)G\left(\left|\cos^{-1}\left(\frac{\text{f}}{\text{f}_{\text{d}}}\right)^{-1}\right|\right)+ \right.$$
$$\left. S_A\left(-\left|\cos^{-1}\left(\frac{\text{f}}{\text{f}_{\text{d}}}\right)\right|\right)G\left(-\left|\cos^{-1}\left(\frac{\text{f}}{\text{f}_{\text{d}}}\right)\right|\right)\right\}\frac{-df}{f_d\sqrt{1-(f/f_d)^2}} \tag{2.48}$$

Thus,

$$S_D(f) \;=\; \frac{1}{f_d\sqrt{1-(f/f_d)^2}}S_A\left(\left|\cos^{-1}\left(\frac{\text{f}}{\text{f}_{\text{d}}}\right)\right|\right)G\left(\left|\cos^{-1}\left(\frac{\text{f}}{\text{f}_{\text{d}}}\right)^{-1}\right|\right)+ \ldots$$
$$S_A\left(-\left|\cos^{-1}\left(\frac{\text{f}}{\text{f}_{\text{d}}}\right)\right|\right)G\left(-\left|\cos^{-1}\left(\frac{\text{f}}{\text{f}_{\text{d}}}\right)\right|\right) \tag{2.49}$$

For a uniform distribution in angle and a dipole antenna:

$$S_A(\theta) \;=\; \frac{1}{2\pi}$$
$$G(\theta) \;=\; \frac{3}{2}$$

we have

$$S_D(f) = \begin{cases} \frac{3}{2\pi f_d\sqrt{1-(f/f_d)^2}} & |f|\leq f_d \\ 0 & else \end{cases} \tag{2.50}$$

Finally, it can be shown that there is a close relationship between the Doppler spectrum and the correlation function. Specifically,

$$S_D(f) = \mathcal{F}\left\{\tilde{R}(\tau)\right\} \tag{2.51}$$

An example for $f_d = 100$Hz is plotted in Figure 2.8.

2.7 General Matrix Channel

Up to this point, we have considered a SISO channel (i.e., one transmit antenna and one receive antenna). Now, we will expand our discussion to a MIMO channel. Consider a set of N_t signals sent from N_t antennas written in vector complex baseband form as

$$\tilde{\mathbf{x}}(t) = \begin{bmatrix} \sum_{i=-\infty}^{\infty} s_{1,i}p(t-iT) \\ \sum_{i=-\infty}^{\infty} s_{2,i}p(t-iT) \\ \ldots \\ \sum_{i=-\infty}^{\infty} s_{N_t,i}p(t-iT) \end{bmatrix} \tag{2.52}$$

where $p(t)$ is the pulse shape used, $s_{j,i}$ is the ith symbol in time sent from the jth antenna. Let the channel between any transmit/receive pair be a slow, flat fading channel so that the received
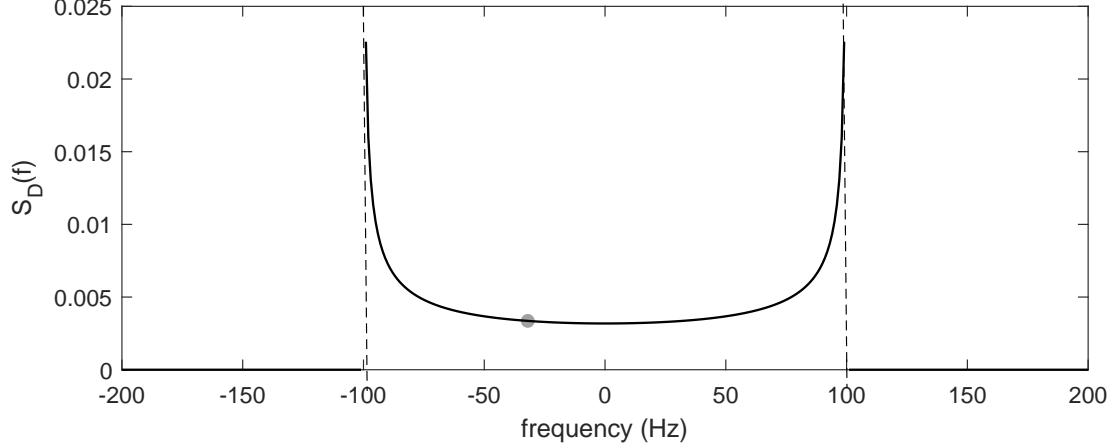
Figure 2.8: Doppler Spectrum in Jakes/Clarkes Model. Note that the plot assumes that the maximum Doppler frequency is f_d=100Hz.

signal on the ith receive antenna is

$$\tilde{r}_i(t) \quad = \quad \sum_{k=1}^{N_t} \int_{-\infty}^{\infty} \tilde{h}_{i,k}(\tau,t)x_k(t-\tau)d\tau + \tilde{n}_i(t) \tag{2.53}$$

$$= \quad \sum_{k=1}^{N_t} \sum_{m=-\infty}^{\infty} \tilde{h}_{i,k}[mT]s_{k,i}p(t-mT) + \tilde{n}_i(t) \tag{2.54}$$

where $h_{i,k}(\tau,t)$ is the channel impulse response between the kth transmit antenna and the ith receive antenna. Our assumption that the channel is flat (i.e., the Narrowband Assumption) means that the transmit signal is not distorted (no inter-symbol interference) and experiences a single time-varying gain. The assumption that the channel is slow means that the channel remains constant over a symbol. Thus, for any particular symbol, the received symbol on the i antenna from kth transmitter can be written as $h_{i,k}[mT]s_{k,m}p(t-mT)$. $\tilde{n}_i(t)$ is the noise process observed on the ith receive antenna (generally assumed to be white Gaussian noise).

The received signal during the mth time slot is passed through a matched filter and sampled:

$$r_i[m] \quad = \quad \int_{(m-1)T/2}^{mT/2} \tilde{r}_i(t)p(t-mT)dt \tag{2.55}$$

$$= \quad \sum_{k=1}^{N_t} s_{k,m}\tilde{h}_{i,k}[m] + \tilde{n}_i[m] \tag{2.56}$$

where $x[n] = x(nT)$. Dropping the dependence on time (and again remembering the Narrowband Assumption), we have

$$\tilde{r}_i \quad = \quad \sum_{k=1}^{N_t} s_k\tilde{h}_{i,k} + \tilde{n}_i \tag{2.57}$$

$$= \quad \tilde{\mathbf{h}}_i^T\mathbf{s} + \tilde{n}_i \tag{2.58}$$

where $\mathbf{s}$ is a vector of symbols transmitted from N_t antennas and $\tilde{\mathbf{h}}_i$ is the complex channel coefficients seen at the ith receiver from each of the N_t antennas. If there are M_r receive antennas, we can write the vector of received signals (after matched filtering and sampling) as:

$$\tilde{\mathbf{r}} = \tilde{\mathbf{H}}\mathbf{s} + \tilde{\mathbf{n}} \qquad (2.59)$$

where $\tilde{\mathbf{r}}$ is the receive signal vector, $\mathbf{s}$ is the vector of transmit symbols, $\tilde{\mathbf{n}}$ is a vector of complex AWGN noise samples with $\mathrm{E}\{\tilde{\mathbf{n}}\} = \mathbf{0}^{M_r \times 1}$ and $\mathrm{E}\{\tilde{\mathbf{n}}\tilde{\mathbf{n}}^H\} = \sigma^2\mathbf{I}$, and $\tilde{\mathbf{H}}$ is the $M_r \times N_t$ channel matrix. $\left[\tilde{\mathbf{H}}\right]_{i,j}$ is the complex channel coefficient from the jth transmit antenna to the ith receive antenna.

If all transmit/receive channels are independent Rayleigh faded channels $\tilde{\mathbf{H}}$ is a matrix of independent complex Gaussian random variables, i.e., $\tilde{\mathbf{H}} = \tilde{\mathbf{H}}_w$.
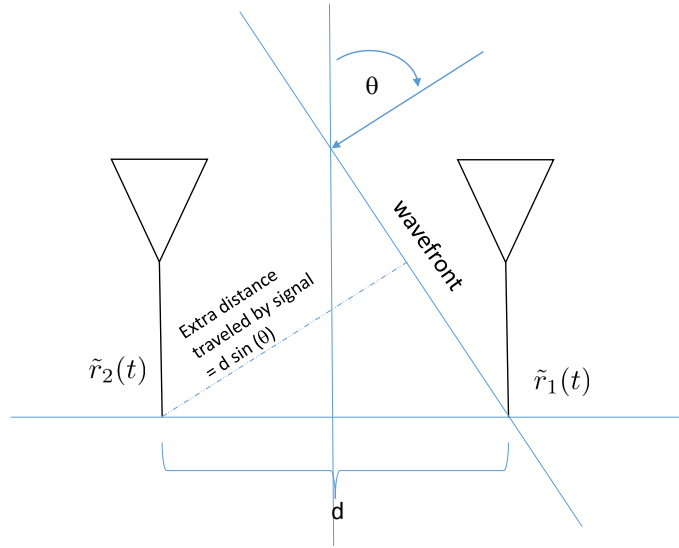
Figure 2.9: Wavefront impinging on two antennas (plane-wave assumption). Note that θ is defined here to be relative to the array normal.

2.8 Spatial Correlation

Consider a single multipath component received at two antennas $\tilde{r}_1(t)$ and $\tilde{r}_2(t)$ separated by distance d. The correlation between the two received signals (ignoring noise) can be written as

$$\rho(d) \quad = \quad \mathrm{E}\{\tilde{r}_1(t)\tilde{r}_2^*(t)\} \qquad (2.60)$$

For a single multipath component arriving from angle θ (note that we have temporarily defined θ to be relative to the array normal as shown in Figure 2.9) we have

$$\tilde{r}_2(t) = \tilde{r}_1(t)e^{-j2\pi\frac{d}{\lambda}\sin(\theta)} \tag{2.61}$$

Thus, we have

$$\rho(d) = \mathrm{E}\left\{\tilde{r}_1(t)\tilde{r}_1^*(t)e^{j2\pi\frac{d}{\lambda}\sin(\theta)}\right\} \tag{2.62}$$

Clearly, the overall correlation will depend on the angular distribution of the arriving multipath. Assuming that the received signal is wide-sense stationary, that $E\{|\tilde{r}_1(t)|\} = P_r$ and the angle of arrival is uniformly distributed on $[0, 2\pi)$:

$$
\begin{aligned}
\rho(d) &= P_r\mathrm{E}\left\{e^{j2\pi\frac{d}{\lambda}\sin(\theta)}\right\} \\
&= \int_{-\pi}^{\pi} \frac{1}{2\pi}e^{j2\pi\frac{d}{\lambda}\sin(\phi)}d\phi \\
&= J_o\left(\frac{2\pi d}{\lambda}\right) \tag{2.63}
\end{aligned}
$$

where we assumed in the second step that $P_r = 1$. NOTE: The imaginary part is zero. We could have also arrived at this result by using $S_A(\theta) = \frac{1}{2\pi}$ in (2.18).

Now, consider a Gaussian distribution[3] for $S_A(\theta)$ with central angle θ_o and standard deviation σ_A

$$
\begin{aligned}
\rho(d) &= \int_{-\pi}^{\pi} S_A(\theta)e^{j2\pi d/\lambda\sin(\theta)}d\theta \\
&= \int_{-\pi}^{\pi} e^{j2\pi\frac{d}{\lambda}\sin(\theta)} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\theta - \theta_o)^2}{2\sigma_A^2}\right)d\theta
\end{aligned}
$$

Let $z = \frac{\theta - \theta_o}{\sigma_A}$ and $dz = \frac{d\theta}{\sigma_A}$.

$$
\begin{aligned}
\rho(d) &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{j2\pi\frac{d}{\lambda}\sin(\sigma_A z + \theta_o)} \exp\left(-\frac{z^2}{2}\right)dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{j2\pi\frac{d}{\lambda}(\sin(\sigma_A z)\cos(\theta_o) + \cos(\sigma_A z)\sin(\theta_o))} \exp\left(-\frac{z^2}{2}\right)dz
\end{aligned}
$$

Assuming relatively small angle spread $\sigma_A z$ will be small where $\exp{-z^2/2}$ is significant. Thus, we can make the approximation

$$
\begin{aligned}
\sin(\sigma_A z) &\approx \sigma_A z \\
\cos(\sigma_A z) &\approx 1
\end{aligned}
$$

---

[3]We note that the distribution is *approximately* Gaussian since it has finite support. Technically it is a truncated Gaussian, but we assume that σ_A is small enough that the assumption is valid.

Thus,

$$
\begin{aligned}
\rho(d) \quad &\approx \quad \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{j2\pi \frac{d}{\lambda}(\sigma_A z \cos(\theta_o) + \sin(\theta_o))} \exp\left(-\frac{z^2}{2}\right) dz \\
&= \quad \frac{e^{j2\pi \frac{d}{\lambda}\sin(\theta_o)}}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{j2\pi \frac{d}{\lambda}\sigma_A z \cos(\theta_o)} \exp\left(-\frac{z^2}{2}\right) dz \qquad (2.64) \\
&= \quad e^{j2\pi \frac{d}{\lambda}\sin(\theta_o)} e^{-\left(\frac{\left(2\pi \frac{d}{\lambda}\sigma_A \cos(\theta_o)\right)^2}{2}\right)} \qquad\qquad\qquad (2.65)
\end{aligned}
$$

As an example, consider $\sigma_A = 5^o$ with a central AOA of zero. (In the equation above, σ is defined in radians. For this example $\sigma_A = 0.0873$ radians.) The spatial correlation is clearly much higher (for the same separation distance) for the Gaussian distribution. In general, we can show an inverse relationship between angular spread (some measure of the width of the angle power distribution) and and spatial correlation. That is, for a given separation distance, a smaller angular spread results in higher correlation. The separation required to obtain a specific correlation (magnitude)
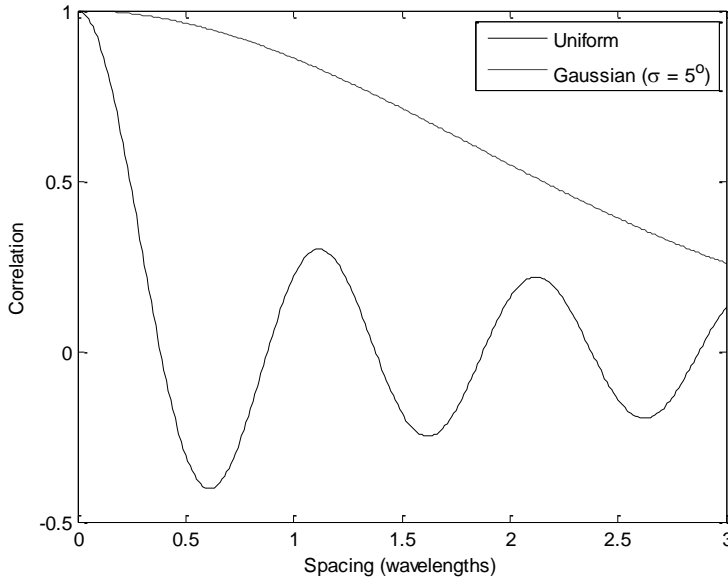
Figure 2.10: Spatial Correlation for Uniform Distribution and Gaussian Distribution ($\sigma = 5^o$)

can be found by taking the magnitude of (2.65) and solving for $\frac{d}{\lambda}$:

$$
\begin{aligned}
|\rho(d)| \quad &\approx \quad e^{-\left(\frac{\left(2\pi \frac{d}{\lambda}\sigma_A \cos(\theta_o)\right)^2}{2}\right)} \\
-\ln\left(|\rho(d)|\right) \quad &= \quad \frac{\left(2\pi \frac{d}{\lambda}\sigma_A \cos\left(\theta_o\right)\right)^2}{2} \\
\frac{d}{\lambda} \quad &= \quad \frac{\sqrt{-2\ln\left(|\rho(d)|\right)}}{2\pi\sigma_A \cos\left(()\,\theta_o\right)} \qquad (2.66)
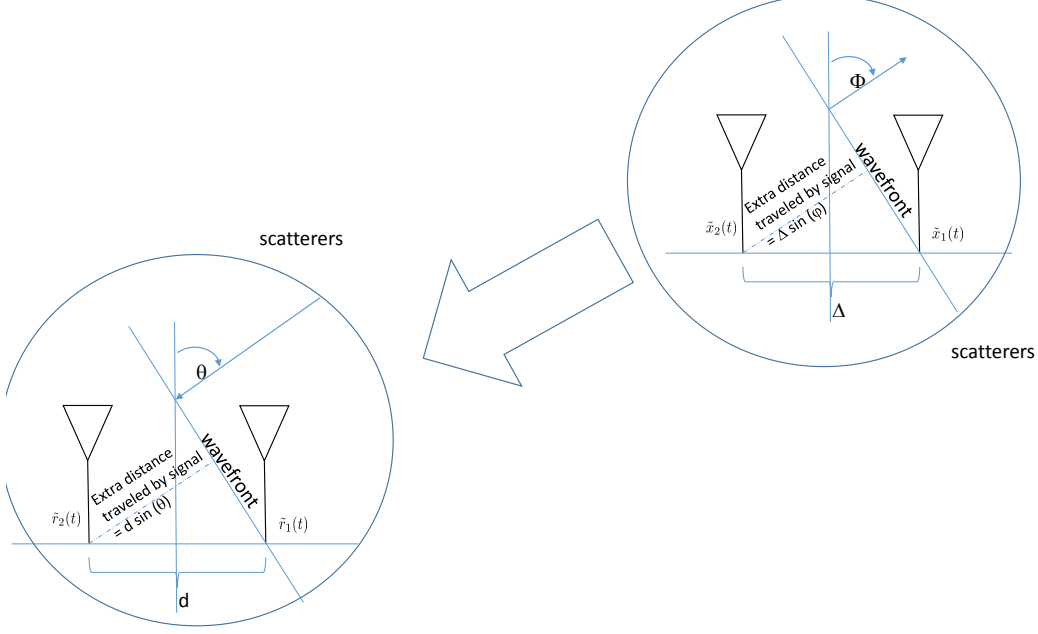\end{aligned}
$$

Figure 2.11: Assumed Geometry for Spatial Jakes Model

2.9   Matrix (Spatial) Version of Clarke's Model

Clarke's narrowband SISO model discussed earlier can be extended to a MIMO channel. To understand this, we will first extend Clarke's model to SIMO channels and then MISO channels. We can then combine those two models into a full MIMO model.

2.9.1   Clarke's Model for SIMO Channels

For SIMO channels, there is a single transmit antenna and multiple receive antennas. The received signal is then an $M_r \times 1$ vector as is the channel $\tilde{\mathbf{h}}$:

$$\tilde{\mathbf{r}}(t) = \tilde{\mathbf{h}}(t)x(t) + \tilde{\mathbf{n}}(t) \tag{2.67}$$

where $x(t)$ is the scalar transmit signal and $\tilde{\mathbf{n}}(t)$ is AWGN with zero mean and covariance $\mathbf{I}_{M_r}$ (i.e., independent noise). The vector channel is

$$\tilde{\mathbf{h}}(t) = \begin{bmatrix} \tilde{h}_1(t) \\ \tilde{h}_2(t) \\ \vdots \\ \tilde{h}_{M_r}(t) \end{bmatrix} \tag{2.68}$$

Now, to see what $\tilde{h}_i(t)$ is let us consider transmitting a tone at a wavelength λ, which in complex baseband is $\tilde{x}(t) = 1$, the received signal on the first antenna based on the ith multipath component is

$$\tilde{r}_1(t) = e^{j(2\pi f_i t + \phi_i)} + \tilde{n}_1(t) \tag{2.69}$$

where $f_i = \frac{v}{\lambda}\cos\left(()\,\theta_i\right)$ is the Doppler shift due to velocity v of the ith component which is arriving from angle θ_i. Assuming a linear array with the first element serving as the phase reference[4], the same multipath component at the second antenna d m away can be written as

$$\tilde{r}_2(t) = e^{j\left(2\pi f_i t + \phi_i - 2\pi \frac{d}{\lambda}\sin(()\theta_i)\right)} + \tilde{n}_2(t) \tag{2.70}$$

Generalizing we can write the received vector based on the ith multipath component as

$$\tilde{\mathbf{r}}(t) = \begin{bmatrix} e^{j(2\pi f_i t + \phi_i)} + \tilde{n}_1(t) \\ e^{j\left(2\pi f_i t + \phi_i - 2\pi \frac{d}{\lambda}\sin(()\theta_i)\right)} + \tilde{n}_2(t) \\ e^{j\left(2\pi f_i t + \phi_i - 2\pi \frac{2d}{\lambda}\sin(()\theta_i)\right)} + \tilde{n}_3(t) \\ \vdots \\ e^{j\left(2\pi f_i t + \phi_i - 2\pi \frac{(M_r-1)d}{\lambda}\sin(()\theta_i)\right)} + \tilde{n}_{M_r}(t) \end{bmatrix} \tag{2.71}$$

The vector SIMO channel (again making a Narrowband Assumption) is then the normalized sum of N multipath components:

$$\tilde{\mathbf{h}}(t) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} e^{j\left(2\pi f_i t + \phi_i - 2\pi \frac{\mathbf{d}}{\lambda}\sin(()\theta_i)\right)} \tag{2.72}$$

where $\mathbf{d}$ is a vector of distances from the first antenna and we define

$$e^{\mathbf{x}} = \begin{bmatrix} e^{[\mathbf{x}]_1} \\ e^{[\mathbf{x}]_2} \\ \vdots \\ e^{[\mathbf{x}]_n} \end{bmatrix} \tag{2.73}$$

$[\mathbf{x}]_i$ is the ith element of the length n vector $\mathbf{x}$. The covariance matrix for this channel is

$$\mathbf{R}_{rx} = E\left\{\tilde{\mathbf{h}}(t_o)\tilde{\mathbf{h}}^H(t_o)\right\} \tag{2.74}$$

and the expectation is over the random angles of arrival at time t_o. However, the channel is can be shown to be ergodic and thus the expectation can also be taken over time t for a fixed set of AOAs.

2.9.2 Clarke's Model for MISO Channels

The MISO extension of Clarke's model is similar to the SIMO case. The received signal can be written as a scalar

$$\tilde{r}(t) = \tilde{\mathbf{h}}^T(t)\tilde{\mathbf{x}}(t) + \tilde{n}(t) \tag{2.75}$$

where $\tilde{\mathbf{h}}(t)$ is now an $N_t \times 1$ vector. The transmit signal $\tilde{\mathbf{x}}(t)$ is also an $N_t \times 1$ vector. We will discuss what such a transmit vector might be later in the text. The vector channel in this case can be written as

$$\tilde{\mathbf{h}}(t) = \frac{1}{\sqrt{N}}\sum_{i=1}^{N} e^{j\left(2\pi f_i t + \phi_i - 2\pi \frac{\mathbf{\Delta}}{\lambda}\sin(()\Phi_i)\right)} \tag{2.76}$$

---

[4]The extension to other array geometries is straight-forward. The key is defining the phase reference point.

where $\boldsymbol{\Delta}$ is the vector of distances of the transmit antennas from the first (reference) antenna, again assuming a linear array. Φ_i is the angle of departure of the ith component from the transmit array. Note that the Doppler shift $f_i = \frac{v}{\lambda}\cos{(()\,\theta_\mathrm{i})}$ and θ_i is the angle of arrival assuming that the receiver is in motion. If the transmitter is in motion, θ_i should be replaced by Φ_i.

2.9.3   Clarke's Model for MIMO Channels

Now, let's consider the scenario depicted in Figure 2.11. If we consider a sinusoidal transmit signal and examine one multipath component from the first transmit antenna to the first receive antenna:

$$\tilde{r}_{11}(t) = e^{j(2\pi f_i t + \phi_i)} \tag{2.77}$$

where $f_i = \frac{v}{\lambda}\cos{(\theta_\mathrm{i})}$ assuming that the array baseline is perpendicular to the velocity vector and that the receiver is mobile (not the transmitter). Note that if the velocity were due to the transmitter movement, the angle-of-arrival θ_i should be replaced by the angle of departure at the transmitter, Φ_i. If we have a uniformly spaced linear array, the received signal at the other antennas can be written:

$$\begin{aligned}
\tilde{r}_{21}(t) &= e^{j(2\pi f_i t + \phi_i - 2\pi \frac{d}{\lambda}\sin(\theta_\mathrm{i}))} \\
\tilde{r}_{31}(t) &= e^{j(2\pi f_i t + \phi_i - 2\pi \frac{2d}{\lambda}\sin(\theta_\mathrm{i}))} \\
&\;\;\vdots \\
\tilde{r}_{M_r 1}(t) &= e^{j(2\pi f_i t + \phi_i - 2\pi \frac{(M_r - 1)d}{\lambda}\sin(\theta_\mathrm{i}))}
\end{aligned}$$

$$\tag{2.78}$$

Similarly, we can write the signal from the multiple transmit antennas (assuming a uniform linear array with spacing Δ) and a departure angle Φ_i seen at first receive antenna:

$$\begin{aligned}
\tilde{r}_{12}(t) &= e^{j(2\pi f_i t + \phi_i - 2\pi \frac{\Delta}{\lambda}\sin(\Phi_\mathrm{i}))} \\
\tilde{r}_{13}(t) &= e^{j(2\pi f_i t + \phi_i - 2\pi \frac{2\Delta}{\lambda}\sin(\Phi_\mathrm{i}))} \\
&\;\;\vdots \\
\tilde{r}_{1 N_t}(t) &= e^{j(2\pi f_i t + \phi_i - 2\pi \frac{(N_t - 1)\Delta}{\lambda}\sin(\Phi_\mathrm{i}))}
\end{aligned}$$

$$\tag{2.79}$$

In general,

$$x_{ab}(t) = e^{j\left(2\pi f_i t + \theta_i - \frac{2\pi}{\lambda}(a-1)d\sin(\phi_\mathrm{i}) - \frac{2\pi}{\lambda}(b-1)\Delta\sin(\Phi_\mathrm{i})\right)} \tag{2.80}$$

If we define the $M_r \times 1$ vector

$$\mathbf{d} = \begin{bmatrix} 0 \\ d \\ 2d \\ \vdots \\ (M_r - 1)d \end{bmatrix} \tag{2.81}$$

and $N_t \times 1$ vector

$$\overline{\Delta} = \begin{bmatrix} 0 \\ \Delta \\ 2\Delta \\ \vdots \\ (N_t - 1)\Delta \end{bmatrix} \tag{2.82}$$

we can write one component in the matrix channel as

$$\mathbf{H}_i(t) = \underbrace{e^{j(2\pi f_i t + \phi_i)}}_{\text{scalar,time−varyingcomponent}} \underbrace{e^{-j\left(\frac{2\pi}{\lambda} \mathbf{d} \sin(\theta_i)\right)}}_{\text{rxantennas}} \underbrace{e^{-j\left(\frac{2\pi}{\lambda} \overline{\Delta}^T \sin(\Phi_i)\right)}}_{\text{txantennas}} \tag{2.83}$$

Note that for a length N column vector $\mathbf{x}$, we define $e^{\mathbf{x}} = \begin{bmatrix} e^{x_1} \\ e^{x_1} \\ e^{x_1} \\ \vdots \\ e^{x_N} \end{bmatrix}$. Thus, the first term in (2.83)

is a scalar (representing the time varying component), the second term is $M_r \times 1$ (representing the impact of the receive antennas) and the third term is $1 \times N_t$ (representing the impact of the transmit antennas) meaning that each mulipath component results in an $M_r \times N_t$ term in the sum. Finally, the narrowband Clarkes $M_r \times N_t$ Matrix channel can be written as

$$\begin{aligned}
\mathbf{H}(t) &= \sqrt{\frac{1}{N}} \sum_{i=1}^{N} \mathbf{H}_i(t) \\
&= \sqrt{\frac{1}{N}} \sum_{i=1}^{N} e^{j(2\pi f_i t + \phi_i)} e^{-j\left(\frac{2\pi}{\lambda} \mathbf{d} \sin(\theta_i)\right)} e^{-j\left(\frac{2\pi}{\lambda} \overline{\Delta}^T \sin(\Phi_i)\right)}
\end{aligned} \tag{2.84}$$

where $f_i = \frac{v}{\lambda} \cos(\theta_i)$ (assuming that the receiver is moving. The distribution of θ_i and Φ_i depend on many factors. Typical assumptions include uniform (mobile), Gaussian, Laplacian (base station). The angle spread of the distribution determines the spatial correlation.

2.10   Matrix Correlation

Define matrix channel $\mathbf{H} = \begin{bmatrix} h_{11} & \cdots & \cdots & h_{1N_t} \\ h_{21} & \cdots & \cdots & h_{2N_t} \\ \vdots & & & \vdots \\ h_{M_r 1} & \cdots & \cdots & h_{M_r N_t} \end{bmatrix}$. We define the stacked vector

$$\mathbf{h}_{st} = vec\left\{\mathbf{H}\right\} = \begin{bmatrix} h_{11} \\ h_{21} \\ \vdots \\ h_{M_r 1} \\ h_{12} \\ h_{22} \\ \vdots \\ h_{M_r 2} \\ \vdots \\ h_{M_r N_t} \end{bmatrix} \tag{2.85}$$

The correlation matrix is defined as

$$\mathbf{R} = \mathrm{E}\left\{\mathbf{h}_{st}\mathbf{h}_{st}^{H}\right\} \tag{2.86}$$

2.10.1   Kronecker Model

The Kronecker model assumes that

$$\mathbf{R} = \mathbf{R}_{tx} \otimes \mathbf{R}_{rx} \tag{2.87}$$

where $\mathbf{R}_{tx}$ is the correlation across transmit antennas (termed *transmit correlation*), $\mathbf{R}_{rx}$ is the correlation across receive antennas (termed *receive correlation*), and $\otimes$ is the Kronecker product.

2.10.2   Example

Define a 2×2 channel matrix with $\mathrm{E}\left\{|h_{i,j}|^2\right\} = 1$

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \tag{2.88}$$

Determine the full correlation matrix $\mathbf{R}$ if the correlation between receive antennas is ρ and the correlation between transmit antennas is λ with and without the Kronecker assumption.

First, we can write

$$\mathbf{R}_{rx} = \mathrm{E}\left\{ \begin{bmatrix} h_{11} \\ h_{21} \end{bmatrix} \begin{bmatrix} h_{11}^* & h_{21}^* \end{bmatrix} \right\} = \begin{bmatrix} 1 & \rho \\ \rho^* & 1 \end{bmatrix} \tag{2.89}$$

$$\mathbf{R}_{rx} = \mathrm{E}\left\{\begin{bmatrix} h_{12} \\ h_{22} \end{bmatrix} \begin{bmatrix} h_{12}^* & h_{22}^* \end{bmatrix}\right\} = \begin{bmatrix} 1 & \rho \\ \rho^* & 1 \end{bmatrix} \tag{2.90}$$

$$\mathbf{R}_{tx} = \mathrm{E}\left\{\begin{bmatrix} h_{11} \\ h_{12} \end{bmatrix} \begin{bmatrix} h_{11}^* & h_{12}^* \end{bmatrix}\right\} = \begin{bmatrix} 1 & \lambda \\ \lambda^* & 1 \end{bmatrix} \tag{2.91}$$

$$\mathbf{R}_{tx} = \mathrm{E}\left\{\begin{bmatrix} h_{21} \\ h_{22} \end{bmatrix} \begin{bmatrix} h_{21}^* & h_{22}^* \end{bmatrix}\right\} = \begin{bmatrix} 1 & \lambda \\ \lambda^* & 1 \end{bmatrix} \tag{2.92}$$

However, this doesn't completely define the channel correlation. We can see this by looking at $\mathbf{R}$:

$$\mathbf{R} = \mathrm{E}\left\{\begin{bmatrix} h_{11} \\ h_{21} \\ h_{12} \\ h_{22} \end{bmatrix} \begin{bmatrix} h_{11}^* & h_{21}^* & h_{12}^* & h_{22}^* \end{bmatrix}\right\} = \begin{bmatrix} 1 & \rho & \lambda & ? \\ \rho^* & 1 & ? & \lambda \\ \lambda^* & ? & 1 & \rho \\ ? & \lambda^* & \rho^* & 1 \end{bmatrix} \tag{2.93}$$

Thus, without the Kronecker assumption, we cannot complete the correlation matrix. The Kronecker Model thus assumes that

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_{rx} & \lambda\mathbf{R}_{rx} \\ \lambda^*\mathbf{R}_{rx} & \mathbf{R}_{rx} \end{bmatrix} \tag{2.94}$$

$$= \begin{bmatrix} 1 & \rho & \lambda & \lambda\rho \\ \rho^* & 1 & \lambda\rho^* & \lambda \\ \lambda^* & \lambda^*\rho & 1 & \rho \\ \lambda^*\rho^* & \lambda^* & \rho^* & 1 \end{bmatrix} \tag{2.95}$$

Thus, with the Kronecker assumption, we can complete the correlation matrix.

2.11 Conclusion

Overcoming the wireless channel is the dominant challenge in wireless communications. Understanding multipath fading is a key to understanding the benefits (and thus prevalence) of MIMO-OFDM communications. In this chapter we have examined the key aspects of the MIMO wireless channel including the temporal, spatial and frequency statistics of the channel. We have primarily focused on the first two aspects, but will return to the frequency statistics later in this manuscript.

Chapter 3

Introduction to MIMO

3.1   Introduction

What is MIMO communications? As we have already defined it, MIMO is "Multi-input Multi-Output" communications, i.e., a communication system that uses mutiple antennas at the transmitter and/or receiver. In this brief chapter, we will first introduce the primary uses of multiple antennas at the transmitter and receiver. In the previous chapter we spent considerable time examing the wireless channel, especially that wireless channel in the presence of multiple antennas. It is the wireless channel that motivates the use of multiple antennas in modern communication systems. However, there are benefits to MIMO even in the absence of multipath, as we will discuss in this chapter. Specifically, there are four primary benefits to the use of MIMO: (a) average SNR gain, (b) diversity gain, (c) spatial multiplexing, and (d) interference mitigation. While the second and third benefits are exclusive to the multipath channel, the first and last gains can be obtained even in the absence of multipath.

We will begin by looking at uses of multiple antennas at the receiver only (commonly referred to as SIMO - single input, multiple output links). When we have multiple antennas at the receiver but only a single antenna at the transmitter, we can use those multiple receive antennas for three purposes: (1) SNR gain, (2) diversity gain, and (3) interference mitigation. Note that spatial multiplexing is not possible in SIMO, as we will discuss later. These three gains are not necessarily mutually exclusive when applied at the receiver. For example, SNR gain and diversity gain can typically be obtained simultaneously (note that this is not always the case at the transmitter). On the other hand, diversity gain and mitigation of interference cannot necessarily be obtained simultaneously.

It is also important to note that the relationship between the channel and the antenna array plays an major role in the gains obtainable. Specifically, in order for diversity gain to be realized, the antennas must be spaced at distances greater than the spatial coherence distance.

After discussing the use of multiple antennas at the receiver, we will discuss the use of multiple antennas at the transmitter only (MISO links). Generally speaking, the same gains can be ob-

tained (i.e., SNR gain, diversity gain and interference mitigation) using multiple antennas at the transmitter, although channel knowledge is required at the transmitter.

Finally, we will discuss spatial multiplexing which requires multiple antennas at both the transmitter and receiver. For MIMO channels, SNR gain, diversity gain and interference mitigation are all possible. However, unlike the MISO and SIMO cases, MIMO also offers the opportunity to increase throughput through spatial multiplexing, *i.e.,* the transmission of multiple parallel data streams (i.e., independent data signals in the same frequency band at the same time). Note that in this chapter we are simply introducing the concepts. We will explore each of these topics in greater detail in the following chapters.

3.2 Multiple Antennas at the Receiver (SIMO)

Our first discussion will focus on the use of multiple antennas at the receiver, the so-called "single input multiple output" or SIMO system. We first briefly investigate the benefit in the absence of multipath (at least absence at the receiver).

3.2.1 SIMO in the Absence of Multipath - Receive Beamforming

The first case to consider is the absence of multipath at the receiver, i.e., where the multiple antennas are located. This case represents a plane wave arriving at the array and is shown in Figure 3.1. More specifically, consider that the channel is constant (i.e., non time-varying) and equal to $\tilde{\alpha}\tilde{\mathbf{h}}$. The received signal (in complex baseband) at multiple antennas can be written as a vector

$$\tilde{\mathbf{r}}(t) = \tilde{\alpha}\tilde{\mathbf{h}}x(t) + \mathbf{n}(t) \tag{3.1}$$

where $\tilde{\alpha} = \alpha e^{j\phi}$ represents the impact of the channel between the transmitter and receive antenna array, $\mathbf{n}(t)$ is a complex vector Gaussian random process representing AWGN with zero mean and $\mathrm{E}\left\{\mathbf{n}(t)\mathbf{n}^H(t)\right\} = \sigma^2\mathbf{I}$, and the transmit signal is a pulse modulated pulse stream $x(t) = \sum_{i=-\infty}^{\infty} s_i p(t - iT)$, $\mathrm{E}\left\{|s_i|^2\right\} = 1$, $\int_{-\infty}^{\infty} p^2(t)dt = 1$. The vector $\tilde{\mathbf{h}}$ represents the impact of the receive array. More specifically, h_i ($|h_i| = 1$) is a complex phasor representing the phase shift observed at the ith antenna. At the output of a matched filter at the νth symbol time we have

$$\tilde{\mathbf{r}}[\nu] = \tilde{\alpha}s_\nu\tilde{\mathbf{h}} + \mathbf{n}[\nu] \tag{3.2}$$

The SNR at each antenna is the same and found to be

$$\gamma_i = \frac{\mathrm{E}\left\{|r_i|^2\right\}}{\sigma^2} = \frac{|\tilde{\alpha}|^2}{\sigma^2} \tag{3.3}$$

Consider that we have a plane wave impinging on a uniform linear array (i.e., the antennas are evenly spaced in a line). The channel vector elements are then

$$h_i = e^{-j2\pi\frac{d}{\lambda}(i-1)\sin(\theta)} \tag{3.4}$$

where θ is defined according to Figure 3.1, i.e., relative to the normal of the array. Now in receive beamforming, we combine the outputs of each antenna using a set of complex weights. That is, let the output from each antenna be weighted with a vector of weights $\mathbf{w}$ to give a decision variable
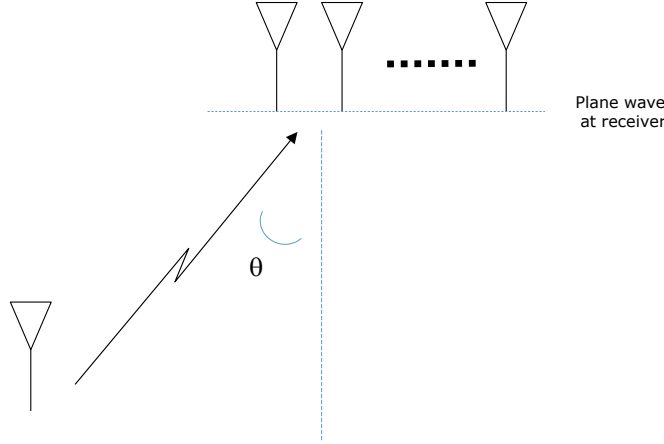
Figure 3.1: Signal Received at a Uniform Linear Array (SIMO) in Absence of Multipath

$z[\nu]$:

$$z[\nu] = \mathbf{w}^H \tilde{\mathbf{r}}[\nu] \tag{3.5}$$

The weights are typically chosen to optimize some criterion, as we will discuss in a later chapter. For now, let $\mathbf{w} = \tilde{\mathbf{h}}\tilde{\alpha}$:

$$
\begin{aligned}
z[\nu] &= \tilde{\alpha}^* \sum_{i=1}^{M_r} e^{j2\pi \frac{d}{\lambda}(i-1)\sin(\theta)} r_i \\
&= M_r |\tilde{\alpha}|^2 s_\nu + \tilde{\alpha}^* \underbrace{\sum_{i=1}^{M_r} e^{j2\pi \frac{d}{\lambda}(i-1)\sin(\theta)} n_i[\nu]}_{GRV(0, M_r |\tilde{\alpha}|^2 \sigma^2)}
\end{aligned}
\tag{3.6}
$$

The SNR the resulting output is then found to be

$$\gamma = \frac{M_r^2 |\tilde{\alpha}|^4}{M_r \sigma^2 |\tilde{\alpha}|^2} = \frac{M_r |\tilde{\alpha}|^2}{\sigma^2} \tag{3.7}$$

Thus, we obtain an SNR improvement of M_r, which is typically called **beamforming gain**. Note that this provides the maximum SNR at the output of the combiner, as we will show later. Thus, in the absence of multipath and interference at the receiver, we can still obtain a benefit from the use of multiple antennas at the receiver.

3.2.2   SIMO Irrespective of Multipath - Interference Mitigation

A second benefit of using multiple antennas at the receiver is interference mitigation. This benefit can be achieved irrespective of the presence of multipath, and uses the degrees of freedom to

mitigate interference. To see this, consider the received signal on two antennas $r_1(t)$, $r_2(t)$ as

$$
\begin{aligned}
r_1(t) &= h_1 s(t) + h_1^i(t)i(t) + n_1(t) \\
r_2(t) &= h_2 s(t) + h_2^i(t)i(t) + n_2(t)
\end{aligned}
$$

where $s(t)$ is the signal of interest and $i(t)$ is an interfering signal. h_1 and h_2 represent the static channel between the transmitter of interest and the two receive antennas, while h_1^i and h_2^i represent the static channel between the interfering transmitter and the two receive antennas. $n_i(t)$ represents AWGN on the ith receive antenna. The two receive antennas could be used for SNR gain or diversity gain as discussed above. However, in this case, the interference $i(t)$ may be very detrimental to performance, and mitigating its effects my provide much bigger gains than either SNR or diversity gain.

If we combine the two signals with weights w_1 and w_2

$$
r_T(t) = w_1 r_1(t) + w_2 r_2(t) \tag{3.8}
$$

we can choose the weights to mitigate the interference. An obvious choice (assuming that we knew or could estimate the interference channel) would be to choose

$$
\begin{aligned}
w_1 &= -h_2^i \\
w_2 &= h_1^i
\end{aligned}
$$

which results in

$$
\begin{aligned}
r_T(t) &= -h_2^i r_1(t) + h_1^i r_2(t) \\
&= \left(-h_2^i h_1 + h_i^i h_2\right) s(t) + \left(-h_2^i h_1^i + h_2^i h_1^i\right) i(t) + \left(-h_2^i + h_i^i\right) n(t) \\
&= \left(-h_2^i h_1 + h_i^i h_2\right) s(t) + \left(-h_2^i + h_i^i\right) n(t) \tag{3.9}
\end{aligned}
$$

Thus, we can see that the interference is eliminated. We can then apply the gain $\left(\left(-h_2^i h_1 + h_i^i h_2\right)\right)^{-1}$ to eliminate the effects of the channel on the desired signal. This allows proper detection, but note that it does not allow for SNR gain, or as we will show in the next section, diversity gain. This is a general truth in MIMO communications: the degrees of freedom can be used for various gains, but not for all gains simultaneously. Thus, the system designer must consider what the best use of the antennas are.

3.2.3   SIMO For Combatting Multipath - Receive Diversity

Now let's consder the case where there is rich multipath scattering at the receiver. In this case we write the received signal vector as

$$
\tilde{\mathbf{r}}(t) = x(t)\tilde{\mathbf{h}} + \mathbf{n}(t) \tag{3.10}
$$

where

$$\tilde{\mathbf{h}} = \begin{bmatrix} \tilde{\alpha}_1 \\ \tilde{\alpha}_2 \\ \vdots \\ \tilde{\alpha}_{M_r} \end{bmatrix} \tag{3.11}$$

and $\mathbf{n}(t)$ is a complex vector Gaussian random process with zero mean and $\mathrm{E}\left\{\mathbf{n}(t)\mathbf{n}^H(t)\right\} = \sigma^2\mathbf{I}$,

$$x(t) = \sum_{n=-\infty}^{\infty} s_n p(t - nT) \tag{3.12}$$

At the output of M_r matched filters sampled at the optimal symbol time we have

$$\tilde{\mathbf{r}}[\nu] = s_v \tilde{\mathbf{h}} + \mathbf{n}[\nu] \tag{3.13}$$

The SNR on the ith antenna is

$$\gamma_i = \frac{|\tilde{\alpha}_i|^2}{\sigma^2} \tag{3.14}$$

Using Maximal Ratio combining (we will describe this in detail in a later chapter):

$$z_\nu = \mathbf{w}^H \tilde{\mathbf{r}}[\nu] \tag{3.15}$$

where $\mathbf{w} = \tilde{\mathbf{h}}$. The SNR after MRC is

$$\begin{aligned} \gamma &= \frac{\left(\mathrm{E}\left\{\left.\sum_{i=1}^{M_r} |\tilde{\alpha}_i|^2 s_\nu \right| x_\nu\right\}\right)^2}{\sigma^2 \sum_{i=1}^{M_r} |\tilde{\alpha}_i|^2} \\ &= \frac{\left(\sum_{i=1}^{M_r} |\tilde{\alpha}_i|^2\right)^2}{\sigma^2 \sum_{i=1}^{M_r} |\tilde{\alpha}_i|^2} \\ &= \frac{\sum_{i=1}^{M_r} |\tilde{\alpha}_i|^2}{\sigma^2} \end{aligned}$$

The SNR now benefits from both diversity and SNR gain. We will determine the statistics of this later, but for now we can say that the fading statistics of the combined channel are much less severe than an individual channel, provided that the channels are independent.

3.3   Multiple Antennas at the Transmitter (MISO)

The benefits of using multiple antennas at the receiver can generally be realized by using multiple antennas at the transmitter, albeit with more difficulty due to the channel knowledge required. That is, whereas at the receiver we can usually estimate the channel to determine the optimal antenna weights, that knowledge is more difficult to obtain at the transmitter.

3.3.1   MISO in the Absence of Multipath - Transmit Beamforming

In general, the gains discussed above for the SIMO channel are also applicable to the MISO channel. To see this, let's consider the received signal

$$\tilde{r}(t) = \tilde{\alpha}\mathbf{w}^H\tilde{\mathbf{h}}x(t) + n(t) \tag{3.16}$$

where $x(t) = \sum_{n=-\infty}^{\infty} s_n p(t - nT)$, $n(t)$ is complex Gaussian random process with power σ^2, and again the vector channel is defined as $h_i = e^{-j2\pi \frac{d}{\lambda}(i-1)\sin(\theta)}$ (not including the scalar fading term).

If at the transmitter we choose

$$w_i = \frac{1}{\sqrt{N_t}} e^{-j2\pi \frac{d}{\lambda}(i-1)\sin(\theta)} \tag{3.17}$$

where the term $\frac{1}{\sqrt{N_t}}$ is due to normalizing the transmit power. Note that choosing the weights requires knowledge of the angle-of-departure θ *at the transmitter*. When applying this at the receiver, we can estimate θ. This is more difficult to obtain at the transmitter. At the output of a pulse matched filter sampled at the optimal symbol time

$$r[\nu] = \tilde{\alpha} \underbrace{\mathbf{w}^T \tilde{\mathbf{h}}}_{G} x_\nu + n[\nu] \tag{3.18}$$

Now $G = \sum_{i=1}^{N_t} w_i h_i = \frac{N_t}{\sqrt{N_t}} = \sqrt{N_t}$. Thus, the SNR at the receiver is

$$\gamma = \frac{N_t |\tilde{\alpha}|^2}{\sigma^2} \tag{3.19}$$

where N_t is again the **beamforming gain**.

Similarly, we can show that diversity gain is achievable using multiple transmit antennas and even interference mitigation is possible (typically termed interference avoidance) using multiple antennas at the transmitter instead of the receiver. However, knowledge of the appropriate channels is required at the transmitter instead of the receiver, which typically requires some form of feedback or time division duplexing.

3.4   True MIMO - Exploiting Multipath through Spatial Multiplexing

When the link has multiple antennas at both the transmitter and receiver, another type of benefit is possible: spatial multiplexing. Spatial multiplexing is the transmission of mulitple simultaneous data streams in the same frequency band. To see this, consider the received signal $r_i(t)$ on M_r receive antennas (written in vector form)

$$\tilde{\mathbf{r}}(t) = \mathbf{H}\mathbf{x}(t) + \mathbf{n}(t) \tag{3.20}$$

where

$$x_i(t) = \sum_{n=-\infty}^{\infty} s_{i,n} p(t - nT) \tag{3.21}$$

and $\mathbf{H}$ is the static matrix channel (assuming flat fading) between N_t transmit antennas and M_r receive antennas. At the output of M_r matched filters sampled at the optimal sampling time:

$$\tilde{\mathbf{r}}[\nu] = \mathbf{H}\mathbf{x}[\nu] + \mathbf{n}[\nu] \tag{3.22}$$

where $\mathbf{x}[\nu] = \begin{bmatrix} s_{1,\nu} \\ \cdots \\ s_{N_t,\nu} \end{bmatrix}$. If the channel matrix $\mathbf{H}$ is known *and is square*, we could formulate

$$\mathbf{z}[\nu] = \mathbf{H}^{-1}\tilde{\mathbf{r}}[\nu] = \mathbf{x}[\nu] + \mathbf{H}^{-1}\mathbf{n}[\nu] \tag{3.23}$$

Note, if $\mathbf{H}$ is not square, we could still use the pseudo-inverse. However, as we will show later, there are actually better ways to do the signal processing at the receiver. The problem with this approach is that $\mathbf{H}$ could be singular or poorly conditioned (very common case). Thus, the noise power can be enhanced since it is pre-multiplied by $\mathbf{H}^{-1}$. This can be seen in Figure 3.2 where we plot an example received QPSK constellation in a SISO channel (left) and MIMO channel (right). Clearly the constellation on the right is more spread out (i.e., adversely affected by noise) due to the application of the matrix inverse. This would lead to a higher BER in the MIMO case, relative to the SISO case. However, downside is counterbalanced by the fact that $N_t = M_r$ parallel channels are transmitted in the same frequency band/time slot. More sophisticated processing will also reduce the negative impact.
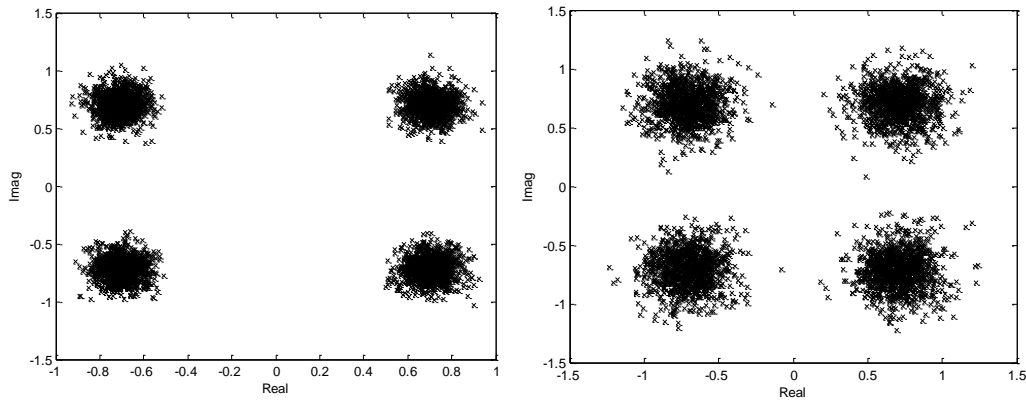
Figure 3.2: Example QPSK Constellation Received in SISO (left) and MIMO (right) Channels

3.4.1 More General Matrix Channel

Again, consider the received signal

$$\tilde{\mathbf{r}}[\nu] = \mathbf{H}\mathbf{x}[\nu] + \mathbf{n}[\nu] \tag{3.24}$$

where $\mathbf{H}$ is an $M_r \times N_t$ matrix channel and $\mathrm{E}\left\{\mathbf{n}(t)\mathbf{n}^H(t)\right\} = \sigma^2\mathbf{I}$. Now, using the psuedo-inverse and assuming that $rank(\mathbf{H}) = N_t$, we can create the decision variables as

$$\mathbf{z}[\nu] = \mathbf{T}\tilde{\mathbf{r}}[\nu] \tag{3.25}$$

with

$$\mathbf{T} = (\mathbf{H}^H \mathbf{H})^{-1} \mathbf{H}^H \tag{3.26}$$

which results in

$$\begin{aligned}
\mathbf{z}[\nu] &= (\mathbf{H}^H \mathbf{H})^{-1} \mathbf{H}^H \tilde{\mathbf{r}}[\nu] \\
&= (\mathbf{H}^H \mathbf{H})^{-1} \mathbf{H}^H \mathbf{H} \mathbf{x}[\nu] + (\mathbf{H}^H \mathbf{H})^{-1} \mathbf{H}^H \mathbf{n}[\nu] \tag{3.27} \\
&= \mathbf{x}[\nu] + (\mathbf{H}^H \mathbf{H})^{-1} \mathbf{H}^H \mathbf{n}[\nu] \tag{3.28}
\end{aligned}$$

Now,

$$\begin{aligned}
\mathrm{E}\left\{ (\mathbf{H}^H \mathbf{H})^{-1} \mathbf{H}^H \mathbf{n}[\nu] \left((\mathbf{H}^H \mathbf{H})^{-1} \mathbf{H}^H \mathbf{n}[\nu] \right)^H \right\} &= (\mathbf{H}^H \mathbf{H})^{-1} \mathbf{H}^H \mathrm{E}\left\{ \mathbf{n}[\nu] \mathbf{n}^H[\nu] \right\} \mathbf{H} (\mathbf{H}^H \mathbf{H})^{-H} \tag{3.29} \\
&= (\mathbf{H}^H \mathbf{H})^{-1} \mathbf{H}^H (\sigma^2 \mathbf{I}) \mathbf{H} (\mathbf{H}^H \mathbf{H})^{-H} \tag{3.30} \\
&= \sigma^2 \underbrace{(\mathbf{H}^H \mathbf{H})^{-1} \mathbf{H}^H \mathbf{H}}_{\mathbf{I}} (\mathbf{H}^H \mathbf{H})^{-H} \tag{3.31} \\
&= \sigma^2 (\mathbf{H}^H \mathbf{H})^{-H} \tag{3.32}
\end{aligned}$$

where $\mathbf{A}^{-H}$ is the Hermitian of the inverse $\mathbf{A}$. In general, the diagonal elements of the matrix $(\mathbf{H}^H \mathbf{H})^{-H}$ will be greater than 1 resulting in noise enhancement.

3.5   Conclusions

In this brief chapter we have quickly summarized the primary benefits of employing multiple antennas at the receiver (SIMO), at the transmitter (MISO) or both (MIMO). Again, in SIMO and MISO channels SNR gains, diversity gains, and interference mitigation are all possible depending on the nature of the multipath and size the antenna array used. Further, in the case of MIMO we can also achieve a spatial multiplexing gain. In the MIMO case we can also achieve combinations of these gains, although we cannot obtain the full benefit of all gains simultaneously. There is always some trade-off that must be considered.

Chapter 4

MIMO Link Capacity

For many years the primary use of multiple antennas in a fading environment was to *combat* multipath fading. Specifically, multiple antennas were a primary source of diversity which mitigated the effect of fading. But in the last 20 years it was realized that multiple antennas were also a means for *exploiting* multipath. Specifically, as we will show in this chapter, the use of multiple antennas at the transmitter and receiver allows for an increase in spectral efficiency (i.e., capacity) in the presence of rich scattering.

4.1   Entropy and Uncertainty

To understand channel capacity, we must first describe a couple of basic concepts in information theory [2]. First, the measure of uncertainty of a random variable X that is used in information theory is known as *entropy*. For continuous random variables this is termed differential entropy and is defined as

$$
\begin{aligned}
H(X) &= -E\left\{\log f_X(x)\right\} \\
&= -\int_{S_X} f_X(x)\left(\log f_X(x)\right) dx
\end{aligned}
\tag{4.1}
$$

where $f_X(x)$ is the probability density function of X and S_X is the support of X. Mutual information between two random variables X and Y is the reduction in uncertainty of one random variable when observing the other. Note that the base of log determines the units of information. When we use base 2, the units are in *bits*, while when we used base e, the units are *nats*. Formally, mutual information is defined as

$$
\begin{aligned}
I(X;Y) &= H(X) - H(X|Y) \\
&= H(Y) - H(Y|X)
\end{aligned}
$$

Example 4.1.1. Find the differential entropy of a (scalar) complex Gaussian random variable in bits.

Solution: The differential entropy of a complex Gaussian random variable with mean μ_x and variance σ_x^2 is

$$
\begin{aligned}
H(X) &= -E\left\{\log_2 f_X(x)\right\} \\
&= -E\left\{\log_2\left(\frac{1}{\pi\sigma_x^2}e^{-\frac{|x-\mu_x|^2}{\sigma_x^2}}\right)\right\} \\
&= \log_2(\pi\sigma_x^2) - E\left\{\left(-\frac{|x-\mu_x|^2}{\sigma_x^2}\right)\right\}\log_2(e) \\
&= \log_2(\pi\sigma_x^2) + \log_2(e) \\
&= \log_2(e\pi\sigma_x^2)
\end{aligned}
$$

Note that the mean is irrelevant. Only the variance determines the differential entropy. Further, it can be shown that this is the maximum possible differential entropy for any random variable. In other words, $H(X) \leq \log_2(e\pi\sigma_x^2)$ for all possible X.

Example 4.1.2. Find the differential entropy of a complex Gaussian random vector in bits.

Solution: The differential entropy of a complex Gaussian random vector with mean $\overline{\mu}_x$ and covariance matrix $\boldsymbol{\Sigma_x}$ is

$$
\begin{aligned}
H(X) &= -E\left\{\log_2 f_{\mathbf{X}}(\mathbf{x})\right\} \\
&= -E\left\{\log_2\left(\frac{1}{\det(\pi\boldsymbol{\Sigma_x})}e^{-(\mathbf{x}-\overline{\mu}_x)^H\Sigma_x^{-1}(\mathbf{x}-\overline{\mu}_x)}\right)\right\} \\
&= \log_2\left(\det(\pi\boldsymbol{\Sigma_x})\right) - E\left\{-(\mathbf{x}-\overline{\mu}_x)^H\boldsymbol{\Sigma_x}^{-1}(\mathbf{x}-\overline{\mu}_x)\right\} \\
&= \log_2\left(\det(e\pi\boldsymbol{\Sigma_x})\right)
\end{aligned}
\tag{4.2}
$$

Example 4.1.3. Find the mutual information in bits between random variables X and Y where $y = x + n$ when X and N are both zero-mean complex Gaussian random variables with variances σ_x^2 and σ^2 respectively.

Solution: The mutual information is defined as

$$
I(X;Y) = H(Y) - H(Y|X)
\tag{4.3}
$$

Since X and N are both Gaussian, Y is also Gaussian. Thus we have:

$$
H(Y) = \log_2(e\pi\sigma_y^2)
$$

Further, if X is known, Y is a Gaussian random variable with mean x and variance σ^2. Thus,

$$
H(Y|X) = \log_2(e\pi\sigma^2)
$$

Thus, the mutual information is

$$
\begin{aligned}
I(X;Y) &= \log_2(e\pi\sigma_y^2) - \log_2(e\pi\sigma^2) \\
&= \log_2\left(\frac{e\pi\sigma_y^2}{e\pi\sigma^2}\right) \\
&= \log_2\left(\frac{\sigma_x^2 + \sigma^2}{\sigma^2}\right) \\
&= \log_2(1+\gamma)
\end{aligned}
\tag{4.4}
$$

where $\gamma = \frac{\sigma_x^2}{\sigma^2}$ is the ratio of the variances. Note that this is the mutual information for the model $y = x + n$ if n is known to be Gaussian but x can be chosen to follow any distribution. In other words, we achieve the maximum mutual information if x is also Gaussian as that makes y Gaussian.

4.2 Capacity

We are interested in a measured known as *capacity* which is the maximum mutual information provided by a channel. Consider a random channel input X and a random channel output Y where $y = x + n$. Further, assume that X has a distribution $f_X(x)$ and Y has a distribution $f_Y(y)$. The uncertainty of the input X is the entropy, defined above. As also mentioned above, the mutual information between the input X and channel output Y is the reduction in uncertainty in one random variable due to observation of the other. In our case, we observe the output Y and are interested in the input X. Mutual information can also be seen as a measure of the dependence of two random variables. For continuous random variables, using the definitions above, the mutual information for two continuous random variables X and Y can be written as

$$
I(X;Y) = \int_{S_X,S_Y} f_{X,Y}(x,y)\log\left(\frac{f_{X,Y}(x,y)}{f_X(x)f_Y(y)}\right)dxdy
\tag{4.5}
$$

where S_X and S_Y are the supports for the random variables X and Y respectively. The capacity of a channel is the maximum mutual information where the maximization occurs over the distribution of the input:

$$
C = \max_{f_X(x)} I(X;Y)
\tag{4.6}
$$

If $y = x + n$ and n has a complex Gaussian distribution, the capacity of the channel is achieved when X also has a complex zero-mean Gaussian distribution as discussed above. Further, using the example above, the capacity is well-known to be (in bits per channel use):

$$
C = \log_2\left(1 + \frac{P}{\sigma_2}\right)
\tag{4.7}
$$

where $P = \sigma_x^2$ and σ^2 is the variance of the noise n. Now, if we have a continuous time AWGN channel with bandpass bandwidth B and AWGN power spectral density $\frac{N_o}{2}$, the output after bandpass to baseband conversion, can be represented by samples taken at a rate of $1/B$. Thus, the capacity of this channel is

$$
C = B\log_2(1+\gamma)
\tag{4.8}
$$

where $\gamma = \frac{P}{N_o B}$ is the signal-to-noise ratio and the capacity is measured in bits per second (b/s). The capacity C/B (b/s/Hz) for an AWGN channel is plotted in Figure 4.1. We can see that for an SNR of 15dB, we can obtain a spectral efficiency of 5b/s/Hz.
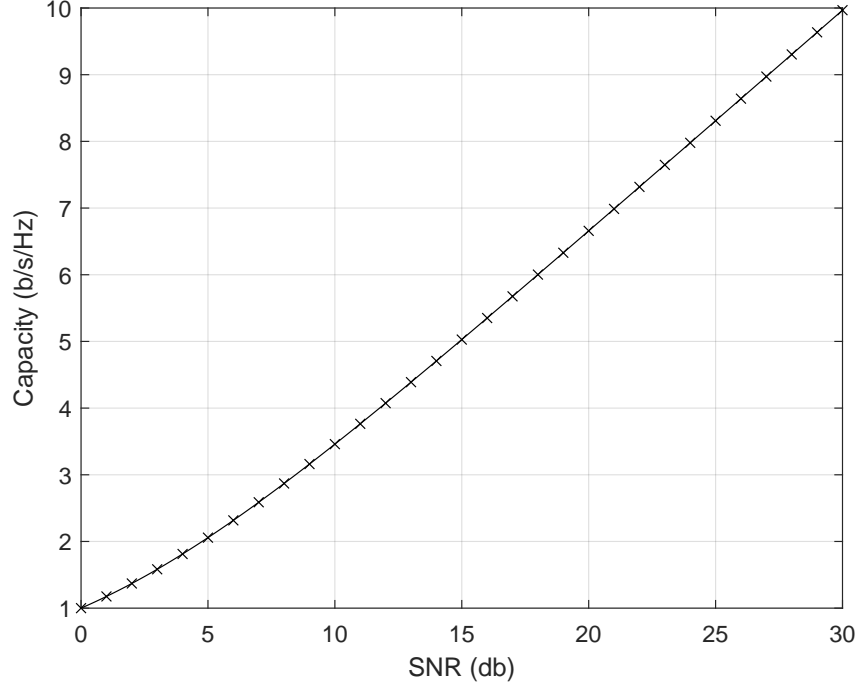

Figure 4.1: Channel Capacity (b/s/Hz) versus SNR in an AWGN Channel

4.3   Channel Capacity in MISO and SIMO Channels

For a fixed scalar channel h, [**NOTE: Although we don't use the tilde notation here, we can assume that all vectors can be either real or complex.**] we have (at the output of a pulse-matched filter):

$$y = hx + n \tag{4.9}$$

where x is the transmit symbol (i.e., the channel input) and n is AWGN (zero mean, variance σ^2). Shannon's capacity theorem above still applies for a fixed value of h. Specifically, the maximum error free transmission rate (i.e., capacity) is

$$C = B \log_2 \left(1 + \gamma |h|^2\right) b/s \tag{4.10}$$

where B is the system bandwidth, $\gamma = \frac{\mathrm{E}\{|x|^2\}}{\sigma^2}$ (i.e., SNR). Note that B has two effects since $\sigma^2 = N_o B$. In terms of spectral efficiency we have

$$\frac{C}{B} = \log_2 \left(1 + \gamma |h|^2\right) b/s/Hz \tag{4.11}$$

Now, if we the symbol transmit on one antenna, but receive the signal on M_r antennas the received signal vector is

$$\mathbf{y} = \tilde{\mathbf{h}}x + \mathbf{n} \tag{4.12}$$

where $\tilde{\mathbf{h}}$ is the $M_r \times 1$ fixed channel vector. Despite the multiple receive antennas, the channel still permits only a single input. The received vector $\mathbf{y}$ has M_r dimensions, but the information is only along $\tilde{\mathbf{h}}$. Thus, we can project $\mathbf{y}$ along $\tilde{\mathbf{h}}$ and retain all of the information while rejecting $M_r - 1$ dimensions of the noise vector $\mathbf{n}$:

$$
\begin{aligned}
z &= \tilde{\mathbf{h}}^H \mathbf{y} \\
&= \tilde{\mathbf{h}}^H \tilde{\mathbf{h}} x + \tilde{\mathbf{h}}^H \mathbf{n}
\end{aligned}
$$

This corresponds to a SISO channel with SNR $\frac{E\{|x|^2\}\tilde{\mathbf{h}}^H\tilde{\mathbf{h}}}{\sigma^2}$. Thus, the capacity that corresponds to a SISO channel with a modified SNR:

$$
\frac{C}{B} = \log_2 \left(1 + \gamma \sum_{i=1}^{M_r} |h_i|^2 \right) b/s/Hz \tag{4.13}
$$

On the other hand, if we transmit the signal from N_t antennas (without channel knowledge) but only have one output (i.e., one receive antenna), the received signal is

$$
y = \tilde{\mathbf{h}}^T \mathbf{x} + n \tag{4.14}
$$

Again, this is similar to a SISO channel, except that the input is now a vector $\mathbf{x}$ which is projected onto a channel $\tilde{\mathbf{h}}$. Without channel knowledge, the optimal transmit signal is a Gaussian random vector with covariance $\Sigma_x = \frac{E\{|x|^2\}}{N_t}\mathbf{I}$. Again, the capacity is the similar to SISO channels with a modified SNR. However, unlike the SIMO case there is a normalizing power constant $1/\sqrt{N_t}$ since we must split the power over multiple antennas:

$$
\frac{C}{B} = \log_2 \left(1 + \frac{\gamma}{N_t} \sum_{i=1}^{N_t} |h_i|^2 \right) b/s/Hz \tag{4.15}
$$

We can also see this by viewing the received signal as $y = v + n$ where $v = \sum_{i=1}^{N_t} h_i x_i$ for fixed values of h_i. Capacity is maximized by making v Gaussian with the power constraint that $\sum_i \sigma_{x_i}^2 = 1$. Without knowledge of h_i, we simply use equal power in each transmit signal and thus $\sigma_{x_i}^2 = \frac{1}{N_t}$.

For M_r receive antennas, capacity goes up due to increased capture of signal energy. For N_t transmit antennas, capacity doesn't necessarily go up since the transmit power is fixed regardless of the number of antennas.

Now, if the transmitter has channel knowledge in the MISO case, we can transmit $\frac{\tilde{\mathbf{h}}}{||\tilde{\mathbf{h}}||} x$ and receive

$$
y = \left(\frac{\tilde{\mathbf{h}}}{\left\|\tilde{\mathbf{h}}\right\|} \right)^H \tilde{\mathbf{h}} x + n \tag{4.16}
$$

Now in this case we again have a modified AWGN SISO channel and the capacity (using a Gaussian distribution for x) is

$$
\frac{C}{B} = \log_2 \left(1 + \gamma \sum_{i=1}^{N_t} |h_i|^2 \right) b/s/Hz \tag{4.17}
$$

which is equivalent to the SIMO case.

4.4   Fading Channels

The previous sections assumed that the channel was *static*. However, in fading channels the channel is not assumed to be constant, but time-varying. The capacity in the fading case differs substantially from the static channel case. The capacity of the fading channel depends on the information available. There are two primary cases of interest:

1. The channel is known by the receiver.

2. The channel is known by the transmitter and receiver.

We already saw that in the case of MISO channels, knowledge of the channel made a difference in the channel capacity. That is also true for fading channels, although the gain achieved depends on the distribution of the channel.

4.4.1   Channel Known at the Receiver

There are two definitions of capacity that are relevant in fading channels when the transmitter does not know the channel (but the receiver does). Those two definitions are

1. Ergodic (Shannon) Capacity

2. Outage Capacity

Because the transmitter does not know the channel, the rate cannot be adapted relative to the channel. As a result, we must pick a constant rate for transmission. There are two general assumptions applied to understanding the capacity in this case. The first assumption is that the channel varies quickly allowing for a sufficiently long code word to experience all channel states. In this case, the constant rate is known as the *ergodic* or Shannon capacity and is defined as

$$C_{erg} = \int_0^\infty B \log_2(1 + \gamma)p(\gamma)d\gamma \tag{4.18}$$

where $p(\gamma)$ is the distribution of the SNR due to channel fading. Note that although this is the AWGN capacity averaged over the SNR distribution we cannot think of this as the rate being modified to match the channel conditions. The transmitter in fact does not know the channel and thus cannot adapt the rate. The rate is fixed. If it is fixed at a rate equal to the ergodic capacity or lower, a bit error rate approaching zero can be achieved. Thus, to achieve this capacity, the transmitter and receiver must know the distribution of the channel to determine this rate, although the transmitter doesn't know the channel at any given instant. As an example consider the ergodic capacity of a Rayleigh fading channel shown in Figure 4.2. Again, noting a 15dB SNR (average in the fading case), we can see that as opposed to the 5b/s/Hz achievable in an AWGN channel, the ergodic capacity in the case of Rayleigh fading is only 4.3b/s/Hz. Thus, there is a nearly 15% reduction in the capacity due to fading.

On the other hand, if the channel varies slowly, such that a code word experiences a constant channel, the capacity will depend on specific channel conditions. Again, since the rate cannot be adapted (since the transmitter does not know the channel), the receiver will experience an outage whenever the channel capacity is less than the constant rate chosen by the transmitter. Thus,
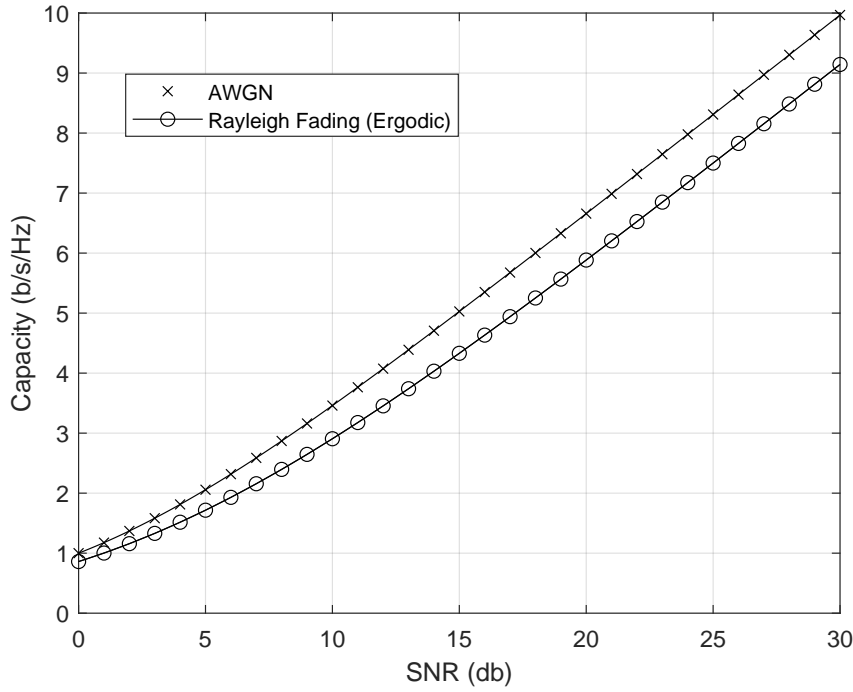
Figure 4.2: Ergodic Capacity (b/s/Hz) versus SNR in an Rayleigh Fading Channel

the higher the rate chosen, the higher the outage probability. If the distribution is known the transmitter can choose the constant rate based on a desired outage probability. This is known as the *outage capacity*. For formally, we define the $p \times 100\%$ outage capacity as the capacity C_{out} that satisfies

$$p = \Pr\left\{B\log_2(1+\gamma) < C_{out}\right\} \tag{4.19}$$

As an example, consider a Rayleigh fading channel with an average SNR of 15dB. The probability of outage versus outage capacity is plotted in Figure 4.3. For example, if a 1% outage is desired, the capacity is approximately 0.4b/s/Hz. On the other hand if a 10% outage probability can be tolerated, the capacity is a good bit higher at approximately 2.1 b/s/Hz. The outage capacity for 1% and 10% outage probabilities are compared with the ergodic capacity and AWGN capacity in Figure 4.4. We can see that for moderate to low outage probabilities the outage capacity is significantly lower than the ergodic capacity in a Rayleigh fading channel. This can be ameliorated by adding antennas at the transmitter or receiver as we will see shortly.

4.4.2   Channel Known at the Transmitter and Receiver

When the transmitter has knowledge of the channel there are two options that the transmitter can choose:

1. Adapt rate only

2. Adapt power and rate
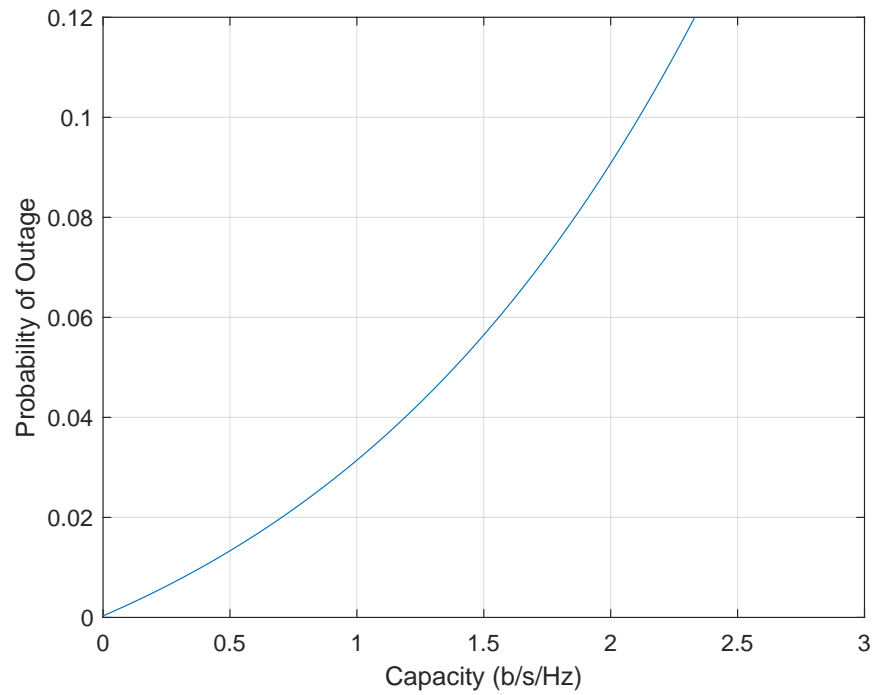
Figure 4.3: Outage Capacity (b/s/Hz) versus Outage Probability in an Rayleigh Fading Channel (average SNR = 15dB)
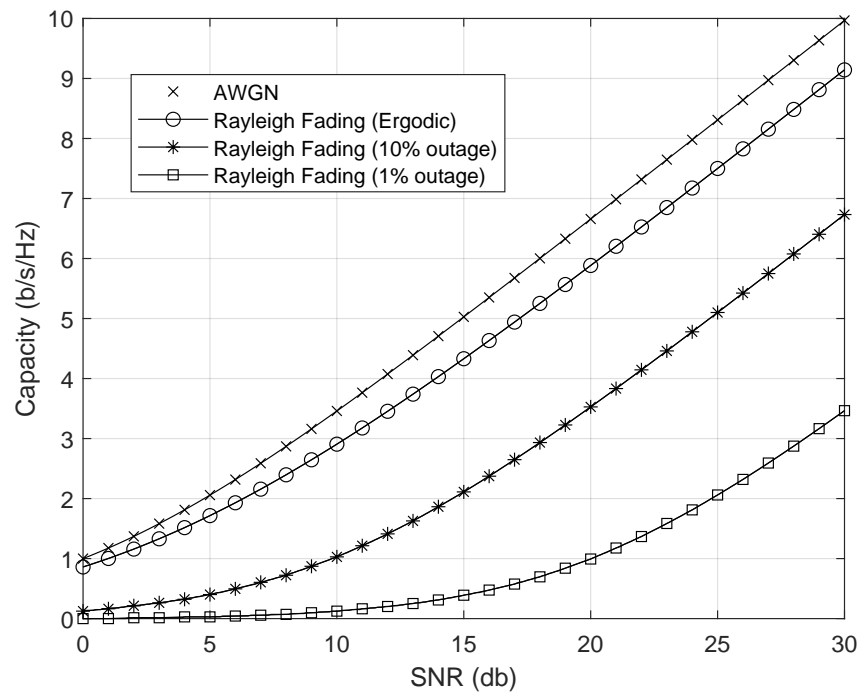
Figure 4.4: Capacity (b/s/Hz) versus SNR in an AWGN or Rayleigh Fading Channel

In the case where rate is adapted, but power isn't, the transmitter choose the rate which corresponding to the capacity at full transmit power. In this case there is not outage capacity, since the transmitter always transmits at the rate supported by the known channel. However, the average rate achieved by adapting the rate based on the known channel is the ergodic capacity and is calculated

$$C = \int_0^\infty B \log_2 (1 + \gamma) p(\gamma) d\gamma \tag{4.20}$$

which is identical to the case where the channel is unknown. Thus channel knowledge at the transmitter is not helpful if power adaptation is not used. On the other hand, if power is adapted, the capacity is calculated as

$$C = \max_{P(\gamma):\int_0^\infty P(\gamma)p(\gamma)d\gamma=\overline{P}} \int_0^\infty B \log_2 \left(1 + \frac{P(\gamma)\gamma}{\overline{P}} \right) p(\gamma) d\gamma \tag{4.21}$$

where $\overline{P}$ is the average power available and $P(\gamma)$ is the mapping between the channel SNR and the transmit power. The solution to this is found to be based on the concept of *water-filling* where the power is chosen as

$$\frac{P(\gamma)}{\overline{P}} = \begin{cases} \frac{1}{\gamma_o} - \frac{1}{\gamma} & \gamma \geq \gamma_o \\ 0 & \gamma < \gamma_o \end{cases} \tag{4.22}$$

where γ_o is known as the cut-off value since the transmitter will not transmit for SNR values less than γ_o. The cut-off value must be found numerically using the constraint

$$\int_{\gamma_o}^\infty \left(\frac{1}{\gamma_o} - \frac{1}{\gamma} \right) d\gamma = 1 \tag{4.23}$$

4.5 Random MISO and SIMO Channels

In the case of random MISO and SIMO channels, we can again define either ergodic capacity or outage capacity. The ergodic capacity is defined

$$C_{erg} = \mathrm{E_h} \left\{ B \log_2 \left(1 + \gamma \tilde{\mathbf{h}}^H \tilde{\mathbf{h}} \right) \right\} b/s/Hz \tag{4.24}$$

in the case of SIMO (or MISO with channel knowledge at the transmitter) and

$$C_{erg} = \mathrm{E_h} \left\{ B \log_2 \left(1 + \frac{\gamma}{N_t} \tilde{\mathbf{h}}^H \tilde{\mathbf{h}} \right) \right\} b/s/Hz \tag{4.25}$$

in the case of MISO in the absence of channel knowledge at the transmitter. The impact of multiple antennas to ergodic capacity with channel information at the receiver can be seen in Figures 4.5 and 4.6. We can see that adding mutliple antennas to either the transmitter or receiver improves ergodic capacity. However, the benefit is much greater when adding them at the receiver, in the absence of transmitter side channel knowledge. Without transmitter side channel knowledge, adding antennas to the transmitter only will provide small increases in ergodic capacity, approaching that of an AWGN channel, but no more. At the receiver, however, we can see that capacity is increased much more from additional antennas, and in fact can exceed that of an AWGN channel even with only two antennas. This is due to the additional received energy available from the additional receive antennas. This same benefit is possible from transmit antennas, but only if

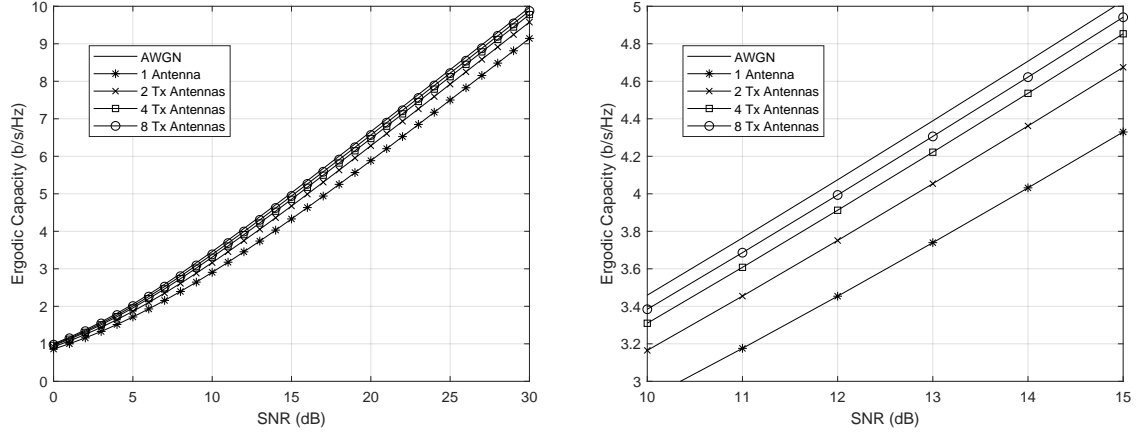transmitter side channel knowledge is available.

Figure 4.5: Ergodic Capacity (b/s/Hz) versus SNR in MISO Rayleigh (independent) Fading Channel with Channel Information at the Receiver

Mutltiple antennas at the transmitter and receiver can have an even greater impact on outage capacity due to the diversity provided. This can be seen in Figure 4.7 for the MISO case in independent Rayleigh fading and 4.8 for the SIMO case, also independent Rayleigh fading. These figures show the outage probability versus capacity in the case of no transmitter channel knowledge when SNR=15dB. In the case of MISO the outage capacity at 10% outage probability increases from 2.1b/s/Hz with one antenna to 3.2b/s/Hz, 3.9b/s/Hz and 4.3b/s/Hz for 2, 4, and 8 transmit antennas respectively. At 1% outage the gains are even more dramatic going from 0.4b/s/Hz with one antenna to 1.8b/s/Hz, 2.9b/s/Hz and 3.7b/s/Hz using 2, 4, and 8 antennas respectively. Note that AWGN capacity is 5b/s/Hz. Thus, just like in the ergodic capacity case, the outage capacity (regardless of outage probability) is upper-bounded by the capacity of the AWGN channel. This is not true if the transmitter has channel knowledge, since that matches the SIMO case which is not bounded by (single antenna) AWGN capacity due to the average SNR gain.

The gains in outage probability for the SIMO case are even greater due to the natural SNR gain from coherent combining. The outage capacity at 10% outage increases to 4.2b/s/Hz, 5.8b/s/Hz and 7.2b/s/Hz for 2, 4 and 8 receive antennas respectively, while the 1% outage capacity increases to 2.5b/s/Hz, 4.8b/s/Hz and 6.5b/s/Hz for those same numbers of antennas. As mentioned above, the outage capacity for SIMO (like the ergodic capacity) is not bounded by the single antenna AWGN case.
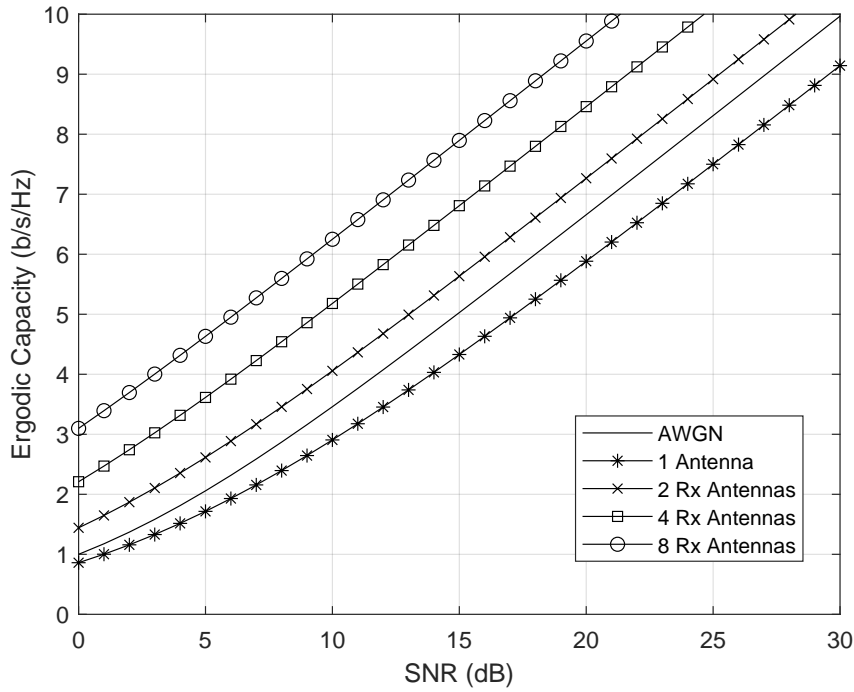
Figure 4.6: Ergodic Capacity (b/s/Hz) versus SNR in SIMO Rayleigh (independent) Fading Channel with Channel Information at the Receiver
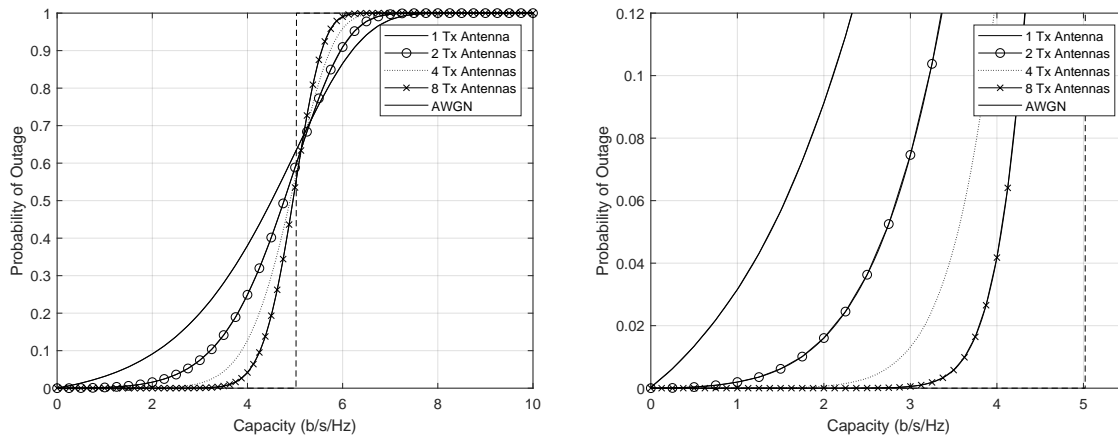

Figure 4.7: Outage Capacity (b/s/Hz) versus SNR in MISO Rayleigh (independent) Fading Channel with Channel Information at the Receiver
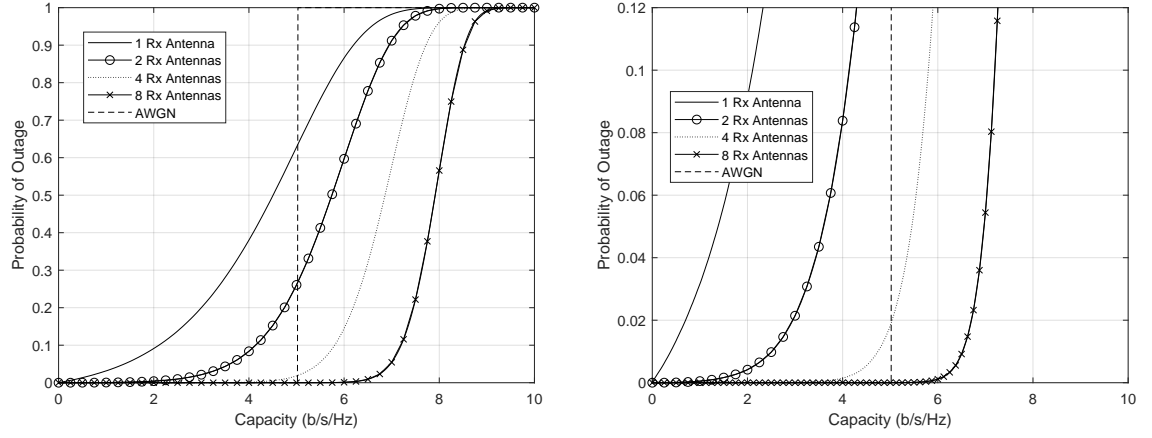
Figure 4.8: Outage Capacity (b/s/Hz) versus SNR in SIMO Rayleigh (independent) Fading Channel with Channel Information at the Receiver

4.6   Matrix Channel

Now let us consider a matrix (i.e., MIMO) channel. Specifically, let

$$\mathbf{y} = \mathbf{Hx} + \mathbf{n} \tag{4.26}$$

where $\mathbf{H}$ is fixed or random and $\mathbf{n}$ is Gaussian circular symmetric noise $\mathbf{n}\ GRV(\mathbf{0}, \sigma^2\mathbf{I})$. For a fixed channel the capacity can be found as

$$C = \max_{f_X(\mathbf{x}):tr(\mathbf{\Sigma_x})\leq P_t} I(\mathbf{x}, \mathbf{y}) \tag{4.27}$$

where $\mathbf{\Sigma_x} = \mathrm{E}\left\{\mathbf{xx}^T\right\} - \mathrm{E}\left\{\mathbf{x}\right\}\mathrm{E}\left\{\mathbf{x}\right\}^H$. For random channels, we again define ergodic capacity or outage capacit. Ergodic capacity is defined as

$$C_{erg} = \mathrm{E}_H\left\{\max_{f_X(\mathbf{x}):tr(\mathbf{\Sigma_x})\leq P_t} I(\mathbf{x}, \mathbf{y})\right\} \tag{4.28}$$

Now,

$$
\begin{aligned}
I(\mathbf{x}, \mathbf{y}) &= H(\mathbf{y}) - H(\mathbf{y}|\mathbf{x}) \\
&= H(\mathbf{y}) - H(\mathbf{Hx} + \mathbf{n}|\mathbf{x}) \\
&= H(\mathbf{y}) - H(\mathbf{n})
\end{aligned} \tag{4.29}
$$

where we have assumed that $\mathbf{x}$ and $\mathbf{n}$ are independent and

$$H(\mathbf{x}) = -\int_{\mathbf{x}} f_X(\mathbf{x}) \log_2\left(f_X(\mathbf{x})\right) d\mathbf{x} \tag{4.30}$$

Now, it is well known that this is maximized when $\mathbf{y}$ and thus $\mathbf{x}$ is Gaussian. Further, for Gaussian random vectors, we can show that

$$H(\mathbf{y}) = \log_2\left(\det\left(\pi e \Sigma_{\mathbf{y}}\right)\right) \tag{4.31}$$

where $\Sigma_{\mathbf{y}}$ is the covariance matrix of $\mathbf{y}$. Now, since $\mathbf{y}$ is zero mean

$$
\begin{aligned}
\Sigma_{\mathbf{y}} &= \mathrm{E}\left\{\mathbf{y}\mathbf{y}^H\right\} \\
&= \mathrm{E}\left\{(\mathbf{H}\mathbf{x}+\mathbf{n})(\mathbf{H}\mathbf{x}+\mathbf{n})^H\right\} \\
&= \mathrm{E}\left\{\mathbf{H}\mathbf{x}\mathbf{x}^H\mathbf{H}^H\right\} + \mathrm{E}\left\{\mathbf{n}\mathbf{n}^H\right\} \\
&= \mathbf{H}\Sigma_{\mathbf{x}}\mathbf{H}^H + \sigma^2\mathbf{I}
\end{aligned}
\tag{4.32}
$$

Thus, we have

$$
\begin{aligned}
I(\mathbf{x},\mathbf{y}) &= \log_2\left(\det\left(\pi e(\mathbf{H}\Sigma_{\mathbf{x}}\mathbf{H}^H + \sigma^2\mathbf{I})\right)\right) - \log_2\left(\det\left(\pi e\sigma^2\mathbf{I}\right)\right) \\
&= \log_2\left[\frac{\det\left(\pi e(\mathbf{H}\Sigma_{\mathbf{x}}\mathbf{H}^H + \sigma^2\mathbf{I})\right)}{\det\left(\pi e\sigma^2\mathbf{I}\right)}\right] \\
&= \log_2\left[\det\left((\mathbf{H}\Sigma_{\mathbf{x}}\mathbf{H}^H + \sigma^2\mathbf{I})\frac{1}{\sigma^2}\mathbf{I}\right)\right] \\
&= \log_2\left[\det\left(\left(\frac{1}{\sigma^2}\mathbf{H}\Sigma_{\mathbf{x}}\mathbf{H}^H + \mathbf{I}\right)\right)\right]
\end{aligned}
\tag{4.33}
$$

Thus, the capacity (in b/s/Hz) can be written as

$$\frac{C}{B} = \log_2\left[\det\left(\left(\frac{1}{\sigma^2}\mathbf{H}\Sigma_{\mathbf{x}}\mathbf{H}^H + \mathbf{I}\right)\right)\right] \tag{4.34}$$

We have already said that $f_X(\mathbf{x})$ is Gaussian. Further, if we have no channel knowledge, it is optimal to choose $\Sigma_{\mathbf{x}} = \frac{P_t}{N_t}\mathbf{I}$, i.e., we put equal power on all transmit antennas. That results in

$$\frac{C}{B} = \log_2\left[\det\left(\left(\frac{P_t}{N_t\sigma^2}\mathbf{H}\mathbf{H}^H + \mathbf{I}\right)\right)\right] \tag{4.35}$$

Let $\gamma = \frac{P_t}{\sigma^2}$:

$$\frac{C}{B} = \log_2\left[\det\left(\left(\frac{\gamma}{N_t}\mathbf{H}\mathbf{H}^H + \mathbf{I}\right)\right)\right] \tag{4.36}$$

Further, since $\mathbf{H}\mathbf{H}^H$ is a Hermitian matrix, it has an eigenvalue decomposition

$$\mathbf{H}\mathbf{H}^H = \mathbf{Q}\Lambda\mathbf{Q}^H \tag{4.37}$$

where $\mathbf{Q}$ is a unitary matrix of orthonormal eigenvectors and Λ is a diagonal matrix of eigenvalues. The diagonal of λ will have r positive eigenvalues and $\min(N_t, M_r) - r$ zeros where $r = rank(\mathbf{H})$:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \lambda_r & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{4.38}$$

Further,

$$det(\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H) = \underbrace{det(\mathbf{Q})}_{=1} \underbrace{det(\mathbf{\Lambda})}_{=\prod_i \lambda_i} \underbrace{det(\mathbf{Q}^H)}_{=1} \tag{4.39}$$

NOTE: $det(\mathbf{Q})det(\mathbf{Q}^H) = 1$. PROOF:We know that (a) $det(\mathbf{A}^*) = (det(\mathbf{A}))^*$, (b) $det(\mathbf{AB}) = det(\mathbf{A})det(\mathbf{B})$ and (c) $det(\mathbf{I}) = 1$. Thus defining $det(\mathbf{Q}) = \lambda$ and using $\mathbf{Q}\mathbf{Q}^H = \mathbf{I}$

$$
\begin{aligned}
det(\mathbf{Q})det(\mathbf{Q}^H) &= det(\mathbf{I}) \\
\lambda\lambda^* &= 1 \\
|\lambda|^2 &= 1
\end{aligned}
\tag{4.40}
$$

Thus, we can write:

$$\boxed{\frac{C}{B} = \log_2\left(\prod_{i=1}^{r}\left(1 + \lambda_i \frac{\gamma}{N_t}\right)\right)} \tag{4.41}$$

since $\lambda_{r+1}, \lambda_{r+1}, \ldots \lambda_{M_r} = 0$ Thus, the matrix channel $\mathbf{H}$ has r parallel channels where r is the rank of $\mathbf{H}$. The SNR per channel is $\frac{\gamma}{N_t}\lambda_i$.

For random channels, the ergodic capacity (in b/s) can be written as

$$C_{erg} = \mathrm{E}_{\mathbf{H}}\left\{B\log_2\left(\prod_{i=1}^{r}\left(1 + \lambda_i \frac{\gamma}{N_t}\right)\right)\right\} \tag{4.42}$$

and the outage capacity can be defined similar to (4.19)

$$p = \Pr\left\{B\log_2\left(\prod_{i=1}^{r}\left(1 + \lambda_i \frac{\gamma}{N_t}\right)\right) < C_{out}\right\} \tag{4.43}$$

The improvement in capacity for rich scattering can be dramatic. Consider a 4×4 channel (i.e., 4 transmit and 4 receive antennas) where all channels are independent (this is the ideal case). The ergodic capacity is shown in Figure 4.9 for an AWGN channel, a one antenna channel with Rayleigh fading, MISO, SIMO and MIMO (all in independent Rayleigh fading). We can see that Rayleigh fading causes a loss in capacity relative to AWGN, although the losses are small at low SNR. Using four antennas at the transmitter pushes capacity nearly back to the AWGN capacity, while four at the receiver allows for a roughly 35% improvement as compared to AWGN. However, adding antennas to both links (assuming independent channels) results in dramatic improvements, with the improvements increasing for higher SNRs. For example, at SNR = 15dB, four antennas at the

receiver only allows for roughly 6.8b/s/Hz as compared to AWGN which allows for 5b/s/Hz. With four antennas at both ends of the link the capacity increases to 16.3b/s/Hz! Thus, while MISO and SIMO permit modest gains in ergodic capacity, true MIMO offers dramatic improvement.
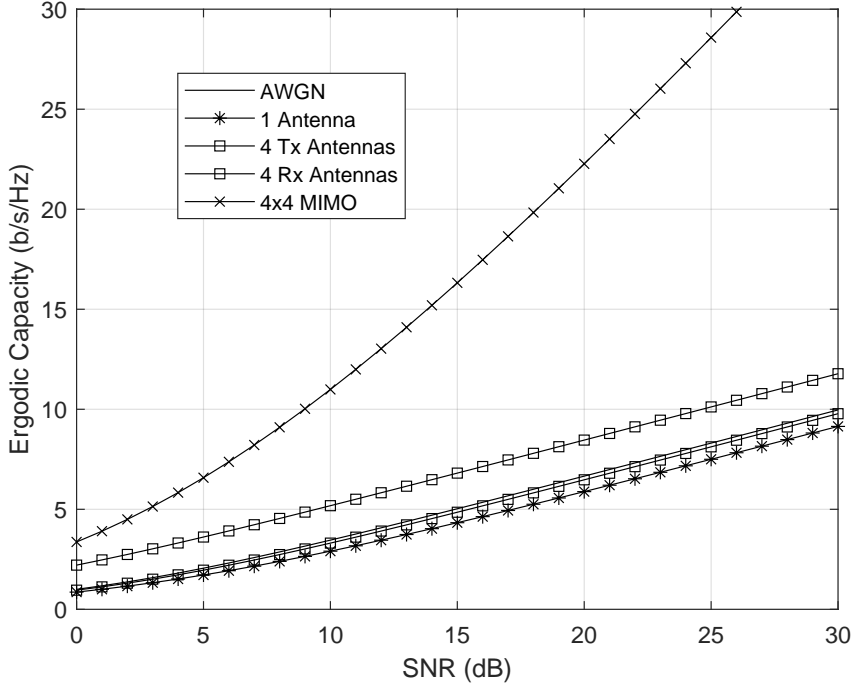

Figure 4.9: Ergodic Capacity (b/s/Hz) versus SNR in Rayleigh (independent) Fading Channel with Channel Information at the Receiver

The capacity improvement can be seen in terms of the outage capacity as well. The outage probability for the same five scenarios is plotted versus outage capacity for SNR=15dB in Figure 4.10. The 1% outage capacity increases from 0.4b/s/Hz to 2.9b/s/Hz or 4.8b/s/Hz for four transmit or four receive antennas respectively. But the 1% outage capacity increases to an amazing 12.6b/s/Hz with four transmit and four receive antennas. Similarly the 10% outage capacity goes from 2.1b/s/Hz to 14.2b/s/Hz when using MIMO as opposed to a single antenna on each end of the link.

4.7 Capacity with Channel Information at the Transmitter: Water-filling

If the channel is known, the transmit vectors $\mathbf{x}$ are still Gaussian, but we choose

$$\Sigma_{\mathbf{x}} = \begin{bmatrix} P_1 & 0 & \cdots & 0 \\ 0 & P_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & P_{N_t} \end{bmatrix} \tag{4.44}$$
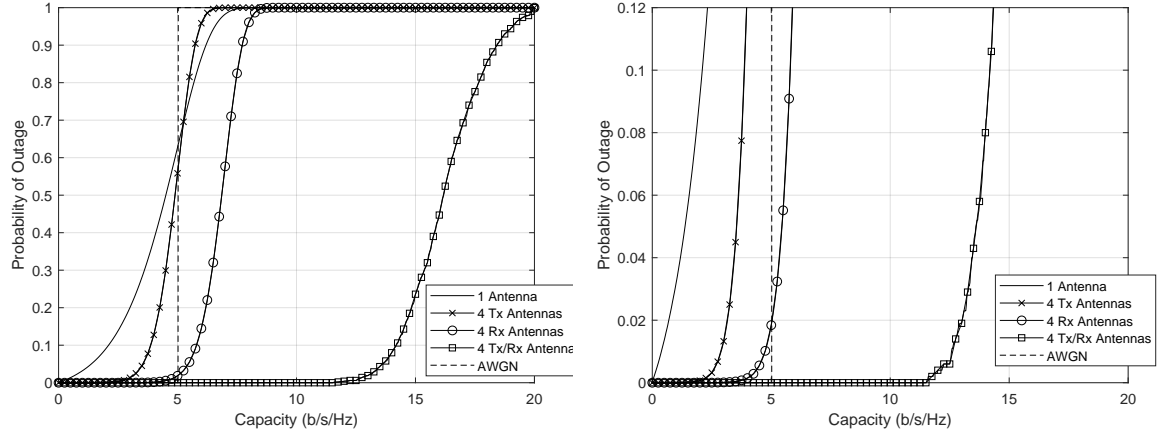
Figure 4.10: Outage Capacity (b/s/Hz) versus SNR in MIMO Rayleigh (independent) Fading Channel with Channel Information at the Receiver

where $P_i = \left(\mu - \frac{1}{\gamma\lambda_i}\right)^+$ and $(x)^+ = \max(x, 0)$. μ is the water-filling level such that $\sum_i P_i = P_t$. Note that for some weak eigenmodes, $P_i = 0$. If we define $\epsilon_i = \frac{P_i}{P_t}$ the resulting capacity is

$$\frac{C}{B} = \sum_{i=1}^{r} \log_2 \left(1 + \epsilon_i \gamma \lambda_i\right) \tag{4.45}$$

The ergodic capacity is

$$\left(\frac{C}{B}\right)_{erg} = \mathrm{E_H} \left\{ \sum_{i=1}^{r} \log_2 \left(1 + \epsilon_i \gamma \lambda_i\right) \right\} \tag{4.46}$$

4.7.1   Lagrangian Optimization

We can prove the above statement about the known channel using the method of Lagrange multipliers. Specifically, we wish to maximize $\frac{C}{B}$ over $\epsilon = \frac{P_i}{P_t}$ subject to the constraint that $\sum_i \epsilon_i = 1$ or $\sum_i P_i = P_t$. We can use the method of Largrange multipliers:

$$F = \sum_{i=1}^{r} \log_2 \left(1 + \epsilon_i \gamma \lambda_i\right) + \eta \left(P_t - P_t \sum_i \epsilon_i\right) \tag{4.47}$$

where η is the Lagrangian multiplier. We can find the optimal ϵ_i by taking the derivative of F with respect to ϵ_i and setting them equal to zero:

$$
\begin{aligned}
0 &= \frac{\partial}{\partial \epsilon_i} \left\{ \log_2 \left(1 + \epsilon_i \gamma \lambda_i \right) - \eta P_t \epsilon_i \right\} \\
&= \frac{1}{\ln(2)} \frac{1}{1 + \gamma \lambda_i \epsilon_i} \gamma \lambda_i - \eta P_t \\
&= \frac{\gamma \lambda_i}{1 + \gamma \lambda_i \epsilon_i} - \eta P_t \ln(2) \\
\epsilon_i &= \frac{1}{\eta \ln(2)} - \frac{1}{\gamma \lambda_i P_t} \\
&= \left[\mu - \frac{1}{\gamma \lambda_i} \right]^+
\end{aligned}
$$

where we have set the water level $\mu = \frac{1}{\eta \ln(2) P_t}$ and enforced the condition that $\epsilon_i \geq 0$.

4.7.2   Iterative Algorithm

1. set $i = 1$

2. $\mu = \frac{1}{\nu} = \frac{1}{n-i+1} \left(1 + \sum_{k=1}^{n-i+1} \frac{1}{\gamma \lambda_k} \right)$

3. $p_k(i) = \mu - \frac{1}{\gamma \lambda_k}$ for $k = 1. \ldots n - i + 1$

4. If all $p_k > 0$ stop; else set $i = i + 1$ and $p_{n-i+1} = 0$ and go to 2.

In general, channel knowledge is only beneficial at low signal to noise ratios. This can be seen in the example shown in Figure 4.11. In the plot we show the capacity of a 4×4 link (independent channels) with and without channel knowledge at the transmitter. As a reference, we also plot the capacity of inverse filling (i.e., attempting to maintain equal SNR). As mentioned, the impact of waterfilling is negligible, with only slight gains at low SNR. Additionally, we see that attempting to maintain constant SNR (and capacity) causes a large reduction in capacity. Thus, we can conclude that equal power transmission is generally an acceptable sub-optimal approach, unless the channel exhibits strong correlation which we examine next.

4.8   Impact of Eigenvalue Distribution

Assume that $N_t = M_r$ and that

$$
\sum_{i=1}^{M_r} \lambda_i = N_t M_r \tag{4.48}
$$

Let's examine two extreme cases: Extreme #1: $\lambda_1 = N_t M_r$ and $\lambda_2 = \lambda_3 = \ldots \lambda_{M_r} = 0$.

$$
\begin{aligned}
C &= log_2(1 + \gamma M_r N_t / N_t) \\
&= \log_2(1 + \gamma M_r) \tag{4.49}
\end{aligned}
$$

For high SNR

$$
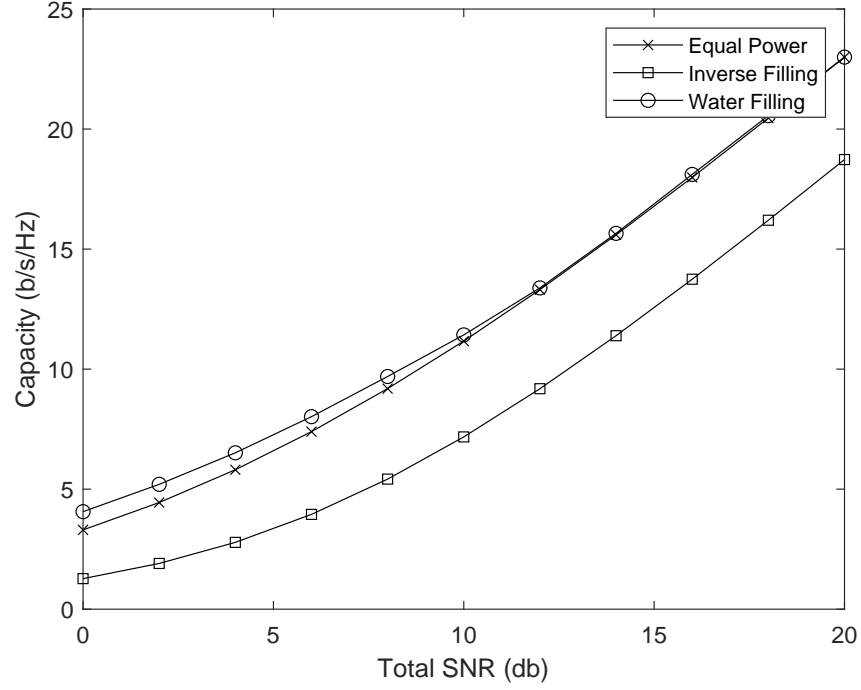C \approx \log_2(\gamma) + \log_2(M_r) \tag{4.50}
$$

Figure 4.11: Ergodic Capacity (b/s/Hz) versus SNR in Rayleigh (independent) Fading Channel with Channel Information at the Receiver

Thus, we get an *additive* effect.

Extreme #2: $\lambda_1 = \lambda_2 = \lambda_3 = \ldots \lambda_{M_r} = \frac{N_t M_r}{M_r} = N_t$.

$$
\begin{aligned}
C &= \sum_{i=1}^{M_r} log_2 \left(1 + \gamma \frac{N_t}{N_t} \right) \\
&= M_r \log_2(1 + \gamma)
\end{aligned}
\tag{4.51}
$$

High SNR

$$
C \approx M_r \log_2(\gamma)
\tag{4.52}
$$

Here we get a *multiplicative effect*!

As another example, consider a 4×4 system with independent Rayleigh fading as compared to a channel with 5° angle spread (and $\lambda/2$ element spacing) on one end of the link. The outage probability versus outage capacity is plotted in Figure 4.12 for the two cases with SNR = 15dB. The 10% outage capacity drops from 14.2b/s/Hz in independent channels to 5.9b/s/Hz with strong correlation on one end of the link. This again highlights the need for rich scattering to realize the gains promised by MIMO.
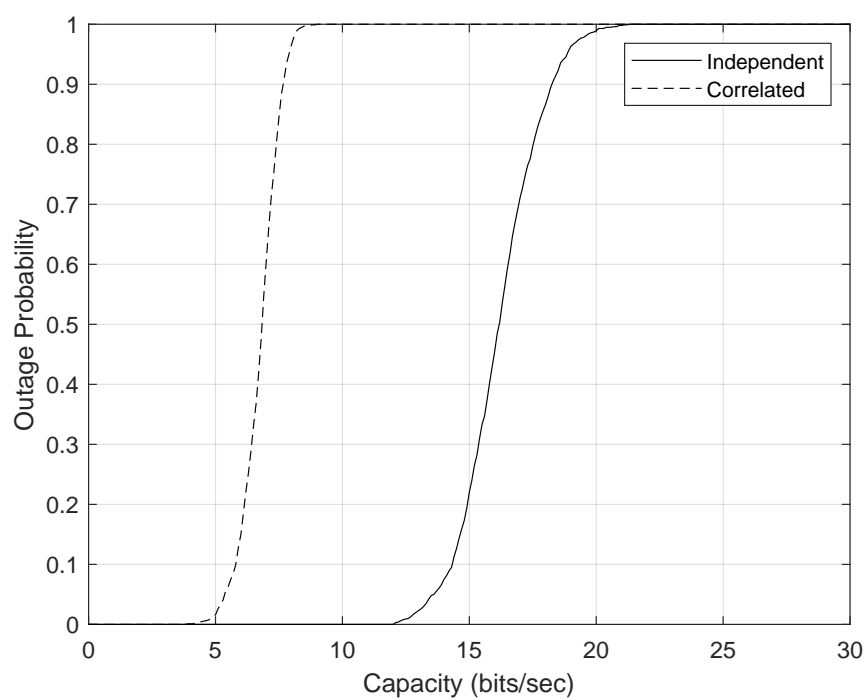
Figure 4.12: Outage Probability versus Capacity (b/s/Hz) SNR in Rayleigh Fading Channel with Channel Information at the Receiver

Chapter 5

Beamforming

The first application of mutliple antennas that we will consider is knwon as beamforming. In the development of beamforming we will assume that the channel is free of multipath at the array. That is, we will make what is known as the plane wave assumption. Note that there are forms of "beamforming" that do not require this assumption, but classical beamforming does. By classical beamforming we simply mean that there exists a meaningful array pattern ("beampattern") that describes the gain of the array response to signals arriving at various angles (or equivalently the transmit array gain in a particular direction). The primary benefit of beamforming is an SNR gain, sometimes termed "beamforming gain" or equivalently the "array gain". However, beamforming can also be formulated to mitigate interference as we will see.

5.1   Concept

Consider a linear array at the receiver as shown in Figure 5.1. Let the received signals on each antenna be combined using weights w_i. The received signal in complex baseband after combining can be written as
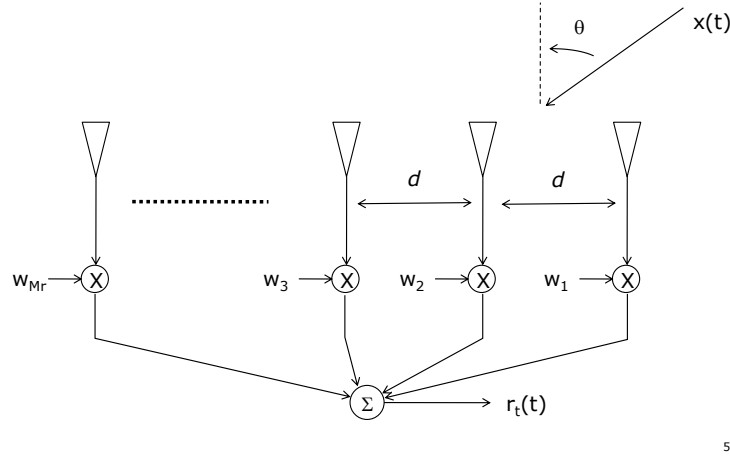
$$r_t(t) = \sum_{i=1}^{M_r} w_i x(t) e^{-j \frac{2\pi d(i-1)}{\lambda} \sin(\theta)} \tag{5.1}$$

where we have ignored noise for the moment, i.e., $r(t) = x(t)$. If we let $w_i = 1$, the received signal becomes:

$$\begin{aligned} r_t(t) &= \sum_{i=1}^{M_r} x(t) e^{-j \frac{2\pi d(i-1)}{\lambda} \sin(\theta)} \\ &= x(t) G(\theta) \end{aligned}$$

where $G(\theta)$ is the beam-pattern as a function of the angle-of-arrival θ:

$$G(\theta) = \sum_{i=1}^{M_r} e^{-j \frac{2\pi d(i-1)}{\lambda} \sin(\theta)} \tag{5.2}$$

Figure 5.1: Wavefront impinging on M_r receive antennas in a beamforming array

To get a better understanding of the beampattern created by this array, let's define a new variable $\psi = \frac{2\pi d}{\lambda}\sin(\theta)$ we can write the pattern as a function of ψ:

$$f(\psi) \quad = \quad 1 + e^{-j\psi} + e^{-j2\psi} \ldots + e^{-j(M_r-1)\psi}$$

$$(5.3)$$

To simplify this expression, first define a second relation by multiplying $f(\psi)$ by $e^{-j\psi}$:

$$f(\psi)e^{-j\psi} \quad = \quad e^{-j\psi} + e^{-j2\psi} \ldots + e^{-j(M_r)\psi}$$

Now, subtracting the last equation from the previous we obtain:

$$
\begin{aligned}
f(\psi)(1 - e^{-j\psi}) &= (1 - e^{-jM_r\psi}) \\
f(\psi) &= \frac{1 - e^{-jM_r\psi}}{1 - e^{-j\psi}} \\
&= \frac{e^{-jM_r/2\psi}}{e^{-j\psi/2}}\frac{e^{jM_r/2\psi} - e^{-jM_r/2\psi}}{e^{j\psi/2} - e^{-j\psi/2}} \\
&= e^{-j(M_r-1)/2\psi}\frac{\sin(M_r\psi/2)}{\sin(\psi/2)}
\end{aligned}
$$

$$(5.4)$$

By making this substitution we have removed the antenna spacing (d/λ) from the pattern expression, which allows us to focus on the impact of the number of elements M_r. From equation (5.4) we can see that (ignoring the leading phase term) with M_r antennas there are $M_r - 1$ lobes in the function $f(\psi)$ over the range of $-\pi \leq \psi \leq \pi$. Since $\psi = \frac{2\pi d}{\lambda}\sin(\theta)$, the number of lobes in the pattern as $-\pi \leq \theta \leq \pi$ depends on the the value of $\frac{d}{\lambda}$. For example, if $\frac{d}{\lambda} = \frac{1}{2}$ there are $M_r - 1$ lobes in $G(\theta)$, $M_r - 2$ sidelobes and one main lobe. More specifically, the value of $\frac{d}{\lambda}$ dictates the range of the generic pattern $f(\psi)$ that appears in the array pattern $G(\theta)$, which we explore in a moment.

Example patterns for two, four, six and eight elements are shown in Figure 5.2. We can clearly see that for M_r elements, there are $M_r - 1$ lobes. The width of the sidelobes in ψ is $\frac{2\pi}{M_r}$ and the width of the main beam is $\frac{4\pi}{M_r}$. As expected,

$$(M_r - 2)\frac{2\pi}{M_r} + \frac{4\pi}{M_r} = 2\pi \tag{5.5}$$

Now, since 2π in ψ is mapped to $\frac{2\pi d}{\lambda}\sin(\theta)$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, the beamwidth will also decrease with M_r and central AOA. If $\frac{d}{\lambda} = \frac{1}{2}$, the beamwidth in θ is $\frac{2\pi}{M_r}$ when the AOA is 0 degrees.

We now consider some important implications of this relationship. First, we can see from (5.4) that $f(\psi)$ repeats when ψ is a multiple of 2π. In other words, the pattern has a periodicity of 2π. Since $\psi = \frac{2\pi d}{\lambda}\sin(\theta)$, for the range of arrival angles $-\pi \le \theta \le \pi$ we have $-\frac{2\pi d}{\lambda} \le \psi \le \frac{2\pi d}{\lambda}$. Thus, the value of $\frac{d}{\lambda}$ will dictate the amount of the pattern $f(\psi)$ seen by the array. For example consider the following scenarios for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$:

* If $\frac{d}{\lambda} = \frac{1}{4}$, then $-\frac{\pi}{2} \le \psi \le \frac{\pi}{2}$

* If $\frac{d}{\lambda} = \frac{1}{2}$, then $-\pi \le \psi \le \pi$

* If $\frac{d}{\lambda} = \frac{3}{4}$, then $-\frac{3\pi}{2} \le \psi \le \frac{3\pi}{2}$.

* If $\frac{d}{\lambda} = 1$, then $-2\pi \le \psi \le 2\pi$.

The gain patterns $g(\theta)$ for these four cases are plotted in Figure 5.3 for $M_r = 8$. In the first case, a spacing of one quarter of a wavelength results in a gain pattern that corresponds to a fraction of the overall pattern $f(\psi)$. This means that the main beamwidth is wider than it could be based on the number of antenna elements. On the other hand, a spacing of one-half a wavelength allows the gain pattern to correspond to the entire pattern $f(\psi)$ and provides a main beamwidth commensurate with the number of antenna elements (i.e., 45°). Third, using a spacing of three-quarters of a wavelength causes the pattern repeat, although the beamwidth continues to narrow. Finally, using a spacing of λ causes sufficient repetition that what are known as "grating lobes" appear, since ψ spans more than 2π.

Finally, it should be noted that θ and $\pi - \theta$ give the same value of ψ. This results from the fact that for a linear array, there is an ambiguity between signals arriving from the angle θ and the angle $\pi - \theta$. This means that the array cannot distinguish whether a signal is arriving from the angle θ or $\pi - \theta$ and the gain pattern repeats on both sides of the array.
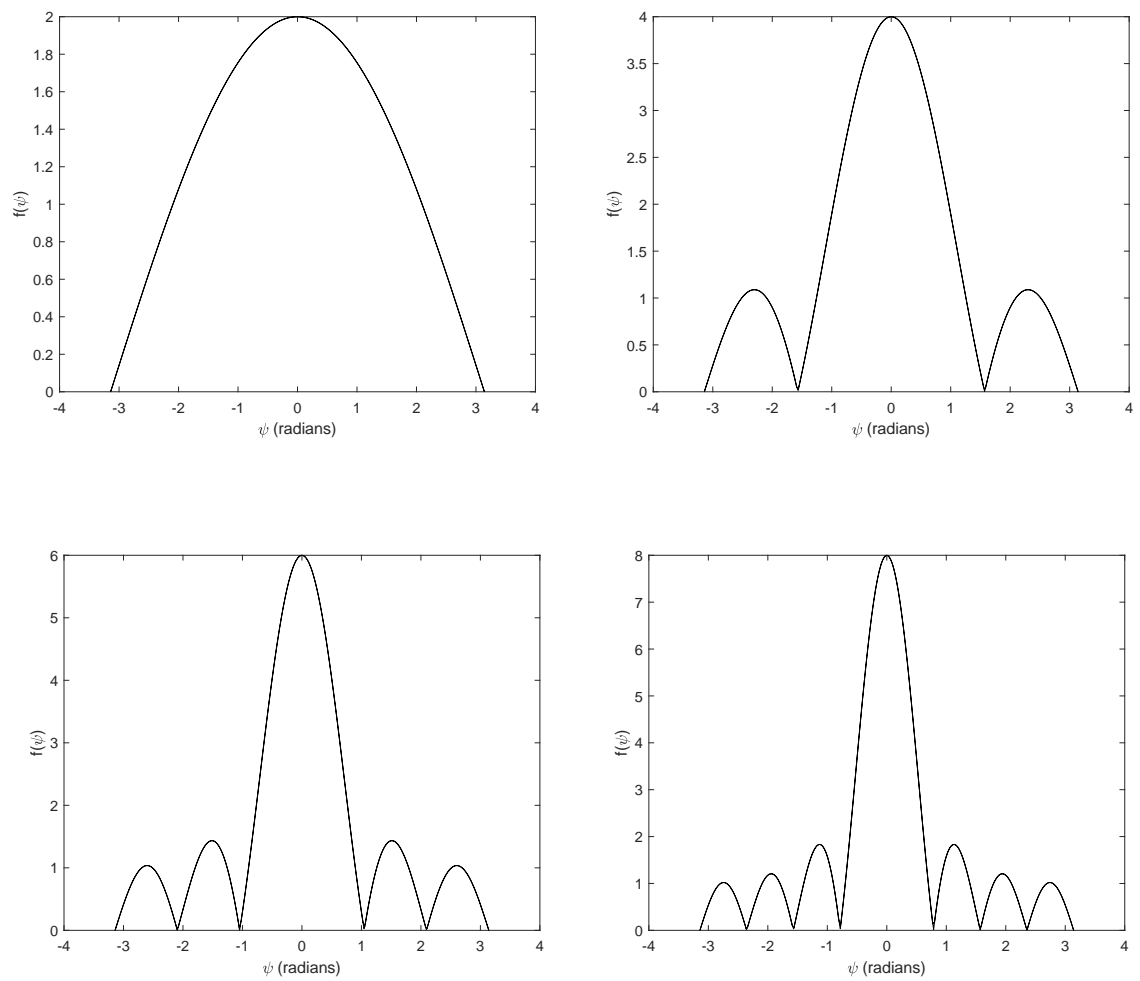
Figure 5.2: The Universal Pattern $f(\psi)$ for Two, Four, Six and Eight Antenna Elements
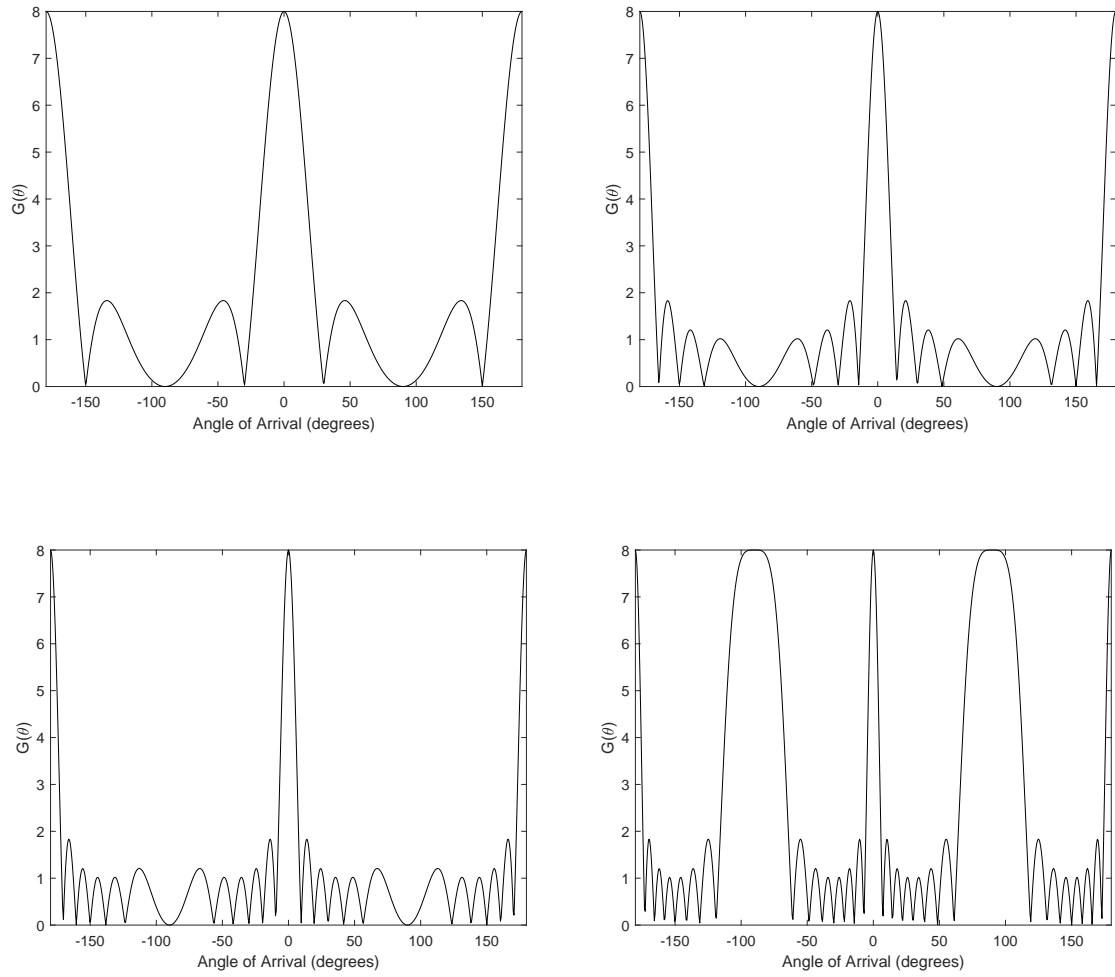
Figure 5.3: The Gain Pattern $g(\theta)$ for Eight Antenna Elements with $\frac{d}{\lambda} = \frac{1}{4}, \frac{d}{\lambda} = \frac{1}{2}, \frac{d}{\lambda} = \frac{3}{4}$, and $\frac{d}{\lambda} = 1$

5.2   Beam-steering

In the previous discussion, we assumed that the weights were all ones. This is helpful if the signal is arriving from boresight (broad-side or $\theta = 0$). If we wish to maximize the array gain in a direction θ_o other than 0°, we must consider the the channel resulting from that particular angle. To see this, the received signal can be written (ignoring any Rayleigh fading and assuming that the channel is constant across antennas except for a phase shift - i.e., the plane wave assumption) as

$$\mathbf{r}(t) = \mathbf{a}(\theta_o)x(t) + \mathbf{n}(t) \tag{5.6}$$

where we shall define $\mathbf{a}(\theta_o) = \begin{bmatrix} 1 \\ e^{-j\psi_o} \\ e^{-j2\psi_o} \\ e^{-j3\psi_o} \\ \vdots \\ e^{-j(M_r-1)\psi_o} \end{bmatrix}$ as the array factor (or array manifold vector) for

our uniform linear array, $\psi_o = \frac{2\pi d}{\lambda}\sin(\theta_o)$ and θ_o is the angle of arrival. After combining we have (ignoring noise)

$$
\begin{aligned}
r_t(t) &= \sum_{i=1}^{M_r} w_i^* a_i(\theta_o)x(t) \\
&= \underbrace{\mathbf{w}^H \mathbf{a}(\theta_o)}_{=\eta} x(t)
\end{aligned}
\tag{5.7}
$$

The array factor is equivalent to the channel experienced by the signal across the array. Note that the gain η is maximized when $\mathbf{w} = \mathbf{a}(\theta_o)$. In this case

$$r_t(t) = M_r x(t) \tag{5.8}$$

where M_r is our *array gain* and is the gain in SNR we will observe (as we will shown in Section 5.5.1).

5.2.1   Beam pattern

For an array tuned to θ_o, we can write the expression for the gain as function of θ, i.e., the beampattern as

$$G(\theta) = \mathbf{w}^H(\theta_o)\mathbf{a}(\theta) \tag{5.9}$$

By choosing the appropriate weight vector we can steer (tune) the beam to any direction θ_o. This is often termed the "steering vector". For example, but choosing

$$\mathbf{w} = \mathbf{a}(\theta) \tag{5.10}$$

for different values of θ, we can move the gain of the array to different locations. This can be seen in the three examples of Figure 5.4 with $\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}$. We can see that changing the value of θ steers the gain to the location of θ. Additionally, steering the gain away from 0 towards $\frac{\pi}{2}$ (i.e., from broadside to endfire) broadens the beam.

5.2.2   Ambiguity with a linear array

Note that we we hinted at earlier,

$$
\mathbf{a}(\pi - \theta_o) = \begin{bmatrix} 1 \\ e^{-j\frac{2\pi d}{\lambda}\sin(\pi-\theta_o)} \\ e^{-j\frac{4\pi d}{\lambda}\sin(\pi-\theta_o)} \\ e^{-j\frac{6\pi d}{\lambda}\sin(\pi-\theta_o)} \vdots \\ e^{-j\frac{2(M_r-1)\pi d}{\lambda}\sin(\pi-\theta_o)} \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 \\ e^{-j\frac{2\pi d}{\lambda}\sin(\theta_o)} \\ e^{-j\frac{4\pi d}{\lambda}\sin(\theta_o)} \\ e^{-j\frac{6\pi d}{\lambda}\sin(\theta_o)} \vdots \\ e^{-j\frac{2(M_r-1)\pi d}{\lambda}\sin(\theta_o)} \end{bmatrix} \tag{5.11}
$$

where we have used the fact that $\sin(\pi - \theta_o) = \cos(\pi)\sin(-\theta_o) + \cos(\theta_o)\sin(\pi) = \sin(\theta_o)$. Thus, the array cannot differentiate between a signal arriving from θ_o and $\pi - \theta_o$. Thus, the gain pattern repeats as seen in Figure 5.4. Finally, it should be noted that this array is often termed a *phased array* since the array weights are simple phase shifts.
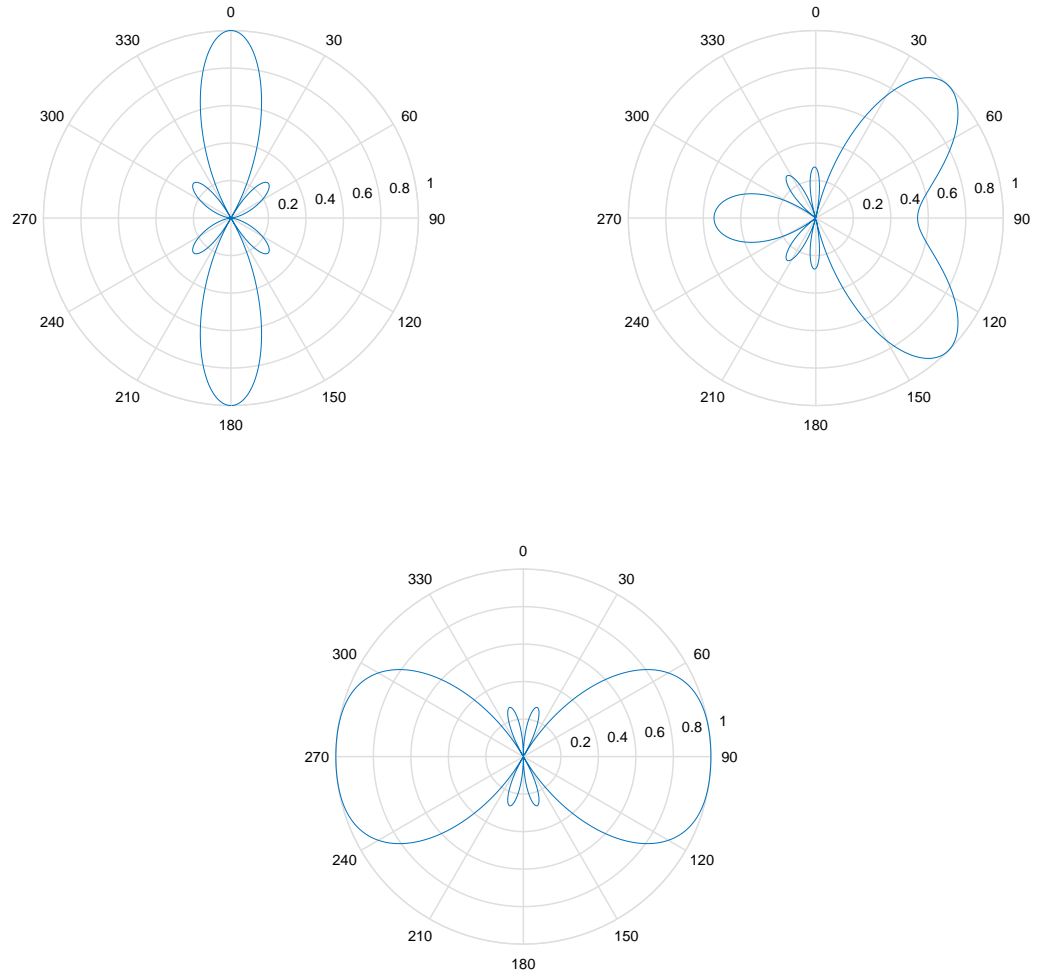
Figure 5.4: The Gain Pattern $g(\theta)$ for Four Antenna Elements with $\frac{d}{\lambda} = \frac{1}{2}$ and steering angles $\theta = 0$ (top, left), $\theta = \frac{\pi}{4}$ (top, right), and $\theta = \frac{\pi}{2}$ (bottom)

5.3 Directivity

Directivity is a measure of an antenna's ability to direct energy in a desired direction relative to a point source. It is defined as

$$D = \frac{4\pi}{\Omega_A} \tag{5.12}$$

where Ω_A is the solid beam angle defined as

$$\Omega_A = \int \int \underbrace{|g_a(\theta,\phi)|^2}_{\text{antenna pattern}} \underbrace{\left|\hat{f}(\theta)\right|^2}_{\text{normalized array factor}} \underbrace{d\Omega}_{\cos(\theta)\mathrm{d}\theta\mathrm{d}\phi} \tag{5.13}$$

Assumes that the antenna elements are point sources which is an OK assumption if the element pattern is much more broad than the array factor ($g_A(\theta,\phi) = 1$). Further, we know that

$$\left|\hat{f}(\psi)\right|^2 = \left|\frac{\sin(M_\mathrm{r}\psi/2)}{M_r\sin(\psi/2)}\right|^2$$

$$= \frac{1}{M_r} + \frac{2}{M_r^2}\sum_{i=1}^{M_r}(M_r - i)\cos(i\psi)$$

Let the pointing angle of the array weights be θ_o. Then

$$\psi = \frac{2\pi d}{\lambda}\sin(\theta) + \psi_\mathrm{o} \tag{5.14}$$

where $\psi_o = \frac{2\pi d}{\lambda}\sin(\theta_o)$. Now $\cos(\theta)\,\mathrm{d}\theta = \frac{\lambda}{2\pi d}\mathrm{d}\psi$. Thus,

$$\begin{aligned}
\Omega_A &= \int_0^{2\pi} d\phi \int_{-\pi/2}^{\pi/2} |f(\theta)|^2 \cos(\theta)\,\mathrm{d}\theta \\
&= 2\pi \int_{-\frac{2\pi d}{\lambda}+\psi_o}^{\frac{2\pi d}{\lambda}+\psi_o} |f(\psi)|^2 \left(\frac{\lambda}{2\pi d}\right) d\psi \\
&= \frac{\lambda}{d} \int_{-\frac{2\pi d}{\lambda}+\psi_o}^{\frac{2\pi d}{\lambda}+\psi_o} |f(\psi)|^2 \, d\psi \\
&= \frac{\lambda}{d} \left[\frac{1}{M_r}\int_{-\frac{2\pi d}{\lambda}+\psi_o}^{\frac{2\pi d}{\lambda}+\psi_o} d\psi + \frac{2}{M_r^2}\sum_{i=1}^{M_r-1}(M_r - i)\int_{-\frac{2\pi d}{\lambda}+\psi_o}^{\frac{2\pi d}{\lambda}+\psi_o} \cos(i\psi)\,\mathrm{d}\psi\right] \\
&= \frac{\lambda}{d} \left[\frac{\psi}{M_r}\bigg|_{-\frac{2\pi d}{\lambda}+\psi_o}^{\frac{2\pi d}{\lambda}+\psi_o} + \frac{2}{M_r^2}\sum_{i=1}^{M_r-1}(M_r - i)\frac{\sin(i\psi)}{i}\bigg|_{-\frac{2\pi d}{\lambda}+\psi_o}^{\frac{2\pi d}{\lambda}+\psi_o}\right] \\
&= \frac{4\pi}{M_r} + \frac{4\pi}{M_r^2}\sum_{i=1}^{M_r-1}\frac{M_r - i}{i 2\pi d/\lambda}2\cos(i\psi_o)\sin\left(\frac{2\pi \mathrm{d}i}{\lambda}\right) \tag{5.15}
\end{aligned}$$

And thus we have the directivity is equal to

$$\boxed{D = \frac{1}{\frac{1}{M_r} + \frac{1}{M_r^2}\sum_{i=1}^{M_r-1}\frac{M_r-i}{i2\pi d/\lambda}2\cos(i\psi_o)\sin\left(\frac{2\pi \mathrm{d}i}{\lambda}\right)}} \tag{5.16}$$

Note that if $d = \frac{\lambda}{2}$ then $D = M_r$ irrespective of θ_o. This is because $\sin\left(\frac{2\pi \mathrm{d}i}{\lambda}\right) = \sin(i\pi) = 0$

5.4   Circular Array

For a any array we can define the received signal as $\mathbf{r}(t) = \mathbf{a}(\theta)r(t)$ where $\mathbf{a}(\theta)$ is the array factor (also called the array manifold vector). For a linear array we have

$$
\mathbf{a}(\theta) = \begin{bmatrix} 1 \\ e^{-j\frac{2\pi d}{\lambda}\sin(\theta)} \\ e^{-j2\frac{2\pi d}{\lambda}\sin(\theta)} \\ e^{-j3\frac{2\pi d}{\lambda}\sin(\theta)} \\ \vdots \\ e^{-j(M_r-1)\frac{2\pi d}{\lambda}\sin(\theta)} \end{bmatrix} \tag{5.17}
$$

where the first element is the phase reference. For a uniformly space circular array, we define the phase reference (phase center) to be the center of the array. Further, let R be the radius of the array. The new array factor is then:

$$
\mathbf{a}(\theta) = \begin{bmatrix} e^{-j\frac{2\pi R}{\lambda}\sin(\theta-\zeta_1)} \\ e^{-j\frac{2\pi R}{\lambda}\sin(\theta-\zeta_2)} \\ e^{-j\frac{2\pi R}{\lambda}\sin(\theta-\zeta_3)} \\ e^{-j\frac{2\pi R}{\lambda}\sin(\theta-\zeta_4)} \\ \vdots \\ e^{-j\frac{2\pi R}{\lambda}\sin(\theta-\zeta_{M_r})} \end{bmatrix} \tag{5.18}
$$

where $\zeta_i = \left(\frac{i-1}{M_r}\right) 2\pi$ is the location (in angle) of the ith antenna element on the array. As with a linear array, we can maximize the gain when we set $\mathbf{w} = \mathbf{a}^*(\theta_o)$ in which case the gain is M_r.

As an example, consider an eight element circular array with $R = \frac{\lambda}{2}$. The beampatterns for this array when the steering angle is $\theta = 0$ and $\theta = \frac{\pi}{4}$ are shown in Figure 5.5. We can see that the pattern does not repeat on the backside of the array, and thus there is no ambiguity for angles greater than $\pi/2$ or less than $-\pi/2$. Further, we see that in this example the beampattern doesn't fundamentally change, but simply rotates.

5.5   Optimal Beamforming

In the previous section we examined the antenna weights that steered a beam in a particular direction. This can also be thought of as maximizing the gain in a particular direction. We will also show in this section that these are the weights which maximize the SNR of the signal post-combining, when the signal is arriving from the direction of the maximum pattern gain. In other words, setting $\mathbf{w} = \mathbf{a}$ maximizes SNR. However, what if there are other signals in the environment? Does maximizing SNR still provide the best results? It is unlikely that maximizing SNR will be the best option. Thus, we can also set the weights to eliminate the interference (known as "zero-forcing"), maximize SINR or minimize the mean square error between a desired signal and the output of the array. We will examine each of these different optimality criteria in the following subsections.
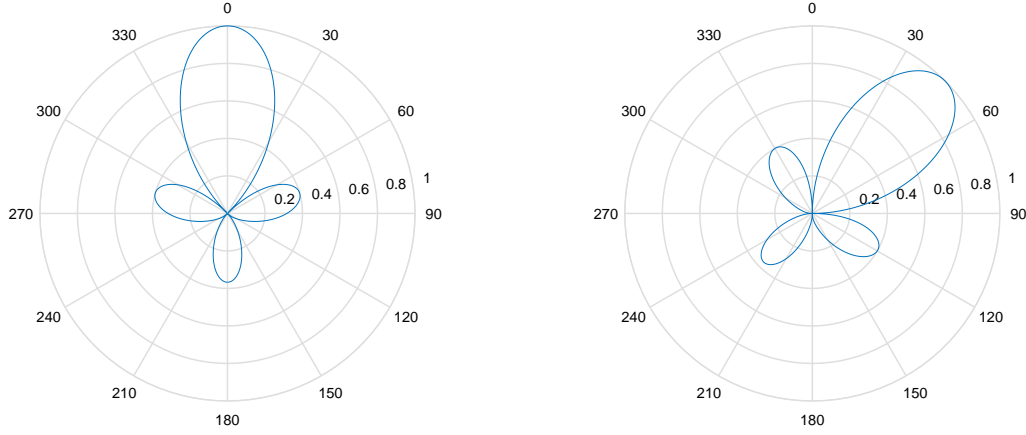
Figure 5.5: The Gain Pattern $g(\theta)$ for an Eight Element Circular Array with $\frac{R}{\lambda} = \frac{1}{2}$ and steering angles $\theta = 0$ (left) and $\theta = \frac{\pi}{4}$ (right)

5.5.1 Maximum SNR

Consider the received signal vector in compex baseband

$$\tilde{\mathbf{r}}(t) = \mathbf{a}x(t) + \mathbf{n}(t) \tag{5.19}$$

where $\mathbf{a} = \mathbf{a}(\theta)$ (we have dropped the function of θ for notational simplicity), $\mathbf{n}(t)$ is a Gaussian random process with $\mathrm{E}\left\{\mathbf{n}(t)\right\} = \bar{0}$ and $\mathrm{E}\left\{\mathbf{n}(t)\mathbf{n}^H(t)\right\} = \sigma_n^2 \mathbf{I}^{M_r \times M_r}$. Assume that the desired signal is assumed to be a modulated pulse stream: $x(t) = \sum_{i=-\infty}^{\infty} s_i p(t - iT)$. After pulse matched filtering and sampling, we have for an arbitrary symbol the vector

$$\tilde{\mathbf{r}} = \mathbf{a}s + \mathbf{n} \tag{5.20}$$

where $\mathrm{E}\left\{|s|^2\right\} = E_s$ (the energy per symbol). At the output of a combiner using weights $\mathbf{w}$ we have the decision variable

$$z = \mathbf{w}^H \mathbf{r} \tag{5.21}$$

The question is, how do we choose $\mathbf{a}$ to maximize the SNR of z? First, we write z in terms of the desired and undesired components (z_s and z_n respectively):

$$
\begin{aligned}
z &= \mathbf{w}^H \tilde{\mathbf{r}} \\
&= z_s + z_n \\
&= \mathbf{w}^H \mathbf{a}s + \mathbf{w}^H \mathbf{n}
\end{aligned}
$$

Now the signal to noise ratio can be written as

$$
\begin{aligned}
\gamma &= \frac{\mathrm{E}\left\{|z_s|^2\right\}}{\mathrm{E}\left\{|z_n|^2\right\}} \\
&= \frac{\mathrm{E}\left\{|\mathbf{w}^H \mathbf{a} s|^2\right\}}{\mathrm{E}\left\{|\mathbf{w}^H \mathbf{n}|^2\right\}} \\
&= \frac{\mathbf{w}^H \mathbf{a} \mathbf{a}^H \mathbf{w} E_s}{\mathbf{w}^H \mathbf{R}_{nn} \mathbf{w}}
\end{aligned}
$$

Since $\mathbf{R}_{nn} = \sigma_n^2 \mathbf{I}$ we have

$$
\gamma = \underbrace{\frac{\mathbf{w}^H \mathbf{R}_{aa} \mathbf{w}}{\mathbf{w}^H \mathbf{w}}}_{\text{Rayleigh Quotient}} \frac{E_s}{\sigma_n^2}
$$

Note that $\mathbf{R}_{aa} = \mathbf{a}\mathbf{a}^H$ is Hermitian and positive semidefinite. Thus, γ is the well-known Rayleigh quotient which is maximized when $\mathbf{w}$ is the maximum eigenvector of $\mathbf{R}_{aa}$. Since $\mathbf{R}_{aa}$ is just the outer product of $\mathbf{a}$, there is only one eigenvector that has a non-zero eigenvalue, which is $\mathbf{a}$. Thus, $\mathbf{w}_{opt} = \mathbf{a}$ and the maximum SNR is

$$
\begin{aligned}
\gamma_{opt} &= \frac{E_s}{\sigma_n^2} \frac{|\mathbf{a}^H \mathbf{a}|^2}{\mathbf{a}^H \mathbf{a}} \\
&= \mathbf{a}^H \mathbf{a} \frac{E_s}{\sigma_n^2} \\
&= M_r \frac{E_s}{\sigma_n^2}
\end{aligned}
$$

and M_r is the beamforming gain, meaning that there is an M_r-fold improvement in SNR in the optimal case. Note that this is the same analaysis that we would apply to a diversity array. Thus **optimal SNR weights are the same as Maximal Ratio Combining** as we will show in the next chapter.

5.5.2   Zero-Forcing

Consider the received signal on $M_r = 4$ antennas where there are $N_I = 3$ interferers. Let's assume that the signals are all synchronous in time and ignore the fading channel and noise. Let the array factor (response) to the desired signal be $\mathbf{a}_0$ and the array response to the interfering signals be $\mathbf{a}_k$ for $k = 1, 2, 3$. The vector of received signals (after matched filtering) can be written as

$$
\begin{aligned}
r_1 &= a_{0,1}s + a_{1,1}i_1 + a_{2,1}i_2 + a_{3,1}i_3 \\
r_2 &= a_{0,2}s + a_{1,2}i_1 + a_{2,2}i_2 + a_{3,2}i_3 \\
r_3 &= a_{0,3}s + a_{1,3}i_1 + a_{2,3}i_2 + a_{3,3}i_3 \\
r_4 &= a_{0,4}s + a_{1,4}i_1 + a_{2,4}i_2 + a_{3,4}i_3
\end{aligned}
$$

$$(5.22)$$

or in vector form

$$\mathbf{r} = \mathbf{A}\mathbf{s} \qquad\qquad (5.23)$$

where $\mathbf{A} = [\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$ and $\mathbf{s} = [s, i_1, i_2, i_3]^T$. We wish to create weights $\mathbf{w}$ such that $\mathbf{w}^T \tilde{\mathbf{r}} = \kappa s$ for arbitrary constant κ. In other words, for four antennas we desire

$$
\begin{aligned}
w_1 a_{0,1} + w_2 a_{0,2} + w_3 a_{0,3} + w_4 a_{0,4} &= \kappa \\
w_1 a_{1,1} + w_2 a_{1,2} + w_3 a_{1,3} + w_4 a_{1,4} &= 0 \\
w_1 a_{2,1} + w_2 a_{2,2} + w_3 a_{2,3} + w_4 a_{2,4} &= 0 \\
w_1 a_{3,1} + w_2 a_{3,2} + w_3 a_{3,3} + w_4 a_{3,4} &= 0
\end{aligned}
$$

(5.24)

or in vector form we desire

$$
\mathbf{A}^T \mathbf{w} = \kappa \bar{\delta}
$$

(5.25)

where $\bar{\delta} = [1, 0, 0, 0]^T$. Thus, we need to set

$$
\mathbf{w} = (\mathbf{A}^T)^{-1} \kappa \bar{\delta}
$$

(5.26)

This solution is termed the "zero-forcing" solution since it forces the interference to zero. Note that the zero-forcing weights are the first comlumn of $(\mathbf{A}^T)^{-1}$. In this case (3 interferers and 1 desired user with 4 antennas), $rank(\mathbf{A}) = N_I + 1 = M_r$. However, if $rank(\mathbf{A}) < M_r$ (i.e., $N_I < M_r - 1$) we have

$$
\mathbf{w} = \mathbf{A}(\mathbf{A}^T \mathbf{A})^{-1} \kappa \bar{\delta} = (\mathbf{A}^T)^\dagger \kappa \bar{\delta}
$$

(5.27)

while if $N_I \geq M_r$ we have

$$
\mathbf{w} = \mathbf{A}^\dagger \kappa \bar{\delta}
$$

(5.28)

Note that this does not eliminate all interferers, but instead is the least squares (LS) solution. Note that although this weight vector eliminates interference, it does not necessarily maximize SNR. In fact, it is very likely that the SNR will be decidedly sub-optimal. From a beampattern perspective, we find that forcing nulls in the directions of all interferers means that a peak gain cannot be guaranteed in the direction of the desired signal, thus resulting in a loss in SNR relative to the optimal SNR case. This can be seen in the examples plotted in Figure 5.6. The first plot shows the beampattern with a four antenna element linear array (half wavelength spacing) and maximum SNR weights. The desired signal is coming from $\theta = \frac{\pi}{4}$. Clearly, the peak gain is in the direction of the desired user. The second plot shows the beampattern formed using zero-forcing weights when the desired signal is coming from $\theta = \frac{\pi}{4}$ and a single interferer is coming from $\theta = \frac{\pi}{2}$. We can see that a null is now placed in the direction of the interferer, but the gain in the direction of the desired user is no longer maximum. As a result, the SNR will suffer. IN the third plot, we add an interferer at $\theta = -\frac{\pi}{4}$ causing a second null and reducing the gain in the direction of the desired signal slightly more. In general, we can use the degrees of freedom in the array weights to either eliminate interference or maximize SNR, but not both.

5.5.3 Max SINR

In the previous two sections we examined the weights which maximized either the SNR or the SIR. However, in the presence of both noise and interference, neither choice may be ideal. Thus, we wish to determine weights which maximize SINR. Consider the following received signal (again
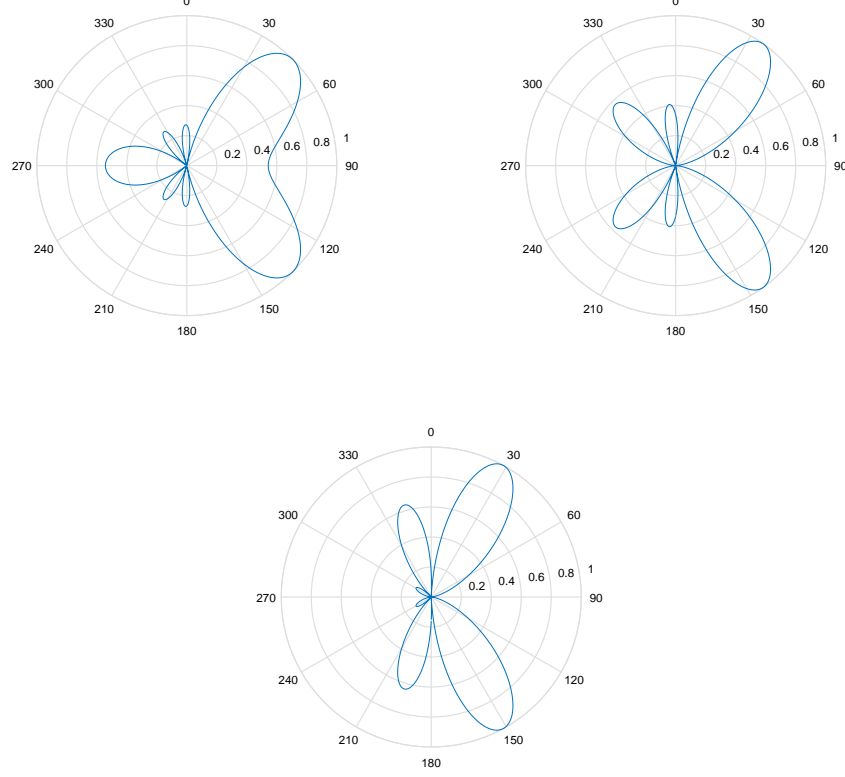
Figure 5.6: The Gain Pattern $g(\theta)$ for Four Antenna Elements with $\frac{d}{\lambda} = \frac{1}{2}$ using Maximum SNR Weights with desired signal at $\theta = \frac{\pi}{4}$ (top, left), Zero-Forcing Weights with One Interferer at $\theta = \frac{\pi}{2}$ (top, right) and Zero-Forcing Weights with Two Interferers at $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{4}$ (bottom)

in complex baseband) on a set of M_r antennas

$$\mathbf{r}(t) = \mathbf{a}_0 x(t) + \sum_{k=1}^{N_I} \mathbf{a}_k i_k(t) + \mathbf{n}(t) \tag{5.29}$$

After matched filtering we obtain a vector $\tilde{\mathbf{r}}$ which is combined using a weight vector $\mathbf{w}$ to obtain a decision variable z:

$$\begin{aligned} z &= \mathbf{w}^H \mathbf{r} \\ &= \underbrace{\mathbf{w}^H \mathbf{a}_0 s}_{z_s} + \underbrace{\mathbf{w}^H \left(\sum_{k=1}^{N_I} \mathbf{a}_k i_k + \mathbf{n} \right)}_{z_n} \end{aligned} \tag{5.30}$$

The SINR can be shown to be

$$
\begin{aligned}
\gamma &= \frac{E\left\{|z_s|^2\right\}}{E\left\{|z_n|^2\right\}} \\
&= \frac{E\left\{|\mathbf{w}^H \mathbf{a}_0 s|^2\right\}}{E\left\{|\mathbf{w}^H \overline{\nu}|^2\right\}} \\
&= \frac{\mathbf{w}^H \mathbf{a}_0 \mathbf{a}_0^H \mathbf{w} E_s}{\mathbf{w}^H \mathbf{R}_{\nu\nu} \mathbf{w}}
\end{aligned}
$$

where $\overline{\nu} = \sum_{k=1}^{N_I} \mathbf{a}_k i_k + \mathbf{n}$. $\mathbf{R}_{\nu\nu}$ is found to be

$$
\begin{aligned}
\mathbf{R}_{\nu\nu} &= E\left\{\overline{\nu}\,\overline{\nu}^H\right\} \\
&= \mathbf{R_{ii}} + \sigma_n^2 \mathbf{I} \tag{5.31}
\end{aligned}
$$

The interference covariance matrix is equal to

$$
\begin{aligned}
\mathbf{R_{ii}} &= E\left\{\left(\sum_{k=1}^{N_I} \mathbf{a}_k i_k\right)\left(\sum_{k=1}^{N_I} \mathbf{a}_k i_k\right)^H\right\} \\
&= \sum_{k=1}^{N_I} \mathbf{a}_k \mathbf{a}_k^H
\end{aligned}
$$

and we have assumed that $E\left\{i_k i_j\right\} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$ (i.e., the interference signals are uncorrelated).

Now we can write the SINR as

$$
\gamma = \frac{\mathbf{w}^H \mathbf{R_{aa}} \mathbf{w}}{\mathbf{w}^H \mathbf{R}_{\nu\nu} \mathbf{w}} E_s \tag{5.32}
$$

where $\mathbf{R_{aa}} = E\{\mathbf{a}_0 \mathbf{a}_0^H\}$ Let's define $\mathbf{z} \triangleq \mathbf{R}_{\nu\nu}^{1/2} \mathbf{w}$. We can then rewrite the SINR as

$$
\gamma = \frac{\mathbf{z}^H \mathbf{R}_{\nu\nu}^{-1/2} \mathbf{R_{aa}} \mathbf{R}_{\nu\nu}^{-1/2} \mathbf{z}}{\mathbf{z}^H \mathbf{z}} E_s \tag{5.33}
$$

where we have used the fact that $\mathbf{R}_{\nu\nu}^{1/2}$ is Hermitian. Thus, we again have a Rayleigh quotient and the SINR is maximized when $\mathbf{z}$ is the maximum eigenvector of $\mathbf{R}_{\nu\nu}^{-1/2} \mathbf{R_{aa}} \mathbf{R}_{\nu\nu}^{-1/2}$. Alternatively, we seek the maximum eigenvector which solves

$$
\mathbf{R_{aa}} \mathbf{w} = \gamma_{max} \mathbf{R}_{\nu\nu} \mathbf{w} \tag{5.34}
$$

Now, we know that (assuming $E_s = 1$ for simplicity of notation)

$$
\gamma_{max} = \frac{\mathbf{w}_{opt}^H \mathbf{R_{aa}} \mathbf{w}_{opt}}{\mathbf{w}_{opt}^H \mathbf{R}_{\nu\nu} \mathbf{w}_{opt}} \tag{5.35}
$$

$$
\begin{aligned}
\mathbf{R_{aa}} \mathbf{w}_{opt} &= \frac{\mathbf{w}_{opt}^H \mathbf{R_{aa}} \mathbf{w}_{opt}}{\mathbf{w}_{opt}^H \mathbf{R}_{\nu\nu} \mathbf{w}_{opt}} \mathbf{R}_{\nu\nu} \mathbf{w}_{opt} \\
\mathbf{a}_0 \mathbf{a}_0^H \mathbf{w}_{opt} &= \frac{\mathbf{w}_{opt}^H \mathbf{a}_0 \mathbf{a}_0^H \mathbf{w}_{opt}}{\mathbf{w}_{opt}^H \mathbf{R}_{\nu\nu} \mathbf{w}_{opt}} \mathbf{R}_{\nu\nu} \mathbf{w}_{opt}
\end{aligned}
$$

$$
\tag{5.36}
$$

Since $\mathbf{a}_0^H \mathbf{w}_{opt}$ is a scalar, we can cancel it from both sides giving

$$\mathbf{a}_0 = \underbrace{\frac{\mathbf{w}_{opt}^H \mathbf{a}_0}{\mathbf{w}_{opt}^H \mathbf{R}_{\nu\nu} \mathbf{w}_{opt}}}_{\text{constant } \kappa} \mathbf{R}_{\nu\nu} \mathbf{w}_{opt}$$

$$= \kappa \mathbf{R}_{\nu\nu} \mathbf{w}_{opt}$$

Thus,

$$\boxed{\mathbf{w}_{opt} = \frac{1}{\kappa} \mathbf{R}_{\nu\nu}^{-1} \mathbf{a}_0} \tag{5.37}$$

where κ can be dropped since it will not affect the final SINR. Further, the maximum SINR can be found as

$$\gamma_{max} = \frac{\mathbf{w}_{opt}^H \mathbf{R_{aa}} \mathbf{w}_{opt}}{\mathbf{w}_{opt}^H \mathbf{R}_{\nu\nu} \mathbf{w}_{opt}}$$

$$= \frac{\left(\frac{1}{\kappa}\mathbf{R}_{\nu\nu}^{-1}\mathbf{a}_0\right)^H \mathbf{R_{aa}} \left(\frac{1}{\kappa}\mathbf{R}_{\nu\nu}^{-1}\mathbf{a}_0\right)}{\left(\frac{1}{\kappa}\mathbf{R}_{\nu\nu}^{-1}\mathbf{a}_0\right)^H \mathbf{R}_{\nu\nu} \left(\frac{1}{\kappa}\mathbf{R}_{\nu\nu}^{-1}\mathbf{a}_0\right)}$$

$$= \frac{\left(\mathbf{R}_{\nu\nu}^{-1}\mathbf{a}_0\right)^H \mathbf{a}_0 \mathbf{a}_0^H \left(\mathbf{R}_{\nu\nu}^{-1}\mathbf{a}_0\right)}{\left(\mathbf{R}_{\nu\nu}^{-1}\mathbf{a}_0\right)^H \mathbf{a}_0}$$

$$= \mathbf{a}_0^H \mathbf{R}_{\nu\nu}^{-1} \mathbf{a}_0$$

Example gain patterns are shown in Figure 5.7. Three plots are provided: (a) desired signal only at $\theta = \frac{\pi}{4}$; (b) desired signal at $\theta = \frac{\pi}{4}$ and an interferer at $\theta = \frac{\pi}{2}$; (c) desired signal at $\theta = \frac{\pi}{4}$ and Two Interferers at $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{4}$. The INR assumed is 0dB. As one would expect, the first case corresponds to the maximum SNR pattern (and weights). This is to be expected since $R_{\nu\nu} = \sigma^2 \mathbf{I}$ and thus the weights correspond to the maximum SNR solution. This can also be seen by comparing with the patterns shown in Figure 5.6. Examining the other two patterns we see that the maximum SINR weights do not necessarily place a null in the direction of the interferers since it also attempts to maintain a strong gain in the direction of the desired signal.

5.5.4 MMSE

Another way to balance noise and interference is to minimize the error between the received signal and the desired signal. A common way to accomplish this is to minimize the mean quare error. Consider the received signal

$$\mathbf{r}(t) = \mathbf{a}_0 x(t) + \sum_{k=1}^{N_I} \mathbf{a}_k i_k(t) + \mathbf{n}(t) \tag{5.38}$$

which after pulse matched filtering on each antenna gives the vector

$$\mathbf{r} = \mathbf{a}_0 s + \sum_{k=1}^{N_I} \mathbf{a}_k i_k + \mathbf{n} \tag{5.39}$$
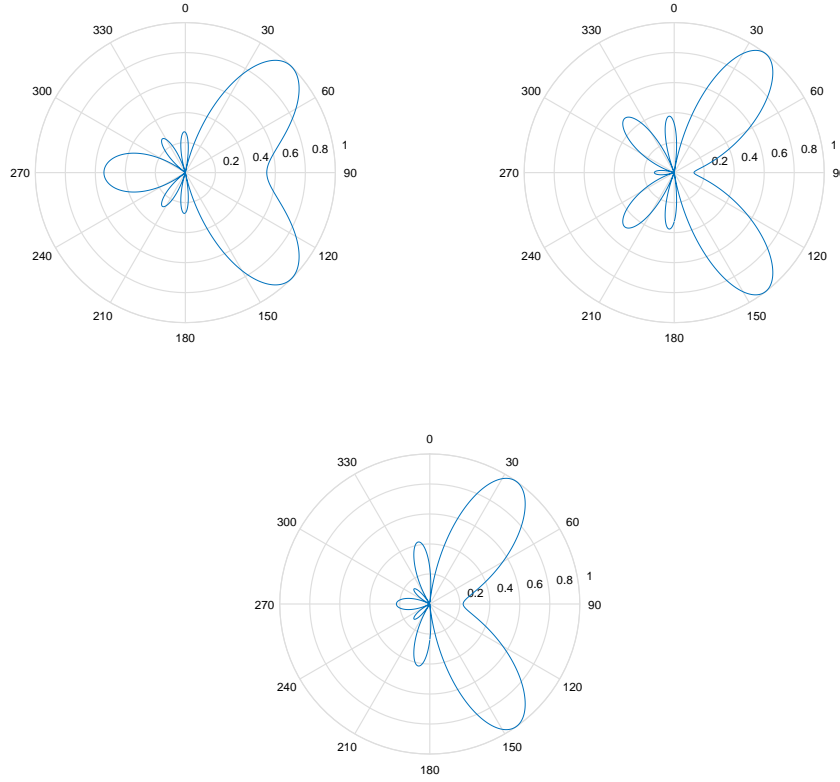
Figure 5.7: The Gain Pattern $g(\theta)$ for Four Antenna Elements with $\frac{d}{\lambda} = \frac{1}{2}$ using Maximum SINR Weights with only a desired signal at $\theta = \frac{\pi}{4}$ (top, left), One Interferer at $\theta = \frac{\pi}{2}$ (top, right) and Two Interferers at $\theta = \frac{\pi}{2}$ and $\theta = -\frac{\pi}{4}$ (bottom)

The decision variable is $z = \mathbf{w}^H \mathbf{r}$. We would like to find the weights $\mathbf{w}$ that minimize the mean squared error:

$$
\begin{aligned}
J(\mathbf{w}) &= \mathrm{E}\left\{|\mathbf{w}^H \mathbf{r} - s|^2\right\} \\
&= \mathbf{w}^H \mathrm{E}\left\{\tilde{\mathbf{r}}\tilde{\mathbf{r}}^H\right\} \mathbf{w} - \mathrm{E}\left\{s^*\tilde{\mathbf{r}}^H\right\} \mathbf{w} - \mathbf{w}^H \mathrm{E}\left\{\tilde{\mathbf{r}}s^*\right\} + \mathrm{E}\left\{ss^*\right\}
\end{aligned}
\tag{5.40}
$$

In order to minimize this function, we need to find the point where the gradient $\nabla J(\mathbf{w}) = \mathbf{0}$. The gradient is

$$
\nabla J(\mathbf{w}) = \begin{bmatrix} \frac{\partial J(\mathbf{w})}{\partial w_{1I}} \\ \frac{\partial J(\mathbf{w})}{\partial w_{2I}} \\ \vdots \\ \frac{\partial J(\mathbf{w})}{\partial w_{M_rI}} \end{bmatrix} + j \begin{bmatrix} \frac{\partial J(\mathbf{w})}{\partial w_{1Q}} \\ \frac{\partial J(\mathbf{w})}{\partial w_{2Q}} \\ \vdots \\ \frac{\partial J(\mathbf{w})}{\partial w_{M_rQ}} \end{bmatrix}
\tag{5.41}
$$

where $w_i = w_{iI} + jw_{iQ}$. In general, the following rules apply to gradients:

$$
\begin{aligned}
\nabla(\mathbf{w}^H \mathbf{A} \mathbf{w}) &= 2\mathbf{A}\mathbf{w}^* \\
\nabla(\mathbf{w}^H \mathbf{c}) &= \mathbf{c} \\
\nabla(\mathbf{c}^H \mathbf{w}) &= \mathbf{c}
\end{aligned}
$$

(5.42)

Thus,

$$
\begin{aligned}
\nabla J(\mathbf{w}) &= 2\mathrm{E}\left\{\tilde{\mathbf{r}}\tilde{\mathbf{r}}^H\right\}\mathbf{w} - 2\mathrm{E}\left\{\tilde{\mathbf{r}}s^*\right\} \\
&= 2\mathbf{R_{rr}}\mathbf{w} - 2\mathbf{a}
\end{aligned}
$$

(5.43)

assuming that $\mathrm{E}\left\{si_k\right\} = 0, \forall k$. Setting the gradient equal to zero gives

$$
\boxed{\mathbf{w}_{mmse} = \mathbf{R_{rr}^{-1}}\mathbf{a}}
$$

(5.44)

which is very similar to the maximum SINR weights. In fact, we will show in the next section that the two weight vectors are equivalent in terms of SINR¿ In practice we can estimate $\mathbf{R_{rr}}$ directly from the received signal. The vector $\mathbf{a}$ must be estimated using known symbols s, i.e., pilot symbols. Alternatively, we can use known symbols to drive an adaptive algorithm as we will discuss in the next section.

5.5.5   Equivalence of Max SINR and MMSE Weights

The two formulations which seek to balance the impact of noise and interference turn out to be very similar. In fact, we can show that from an SINR perspective, the two weight vectors are equivalent. To see this consider the inverse of the signal covariance matrix $\mathbf{R_{rr}^{-1}}$. Using the matrix inversion lemma we can show that it is related to the inverse of the interference plus noise covariance matrix $\mathbf{R}_{\nu\nu}$:

$$
\begin{aligned}
\mathbf{R_{rr}^{-1}} &= \left[\mathbf{R}_{\nu\nu} + \mathbf{a}_0\mathbf{a}_0^H\right]^{-1} \\
&= \mathbf{R}_{\nu\nu}^{-1} - \frac{\mathbf{R}_{\nu\nu}^{-1}\mathbf{a}_0\mathbf{a}_0^H\mathbf{R}_{\nu\nu}^{-1}}{\mathbf{a}_0^H\mathbf{R}_{\nu\nu}^{-1}\mathbf{a}_0 + 1}
\end{aligned}
$$

(5.45)

Now if we multiply $\mathbf{R_{rr}^{-1}}$ by $\mathbf{a}_0$ to obtain the MMSE weights we obtain

$$
\begin{aligned}
\mathbf{R_{rr}^{-1}}\mathbf{a}_0 &= \mathbf{R}_{\nu\nu}^{-1}\mathbf{a}_0 - \frac{\mathbf{R}_{\nu\nu}^{-1}\mathbf{a}_0\mathbf{a}_0^H\mathbf{R}_{\nu\nu}^{-1}\mathbf{a}_0}{\mathbf{a}_0^H\mathbf{R}_{\nu\nu}^{-1}\mathbf{a}_0 + 1} \\
&= \left(1 - \frac{\mathbf{a}_0^H\mathbf{R}_{\nu\nu}^{-1}\mathbf{a}_0}{\mathbf{a}_0^H\mathbf{R}_{\nu\nu}^{-1}\mathbf{a}_0 + 1}\right)\mathbf{R}_{\nu\nu}^{-1}\mathbf{a}_0 \\
&= \left(\frac{1}{\mathbf{a}_0^H\mathbf{R}_{\nu\nu}^{-1}\mathbf{a}_0 + 1}\right)\mathbf{R}_{\nu\nu}^{-1}\mathbf{a}_0 \\
&= c\mathbf{R}_{\nu\nu}^{-1}\mathbf{a}_0
\end{aligned}
$$

(5.46)

where c is a scaling constant that does not affect SINR or the weight vector direction. Thus, the MMSE weights are equivalent to the max SINR weights and we can state that the SINR of the

MMSE solution is identical to the maximum SINR:

$$\gamma_{mmse} = \gamma_{max} = \mathbf{a}_0^H \mathbf{R}_{\nu\nu}^{-1} \mathbf{a}_0 \tag{5.47}$$

5.6   Adaptive Algorithms

5.6.1   Least Mean Squares Algorithm

Each of the choices for optimal weights discussed previously assumes knowledge of the array vectors of the desired signal *and the interference*. Typically, this information is not avaialable *a priori*. Thus, we seek adaptive algorithms that can essentially learn this information (indirectly).

The classic approach is the Least Mean Squares (LMS) Algorithm which is based on the method of *steepest descent*, which is a stochastic approach to find the vector which minimizes an objective function. The approach assumes the availability of a data training sequence (decision-directed approaches can be used after initial convergence).

The iterative technique adapts the weights according to the recursion

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \delta_s \hat{\nabla}(k) \tag{5.48}$$

where δ_s is a *negative* scaling factor which controls convergence and $\hat{\nabla}$ is the estimated gradient of the squared error ϵ^2 with respect to $\mathbf{w}$. Specifically, if the error is defined as

$$\epsilon(k) = s(k) - \mathbf{w}^H(k)\tilde{\mathbf{r}}(k) \tag{5.49}$$

then we have

$$\begin{aligned}
\hat{\nabla}(k) &= \nabla\left[\epsilon^2(k)\right] \\
&= 2\epsilon^*(k)\nabla[\epsilon(k)] \\
&= -2\epsilon^*(k)\mathbf{r}(k) \tag{5.50}
\end{aligned}$$

Does this get us what we want? Let's look at the average value of the estimated gradient:

$$\begin{aligned}
\mathrm{E}\left\{\hat{\nabla}(k)\right\} &= \mathrm{E}\left\{-2\epsilon^*(k)\mathbf{r}(k)\right\} \\
&= \mathrm{E}\left\{-2(s(k) - \mathbf{w}^H(k)\tilde{\mathbf{r}}(k))^*\mathbf{r}(k)\right\} \\
&= -2\mathrm{E}\left\{s^*(k)\mathbf{r}(k)\right\} - 2\mathrm{E}\left\{\mathbf{w}^H(k)\tilde{\mathbf{r}}^*(k)\tilde{\mathbf{r}}(k)\right\} \\
&= -2\mathbf{a} + 2\mathbf{R_{rr}}\mathbf{w}
\end{aligned}$$

Thus, the expected value of the gradient estimate is in fact the gradient of the MSE. The LMS update is then defined as

$$\boxed{\mathbf{w}(k+1) = \mathbf{w}(k) - 2\delta_s\epsilon^*(k)\mathbf{r}(k)} \tag{5.51}$$

where δ_s is a *negative* scaling factor that controls the speed of convergence. For convergence we require that

$$-\frac{1}{\lambda_{max}} < \delta_s < 0 \tag{5.52}$$

where $\lambda_{max} = \max eig\left(\mathbf{R_{rr}}\right)$. Note that smaller $|\delta_s|$ results in slower convergence but less final error.

5.6.2   Recursive Least Squares

Recursive Least Squares is an approach that recursively estimates the weights that minimize the weighted square error. Let us define a weigthed square cost function as follows:

$$J(k) = \sum_{i=1}^{k} \beta(k,i)|\epsilon(i)|^2 \tag{5.53}$$

where $\epsilon(i)$ is the error in the ith estimate:

$$\epsilon(i) = s(i) - \mathbf{w}^H(i)\tilde{\mathbf{r}}(i) \tag{5.54}$$

where $\tilde{\mathbf{r}}(i)$ is the received signal vector (after pulse matched filtering) at time step i, $s(i)$ is the desired (known) signal at time step i, and $\mathbf{w}(i)$ is the weight vector at time step i. The term $\beta(k,i)$ is a forgetting factor that forgets old data. Note that

$$0 \leq \beta(k,i) \leq 1 \tag{5.55}$$

We shall choose $\beta(k,i) = \chi^{k-i}$ for $0 \leq \chi \leq 1$. Thus, we have

$$J(k) = \sum_{i=1}^{k} \chi^{k-i}|\epsilon(i)|^2 \tag{5.56}$$

It can be shown that this squared error is minimized by the least squared solution:

$$\mathbf{w}(k) = \tilde{\mathbf{R}}_{\mathbf{rr}}^{-1}(k)\tilde{\mathbf{x}}(k)$$

$$\tilde{\mathbf{R}}_{\mathbf{rr}}(k) = \sum_{i=1}^{k} \chi^{k-i}\tilde{\mathbf{r}}(i)\tilde{\mathbf{r}}^H(i)$$

$$\tilde{\mathbf{x}}(k) = \sum_{i=1}^{k} \chi^{k-i}\mathbf{r}(i)s^*(i)$$

The RLS method provides an efficient recursive approach to determine $\mathbf{w}(k)$. First, let us rewrite the expression for $\tilde{\mathbf{R}}_{\mathbf{rr}}(k)$:

$$\tilde{\mathbf{R}}_{\mathbf{rr}}(k) = \sum_{i=1}^{k} \chi^{k-i}\tilde{\mathbf{r}}(i)\tilde{\mathbf{r}}^H(i)$$

$$= \chi\underbrace{\left[\sum_{i=1}^{k-1} \chi^{k-i-1}\tilde{\mathbf{r}}(i)\tilde{\mathbf{r}}^H(i)\right]}_{\tilde{\mathbf{R}}_{\mathbf{rr}}(k-1)} + \tilde{\mathbf{r}}(k)\tilde{\mathbf{r}}(k)^H$$

$$= \chi\tilde{\mathbf{R}}_{\mathbf{rr}}(k-1) + \tilde{\mathbf{r}}(k)\tilde{\mathbf{r}}(k)^H$$

Similarly, we can write

$$\tilde{\mathbf{x}}(k) = \chi\tilde{\mathbf{x}}(k-1) + \tilde{\mathbf{r}}(k)s^*(k) \tag{5.57}$$

Now, according to the matrix inversion lemma:

$$\left(\mathbf{A} + \mathbf{u}\mathbf{v}^H\right)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{v}^H\mathbf{A}^{-1}}{1 + \mathbf{v}^H\mathbf{A}^{-1}\mathbf{u}} \tag{5.58}$$

Thus, we can update the inverse as

$$\left(\chi\tilde{\mathbf{R}}_{\mathbf{rr}}(k-1) + \tilde{\mathbf{r}}(k)\tilde{\mathbf{r}}(k)^H\right)^{-1} = \chi^{-1}\tilde{\mathbf{R}}_{\mathbf{rr}}^{-1}(k-1) - \frac{\chi^{-1}\tilde{\mathbf{R}}_{\mathbf{rr}}^{-1}(k-1)\tilde{\mathbf{r}}(k)\tilde{\mathbf{r}}^H(k)\chi^{-1}\tilde{\mathbf{R}}_{\mathbf{rr}}^{-1}(k-1)}{1 + \tilde{\mathbf{r}}^H(k)\chi^{-1}\tilde{\mathbf{R}}_{\mathbf{rr}}^{-1}(k-1)\tilde{\mathbf{r}}(k)} \tag{5.59}$$

For ease of notation we define

$$\begin{aligned}
\mathbf{P}(k) &= \tilde{\mathbf{R}}_{\mathbf{rr}}^{-1}(k) \\
\mathbf{K}(k) &= \frac{\chi^{-1}}{1 + \chi^{-1}\tilde{\mathbf{r}}^H(k)\mathbf{P}(k-1)\tilde{\mathbf{r}}(k)}\mathbf{P}(k-1)\tilde{\mathbf{r}}(k)
\end{aligned} \tag{5.60}$$

Thus, the matrix inversion update can be written as

$$\mathbf{P}(k) = \chi^{-1}\mathbf{P}(k-1) - \chi^{-1}\mathbf{K}(k)\tilde{\mathbf{r}}^H(k)\mathbf{P}(k-1) \tag{5.61}$$

Now, rearranging the equation above we have

$$\begin{aligned}
\mathbf{K}(k)(1 + \chi^{-1}\tilde{\mathbf{r}}^H(k)\mathbf{P}(k-1)\tilde{\mathbf{r}}(k)) &= \chi^{-1}\mathbf{P}(k-1)\tilde{\mathbf{r}}(k) \\
\mathbf{K}(k) &= \underbrace{\left[\chi^{-1}\mathbf{P}(k-1) - \chi^{-1}\mathbf{K}(k)\tilde{\mathbf{r}}^H(k)\mathbf{P}(k-1)\right]}_{\mathbf{P}(k)}\tilde{\mathbf{r}}(k) \\
&= \mathbf{P}(k)\tilde{\mathbf{r}}(k) \tag{5.62}
\end{aligned}$$

Now, examining the weights:

$$\begin{aligned}
\mathbf{w}(k) &= \mathbf{R}_{\mathbf{rr}}^{-1}(k)\tilde{\mathbf{x}}(k) \\
&= \mathbf{P}(k)\tilde{\mathbf{x}}(k) \\
&= \chi\mathbf{P}(k)\tilde{\mathbf{x}}(k-1) + \mathbf{P}(k)\tilde{\mathbf{r}}(k)s^*(k) \\
&= \mathbf{P}(k-1)\tilde{\mathbf{x}}(k-1) - \mathbf{K}(k)\tilde{\mathbf{r}}^H(k)\mathbf{P}(k-1)\tilde{\mathbf{x}}(k-1) + \mathbf{P}(k)\tilde{\mathbf{r}}(k)s^*(k) \\
&= \mathbf{w}(k-1) - \mathbf{K}(k)\tilde{\mathbf{r}}^H(k)\mathbf{w}(k-1) + \underbrace{\mathbf{P}(k)\tilde{\mathbf{r}}(k)}_{\mathbf{K}(k)}s^*(k) \\
&= \mathbf{w}(k-1) - \mathbf{K}(k)\tilde{\mathbf{r}}^H(k)\mathbf{w}(k-1) + \mathbf{K}(k)s^*(k) \\
&= \mathbf{w}(k-1) + \mathbf{K}(k)\epsilon^*(k) \tag{5.63}
\end{aligned}$$

where $\epsilon(k) = s(k) - \mathbf{w}^H(k-1)\tilde{\mathbf{r}}(k)$.

Thus we can define the RLS recursion as

1. $\mathbf{K}(k) = \frac{\chi^{-1}\mathbf{P}(k-1)\tilde{\mathbf{r}}(k)}{1+\chi^{-1}\tilde{\mathbf{r}}^H(k)\mathbf{P}(k-1)\tilde{\mathbf{r}}(k)}$

2. $\epsilon(k) = s(k) - \mathbf{w}^H(k-1)\tilde{\mathbf{r}}(k)$

3. $\mathbf{w}(k) = \mathbf{w}(k-1) + \mathbf{K}(k)\epsilon^*(k)$

4. $\mathbf{P}(k) = \chi^{-1}\mathbf{P}(k-1) - \chi^{-1}\mathbf{K}(k)\tilde{\mathbf{r}}^H(k)\mathbf{P}(k-1)$

with initial conditions $\mathbf{P}(0) = \kappa\mathbf{I}$ and $\mathbf{w}(0) = \mathbf{0}^{M_r \times 1}$ for positive constant κ.

5.7 Direction Finding

A common use of beamforming (coherent) arrays is direction finding - i.e., the estimation of the angles-of-arrival θ_i if one or more signals arriving at the array.

$$\tilde{\mathbf{r}}(t) = \sum_{i=1}^{N_s} \mathbf{a}_i x_i(t) + \mathbf{n}(t) \tag{5.64}$$

where N_s is the total number of signals present each with array response $\mathbf{a}_i$ and signal $x_i(t)$. We shall assume that $\mathrm{E}\left\{|x_i(t)|^2\right\} = 1$ and $\mathrm{E}\left\{\mathbf{n}(t)\mathbf{n}^H(t)\right\} = \sigma_n^2\mathbf{I}$. If we filter and sample the received signal, we can write one sample vector as

$$\tilde{\mathbf{r}} = \mathbf{A}(\overline{\theta})\mathbf{x} + \mathbf{n} \tag{5.65}$$

where $\mathbf{A}(\overline{\theta}) = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), ..\mathbf{a}(\theta_{N_s})]$, $\mathbf{x}$ is a vector of sampled signals and $\mathbf{n}$ is a vector of noise samples. Note that we have used $\mathbf{a}_i = \mathbf{a}(\theta_i)$ to explicitly show the dependence of $\mathbf{a}_i$ on θ_i. Also, we have defined $\overline{\theta} = [\theta_1, \theta_2, \ldots \theta_{N_s}]^T$. If we consider the vector $\tilde{\mathbf{r}}$ as our observables, and we assume Gaussian noise and Gaussian distributed signals, we can write the distribution of $\tilde{\mathbf{r}}$ as

$$f(\tilde{\mathbf{r}}) = \frac{1}{\sqrt{(2\pi)^{M_r}|\mathbf{R_{rr}}|}} exp\left(-\frac{1}{2}\tilde{\mathbf{r}}^H\mathbf{R_{rr}}^{-1}\tilde{\mathbf{r}}\right) \tag{5.66}$$

where $|\mathbf{R}|$ is the determinant of $\mathbf{R}$, $\mathbf{R_{rr}} = \mathbf{A}(\overline{\theta})\mathbf{A}^H(\overline{\theta}) + \sigma_n^2\mathbf{I}$ and we have assumed that $R_{xx} = \mathbf{I}$. Further, if we assume a series of K observed vectors separated in time such that the transmit signals and noise processes are independent from one sample to the next, we can write the distribution of the observed sequences of vectors as:

$$f\left(\tilde{\mathbf{r}}_1, \tilde{\mathbf{r}}_2, \ldots \tilde{\mathbf{r}}_K\right) = \prod_{k=1}^{K} \frac{1}{\sqrt{(2\pi)^{M_r}|\mathbf{R_{rr}}|}} exp\left(-\frac{1}{2}\tilde{\mathbf{r}}_k^H\mathbf{R_{rr}}^{-1}\tilde{\mathbf{r}}_k\right) \tag{5.67}$$

For an observed set of $\{\tilde{\mathbf{r}}_k\}$ with fixed but unknown $\overline{\theta}$, we can view (5.67) as a likelihood function. The Maximum Likelihood estimate of $\overline{\theta}$ can then be found by maximizing the log likelihood function:

$$\hat{\overline{\theta}}_{ML} = \max_{\overline{\theta}} -K\log|\mathbf{R_{rr}}| - \sum_{k=1}^{K} \tilde{\mathbf{r}}^H\mathbf{R_{rr}}^{-1}\tilde{\mathbf{r}} \tag{5.68}$$

The maximum likelihood estimate of $\overline{\theta}$ is straightforward to compute, albeit computationally intensive. Assuming a grid search over N values of θ, the overall estimation process requires calculating the argument of the maximization in (eq:MLAOA) N^{N_s} times. In other words, for a 1 degree granularity over unambiguous angles of a linear array with four signals we would require calculating the metric above $180^4 \approx 10^9$ times. Thus, ML estimation is in general too computationally intensive. Thus, we seek alternative, less complex suboptimal methods.

5.7.1   The Fourier Method

This method is also known as the "Delay-and-Sum Method" or "Classical Beamformer Method". Again, if we let

$$\tilde{\mathbf{r}}(t) = \sum_{i=1}^{N_t} \mathbf{a}_i x_i(t) + \mathbf{n}(t) \tag{5.69}$$

where N_t is the total number of signals present each with array response $\mathbf{a}_i$ and signal $x_i(t)$. We shall assume that $\mathrm{E}\left\{|x_i(t)|^2\right\} = 1$ and $\mathrm{E}\left\{\mathbf{n}(t)\mathbf{n}^H(t)\right\} = \sigma_n^2 \mathbf{I}$. If we know that there are N_t signals and we look for N_t peaks in the angle spectrum

$$P(\theta) = \mathrm{E}\left\{|\mathbf{w}^H(\theta)\tilde{\mathbf{r}}(t)|^2\right\} \tag{5.70}$$

where the peaks will occur at $\mathbf{w}(\theta) = \mathbf{a}_i(\theta)$. Thus, assuming that we know the array response we set the weights to $\mathbf{a}(\theta)$ and vary θ the angle spectrum is

$$\begin{aligned} P(\theta) &= \mathbf{a}^H(\theta)\mathrm{E}\left\{\tilde{\mathbf{r}}(t)\tilde{\mathbf{r}}^H(t)\right\}\mathbf{a}(\theta) \\ &= \mathbf{a}^H(\theta)\mathbf{R_{rr}}\mathbf{a}(\theta) \end{aligned} \tag{5.71}$$

where $\mathbf{R_{rr}}$ is the covariance of the received signal. If there is only one signal in the environment with array response $\mathbf{a}_o = \mathbf{a}(\theta_o)$, the covariance matrix is

$$\mathbf{R_{rr}} = P_o\mathbf{a}_o\mathbf{a}_o^H + \sigma_n^2\mathbf{I} \tag{5.72}$$

Thus, the resulting angle spectrum is

$$P(\theta) = P_o\mathbf{a}^H(\theta)\mathbf{a}_o\mathbf{a}_o^H\mathbf{a}(\theta) + \sigma_n^2\mathbf{a}^H(\theta)\mathbf{I}\mathbf{a}(\theta) \tag{5.73}$$

Clearly, $P(\theta)$ will be maximized at $\mathbf{a}(\theta) = \mathbf{a}(\theta_o)$, i.e., when $\theta = \theta_o$. If there are multiple (N) signals in the environment,

$$P(\theta) = \sum_{i=1}^{N} P_i\mathbf{a}^H(\theta)\mathbf{a}_i\mathbf{a}_i^H\mathbf{a}(\theta) + \sigma_n^2 M_r \tag{5.74}$$

Now there can be up to N peaks although the number of peaks may be less if the $\mathbf{a}_i$ are not spatially resolvable.

Notes:

* We have not made any assumptions about the kind of array we have. It can be linear, circular, planar, etc. However, we **must know** the array manifold $\mathbf{a}(\theta)$ for the array.

* Due to imperfections in the array, typically the array manifold is determined via calibration.

* If the array is linear, there will always be an ambiguity in θ of 180° since the array cannot distinguish between θ and $\pi - \theta$.

5.7.2 Capon's Minimum Variance Approach

The Fourier method can be ineffective when there are multiple signals in the environment. Fundamentally, the Fourier method uses all M_r degrees of freedom (DOF) to form a beam in the desired direction but is not concerned with signals in the environment coming from other directions. Capon's technique (also known as Minimum Variance Distortionless Response) uses some of the DOF to form a beam in the desired direction and some to null the other signals. The technique minimizes output power subject to the constraint that the gain in the desired direction is unity.

Formulation

Again, we wish to create a power spectrum $P(\theta)$ but now with weights chosen to minimize the impact of signals coming from directions other than θ. More specifically, the weights are chosen as

$$
\begin{aligned}
\mathbf{w}_{opt}(\theta) &= \min_{\mathbf{w}} \mathrm{E}\left\{|P(\theta)|^2\right\} \quad s.t. \quad \mathbf{a}^H(\theta)\mathbf{a}(\theta) = 1 \\
&= \min_{\mathbf{w}} \mathbf{w}^H \mathbf{R_{rr}} \mathbf{w} \quad s.t. \quad \mathbf{a}^H(\theta)\mathbf{a}(\theta) = 1
\end{aligned}
\tag{5.75}
$$

Using the technique of Lagrange multipliers the solution is found to be

$$
\mathbf{w}_{opt}(\theta) = \frac{\mathbf{R_{rr}^{-1}}\mathbf{a}(\theta)}{\mathbf{a}^H(\theta)\mathbf{R_{rr}^{-1}}\mathbf{a}(\theta)}
\tag{5.76}
$$

Further, the minimum output is

$$
J_{min} - \frac{1}{\mathbf{a}^H(\theta)\mathbf{R_{rr}^{-1}}\mathbf{a}(\theta)}
\tag{5.77}
$$

In general, the output angle spectrum

$$
P(\theta) = \frac{1}{\mathbf{a}^H(\theta)\mathbf{R_{rr}^{-1}}\mathbf{a}(\theta)}
\tag{5.78}
$$

This is also known as the **Minimum Variance Distortionless Response** (MVDR).

NOTE: This solution will minimize the output power and be distortionless at the desired angle.

5.7.3 MUSIC

Let

$$
\tilde{\mathbf{r}}(t) = \sum_{i=0}^{N_t-1} \mathbf{a}_i x_i(t) + \mathbf{n}(t)
\tag{5.79}
$$

where N_t is the total number of signals present each with array response $\mathbf{a}_i$ and signal $x_i(t)$. We shall assume that $\mathrm{E}\left\{|x_i(t)|^2\right\} = 1$ and $\mathrm{E}\left\{\mathbf{n}(t)\mathbf{n}^H(t)\right\} = \sigma_n^2\mathbf{I}$. Dropping the dependence on time,

after matched filtering and sampling, we can re-write this as

$$\tilde{\mathbf{r}} = [\mathbf{a}_0, \mathbf{a}_1, \ldots \mathbf{a}_{N_t-1}] \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_{N_t-1} \end{bmatrix} + \mathbf{n}$$

$$= \mathbf{As} + \mathbf{n} \tag{5.80}$$

Note that $\mathbf{a}_i$ vectors are vectors in M_r-dimensional space and $\tilde{\mathbf{r}}$ is a linear combination of these vectors. The input covariance matrix $\mathbf{R}_{rr}$ can be written as

$$\begin{aligned} \mathbf{R_{rr}} &= \mathrm{E}\left\{\tilde{\mathbf{r}}\tilde{\mathbf{r}}^H\right\} \\ &= \mathbf{A}\mathrm{E}\left\{\mathbf{ss}^H\right\}\mathbf{A}^H + \mathrm{E}\left\{\mathbf{nn}^H\right\} \\ &= \mathbf{AR_{ss}A}^H + \sigma_n^2\mathbf{I} \end{aligned} \tag{5.81}$$

Now, the eigenvalues of $\mathbf{R_{rr}}$ are such that

$$\begin{aligned} |\mathbf{R_{rr}} - \lambda_i\mathbf{I}| &= 0 \\ |\mathbf{AR_{ss}A}^H + \sigma_n^2\mathbf{I} - \lambda_i\mathbf{I}| &= 0 \\ |\mathbf{AR_{ss}A}^H - (\lambda_i - \sigma_n^2)\mathbf{I}| &= 0 \end{aligned} \tag{5.82}$$

Thus, we can define $v_i = \lambda_i - \sigma_n^2$ as the eigenvalues of $\mathbf{AR_{ss}A}^H$. Now, since $\mathbf{a}_i$ are linearly independent steering vectors, $\mathbf{A}$ has full rank. Assuming that $\mathbf{R_{ss}}$ is non-singular, provided that the number of signals $N_t < M_r$ the $M_r \times M_r$ matrix $\mathbf{AR_{ss}A}^H$ is positive semi-definite with rank N_t. Thus, $M_r - N_t$ eigenvalues v_i of $\mathbf{AR_{ss}A}^H$ are zero.

Thus, $M_r - N_t$ eigenvalues λ_i of $\mathbf{R_{rr}}$ are equal to σ_n^2. Sorting the eigenvalues such that

$$\lambda_0 < \lambda_1 < \ldots < \lambda_{M_r-1} \tag{5.83}$$

we can say that $\lambda_0 = \lambda_1 = \ldots \lambda_{M_r-N_t-1} = \sigma_n^2$.

In a real application, since $\mathbf{R_{rr}}$ is estimated, $M_r - N_t$ eigenvalues will not be *exactly* σ_n^2, but will be close. Using this fact we can determine the number of signals in the environment.

Now, let us define $\mathbf{q}_i$ $0 \le i \le M_r - N_t - 1$ as the eigenvectors of $\mathbf{R_{rr}}$ associated with the $M_r - N_t$ eigenvalues corresponding to the noise subspace. Since the steering vectors $\mathbf{a}_i$ do not correspond to the noise we are guaranteed that $\mathbf{A}^H\mathbf{q}_i = \mathbf{0}$ and $\mathbf{a}_i^H\mathbf{q}_i = 0$.

Proof:

From the definition of eigenvectors

$$(\mathbf{R_{rr}} - \lambda_i\mathbf{I})\mathbf{q}_i = \mathbf{0} \tag{5.84}$$

However, for $\mathbf{q}_i$, the associated eigenvalues $\lambda_i = \sigma_n^2$. Thus,

$$
\begin{aligned}
(\mathbf{R_{rr}} - \sigma_n^2 \mathbf{I})\mathbf{q}_i &= \mathbf{0} \\
(\mathbf{A}\mathbf{R_{ss}}\mathbf{A}^H + \sigma_n^2 \mathbf{I} - \sigma_n^2 \mathbf{I})\mathbf{q}_i &= \mathbf{0} \\
\mathbf{A}\mathbf{R_{ss}}\mathbf{A}^H \mathbf{q}_i &= \mathbf{0}
\end{aligned}
\tag{5.85}
$$

Since $\mathbf{A}$ is full rank and $\mathbf{R_{ss}}$ is non-singular we have

$$
\mathbf{A}^H \mathbf{q}_i = \mathbf{0}
\tag{5.86}
$$

Since all of the steering vectors are orthogonal to the noise subspace, the MUSIC algorithm (MUltiple SIgnal Classification) is

1. Find the eigen-decomposition $\mathbf{R_{rr}} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H$

2. Identify the noise subspace as the $M_r - N$ eigenvectors associated with the weakest eigenvalues:

$$
\mathbf{Q} = \left[\underbrace{\mathbf{q}_0 \mathbf{q}_1 \cdots \mathbf{q}_{M_r - N_t - 1}}_{M_r \times (M_r - N_t) \text{ noise subspace } \mathbf{V}} \quad \cdots \mathbf{q}_{M_r - 1} \right]
\tag{5.87}
$$

3. Calculate the MUSIC angle spectrum as

$$
P(\theta) = \frac{\mathbf{a}^H(\theta)\mathbf{a}(\theta)}{\mathbf{a}^H(\theta)\mathbf{V}\mathbf{V}^H\mathbf{a}(\theta)}
\tag{5.88}
$$

Note that we are guaranteed that $\mathbf{a}_i$ is perpendicular to $\mathbf{V}$. Thus, when $\mathbf{a}(\theta) = \mathbf{a}_i$ the spectrum $P(\theta)$ will have a sharp peak. There aer then peaks in the MUSIC spectrum corresponding to the AOA's of the signals in the environment.

5.7.4   ESPRIT

Estimation of Signal Parameters using Rotational Invariance Techniques (ESPRIT) is another DoA estimation technique which is based on the fact that for a linear array each element in the steering vector is a constant phase shift from the neighboring element. More specifically, defining the $M_r \times K$ matrix of K steering vectors $\mathbf{A}$:

$$
\begin{aligned}
\mathbf{A} &= [\mathbf{a}_0, \mathbf{a}_1, ... \mathbf{a}_{K-1}] \\
&= \begin{bmatrix}
1 & \cdots & 1 \\
e^{-\frac{j2\pi d}{\lambda}\sin(\theta_0)} & \cdots & e^{-\frac{j2\pi d}{\lambda}\sin(\theta_{K-1})} \\
\vdots & \cdots & \vdots \\
e^{-\frac{j2\pi d(M_r - 1)}{\lambda}\sin(\theta_0)} & \cdots & e^{-\frac{j2\pi d(M_r - 1)}{\lambda}\sin(\theta_{K-1})}
\end{bmatrix}
\end{aligned}
\tag{5.89}
$$

we can define two new vectors corresponding to the first $M_r - 1$ rows and the last $M_r - 1$ rows:

$$\mathbf{A}_0 \;=\; \begin{bmatrix} 1 & \cdots & 1 \\ e^{-\frac{j2\pi d}{\lambda}\sin(\theta_0)} & \cdots & e^{-\frac{j2\pi d}{\lambda}\sin(\theta_{K-1})} \\ \vdots & \cdots & \vdots \\ e^{-\frac{j2\pi d(M_r-2)}{\lambda}\sin(\theta_0)} & \cdots & e^{-\frac{j2\pi d(M_r-2)}{\lambda}\sin(\theta_{K-1})} \end{bmatrix} \tag{5.90}$$

$$\mathbf{A}_1 \;=\; \begin{bmatrix} e^{-\frac{j2\pi d}{\lambda}\sin(\theta_0)} & \cdots & e^{-\frac{j2\pi d}{\lambda}\sin(\theta_{K-1})} \\ \vdots & \cdots & \vdots \\ e^{-\frac{j2\pi d(M_r-1)}{\lambda}\sin(\theta_0)} & \cdots & e^{-\frac{j2\pi d(M_r-1)}{\lambda}\sin(\theta_{K-1})} \end{bmatrix} \tag{5.91}$$

Note that

$$\mathbf{A}_1 = \mathbf{A}_0\mathbf{\Phi} \tag{5.92}$$

where

$$\mathbf{\Phi} = \begin{bmatrix} e^{-\frac{j2\pi d}{\lambda}\sin(\theta_0)} & 0 & \cdots & 0 \\ 0 & \ddots & & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & & e^{-\frac{j2\pi d}{\lambda}\sin(\theta_{K-1})} \end{bmatrix} \tag{5.93}$$

Thus, the diagonal values of $\mathbf{\Phi}$ correspond to the constant phase shift across the array due to each signal. If we can determine $\mathbf{\Phi}$, we can find the DoAs. Unfortunately, the matrix of steering vectors is not available. However, we can find the signal subspace which spans the same subspace as the steering vectors. If we define the matrix $\mathbf{Q}$ as the matrix of eigenvectors, we can partition $\mathbf{Q}$ the same way we did with the steering vector matrix to obtain $\mathbf{Q}_0$ and $\mathbf{Q}_1$, we kwow that

$$\begin{aligned} \mathbf{Q}_0 &= \mathbf{A}_0\mathbf{C} & (5.94) \\ \mathbf{Q}_1 &= \mathbf{A}_1\mathbf{C} & (5.95) \end{aligned}$$

$$(5.96)$$

for some unknown matrix $\mathbf{C}$. Further, this implies that

$$\mathbf{Q}_1 = \mathbf{Q}_0\mathbf{\Psi} \tag{5.97}$$

for some matrix $\mathbf{\Psi}$. Now, using the relationships above we can easily show that

$$\mathbf{\Psi} \;=\; \mathbf{C}^{-1}\mathbf{\Phi}\mathbf{C} \tag{5.98}$$

which implies that $\mathbf{\Phi}$ corresponds to the eigenvalues of $\mathbf{\Psi}$. This suggests the following algorithm:

1. Estimate the covariance matrix of the received signal $\mathbf{R}$.

2. Determine the eigenvalue decomposition of $\mathbf{R} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H$.

3. Divide the eigenvectors into the signal subspace $\mathbf{Q}_s$ and the noise subspace $\mathbf{Q}_n$.

4. Create two new matrices $\mathbf{Q}_0$ and $\mathbf{Q}_1$ as the first and last $M_r - 1$ rows of $\mathbf{Q}_s$ respectively.

5. Using equation (5.97) find the least squares estimate of $\mathbf{\Psi}$. This can be found as $\mathbf{Q}_0^\dagger \mathbf{Q}_1$.

6. Determine the eigenvalues of $\hat{\mathbf{\Psi}}$, λ_i, $1 \leq i \leq K$.

7. Noting that $\lambda_i = e^{-\frac{j2\pi d}{\lambda}\sin(()\theta_i)}$, solve for $\hat{\theta}_i$.

Note that the least squares estimate of $\hat{\mathbf{\Psi}}$ assumes that $\mathbf{Q}_0$ is noisy, but that $\mathbf{Q}_1$ is not. Since in reality, both are noisy estimates, a total least squares solution is more appropriate. In that case we can solve for $\mathbf{\Psi}$ by first creating the singular value decomposition of the $(M_r - 1) \times 2K$ matrix $[\mathbf{Q}_0 \mathbf{Q}_1]$:

$$[\mathbf{Q}_0 \mathbf{Q}_1] = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H \tag{5.99}$$

Then, defining the four $M \times M$ sub-matrices of $\mathbf{V}$ as

$$\begin{bmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{bmatrix} \tag{5.100}$$

we finally have

$$\hat{\mathbf{\Psi}} = -\mathbf{V}_{12}\mathbf{V}_{22}^{-1} \tag{5.101}$$

The advantage of this algorithm over MUSIC is that there is no requirement to search for peaks which can be frought with difficulty, and no requirement for choosing a resolution (i.e., number of candidate steering vectors). Example performance of ESPRIT for one signal ($\theta = 0$) with $N_s = 100$ samples and $M_r = 4$ is plotted in Figure 5.8. We can see that for SNR greater than 0dB, ESPRIT approaches the CRLB.

Appendix: Cramer Rao Lower Bound

Consider N_s samples of the receive vector $\tilde{\mathbf{r}}(t)$

$$\tilde{\mathbf{r}}(t) = \mathbf{a}(\theta) + \mathbf{n}(t) \tag{5.102}$$

for $t = 1, 2, \ldots N_s$. The conditional probability function is written as

$$f_r(\tilde{\mathbf{r}}|\theta) = \frac{1}{(2\pi\sigma^2)^{M_r N_s}} e^{-\sum_{t=1}^{N_s}(\tilde{\mathbf{r}}(t)-\mathbf{a})^H \mathbf{R}^{-1}(\tilde{\mathbf{r}}(t)-\mathbf{a})} \tag{5.103}$$

where the covariance matrix $\mathbf{R} = \sigma^2 \mathbf{I}$. Taking the log of $f_r(\tilde{\mathbf{r}}|\theta)$ we obtain

$$\begin{aligned} \log f_r(\tilde{\mathbf{r}}|\theta) &= -\log\left[(2\pi\sigma^2)^{M_r N_s}\right] - \frac{1}{\sigma^2}\sum_{t=1}^{N_s}(\tilde{\mathbf{r}}(t)-\mathbf{a})^H(\tilde{\mathbf{r}}(t)-\mathbf{a}) \tag{5.104} \\ &= -\log\left[(2\pi\sigma^2)^{M_r N_s}\right] - \frac{1}{\sigma^2}\sum_{t=1}^{N_s}\left\{\tilde{\mathbf{r}}^H(t)\tilde{\mathbf{r}}(t) - \tilde{\mathbf{r}}^H(t)\mathbf{a} - \mathbf{a}^H\tilde{\mathbf{r}}(t) + \mathbf{a}^H\mathbf{a}\right\} \tag{5.105} \end{aligned}$$
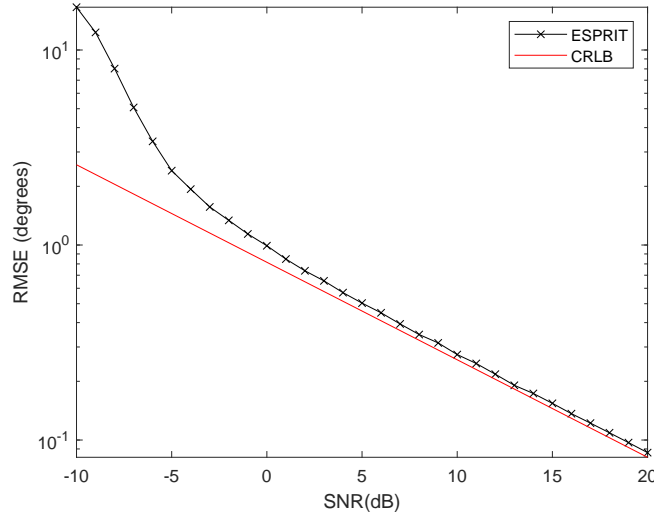
Figure 5.8: Performance of ESPRIT with $M_r = 4$, $N_s = 100$ and $\theta = 0$

Dropping the terms that don't depend on θ we obtain

$$g(\theta) = \frac{1}{\sigma^2} \sum_{t=1}^{N_s} \left\{ \tilde{\mathbf{r}}^H(t)\mathbf{a} + \mathbf{a}^H \tilde{\mathbf{r}}(t) - \mathbf{a}^H \mathbf{a} \right\} \tag{5.106}$$

The Cramer-Rao Lower Bound on the estimation variance is then

$$CRLB(\theta) \geq \mathbf{J}^{-1} \tag{5.107}$$

where

$$\mathbf{J} = E\left\{ \frac{\partial^2 g(\theta)}{\partial \theta^2} \right\} \tag{5.108}$$

Now,

$$\frac{\partial g(\theta)}{\partial \theta} = \frac{1}{\sigma^2} \sum_{t=1}^{N_s} \left\{ \tilde{\mathbf{r}}^H(t)\frac{\partial \mathbf{a}}{\partial \theta} + \frac{\partial \mathbf{a}^H}{\partial \theta}\tilde{\mathbf{r}}(t) - \mathbf{a}^H \frac{\partial \mathbf{a}}{\partial \theta} - \frac{\partial \mathbf{a}^H}{\partial \theta}\mathbf{a} \right\} \tag{5.109}$$

Further,

$$\frac{\partial^2 g(\theta)}{\partial \theta^2} = \frac{1}{\sigma^2} \sum_{t=1}^{N_s} \left\{ \tilde{\mathbf{r}}^H(t)\frac{\partial^2 \mathbf{a}}{\partial \theta^2} + \frac{\partial^2 \mathbf{a}^H}{\partial \theta^2}\tilde{\mathbf{r}}(t) - \mathbf{a}^H \frac{\partial^2 \mathbf{a}}{\partial \theta^2} - \frac{\partial \mathbf{a}^H}{\partial \theta}\frac{\partial \mathbf{a}}{\partial \theta} - \frac{\partial^2 \mathbf{a}^H}{\partial \theta^2}\mathbf{a} - \frac{\partial \mathbf{a}^H}{\partial \theta}\frac{\partial \mathbf{a}}{\partial \theta} \right\} \tag{5.110}$$

Now taking the expectation and realizing that $E\{\tilde{\mathbf{r}}(t)\} = \mathbf{a}$:

$$E\left\{\frac{\partial^2 g(\theta)}{\partial \theta^2}\right\} = -\frac{2N_s}{\sigma^2}\frac{\partial \mathbf{a}^H}{\partial \theta}\frac{\partial \mathbf{a}}{\partial \theta} \tag{5.111}$$

$$= -\frac{2N_s}{\sigma^2}\left(\frac{2\pi d}{\lambda}\right)^2\frac{M_r(M_r-1)^2}{12}\left(\cos\left(\theta\right)\right)^2 \tag{5.112}$$

And thus

$$CRLB(\theta) \geq \frac{6}{N_s\gamma\left(\frac{2\pi d}{\lambda}\right)^2 M_r(M_r-1)^2(\cos\left(\theta\right))^2} \tag{5.113}$$

Chapter 6

Fading and Diversity

6.1 Introduction

In this chapter we examine a fundamental use of multiple antennas - diversity. As we will show shortly, fading is a highly detrimental factor in wireless communications. The primary tool to combat fading is diversity (typically combined with error correction coding). While diversity can be achieved in time and/or frequency, we are focused in this chapter on spatial diversity. However, much of what is developed can be applied to time/frequency with appropriate adjustments.

6.2 Preliminaries

In this section, we provide a few useful mathematical basics that are useful in our study of diversity.

6.2.1 Transformation of Random Variables

Let the random variable X have probability density function (pdf) and cumulative distribution function (cdf) defined as $f_X(x)$ and $F_X(x)$ respectively. Assume further that we have a random variable $Y = g(X)$ where $g(x)$ is a one-to-one transformation and is increasing on the range of X. This implies that g^{-1} is also an increasing function. Now the cdf of Y is

$$
\begin{aligned}
F_Y(y) &= \Pr\{Y \le y\} \\
&= \Pr\{g(X) \le y\} \\
&= \Pr\{X \le g^{-1}(y)\} \\
&= F_X(g^{-1}(y)) \qquad (6.1)
\end{aligned}
$$

The pdf can then be found using the chain rule

$$
\begin{aligned}
f_Y(y) &= \frac{d}{dy} F_Y(y) \\
&= \frac{d}{dy} F_X(g^{-1}(y)) \\
&= f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)
\end{aligned}
\tag{6.2}
$$

Similarly if $g(X)$ is a decreasing function of X we get

$$
f_Y(y) = -f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y)
\tag{6.3}
$$

Combining the two, we arrive at a general rule for one-to-one transformations:

$$
f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|
\tag{6.4}
$$

For transformations that are not one-to-one, we have to be a bit more careful. For example, we are interested in the following transformation:

$$
Y = aX^2
\tag{6.5}
$$

and wish to find $f_Y(y)$. Again, we shall start by finding an expression for $F_Y(y)$, the cdf of Y. Specifically, using Figure 6.1 we have

$$
\begin{aligned}
F_Y(y) &= \Pr\{Y \le y\} \\
&= \Pr\left\{|X| \le \sqrt{\frac{y}{a}}\right\} \\
&= F_X\left(\sqrt{\frac{y}{a}}\right) - F_X\left(-\sqrt{\frac{y}{a}}\right)
\end{aligned}
$$

Taking the derivative to get the pdf:

$$
\begin{aligned}
f_Y(y) &= \frac{d}{dy} F_Y(y) \\
&= \frac{1}{2}\sqrt{\frac{a}{y}}\frac{1}{a} f_X\left(\sqrt{\frac{y}{a}}\right) + \frac{1}{2}\sqrt{\frac{a}{y}}\frac{1}{a} f_X\left(-\sqrt{\frac{y}{a}}\right) \\
&= \frac{1}{2a\sqrt{y/a}} f_X\left(\sqrt{\frac{y}{a}}\right) + \frac{1}{2a\sqrt{y/a}} f_X\left(-\sqrt{\frac{y}{a}}\right)
\end{aligned}
$$

Now, since $f_X(x) = 0$ for $x < 0$ we have finally:

$$
\boxed{f_Y(y) = \frac{1}{2a\sqrt{y/a}} f_X\left(\sqrt{\frac{y}{a}}\right)}
\tag{6.6}
$$

This particular transformation will be useful for converting the distribution of signal amplitude
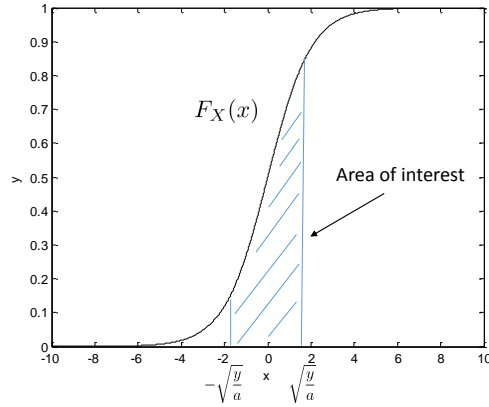
Figure 6.1: Determining the Integration region for the transformation of random variables

(e.g., Rayleigh fading) to the distribution of received signal power (and thus SNR).

6.2.2 Order Statistics

Let $X_1, X_2, \ldots X_N$ be i.i.d. random variables. For any sample of N values let's reorder them as

$$Y_1 < Y_2 < \ldots Y_N \tag{6.7}$$

where $Y_N = \max_j \{X_j\}$ and $Y_1 = \min_j \{X_j\}$. The study of the statistics of Y_r is known as *order statistics*. As an example, the distribution of Y_r is

$$f_{Y_r}(y) = \frac{N!}{(r-1)!(N-r)!} \left[F_X(y)\right]^{r-1} \left[1 - F_X(y)\right]^{N-r} f_X(y) \tag{6.8}$$

Thus, for the maximum value we have

$$f_{\max}(y) = f_{Y_N}(y) = N \left[F_X(y)\right]^{N-1} f_X(y) \tag{6.9}$$

and for the minimum value

$$f_{\min}(y) = f_{Y_1}(y) = N \left[1 - F_X(y)\right]^{N-1} f_X(y) \tag{6.10}$$

6.2.3 The Moment Generating Function

When considering sums of random variables, we often use the property that the distribution of the sum of independent random variables is the convolution of the individual distributions. In general, this convolution is not mathematically convenient. However, an equivalent property is that the moment geenerating function of a sum of independent random variables is the product of the individual moment generating functions. The same is also true of the characteristic function. We will use a slightly modified definition of the moment generating function (in order to exploit

the Laplace Transform). Specifically, we will define it as

$$\Phi_X(s) = \mathrm{E}\left\{e^{-sx}\right\} = \int_{-\infty}^{\infty} e^{sx} f_X(x) dx \tag{6.11}$$

Note that

$$\Phi_X'(s) = -\mathrm{E}\left\{xe^{-sx}\right\} \tag{6.12}$$

and thus

$$\Phi_X'(0) = -\mathrm{E}\left\{x\right\} \tag{6.13}$$

Thus, the kth moment about zero is simply

$$\mu_k = (-1)^k \Phi_X^{(k)}(0) \tag{6.14}$$

The moment generating function is similar to the Laplace transform but replaces s with $-s$. Note that the moment generating function of $\frac{1}{\overline{\gamma}} e^{-\gamma/\overline{\gamma}}$ can be found as

$$
\begin{aligned}
\Phi_\gamma(s) &= \mathrm{E}\left\{e^{s\gamma}\right\} \\
&= \int_0^\infty e^{s\gamma} \frac{1}{\overline{\gamma}} e^{-\gamma/\overline{\gamma}} \\
&= \frac{1}{1 + s\overline{\gamma}}
\end{aligned}
\tag{6.15}
$$

As stated above, an important aspect of the moment generating function is that the moment generating function of the sum of two independent random variables is the product of the individual moment generating functions. To see this, let's assume that we have a random variable $z = x + y$ where x and y are independent random variables. The moment generating function of z is

$$
\begin{aligned}
\Phi_z(s) &= \mathrm{E}\left\{e^{sz}\right\} \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{(x+y)s} f_X(x) f_Y(y) dx dy \\
&= \int_{-\infty}^{\infty} e^{xs} f_X(x) dx \int_{-\infty}^{\infty} e^{ys} f_Y(y) dy \\
&= \Phi_x(s) \Phi_y(s)
\end{aligned}
\tag{6.16}
$$

6.2.4   The Complex Sinusoid

We have already used the complex sinusoid extensively in the study of complex baseband. However, we will formally introduce it here. The complex sinusoid is defined as

$$
\begin{aligned}
e^{jx} &= \cos(x) + j\sin(x) \\
&= \sum_{i=0}^{\infty} \frac{(jx)^i}{i!}
\end{aligned}
\tag{6.17}
$$

$$\begin{aligned} \mathrm{E}\left\{e^{jx}\right\} &= \int_{-\infty}^{\infty}\sum_{i=0}^{\infty}\frac{(jx)^i}{i!}f(x)dx \\ &= \sum_{i=0}^{\infty}\int_{-\infty}^{\infty}\frac{(jx)^i}{i!}f(x)dx \\ &= \sum_{i=0}^{\infty}\frac{j^i}{i!}\underbrace{\int_{-\infty}^{\infty}x^i f(x)dx}_{i\text{th moment about zero}} \\ &= \sum_{i=0}^{\infty}\frac{j^i}{i!}\mu_i \end{aligned} \tag{6.18}$$

6.2.5   The Characteristic Function

Similar to the moment generator function, the characteristic function is defined as

$$\begin{aligned} \mathrm{E}\left\{e^{jx\omega}\right\} &= \int_{-\infty}^{\infty}e^{jx\omega}f_X(x)dx \\ &\triangleq \phi(\omega) \end{aligned}$$

This is essentially the Fourier Transform (with $-\omega$ replaced by ω) of the pdf of X and is analogous to the relationship between the Moment Generating Function and the Laplace Transform. This function can be used to generate either the moments or the cummulants. Specifically for the moments:

$$\left[\frac{d^n}{d\omega^n}\phi(\omega)\right]_{\omega=0} = j^n \mu_n \tag{6.19}$$

and for the Cumulants (κ_n is the nth cumulant)

$$c(\omega) = \ln(\phi(\omega)) = \sum_{k=0}^{\infty}\kappa_k \frac{(j\omega)^k}{k!} \tag{6.20}$$

Thus,

$$\kappa_n = j^{-n}\left[\frac{d^n}{d\omega^n}c(\omega)\right]_{\omega=0} \tag{6.21}$$

6.2.6   Gamma Distribution

In the study of the SNR in fading channels, the Gamma distribution is very useful. The Gamma distribution is defined by two parameters a and b:

$$f_X(x) = \frac{a^b x^{b-1}e^{-ax}}{\Gamma(b)} \tag{6.22}$$

for $x > 0$, $a > 0$, $b > 0$. The moment generating function of the Gamma distribution is easily shown to be

$$\Phi(s) = \left(\frac{a}{a-s}\right)^b \tag{6.23}$$

The cummulative distribution function (CDF) is

$$
\begin{aligned}
F_X(x) &= \int_0^x \frac{a^b y^{b-1} e^{-ay}}{\Gamma(b)} dy \\
&= \frac{1}{\Gamma(b)} \int_0^{ax} z^{b-1} e^{-z} dy \\
&= \frac{\tilde{\gamma}(b, ax)}{\Gamma(b)}
\end{aligned}
\tag{6.24}
$$

where $\tilde{\gamma}(\alpha, \beta)$ is the incomplete gamma function defined as

$$
\tilde{\gamma}(\alpha, \beta) = \int_0^\beta \eta^{\alpha-1} e^{-\eta} d\eta
\tag{6.25}
$$

6.2.7   Chi-Square Distribution

A Gamma distribution where the $a = \frac{1}{2}$ and $b = \frac{k}{2}$ for integer values k is a special case known as the Chi-Square Distribution (or χ^2 Distribution. The pdf and cdf are written as

$$
\begin{aligned}
f_X(x) &= \frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2} \\
F_X(x) &= \frac{\tilde{\gamma}(k/2, x/2)}{\Gamma(k/2)}
\end{aligned}
\tag{6.26}
$$

and k is termed the *degrees of freedom.*

6.2.8   The Cauchy-Schwarz Inequality

The Cauchy-Schwarz Inequality is an incredibly important theorem in mathematics. Sometimes also called simply hte Schwarz Inequality, the theorem states that for two real length-L vectors $\mathbf{x}$ and $\mathbf{y}$

$$
\begin{aligned}
| <\mathbf{x}, \mathbf{y}> |^2 &\leq \langle \mathbf{x}, \mathbf{x} \rangle < \mathbf{y}, \mathbf{y} > \\
\left(\sum_{i=1}^L x_i y_i \right)^2 &\leq \left(\sum_{i=1}^L x_i^2 \right) \left(\sum_{i=1}^L y_i^2 \right)
\end{aligned}
\tag{6.27}
$$

with equality holding if and only if $ax_i = by_i$ for arbitrary a and b. The complex version states that

$$
\left| \sum_{i=1}^L x_i y_i^* \right|^2 \leq \left(\sum_{i=1}^L |x_i|^2 \right) \left(\sum_{i=1}^L |y_i|^2 \right)
\tag{6.28}
$$

with equality occurring when $ax_i = by_i$ for arbitrary real values a and b.

6.3   SNR Distribution in Rayleigh Fading

As discussed in Chapter 2, a common model for the wireless channel when there is no line-of-sight component is Rayleigh fading. Rayleigh fading results from the superposition of a large number of phase shifted received signals of roughly equal power. This complex sum results in Guassian distributed signal voltages in the real and imaginary components and thus a Rayleigh

distributed amplitude. Here we would like to determine the resulting distribution of received signal power (and consequently SNR). Let α be the absolute value (i.e., amplitude) of the channel, i.e., $\alpha = |\tilde{\alpha}| = |\alpha e^{j\phi}|$. For a Rayleigh fading channel the real and imaginary parts of the complex channel are Gaussian distributed and the amplitude distribution is Rayleigh:

$$f_\alpha(a) = \frac{a}{\sigma^2} e^{-\frac{a^2}{2\sigma^2}} \quad 0 \leq a \leq \infty \tag{6.29}$$

where σ is the standard deviation of either of the two i.i.d. Gaussian random variables (i.e., real or imaginary components). Further, $E\left\{\alpha^2\right\} = 2\sigma^2$. When we normalize the channel such that $E\left\{\alpha^2\right\} = 1$, we have $\sigma = \frac{1}{2}$. If the received signal (during one symbol interval) is written as

$$\tilde{r}(t) = \tilde{\alpha}\sqrt{P_r}x(t) + \tilde{n}(t) \tag{6.30}$$

where P_r is the *average* received power and $\tilde{\alpha} = \alpha e^{j\phi}$ is the normalized complex channel, then at the output of a pulse-matched filter (assuming pulse-amplitude modulated signaling with unit energy symbols s_ν) we have at the νth symbol

$$r[\nu] = \tilde{\alpha}E_s s_\nu + \tilde{n}[\nu] \tag{6.31}$$

where $\tilde{n}[\nu]$ is $\mathcal{N}(0, N_o E_s)$ and E_s is the energy per symbol. The average SNR per symbol can be written as

$$\gamma = \alpha^2 \frac{E_s}{N_o} \tag{6.32}$$

What is the distribution of the SNR given that α represents the channel amplitude? We know that the transformation of random variables $Y = aX^2$ is accomplished as

$$f_Y(y) = \frac{1}{2a\sqrt{y/a}} f_X\left(\sqrt{\frac{y}{a}}\right) \tag{6.33}$$

Thus, we have

$$
\begin{aligned}
f_\gamma(\gamma) &= \frac{1}{2\frac{E_s}{N_o}\sqrt{\frac{\gamma}{E_s/N_o}}} f_\alpha\left(\sqrt{\frac{\gamma}{E_s/N_o}}\right) \\
&= \frac{1}{2\frac{E_s}{N_o}\sqrt{\frac{\gamma}{E_s/N_o}}} \frac{\sqrt{\frac{\gamma}{E_s/N_o}}}{\sigma^2} e^{-\frac{\gamma}{2\sigma^2 E_s/N_o}} \\
&= \frac{1}{\bar{\gamma}} e^{-\frac{\gamma}{\bar{\gamma}}} \tag{6.34}
\end{aligned}
$$

where the average SNR $\bar{\gamma} = 2\sigma^2 \frac{E_s}{N_o} = \frac{E_s}{N_o}$ since $\sigma^2 = \frac{1}{2}$.

The CDF of SNR is then

$$
\begin{aligned}
F_\gamma(\gamma) &= \int_0^\gamma f_\gamma(x)dx \\
&= \int_0^\gamma \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}} dx \\
&= 1 - e^{-\frac{\gamma}{\bar{\gamma}}}
\end{aligned}
$$

Note that the probability of outage $P_{out}(\gamma_{th}) = F_\gamma(\gamma_{th})$.

6.4   Probability of Bit Error of BPSK in a Rayleigh Fading Channel

It is well known that the probability of error for BPSK in an AWGN channel can be written as

$$P_b^{awgn} = Q\left(\sqrt{2\frac{E_b}{N_o}}\right) \tag{6.35}$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{y^2}{2}} dy \tag{6.36}$$

If we assume that the channel in a Rayleigh fading environment is constant over one symbol, the *average* probability of error for BPSK in Rayleigh fading can be written as

$$\begin{aligned}
P_b &= \int_0^\infty f_\gamma(\gamma) Q(\sqrt{2\gamma}) d\gamma \\
&= \int_0^\infty \frac{1}{\bar{\gamma}} e^{-\frac{\gamma}{\bar{\gamma}}} Q(\sqrt{2\gamma}) d\gamma
\end{aligned} \tag{6.37}$$

where we have set γ as the SNR per bit which is equivalent to the SNR per symbol for BPSK.
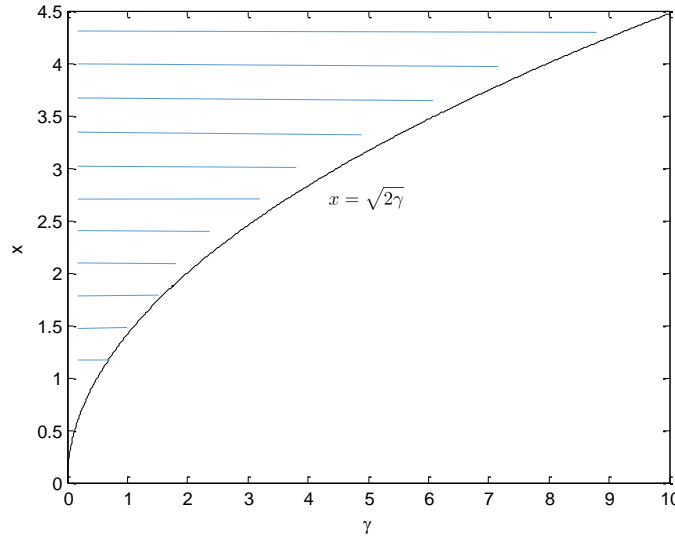

Figure 6.2: Understanding the change in integration limits when reversing the order of integration

Thus, we have (substituting (6.36) for $Q(x)$):

$$P_b = \int_0^\infty \frac{1}{\bar{\gamma}} e^{-\frac{\gamma}{\bar{\gamma}}} \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2\gamma}}^\infty e^{-\frac{x^2}{2}} dx d\gamma$$

Now, to proceed we interchange the order of integration. To understand the new limits see Figure

6.2. This results in

$$
\begin{aligned}
P_b &= \int_0^\infty \int_0^{x^2/2} \frac{1}{\overline{\gamma}} e^{-\frac{\gamma}{\overline{\gamma}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} d\gamma dx \\
&= \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \underbrace{\int_0^{x^2/2} \frac{1}{\overline{\gamma}} e^{-\frac{\gamma}{\overline{\gamma}}} d\gamma}_{=1-e^{-x^2/(2\overline{\gamma})}} dx \\
&= \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left(1 - e^{-x^2/(2\overline{\gamma})}\right) dx \\
&= \underbrace{\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx}_{=\frac{1}{2}} - \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2} - \frac{x^2}{2\overline{\gamma}}} dx \\
&= \frac{1}{2} - \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2} - \frac{x^2}{2\overline{\gamma}}} dx
\end{aligned}
$$

Now, let $z = x\sqrt{1 + \frac{1}{\overline{\gamma}}} = x\sqrt{\frac{1+\overline{\gamma}}{\overline{\gamma}}}$ and $dz = dx\sqrt{\frac{1+\overline{\gamma}}{\overline{\gamma}}}$. This provides

$$
\begin{aligned}
P_b &= \frac{1}{2} - \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \left(\sqrt{\frac{1+\overline{\gamma}}{\overline{\gamma}}}\right)^{-1} dz \\
&= \frac{1}{2} - \sqrt{\frac{\overline{\gamma}}{1+\overline{\gamma}}} \underbrace{\int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz}_{=\frac{1}{2}} \\
&= \frac{1}{2}\left(1 - \sqrt{\frac{\overline{\gamma}}{1+\overline{\gamma}}}\right) \tag{6.38}
\end{aligned}
$$

For large average SNR $\overline{\gamma} \gg 1$ we have the approximation

$$
P_b \approx \frac{1}{4\overline{\gamma}} \tag{6.39}
$$

To see this, consider the Taylor Series expansion

$$
\sqrt{\frac{1}{1+1/\overline{\gamma}}} \approx 1 - \frac{1}{2\overline{\gamma}} \tag{6.40}
$$

Thus,

$$
\begin{aligned}
P_b &= \frac{1}{2}\left(1 - \sqrt{\frac{\overline{\gamma}}{1+\overline{\gamma}}}\right) \\
&\approx \frac{1}{2}\left(1 - \left(1 - \frac{1}{2\overline{\gamma}}\right)\right) \\
&= \frac{1}{4\overline{\gamma}} \tag{6.41}
\end{aligned}
$$

The impact of Rayleigh fading can be seen in Figure 6.3. The probability of bit error (bit error rate or BER) is plotted for both AWGN and Rayleigh fading for BPSK versus the average SNR per bit $\gamma_b = \frac{E_b}{N_o} = \overline{\gamma}$. We can see two things from the plot. First, we can see that the simple approximation in (6.39) matches well for $E_b/N_o > 10$dB. Second, and more importantly, Rayleigh fading has a major impact on performance. For example, if a BER of 1% is desired, approximately 9dB of additional received energy is needed (on average) in the case of Rayleigh fading. For higher

BER values, the needed additional energy is even higher. Thus, combating the impact of fading is major design requirement for wireless communications.
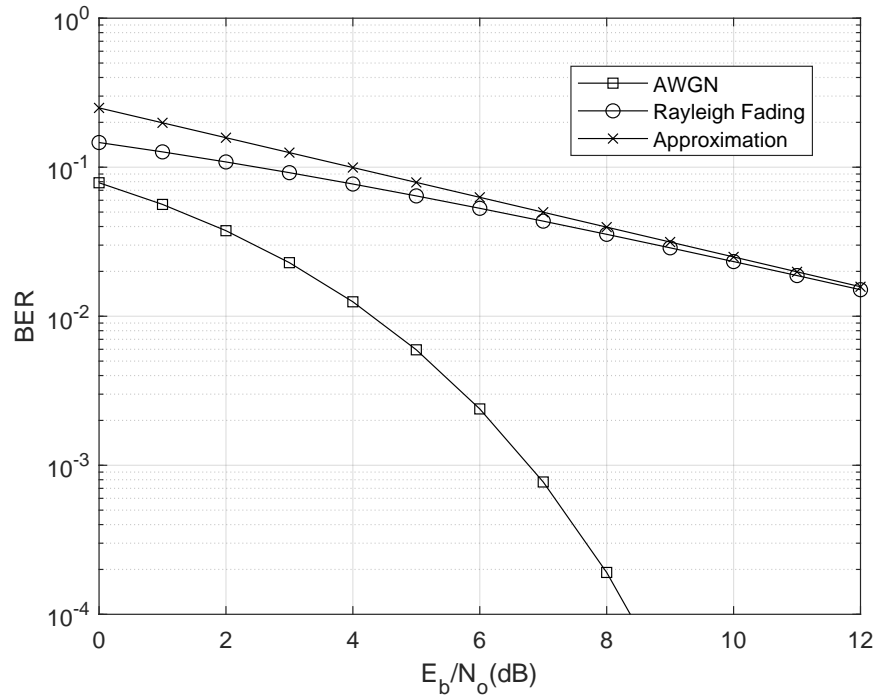

Figure 6.3: Probability of Bit Error for BPSK in AWGN and Rayleigh Fading

6.5 Selection Diversity

In the previous section we saw that Rayleigh fading has a strong negative impact on the bit error rate of wireless communications links. The primary means of mitigating the impact of fading is *diversity*. Diversity is based on the properties of spatial, temporal and/or frequency coherence of the wireless channel. A simple means of achieving diversity is by exploiting the spatial coherence of the wireless channel and using antennas that are spaced greater than that spatial coherence. This is called *spatial diversity*. The simplest form of spatial diveristy is known as *selection diversity*. With selection diversity, the receiver uses M_r receive antennas, but only a single receiver preceded by a switch and a simple form of energy measurement. The switch simply chooses the antenna with the largest received energy.

To understand the impact of selection diversity, assume that we observe L i.i.d. channels (frequency, time or antennas). Let γ_i be the SNR seen on the ith channel. The selection diversity mechanism chooses the SNR with the highest SNR (typically by choosing the largest received signal) so that the SNR at the output of the detector is

$$\gamma_{sel} = \max\{\gamma_1, \gamma_2, \ldots, \gamma_L\} \tag{6.42}$$

From order statistics we know that for i.i.d. random variables the pdf of the maximum is

$$\begin{aligned} f_{\gamma_{\max}}(\gamma) &= L f_\gamma(\gamma) \left(\int_0^\gamma f_\gamma(x)dx\right)^{L-1} \\ &= L f_\gamma(\gamma) \left(F_\gamma(\gamma)\right)^{L-1} \end{aligned}$$

where $f_\gamma(\gamma)$ is the pdf of the SNR for all channels (since iid). For Rayleigh fading γ is exponentially distributed thus

$$f_\gamma(\gamma) = \frac{1}{\bar{\gamma}} e^{-\frac{\gamma}{\bar{\gamma}}} \quad \gamma > 0 \tag{6.43}$$

where $\bar{\gamma}$ is the average SNR per channel. Further,

$$F_\gamma(\gamma) = 1 - e^{-\frac{\gamma}{\bar{\gamma}}} \tag{6.44}$$

Thus, we have

$$f_{\gamma_{\max}}(\gamma) = \frac{L}{\bar{\gamma}} e^{-\frac{\gamma}{\bar{\gamma}}} \left(1 - e^{-\frac{\gamma}{\bar{\gamma}}}\right)^{L-1} \tag{6.45}$$

Now, using the binomial expansion

$$(1-x)^n = 1 - nx + \frac{n(n-1)x^2}{2!} - \frac{n(n-1)(n-2)x^3}{3!} + \ldots \frac{n(n-1)(n-2)\ldots(n-(n-1))x^n}{n!} \tag{6.46}$$

we can write

$$
\begin{aligned}
f_{\gamma_{\max}}(\gamma) &= \frac{L}{\overline{\gamma}} e^{-\frac{\gamma}{\overline{\gamma}}} \left\{ 1 - (L-1)e^{-\frac{\gamma}{\overline{\gamma}}} + \frac{(L-1)(L-2)}{2} e^{-\frac{2\gamma}{\overline{\gamma}}} \ldots \right\} \\
&= \frac{1}{\overline{\gamma}} \left\{ L e^{-\frac{\gamma}{\overline{\gamma}}} - L(L-1)e^{-\frac{2\gamma}{\overline{\gamma}}} + \frac{L(L-1)(L-2)}{2} e^{-\frac{3\gamma}{\overline{\gamma}}} \ldots + \frac{L(L-1)(L-2)\ldots(2)(1)}{(L-1)!} e^{-\frac{L\gamma}{\overline{\gamma}}} \right\} \\
&= \sum_{k=1}^{L} \binom{L}{k} \frac{k}{\overline{\gamma}} (-1)^{k-1} e^{-\frac{k\gamma}{\overline{\gamma}}}
\end{aligned}
\tag{6.47}
$$

6.5.1 Probability of Outage

Bit errors in fading channels are dominated by outages, i.e., drops in SNR. Thus, we are interested in the probability that the SNR drops below a specific threshold. That can be found from the cumulative distribution function (CDF). That is

$$
\begin{aligned}
P_{out} &= Pr(\gamma \le \gamma_{th}) \\
&= F_{\gamma_{max}}(\gamma_{th}) \\
&= (1 - e^{-\frac{\gamma_{th}}{\overline{\gamma}}})^L
\end{aligned}
\tag{6.48}
$$

For large values of $\overline{\gamma}$

$$
(1 - e^{-\frac{\gamma_{th}}{\overline{\gamma}}})^L \approx 1 - L e^{-\frac{\gamma_{th}}{\overline{\gamma}}}
\tag{6.49}
$$

We say that the diversity order is L since on a log plot, the slope will be related to L. The benefit of selection diversity can be seen in Figure 6.4. We see that even though only a single receiver is ultimately used, the diversity benefit is substantial as large improvements in the outage probability are obtained.

6.5.2 Mean SNR

In addition to improving the probability of outage, diversity can improve the average SNR. This is true of selection diversity even though only one receiver is used. To see this, consider the mean of the maximum SNR:

$$
\begin{aligned}
\overline{\gamma}_{max} &= \int_0^\infty x f_{\gamma_{max}}(x) dx \\
&= \sum_{k=1}^{L} \binom{L}{k} \frac{k}{\overline{\gamma}} (-1)^{k-1} \int_0^\infty x e^{-\frac{kx}{\overline{\gamma}}} dx
\end{aligned}
$$

Now

$$
\begin{aligned}
\int_0^\infty x e^{-\frac{kx}{\overline{\gamma}}} dx &= e^{-\frac{kx}{\overline{\gamma}}} \left[\frac{x}{-k/\overline{\gamma}} - \frac{1}{k^2/\overline{\gamma}^2} \right]_0^\infty \\
&= \frac{\overline{\gamma}^2}{k^2}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\overline{\gamma}_{max} &= \sum_{k=1}^{L} \left(\begin{array}{c} L \\ k \end{array} \right) \frac{k}{\overline{\gamma}} (-1)^{k-1} \frac{\overline{\gamma}^2}{k^2} \\
&= \overline{\gamma} \sum_{k=1}^{L} \frac{1}{k}
\end{aligned} \tag{6.50}
$$

Clearly, selection diversity provides a gain in the average SNR of $\sum_{k=1}^{L} \frac{1}{k}$.
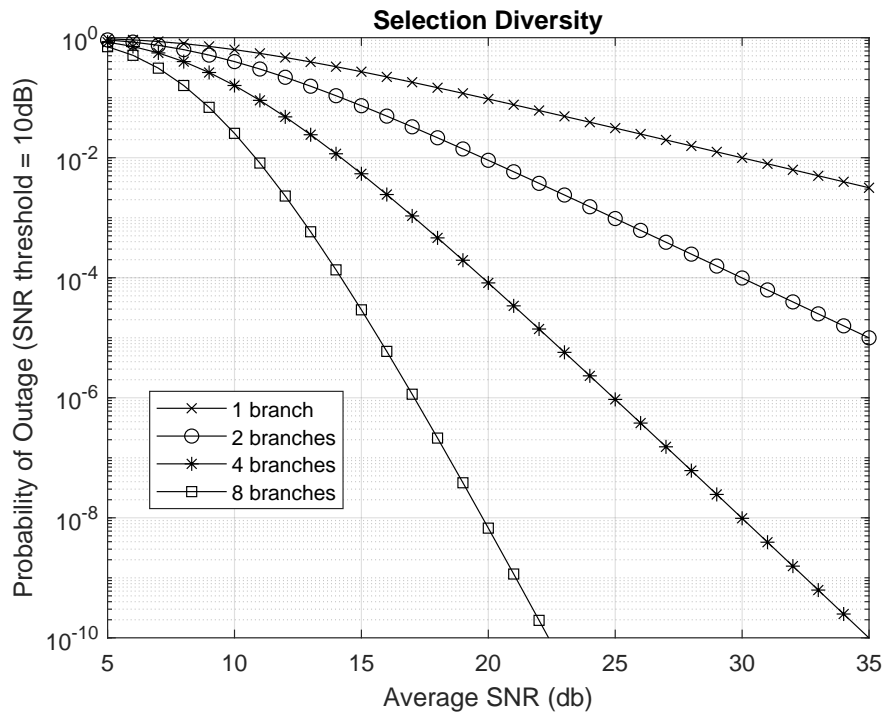

Figure 6.4: Probability of Outage for a Threshold SNR of $\gamma_{th} = 10dB$ in Rayleigh Fading with Selection Diversity

6.5.3   Probability of Bit Error

The probability of bit error can be calculated as

$$P_b = \int_0^\infty f(\gamma) Q(\sqrt{2\gamma}) d\gamma \tag{6.51}$$

Using (6.47):

$$
\begin{aligned}
P_b &= \int_0^\infty \sum_{k=1}^{L} \binom{L}{k} \frac{k}{\overline{\gamma}} (-1)^{k-1} e^{-\frac{k\gamma}{\overline{\gamma}}} Q(\sqrt{2\gamma}) d\gamma \\
&= \sum_{k=1}^{L} \binom{L}{k} (-1)^{k-1} \int_0^\infty \frac{k}{\overline{\gamma}} e^{-\frac{k\gamma}{\overline{\gamma}}} Q(\sqrt{2\gamma}) d\gamma
\end{aligned} \tag{6.52}
$$

Using our result from the probability of bit error for Rayleigh fading without diversity, we obtain the final probability of bit error

$$P_b = \frac{1}{2} \sum_{k=1}^{L} \binom{L}{k} (-1)^{k-1} \left(1 - \sqrt{\frac{\overline{\gamma}}{k + \overline{\gamma}}} \right) \tag{6.53}$$

The probability of bit error for $L = 1, 2, 4$ is plotted in Figure 6.5. Note that the probability of error is plotted versus the average SNR per bit $\gamma_b = \frac{E_b}{N_o} = \frac{\overline{\gamma}}{\sum_{k=1}^{L} \frac{1}{k}}$. As with the outage probability, the probability of bit error improves substantially with two or four diversity branches.
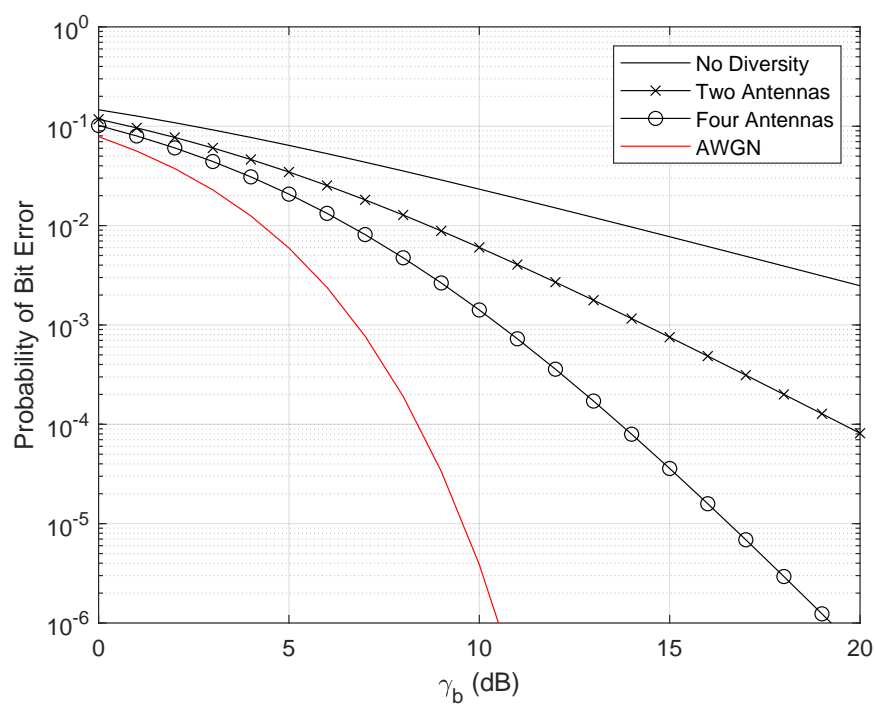
Figure 6.5: Probability of Bit Error (BPSK) in Rayleigh Fading with Selection Diversity

6.6   Equal Gain Combining

Instead of simply selecting the largest SNR, we can improve performance by combining signals. Specifically, we consider coherently combining the signals on L diversity channels. The desired signal adds up coherently while the noise adds up incoherently. If the channel on the ith channel is $\tilde{\alpha}_i = \alpha_i e^{j\phi_i}$, the decision variable is created as

$$z[\nu] = \sum_{i=1}^{L} r_i[\nu]e^{-j\phi_i} \tag{6.54}$$

The SNR per channel is $\gamma_i = \alpha_i^2 \frac{E_s}{N_o}$. The total SNR is then

$$\gamma_{egc} = \frac{1}{L}\frac{E_s}{N_o}\left(\sum_{i=1}^{L}\alpha_i^2\right)^2 \tag{6.55}$$

If we normalize the channel such that $\mathrm{E}\left\{\alpha^2\right\} = 1$ and $\overline{\gamma}_i = \frac{E_s}{N_o}$. Further, assuming iid channels, we can show that in the case of Rayleigh fading we can approximate the SNR of equal gain combining as

$$f_{\gamma_{egc}}(\gamma) \approx \frac{1}{(L-1)!}\frac{\eta^L}{\overline{\gamma}^L}\gamma^{L-1}e^{-\eta\frac{\gamma}{\overline{\gamma}}} \tag{6.56}$$

where

$$\eta = \frac{L}{2}\left(\frac{(2L-1)!}{(L-1)!2^{2L-1}}\right)^{-\frac{1}{L}} \tag{6.57}$$

Note that this is a Chi-Square distribution. Further, we can write the average SNR as

$$\overline{\gamma}_{egc} = L\frac{\overline{\gamma}}{\eta} \tag{6.58}$$

The probability of outage is simply

$$
\begin{aligned}
P_{out}^{egc}(\gamma) &= F_{\gamma_{egc}}(\gamma) \\
&= 1 - e^{-\gamma\frac{\eta}{\overline{\gamma}}}\sum_{k=0}^{L-1}\frac{1}{k!}\left(-\gamma\frac{\eta}{\overline{\gamma}}\right)^k
\end{aligned}
\tag{6.59}
$$

Recalling the series expansion $e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$

$$
\begin{aligned}
P_{out}^{egc}(\gamma) &= 1 - e^{-\gamma\frac{\eta}{\overline{\gamma}}}\left(e^{-\gamma\frac{\eta}{\overline{\gamma}}} - \sum_{k=L}^{\infty}\frac{1}{k!}\left(-\gamma\frac{\eta}{\overline{\gamma}}\right)^k\right) \\
&= e^{-\gamma\frac{\eta}{\overline{\gamma}}}\sum_{k=L}^{\infty}\frac{1}{k!}\left(-\gamma\frac{\eta}{\overline{\gamma}}\right)^k \\
&\approx e^{-\gamma\frac{\eta}{\overline{\gamma}}}\frac{1}{L!}\left(-\gamma\frac{\eta}{\overline{\gamma}}\right)^L
\end{aligned}
\tag{6.60}
$$

where we can see that the outage probability drops to the Lth power (or has a slope of L in the log domain). We can this the diversity order or diversity gain. An example for a threshold of 10dB is plotted in Figure 6.6. Note that this is somewhat better than selection diversity due to the increase in average SNR, since both have the same diversity gain.
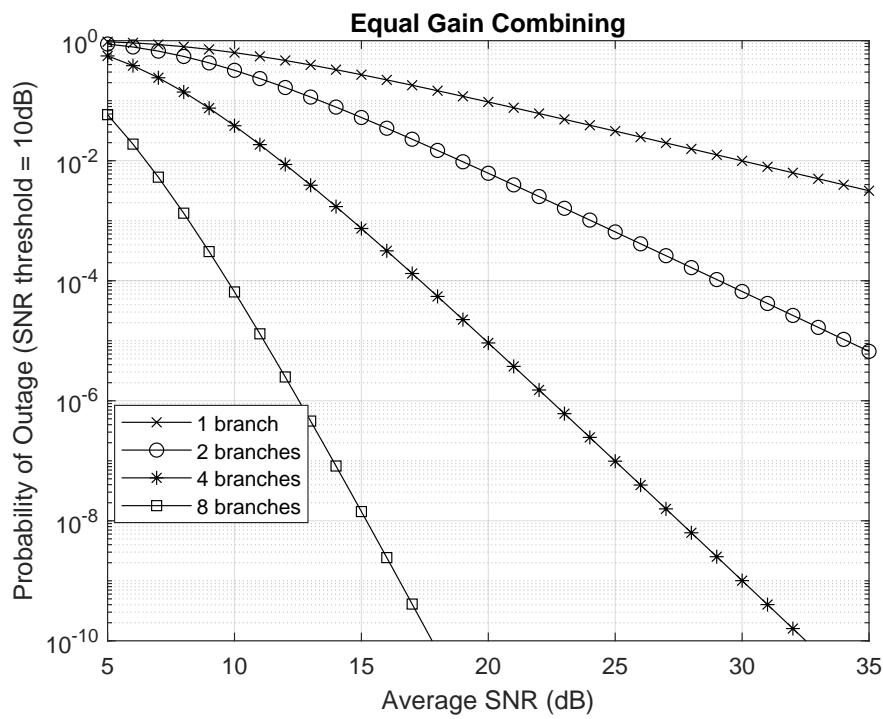
Figure 6.6: Probability of Outage for a Threshold SNR of $\gamma_{th} = 10dB$ in Rayleigh Fading with Equal Gain Combining

6.7   Optimal Combining

It is intuitive that equal gain combining cannot be optimal, since it includes low SNR signals and high SNR signals with equal weight. We would like to determine the optimal weights for combining. To do this, let the received signal on the ith antenna be (assuming all real signals for ease of notation)

$$r_i(t) = \tilde{\alpha}_i(t)x(t) + n_i(t) \tag{6.61}$$

with AWGN $n_i(t)$ having variance σ_n^2. Further, the instantaneous SNR per channel is

$$\gamma_i = \frac{\alpha_i^2}{\sigma^2} \tag{6.62}$$

After matched filtering we have

$$r_i[\nu] = \tilde{\alpha}_i s_\nu + n_i[\nu] \tag{6.63}$$

or dropping dependence on time

$$r_i = \tilde{\alpha}_i s + n_i \tag{6.64}$$

The decision statistic is formed as

$$
\begin{aligned}
z &= \sum_{i=1}^{L} w_i^* r_i \\
&= \sum_{i=1}^{L} (w_i^* \tilde{\alpha}_i s + w_i^* n_i) \\
&= \underbrace{\sum_{i=1}^{L} w_i^* \tilde{\alpha}_i s}_{\text{desiredterm}} + \underbrace{\sum_{i=1}^{L} w_i^* n_i}_{\text{undesiredterm}}
\end{aligned}
$$

The resulting SNR (for fixed channels $\tilde{\alpha}_i$) is

$$
\begin{aligned}
\gamma &= \frac{\mathrm{E}\left\{ \left| \sum_{i=1}^{L} w_i^* \tilde{\alpha}_i s_i \right|^2 \right\}}{\mathrm{E}\left\{ \left| \sum_{i=1}^{L} w_i^* n_i \right|^2 \right\}} \\
&= \frac{\left| \sum_{i=1}^{L} w_i^* \tilde{\alpha}_i \right|^2}{\sum_{i=1}^{L} |w_i|^2 \sigma_n^2}
\end{aligned} \tag{6.65}
$$

Now, define $x_i = \frac{\tilde{\alpha}_i}{\sigma_n}$ and $y_i = w_i \sigma_n$. From the complex version of Schwarz' Inequality

$$\left| \sum_{i=1}^{L} w_i^* \tilde{\alpha}_i \right|^2 \leq \sum_{i=1}^{L} |w_i|^2 \sigma_n^2 \sum_{i=1}^{L} \underbrace{\frac{|\tilde{\alpha}_i|^2}{\sigma_n^2}}_{\gamma_i} \tag{6.66}$$

Or in other words

$$\gamma = \frac{\left(\sum_{i=1}^{L} w_i^* \tilde{\alpha}_i \right)^2}{\sum_{i=1}^{L} |w_i|^2 \sigma_n^2} \leq \sum_{i=1}^{L} \gamma_i \tag{6.67}$$

with equality occurring iff $aw_i\sigma_n = b\tilde{\alpha}_i/\sigma_n$ or in other words $w_i = c\frac{\tilde{\alpha}_i}{\sigma_n^2}$ since $c = b/a$ is arbitrary, we choose it to be $c = \sigma_n^2$ and thus,

$$\boxed{w_i = \tilde{\alpha}_i} \tag{6.68}$$

We term this **Maximal Ratio Combining**. Further, we see that the optimal SNR as

$$\gamma_{mrc} = \sum_{i=1}^{L} \gamma_i \tag{6.69}$$

6.7.1 SNR Distribution

For a Rayleigh channel with iid channels (each having average SNR $\overline{\gamma}$), we know that the SNR is the sum of the SNRs per channel. The moment generating function can be written as:

$$\begin{aligned}
\Phi_{\gamma_{mrc}}(s) &= \prod_{i=1}^{L} \Phi_{\gamma_i}(s) \\
&= \prod_{i=1}^{L} \frac{1}{1 + s\overline{\gamma}} \\
&= \frac{1}{(1 + s\overline{\gamma})^L} \tag{6.70}
\end{aligned}$$

Taking the inverse Laplace Transform we have

$$f_{\gamma_{mrc}}(\gamma) = \frac{1}{(L-1)!} \frac{\gamma^{L-1}}{\overline{\gamma}^L} e^{-\frac{\gamma}{\overline{\gamma}}} \quad \gamma > 0 \tag{6.71}$$

Note, that this is a Chi-Square distribution with $2L$ degrees of freedom. Further, the mean SNR is

$$\overline{\gamma}_{mrc} = L\overline{\gamma} \tag{6.72}$$

6.7.2 Probability of Outage

For Rayleigh fading channels, the probability of outage is

$$\begin{aligned}
P_{out}^{mrc}(\gamma_{th}) &= F_{\gamma_{mrc}}(\gamma_{th}) = 1 - e^{-\frac{\gamma_{th}}{\overline{\gamma}}} \sum_{k=0}^{L-1} \frac{1}{k!} \left(\frac{\gamma_{th}}{\overline{\gamma}} \right)^k \\
&\approx e^{-\frac{\gamma_{th}}{\overline{\gamma}}} \frac{1}{L!} \left(\frac{\gamma_{th}}{\overline{\gamma}} \right)^L \tag{6.73}
\end{aligned}$$

for large values of $\overline{\gamma}$. The probability of outage is plotted in Figure fig:mrc for $L = 1, 2, 4$. We can see that MRC provides a tremendous improvement over no diversity, especially in terms of the average SNR improvement.
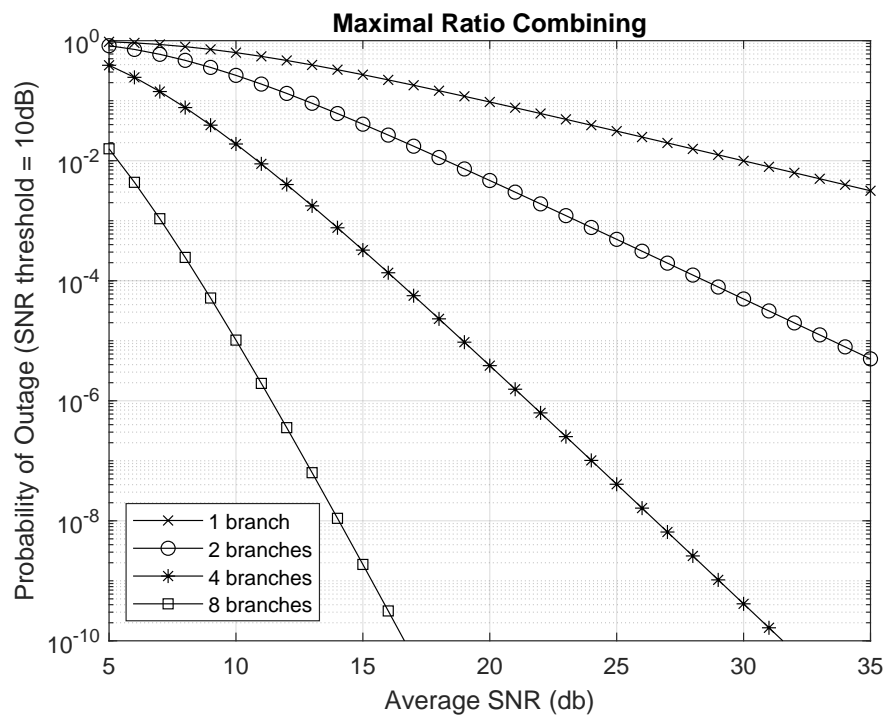
Figure 6.7: Probability of Outage for a Threshold SNR of $\gamma_{th} = 10dB$ in Rayleigh Fading with Maximal Ratio Combining

6.7.3   Bit Error Rate for BPSK

We can find the probabilty of bit error for BPSK by using (6.71) in (6.37) which gives

$$P_b = \int_0^\infty \frac{1}{(L-1)!} \frac{\gamma^{L-1}}{\overline{\gamma}^L} e^{-\gamma/\overline{\gamma}} Q\left(\sqrt{2\gamma}\right) d\gamma \tag{6.74}$$

Now, we can integrate this as we did previously, but the integration is a bit more difficult. Again, replacing $Q(x)$ with its integral definition:

$$
\begin{aligned}
P_b &= \int_0^\infty \int_{-\infty}^{\sqrt{2\gamma}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{(L-1)!} \frac{\gamma^{L-1}}{\overline{\gamma}^L} e^{-\gamma/\overline{\gamma}} dx d\gamma \\
&= \int_0^\infty \left\{ \int_0^{x^2/2} \frac{1}{(L-1)!} \frac{\gamma^{L-1}}{\overline{\gamma}^L} e^{-\gamma/\overline{\gamma}} d\gamma \right\} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \\
&= \frac{1}{(L-1)!} \frac{1}{\overline{\gamma}^L} \int_0^\infty \left\{ \int_0^{x^2/2} \gamma^{L-1} e^{-\gamma/\overline{\gamma}} d\gamma \right\} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx
\end{aligned}
\tag{6.75}
$$

The inner integral can be solved using integration by parts:

$$\int \gamma^{L-1} e^{-\gamma/\overline{\gamma}} d\gamma = -e^{-\gamma/\overline{\gamma}} \sum_{i=0}^{L-1} \overline{\gamma}^{L-i} \gamma^i \frac{(L-1)!}{i!} \tag{6.76}$$

Applying the limits we get for the inner integral

$$\overline{\gamma}^L (L-1)! - e^{-x^2/(2\overline{\gamma})} \sum_{i=0}^{L-1} \overline{\gamma}^{L-i} \left(\frac{x^2}{2}\right)^i \frac{(L-1)!}{i!} \tag{6.77}$$

Using this in (6.75) gives

$$P_b = \frac{1}{(L-1)!} \frac{1}{\overline{\gamma}^L} \left\{ \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \overline{\gamma}^L (L-1)! dx - \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^{-x^2/(2\overline{\gamma})} \sum_{i=0}^{L-1} \overline{\gamma}^{L-i} \left(\frac{x^2}{2}\right)^i \frac{(L-1)!}{i!} dx \right\} \tag{6.78}$$

Simplifying,

$$
\begin{aligned}
P_b &= \frac{1}{2} - \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\left(x^2/2 + x^2/(2\overline{\gamma})\right)} \sum_{i=0}^{L-1} \frac{1}{\overline{\gamma}^i} \left(\frac{x^2}{2}\right)^i \frac{1}{i!} dx \tag{6.79} \\
&= \frac{1}{2} - \sum_{i=0}^{L-1} \frac{1}{\overline{\gamma}^i} \frac{1}{i!} \frac{1}{2^i} \int_0^\infty \frac{x^{2i}}{\sqrt{2\pi}} e^{-(x^2/2)(1+1/\overline{\gamma})} dx \tag{6.80}
\end{aligned}
$$

Now, using integration tables we can find

$$\int_0^\infty x^{2i} e^{-kx^2} dx = \frac{(2i-1)!!}{2^{i+1} k^i} \sqrt{\frac{\pi}{k}} \tag{6.81}$$

Thus,

$$P_b = \frac{1}{2} - \sum_{i=0}^{L-1} \frac{1}{\overline{\gamma}^i} \frac{1}{i!} \frac{1}{2^i} \frac{1}{\sqrt{2\pi}} \frac{(2i-1)!!}{2^{i+1} \left(\frac{1}{2}\left(1+\frac{1}{\overline{\gamma}}\right)\right)^i} \sqrt{\frac{2\pi}{1+\frac{1}{\overline{\gamma}}}} \tag{6.82}$$

Using $(2i - 1)!! = \frac{(2i)!}{2^i i!}$ we obtain

$$P_b = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\overline{\gamma}}{1+\overline{\gamma}}} \sum_{i=0}^{L-1} \binom{2i}{i} \left(\frac{1}{4}\frac{1}{1+\overline{\gamma}}\right)^i \tag{6.83}$$

Alternate Derivation

Another way of solving the problem is as follows. First, Alouini and Simon show that the following integral can be used for the Q-function[8]:

$$Q(x) = \frac{1}{\pi}\int_0^{\pi/2} \exp\left(-\frac{x^2}{2\sin^2(\theta)}\right) d\theta \tag{6.84}$$

Thus, we can write the probability of error as

$$
\begin{aligned}
P_b &= \int_0^\infty \frac{1}{\pi}\int_0^{\pi/2} \exp\left(-\frac{\gamma}{\sin^2(\theta)}\right) f_\gamma(\gamma) d\theta d\gamma \\
&= \frac{1}{\pi}\int_0^{\pi/2}\int_0^\infty \exp\left(-\frac{\gamma}{\sin^2(\theta)}\right) f_\gamma(\gamma) d\gamma d\theta
\end{aligned}
$$

However, the definition of the moment-generating function

$$\Phi(s) = \int_{-\infty}^\infty e^{s\gamma} f_\gamma(\gamma) d\gamma \tag{6.85}$$

Thus,

$$\int_0^\infty \exp\left(-\frac{\gamma}{\sin^2(\theta)}\right) f_\gamma(\gamma) d\gamma = \Phi\left(-\frac{1}{\sin^2(\theta)}\right) \tag{6.86}$$

and

$$P_b = \frac{1}{\pi}\int_0^{\pi/2} \Phi\left(-\frac{1}{\sin^2(\theta)}\right) d\theta \tag{6.87}$$

Now, we know that for MRC

$$\Phi(s) = \left(\frac{1}{1-s\overline{\gamma}}\right)^L \tag{6.88}$$

Thus,

$$
\begin{aligned}
P_b &= \frac{1}{\pi}\int_0^{\pi/2} \left(\frac{1}{1+\frac{\overline{\gamma}}{\sin^2(\theta)}}\right)^L d\theta \tag{6.89} \\
&= \frac{1}{\pi}\int_0^{\pi/2} \left(\frac{\sin^2(\theta)}{\sin^2(\theta)+\overline{\gamma}}\right)^L d\theta \tag{6.90}
\end{aligned}
$$

It is shown in [8] that

$$\frac{1}{\pi}\int_0^{\pi/2} \left(\frac{\sin^2(\theta)}{\sin^2(\theta)+a}\right)^m d\theta = \frac{1}{2}\left[1 - \mu(a)\sum_{k=0}^{m-1}\binom{2k}{k}\left(\frac{1-\mu^2(a)}{4}\right)^k\right] \tag{6.91}$$

where

$$\mu(a) = \sqrt{\frac{a}{1+a}} \qquad (6.92)$$

Further, it is also shown in [8] the integral can also be written as

$$\frac{1}{\pi}\int_0^{\pi/2}\left(\frac{\sin^2(\theta)}{\sin^2(\theta)+a}\right)^m d\theta = \left(\frac{1-\mu(a)}{2}\right)^m \sum_{k=0}^{m-1}\binom{m-1+k}{k}\left(\frac{1+\mu(a)}{2}\right)^k \qquad (6.93)$$

Using the first solution of (6.91) in (6.90) results in,

$$P_b = \frac{1}{\pi}\int_0^{\pi/2}\left(\frac{\sin^2(\theta)}{\sin^2(\theta)+\overline{\gamma}}\right)^L d\theta \qquad (6.94)$$

$$= \frac{1}{2}\left[1-\sqrt{\frac{\overline{\gamma}}{1+\overline{\gamma}}}\sum_{k=0}^{L-1}\binom{2k}{k}\left(\frac{1-\frac{\overline{\gamma}}{1+\overline{\gamma}}}{4}\right)^k\right] \qquad (6.95)$$

$$= \frac{1}{2}\left[1-\sqrt{\frac{\overline{\gamma}}{1+\overline{\gamma}}}\sum_{k=0}^{L-1}\binom{2k}{k}\left(\frac{1}{4}\frac{1}{1+\overline{\gamma}}\right)^k\right] \qquad (6.96)$$

which is equivalent to our previous result.

Using the second integral solution from (6.93) in (6.90) results in

$$P_b = \left[\frac{1}{2}\left(1-\sqrt{\frac{\overline{\gamma}}{1+\overline{\gamma}}}\right)\right]^L \sum_{k=0}^{L-1}\binom{L-1+k}{k}\left[\frac{1}{2}\left(1+\sqrt{\frac{\overline{\gamma}}{1+\overline{\gamma}}}\right)\right]^k \qquad (6.97)$$

The performance for $L = 1,2,4$ branches is plotted in Figure and 6.8. We can see that on an *SNR per bit* basis, the difference between MRC and selection diversity is very small. This tells us that the *diversity gain* is essentially the same. Note that even with four branches, there is still a large difference between the fading case and AWGN (no fading). Comparing the performance from an SNR per branch perspective, we can see more difference between MRC and selection diversity due to the better average SNR improvement from coherent combining. There is also a larger improvement when adding more antennas.
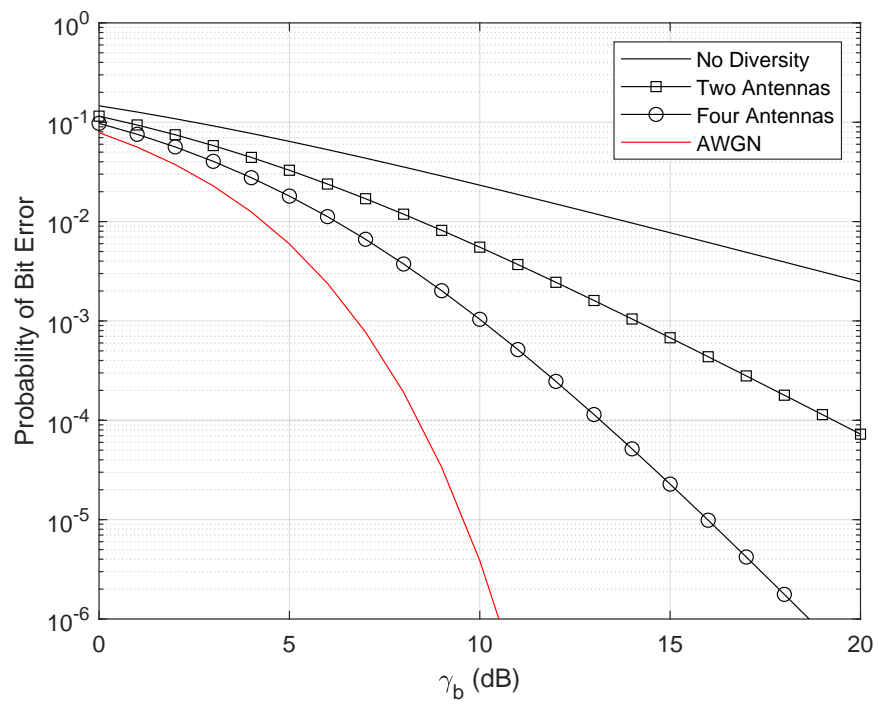
Figure 6.8: Probability of Bit Error (BPSK) in Rayleigh Fading with Maximal Ratio Combining

6.7.4 Impact of Unequal Powers on MRC

To this point we have assumed that each branch has equal average power and thus equal average SNR $\overline{\gamma}$. What if the average SNRs are different? We will show that this does have an effect on overall performance. To see this recall that the SNR of MRC is

$$\gamma_{mrc} = \sum_{i=1}^{L} \gamma_i \tag{6.98}$$

where $\gamma_i = \frac{\alpha_i^2}{\sigma_n^2}$. The SNR per channel (Rayleigh fading):

$$f_{\gamma_i}(\gamma) = \frac{1}{\overline{\gamma}_i} e^{-\frac{\gamma}{\overline{\gamma}_i}} \quad \gamma > 0 \tag{6.99}$$

The Moment-Generating Function for the SNR per channel is defined as

$$\begin{aligned}
\Phi(s) &= \mathrm{E}_\gamma \left\{ e^{-s\gamma} \right\} \\
&= \int_0^\infty e^{-s\gamma} \frac{1}{\overline{\gamma}_i} e^{-\frac{\gamma}{\overline{\gamma}_i}} d\gamma \\
&= \frac{1}{1 + s\overline{\gamma}_i}
\end{aligned} \tag{6.100}$$

The Moment Generating Function of γ_{mrc} is the product of L moment generating functions

$$\begin{aligned}
\Phi(s) &= \frac{1}{\prod_{i=1}^{L}(1 + s\overline{\gamma}_i)} \\
&= \sum_{i=1}^{L} \frac{\eta_i}{1 + s\gamma_i} \\
&= \sum_{i=1}^{L} \frac{\overline{\gamma}_i^{L-1}}{\prod_{k \neq i}(\overline{\gamma}_i - \overline{\gamma}_k)} \frac{1}{1 + s\overline{\gamma}_i}
\end{aligned} \tag{6.101}$$

Taking the inverse Laplace Transform:

$$f_\gamma(\gamma) = \sum_{i=1}^{L} \frac{\overline{\gamma}_i^{L-2}}{\prod_{k \neq i}(\overline{\gamma}_i - \overline{\gamma}_k)} e^{-\frac{\gamma}{\overline{\gamma}_i}} \tag{6.102}$$

The outage probability (cdf) is

$$P_{out} = F_\gamma(\gamma_{th}) = \sum_{i=1}^{L} \frac{\overline{\gamma}_i^{L-1}}{\prod_{k \neq i}(\overline{\gamma}_i - \overline{\gamma}_k)} \left(1 - e^{-\frac{\gamma_{th}}{\overline{\gamma}_i}}\right) \tag{6.103}$$

The mean SNR is

$$
\begin{aligned}
\overline{\gamma}_{mrc} &= \int_0^\infty \gamma \left(\sum_{i=1}^{L} \frac{\overline{\gamma}_i^{L-2}}{\prod_{k \neq i}(\overline{\gamma}_i - \overline{\gamma}_k)} e^{-\frac{\gamma}{\overline{\gamma}_i}} \right) d\gamma \\
&= \sum_{i=1}^{L} \frac{\overline{\gamma}_i^{L-2}}{\prod_{k \neq i}(\overline{\gamma}_i - \overline{\gamma}_k)} \int_0^\infty \gamma e^{-\frac{\gamma}{\overline{\gamma}_i}} d\gamma \\
&= \sum_{i=1}^{L} \frac{\overline{\gamma}_i^{L-1}}{\prod_{k \neq i}(\overline{\gamma}_i - \overline{\gamma}_k)} \underbrace{\int_0^\infty \frac{\gamma}{\overline{\gamma}_i} e^{-\frac{\gamma}{\overline{\gamma}_i}} d\gamma}_{=\overline{\gamma}_i} \\
&= \sum_{i=1}^{L} \frac{\overline{\gamma}_i^{L}}{\prod_{k \neq i}(\overline{\gamma}_i - \overline{\gamma}_k)}
\end{aligned}
$$

$$(6.104)$$

The probability of bit error for BPSK can be found as

$$
\begin{aligned}
P_b &= \int_0^\infty f_\gamma(\gamma) Q(\sqrt{(2\gamma)}) d\gamma \\
&= \int_0^\infty \sum_{i=1}^{L} \frac{\overline{\gamma}_i^{L-2}}{\prod_{k \neq i}(\overline{\gamma}_i - \overline{\gamma}_k)} e^{-\frac{\gamma}{\overline{\gamma}_i}} Q(\sqrt{(2\gamma)}) d\gamma \\
&= \sum_{i=1}^{L} \frac{\overline{\gamma}_i^{L-2}}{\prod_{k \neq i}(\overline{\gamma}_i - \overline{\gamma}_k)} \int_0^\infty e^{-\frac{\gamma}{\overline{\gamma}_i}} Q(\sqrt{(2\gamma)}) d\gamma \\
&= \sum_{i=1}^{L} \frac{\overline{\gamma}_i^{L-1}}{\prod_{k \neq i}(\overline{\gamma}_i - \overline{\gamma}_k)} \underbrace{\int_0^\infty \frac{1}{\overline{\gamma}_i} e^{-\frac{\gamma}{\overline{\gamma}_i}} Q(\sqrt{(2\gamma)}) d\gamma}_{\frac{1}{2}\left(1 - \sqrt{\frac{\overline{\gamma}_i}{1+\overline{\gamma}_i}}\right)} \\
&= \frac{1}{2} \sum_{i=1}^{L} \frac{\overline{\gamma}_i^{L-1}}{\prod_{k \neq i}(\overline{\gamma}_i - \overline{\gamma}_k)} \left(1 - \sqrt{\frac{\overline{\gamma}_i}{1+\overline{\gamma}_i}} \right)
\end{aligned}
$$

$$(6.105)$$

6.8   Diversity Schemes Compared

In the previous section we compared three different diversity combining techniques: selection, equal gain combining, and maximal ratio combining. All three are fairly similar in terms of the SNR improvement. More specifically, as shown in Tables 6.1 and 6.2, all three combining techniques achieve the same diversity gain and thus show the same slope in log-log plot of BER vs SNR per bit. On the other hand, the three techniques do differ in terms of the mean SNR improvement. Selection combining provides the least improvement followed by equal gain combining with MRC providing the largest improvement. For a small number of antennas, the difference is relatively minor, but as the number of antennas (more generally diversity branches) grows, the difference becomes much larger as seen in Figure **??**. The resulting benefit in terms of the bit error rate is shown in Figure fig:bercomp. Note that the BER is plotted versus SNR per bit $\gamma_b = \frac{E_b}{N_o}$ as opposed to SNR per branch γ_c. If the results were plotted versus SNR per branch (which includes the average SNR improvement, the benefits of MRC would have been larger. Also, note that the performance of equal gain combining falls between the other two techniques.

Table 6.1: General Relationship between Diversity Schemes

	Average SNR Gain	Diversity Order
Selection Combining	$\sum_{k=1}^{L} \frac{1}{k}$	L
Equal Gain Combining	$2\left(\frac{(2L-1)!}{(L-1)!2^{2L-1}}\right)^{\frac{1}{L}}$	L
Maximal Ratio Combining	L	L

Table 6.2: Example with $L = 4$

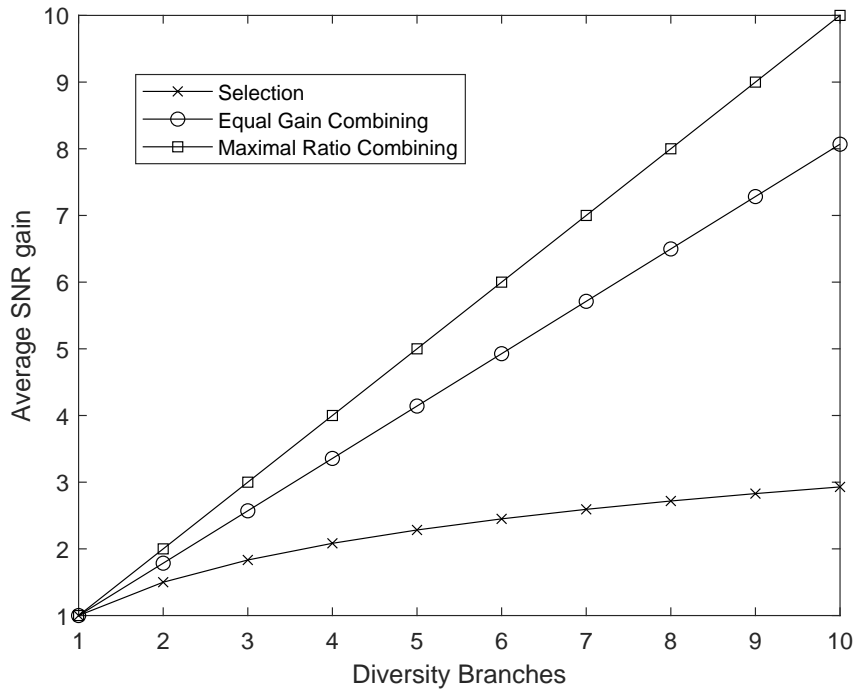|	Average SNR Gain	Diversity Order
Selection Combining	2.08	4
Equal Gain Combining	3.2	4
Maximal Ratio Combining	4	4

Figure 6.9: Comparison of the Average SNR Gains for Different Diversity Combining Techniques
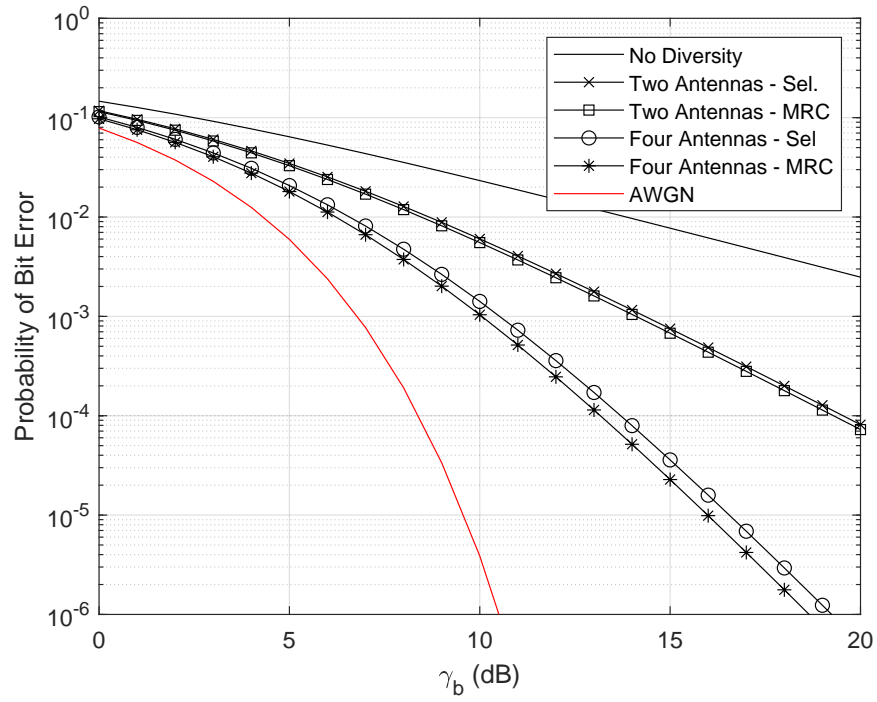
Figure 6.10: Comparison of the Probability of Bit Error (BPSK) in Rayleigh Fading with Selection Diversity and Maximal Ratio Combining

6.9   Generalized Selection Combining

Let $\gamma_1, \gamma_2, \ldots \gamma_L$ be the SNR on L channels. For any instantiation let's redefine them as $g_1, g_2, \ldots g_L$ such that $g_1 > g_2 > \ldots g_L$. Now further, we select the first L_c values: $g_1, g_2, \ldots g_{L_c}$. If we combine these via MRC, we have an SNR

$$\gamma_{gsc} = \sum_{i=1}^{L_c} g_i \tag{6.106}$$

It can be shown that for iid γ_i, the joint pdf can be written as

$$f_{g_1,g_2,\ldots g_{L_c}}(g_1, g_2, \ldots g_{L_c}) = L_c! \begin{pmatrix} L \\ L_c \end{pmatrix} [F_\gamma(g_{L_c})]^{L-L_c} \prod_{i=1}^{L_c} f_\gamma(g_i) \tag{6.107}$$

where $f_\gamma(\gamma)$ and $F_\gamma(\gamma)$ are the pdf and cdf of the SNR per branch. Now, to find the pdf of γ_{gsc} we can use the moment generating function approach (as before):

$$
\begin{aligned}
\Phi(s) &= \mathrm{E}_{\gamma_{gsc}} \left\{ e^{s\gamma_{gsc}} \right\} \\
&= \mathrm{E}_{\gamma_1,\gamma_2,\ldots\gamma_L} \left\{ e^{s \sum_{i=1}^{L_c} g_i} \right\} \\
&= \underbrace{\int_0^\infty \int_{g_{L_c}}^\infty \int_{g_{L_c-1}}^\infty \cdots \int_{g_2}^\infty f_{g_1,g_2,\ldots g_{L_c}}(g_1, g_2, \ldots g_{L_c}) e^{s \sum_{i=1}^{L_c} g_i} dg_1 dg_2 \ldots dg_{L_c}}_{L_c - fold}
\end{aligned}
$$

Unfortunately, this isn't separable as in the case of MRC (since g_i are not independent). Trick: we will define

$$
\begin{aligned}
x_k &\triangleq g_k - g_{k+1} \\
x_L &\triangleq g_L
\end{aligned}
$$

It can be shown that the x_i are independent with exponential distribution:

$$f_{x_i}(x) = \tfrac{1}{\gamma} e^{-\frac{ix}{\gamma}} \quad x > 0 \quad i = 1, 2 \ldots L \tag{6.108}$$

Now,

$$
\begin{aligned}
\gamma_{gsc} &= \sum_{i=1}^{L_c} g_i \\
&= \sum_{i=1}^{L_c} \sum_{k=i}^{L} x_k
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\Phi_{\gamma_{gsc}}(s) &= \underbrace{\int_0^\infty \int_0^\infty \int_0^\infty \cdots \int_0^\infty f_{x_1,x_2,\ldots x_L}(x_1, x_2, \ldots x_L) e^{s \sum_{i=1}^{L_c} \sum_{k=i}^{L} x_i} dx_1 dx_2 \ldots dx_L}_{L-fold} \\
&= \underbrace{\int_0^\infty \int_0^\infty \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{L} \frac{1}{\gamma} e^{-\frac{ix_i}{\gamma}} e^{s \sum_{i=1}^{L_c} \sum_{k=i}^{L} x_i} dx_1 dx_2 \ldots dx_L}_{L-fold}
\end{aligned}
$$

We can write

$$\sum_{i=1}^{L_c} \sum_{k=i}^{L} x_i = x_1 + 2x_2 + 3x_3 + \ldots L_c x_{L_c} + L_c x_{L_c+1} + L_c x_{L_c+2} + \ldots L_c x_L \tag{6.109}$$

Thus, we have

$$\begin{aligned}
\Phi_{\gamma_{gsc}}(s) &= \int_0^\infty e^{sx_1} \frac{1}{\overline{\gamma}} e^{-\frac{x_1}{\overline{\gamma}}} dx_1 \int_0^\infty e^{2sx_2} \frac{1}{\overline{\gamma}} e^{-\frac{2x_2}{\overline{\gamma}}} dx_2 + \ldots \int_0^\infty e^{L_c sx_L} \frac{1}{\overline{\gamma}} e^{-L\frac{x_L}{\overline{\gamma}}} dx_L \\
&= (1 - s\overline{\gamma})^{-L_c+1} \prod_{k=L_c}^{L} (1 - s\overline{\gamma}L_c/k)^{-1}
\end{aligned}$$

Through partial fraction expansion,

$$\Phi_{\gamma_{gsc}}(s) = (1 - s\overline{\gamma})^{-L_c+1} \sum_{k=0}^{L-L_c} (-1)^k \begin{pmatrix} L \\ L_c \end{pmatrix} \begin{pmatrix} L-L_c \\ k \end{pmatrix} \frac{1}{1 + k/L_c - s\overline{\gamma}} \tag{6.110}$$

Using inverse Laplace Transform identities we arrive at

$$\begin{aligned}
f_{\gamma_{gsc}}(\gamma) &= \begin{pmatrix} L \\ L_c \end{pmatrix} \left[\frac{\gamma^{L_c-1} e^{-\gamma/\overline{\gamma}}}{\overline{\gamma}^{L_c}(L_c - 1)!} + \ldots \right. \\
&\qquad \left. \frac{1}{\overline{\gamma}} \sum_{k=1}^{L-L_c} (-1)^{L_c+k-1} \begin{pmatrix} L-L_c \\ k \end{pmatrix} \left(\frac{L_c}{k} \right)^{L_c-1} e^{-\gamma/\overline{\gamma}} \left(e^{-k\gamma/L_c\overline{\gamma}} - \sum_{m=0}^{L_c-2} \frac{1}{m!} \left(-\frac{k\gamma}{L_c\overline{\gamma}} \right)^m \right) \right]
\end{aligned}$$

We can check this and show that it reduces properly for the cases of $L_c = L$ or $L_c = 1$.

6.9.1   Outage Probability

$$\begin{aligned}
F_{\gamma_{gsc}}(\gamma) &= \begin{pmatrix} L \\ L_c \end{pmatrix} \left[1 - e^{-\gamma/\overline{\gamma}} \sum_{k=0}^{L_c-1} \frac{1}{k!} \left(\frac{\gamma_{th}}{\overline{\gamma}} \right)^k + \sum_{k=1}^{L-L_c} (-1)^{L_c+k-1} \begin{pmatrix} L0L_c \\ k \end{pmatrix} \left(\frac{L_c}{k} \right)^{L_c-1} \right. \\
&\qquad \left. \times \left(\frac{1 - e^{-(1+k/L_c)(\gamma_{th}/\overline{\gamma})}}{1 + k/L_c} - \sum_{m=0}^{L_c-2} \left(\frac{-k}{L_c} \right)^m \left(1 - e^{-\gamma_{th}/\overline{\gamma}} \sum_{n=0}^{m} \frac{1}{n!} \left(\frac{\gamma_{th}}{\overline{\gamma}} \right)^n \right) \right) \right] \tag{6.111}
\end{aligned}$$

6.9.2   Mean SNR Analysis

The mean SNR of generalized selection combining is

$$\begin{aligned}
\overline{\gamma}_{gsc} &= L_c\overline{\gamma} + \sum_{k=L_c+1}^{L} \frac{L_c\overline{\gamma}}{k} \\
&= L_c\overline{\gamma} \left(1 + \sum_{k=L_c+1}^{L} \frac{1}{k} \right)
\end{aligned}$$

The improvement in average SNR is

$$\frac{\overline{\gamma}_{gsc}}{\overline{\gamma}} = L_c \left(1 + \sum_{k=L_c+1}^{L} \frac{1}{k} \right)$$

Checking the extreme cases: $L_c = L$ (MRC)

$$\frac{\overline{\gamma}_{gsc}}{\overline{\gamma}} = L$$

Also for $L_c = 1$ (selection combining):

$$\frac{\overline{\gamma}_{gsc}}{\overline{\gamma}} = \left(1 + \sum_{k=2}^{L}\frac{1}{k}\right) = \left(\sum_{k=1}^{L}\frac{1}{k}\right)$$

6.10 Nakagami Fading

Consider the amplitude distribution

$$f_\alpha(\alpha) = \frac{2}{\Gamma(m)}\left(\frac{m}{2\sigma^2}\right)^m \alpha^{2m-1}e^{-m\alpha^2/2\sigma^2} \tag{6.112}$$

Note that $\mathrm{E}\left\{\alpha^2\right\} = 2\sigma^2$. Note that $m = 1 \rightarrow$ Rayleigh. Again, if we let $\gamma = \alpha^2\frac{E_s}{N_o}$ and $\sigma^2 = 1/2$, what is the distribution of γ? Using the transformation of random variables described above we obtain

$$
\begin{aligned}
f_\gamma(\gamma) &= \frac{1}{2E_s/N_o\sqrt{\gamma/(E_s/N_o)}}\frac{2}{\Gamma(m)}\left(\frac{m}{2\sigma^2}\right)^m\left(\frac{\gamma}{E_s/N_o}\right)^{\frac{2m-1}{2}}e^{-\frac{m\gamma}{2\sigma^2 E_s/N_o}} \\
&= \frac{1}{\Gamma(m)E_s/N_o}\left(\frac{m}{2\sigma^2}\right)^m\left(\frac{\gamma}{E_s/N_o}\right)^{m-1}e^{-\frac{m\gamma}{2\sigma^2 E_s/N_o}}
\end{aligned}
\tag{6.113}
$$

Letting $\overline{\gamma} = \frac{E_s}{N_o}$ and $\sigma^2 = 1/2$ we have

$$\boxed{f_\gamma(\gamma) = \frac{1}{\Gamma(m)}\frac{m^m}{\overline{\gamma}^m}\gamma^{m-1}e^{-m\gamma/\overline{\gamma}}} \tag{6.114}$$

which is a Gamma Distribution. If $m = N/2$ for integer N, this is a Chi-Square distribution. In this case, Nakagami behaves like m-fold diversity. If $N = 2$, $m = 1$, Nakagami fading reduces to Rayleigh fading.

6.10.1 Outage Probability

The outage probability in Nakagami fading can be found as

$$
\begin{aligned}
F_\gamma(\gamma) &= \int_0^\gamma \frac{1}{\Gamma(m)}\frac{m^m}{\overline{\gamma}^m}x^{m-1}e^{-mx/\overline{\gamma}}dx \\
&= \frac{1}{\Gamma(m)}\int_0^{m\gamma/\overline{\gamma}} y^{m-1}e^{-y}dy \\
&= \frac{\tilde{\gamma}\left(m,\frac{m\gamma}{\overline{\gamma}}\right)}{\Gamma(m)}
\end{aligned}
\tag{6.115}
$$

An example of the outage probability for SNR in Nakagami fading is given in Figure 6.11 for $m = 1, 2, 4, 10$.
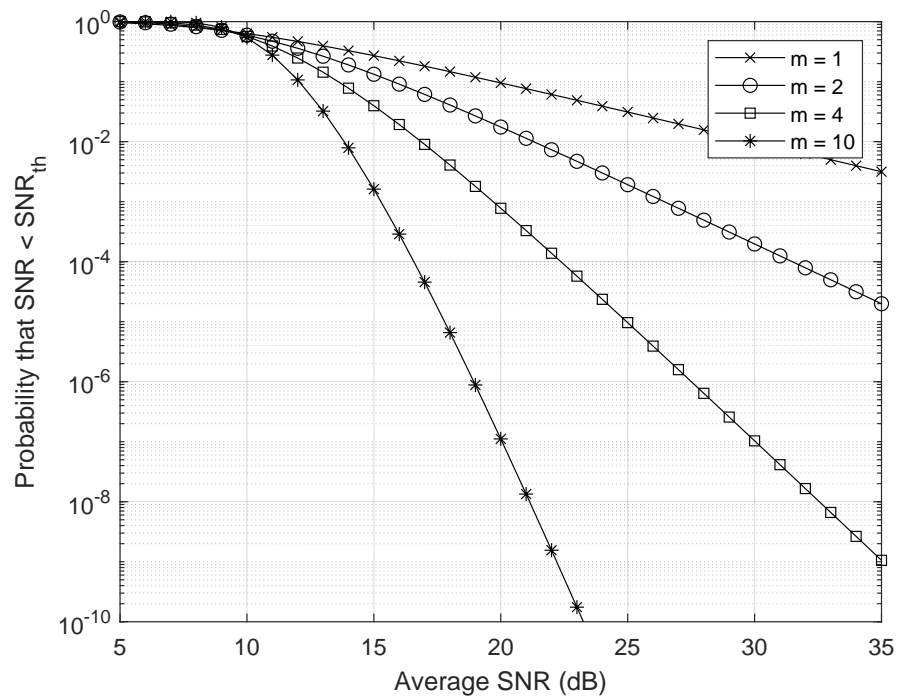
Figure 6.11: Outage Probability of Nakagami Fading for Various m Factors

6.11 MRC with Nakagami Fading

For MRC we know that

$$\gamma_{mrc} = \sum_{i=1}^{L} \gamma_i \tag{6.116}$$

If the γ_i are Gamma distributed and iid, then

$$f_\gamma(\gamma) = \frac{1}{\Gamma(m)} \frac{m^m}{\overline{\gamma}^m} \gamma^{m-1} e^{-m\gamma/\overline{\gamma}} \tag{6.117}$$

Thus, the moment generating function is

$$\Phi_\gamma(s) = \left(\frac{m/\overline{\gamma}}{m/\overline{\gamma} - s} \right)^m \tag{6.118}$$

Now, for MRC of L diversity branches the moment generating function is

$$
\begin{aligned}
\Phi_{\gamma_{mrc}}(s) &= \prod_{i=1}^{L} \Phi_{\gamma_i}(s) \\
&= \prod_{i=1}^{L} \left(\frac{m/\overline{\gamma}}{m/\overline{\gamma} - s} \right)^m \\
&= \left(\frac{m/\overline{\gamma}}{m/\overline{\gamma} - s} \right)^{mL}
\end{aligned} \tag{6.119}
$$

Note that this is the moment generating function of a Gamma random variable with $a = m/\overline{\gamma}$ and $b = mL$. Thus,

$$\boxed{f_{\gamma_{mrc}}(\gamma) = \frac{1}{\Gamma(mL)} \left(\frac{m}{\overline{\gamma}} \right)^{mL} \gamma^{mL-1} e^{-m\gamma/\overline{\gamma}}} \tag{6.120}$$

The cdf is

$$
\begin{aligned}
F_{\gamma_{mrc}}(\gamma) &= \int_0^\gamma \frac{1}{\Gamma(mL)} \left(\frac{m}{\overline{\gamma}} \right)^{mL} x^{mL-1} e^{-mx/\overline{\gamma}} dx \\
&= \frac{1}{\Gamma(mL)} \int_0^{m\gamma/\overline{\gamma}} y^{m-1} e^{-y} dy \\
&= \frac{\tilde{\gamma}(mL, m\gamma/\overline{\gamma})}{\Gamma(mL)}
\end{aligned} \tag{6.121}
$$

where again $\tilde{\gamma}$ is the incomplete gamma function. Examples of the probability of outage are given in Figures 6.12 and 6.13 for $m = 2$ and $m = 10$ respectively. We can see both from the pdf and the outage probability figure that the m-factor has the same impact as the diversity order.
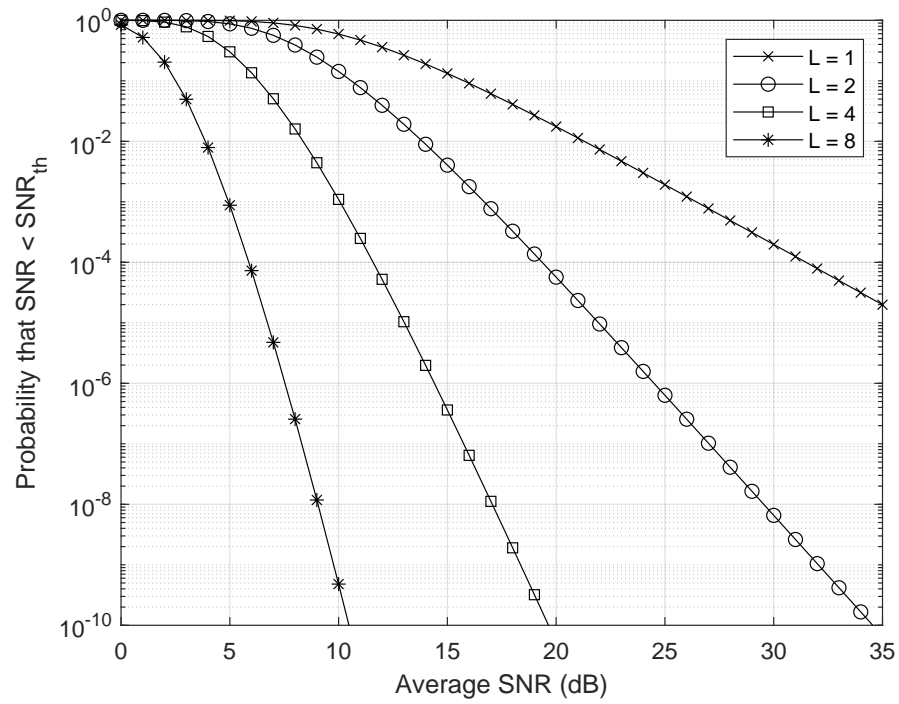
Figure 6.12: Outage Probability of Nakagami Fading for $m = 2$ and Various Numbers of Diversity Branches L
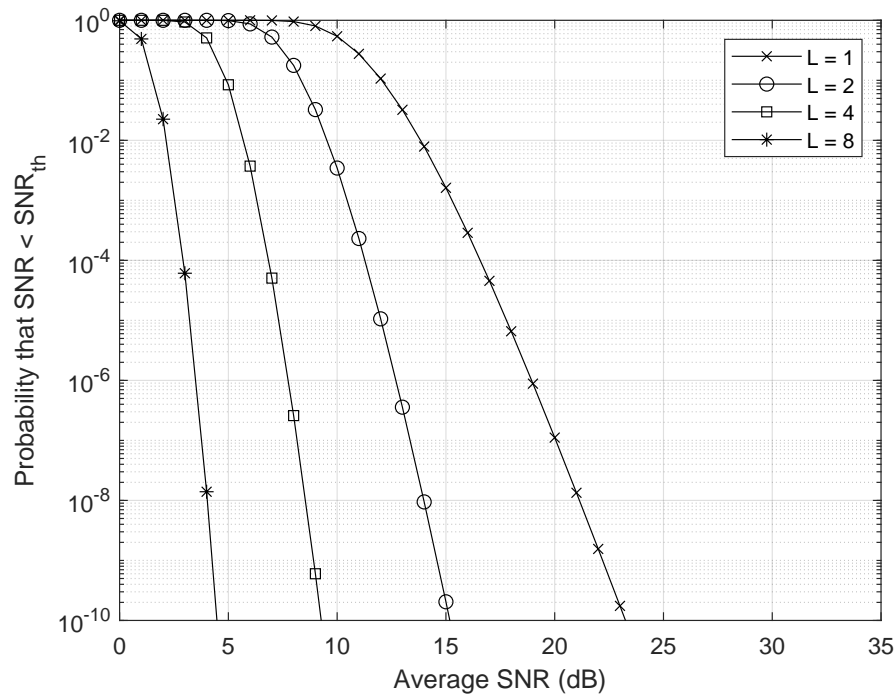
Figure 6.13: Outage Probability of Nakagami Fading for $m = 10$ and Various Numbers of Diversity Branches L

6.11.1 BER of BPSK in Nakagami Fading

It is shown in [11, 10] that the BER of BPSK in Nakagami fading can be written as

$$P_b = \frac{1}{2} \sqrt{\frac{\overline{\gamma}/m}{\pi(1+\overline{\gamma}/m)}} \frac{\Gamma(Lm+1/2)}{Lm+1} \left(\frac{1}{1+\overline{\gamma}/m}\right)^{Lm} {}_2F_1\left(1, Lm+1/2; Lm+1, \frac{1}{1+\overline{\gamma}/m}\right)$$
(6.122)

where ${}_2F_1(a, b; c, z)$ is the Guassian hypergeometric formula. If Lm is an integer, SNR distribution simplifies to a Chi-Square distribution and the bit error rate simplifies to

$$P_b = \frac{1}{2}\left[1 - \sqrt{\frac{\overline{\gamma}/m}{1+\overline{\gamma}/m}} \sum_{k=0}^{Lm-1} \binom{2k}{k} \left(\frac{1}{4(1+\overline{\gamma}/m)}\right)\right]$$
(6.123)

which is essentially the BER of MRC in Rayleigh fading with Lm degrees of freedom and an SNR per branch of $\overline{\gamma}/m$. Expample plots for $L = M_r = 1$ (no diversity) and $m = 1, 2, 4, 10$ are given in Figure 6.14. Note that $m = 1$ corresponds to Rayleigh fading and thus agrees with the plots in previous figures. The error rate improvement with $L = M_r = 2$ is shown in Figure 6.15. Again, note that the improvement due to diversity order and the improvement due to m factor (less severe fading) are analogous. Also note that in order to focus on the diversity benefit (as opposed to the average SNR improvement due to multiple antennas) the error rate is plotted against the SNR per bit $\gamma_b = \frac{E_b}{N_o} = \frac{\overline{\gamma}}{L}$. The performance for $L = M_r = 4$ is given in Figure 6.16.
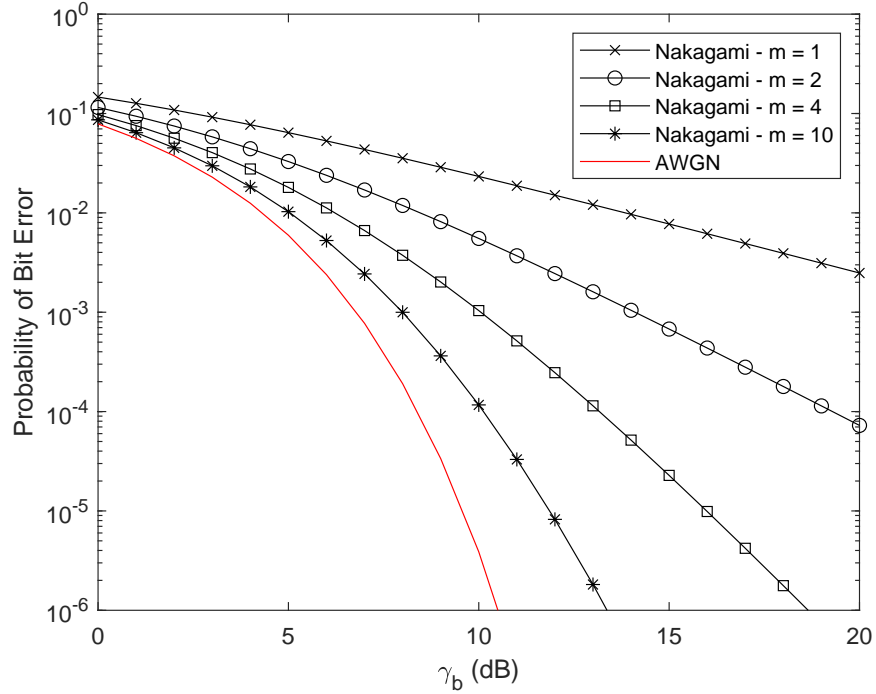

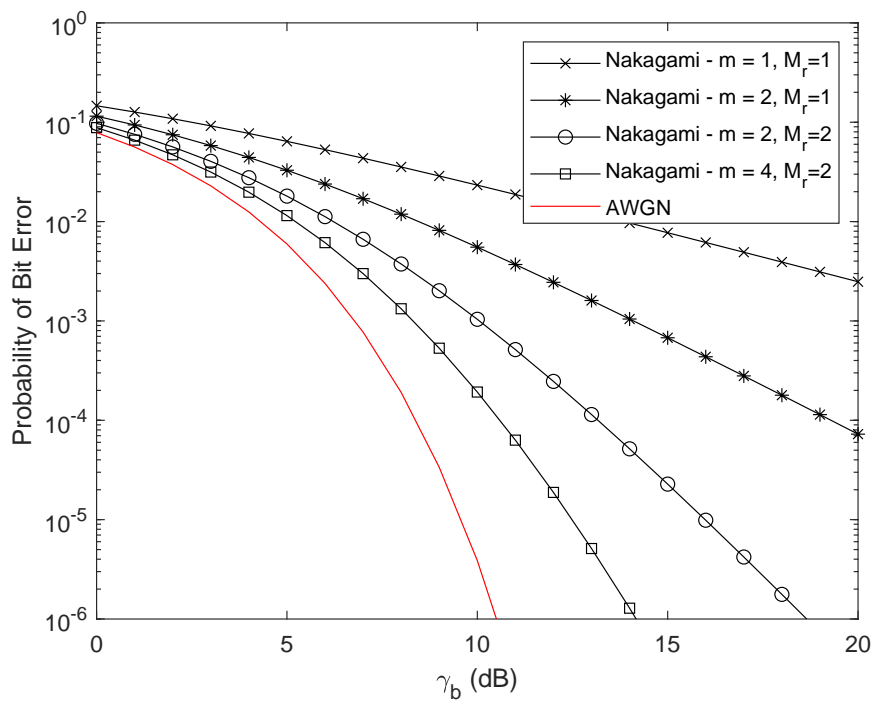Figure 6.14: Bit Error Rate of BPSK in Nakagami Fading for Various m Factors

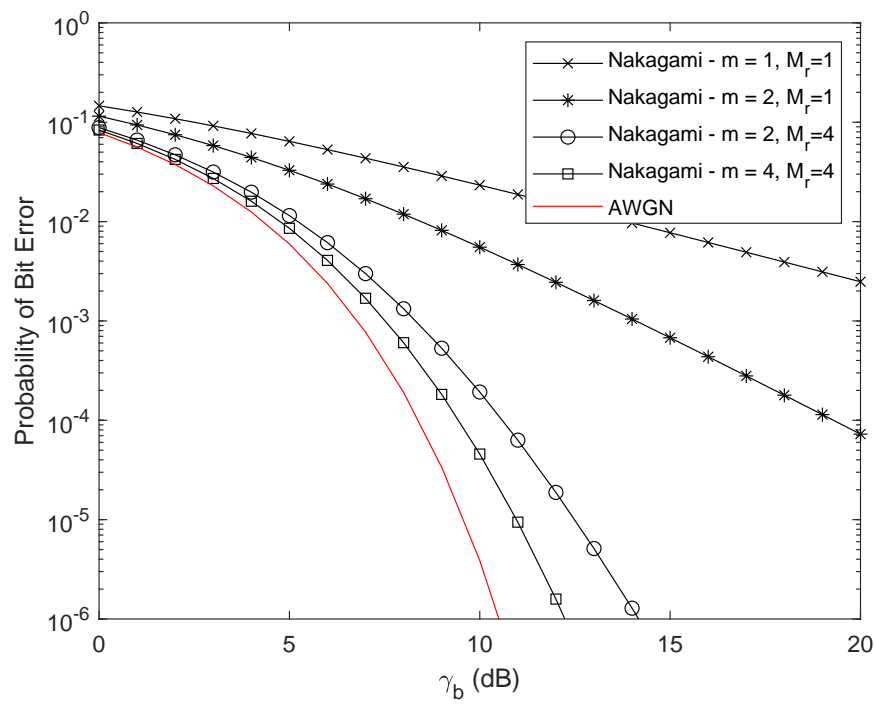Figure 6.15: Bit Error Rate of BPSK in Nakagami Fading with MRC ($M_r = 2$) for Various m Factors

Figure 6.16: Bit Error Rate of BPSK in Nakagami Fading with MRC ($M_r = 4$) for Various m Factors

6.12 Correlated Branches with MRC in Rayleigh Fading

Up to this point we have assumed that all antennas experience independent fading. If fading on each antenna is correlated, we expect that the benefit of diversity will be diminished. To see this, consider MRC of L-branch diversity. The total received power $P = \alpha^2$ can be written as

$$
\begin{aligned}
P_t &= \sum_{i=1}^{L} \alpha_i^2 \\
&= \sum_{i=1}^{L} (X_i^2 + Y_i^2)
\end{aligned}
$$

where X_i and Y_i are the real and imaginary parts of the fading channel. For Rayleigh fading, X_i and Y_i are $GRV(0, \frac{P_i}{2})$. We can then group all $2L$ GRV into a single vector

$$
\mathbf{x} = [X_1 Y_1 X_2 Y_2 \ldots X_L Y_L] \tag{6.124}
$$

The joint distribution function $f_{\mathbf{X}}(\mathbf{x})$ can be written as

$$
f_{\mathbf{X}}(\mathbf{x}) = (2\pi)^{-L} |\mathbf{R}|^{-1/2} exp\left(-\frac{1}{2}\mathbf{x}\mathbf{R}^{-1}\mathbf{x}^T\right) \tag{6.125}
$$

where $\mathbf{R}$ is the covariance matrix

$$
\mathbf{R} = \begin{bmatrix} A_1 & B_{12} & B_{13} & \cdots & B_{1L} \\ B_{12}^T & A_2 & & & \vdots \\ \vdots & & & \vdots & \\ B_{1L}^T & \cdots & \cdots & \cdots & A_L \end{bmatrix} \tag{6.126}
$$

where

$$
A_i = \begin{bmatrix} \frac{1}{2}\overline{P}_i & 0 \\ 0 & \frac{1}{2}\overline{P}_i \end{bmatrix} \tag{6.127}
$$

$$
B_{ij} = \begin{bmatrix} \rho_{ij} & \beta_{ij} \\ -\beta_{ij} & \rho_{ij} \end{bmatrix} \tag{6.128}
$$

$\rho_{ij} = \mathrm{E}\{X_i X_j\}$ and $\beta_{ij} = \mathrm{E}\{X_i Y_j\}$ and $|\mathbf{R}|$ is the determinant of $\mathbf{R}$.

Now, we wish to find the pdf of P_t. We will use the moment generating function approach. To do so, we can write P_t in terms of $\mathbf{x}$:

$$
P_t = \mathbf{x}\mathbf{I}\mathbf{x}^T \tag{6.129}
$$

Now, the moment generating function can be written as

$$
\begin{aligned}
\Phi_{P_t}(s) &= \int_0^\infty e^{-sp} f_P(p)\,dp \\
&= \mathrm{E}\left\{exp(-s\mathbf{x}\mathbf{I}\mathbf{x}^T)\right\} \\
&= \underbrace{\int_{-\infty}^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty}_{2L-fold} (2\pi)^{-L}|\mathbf{R}|^{-1/2} exp\left(-\frac{1}{2}\mathbf{x}(\mathbf{R}^{-1}+2sI)\mathbf{x}^T\right) dX_1 dY_1 \cdots dX_L dY_L \\
&= |\mathbf{R}|^{-1/2}\left|(\mathbf{R}^{-1}+2sI)^{-1}\right|^{1/2} \times \\
&\qquad \underbrace{\int_{-\infty}^\infty \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty (2\pi)^{-L}|(\mathbf{R}^{-1}+2s\mathbf{I})^{-1}|^{-1/2} exp\left(-\frac{1}{2}\mathbf{x}(\mathbf{R}^{-1}+2s\mathbf{I})\mathbf{x}^T\right) dX_1 dY_1 \cdots dX_L dY_L}_{=1\ (\text{integral of Gaussian pdf})} \\
&= |\mathbf{R}|^{-1/2}\underbrace{\left|(\mathbf{R}^{-1}+2s\mathbf{I})^{-1}\right|^{1/2}}_{|A^{-1}|=|A|^{-1}} \\
&= \underbrace{|\mathbf{R}|^{-1/2}|\mathbf{R}^{-1}+2s\mathbf{I}|^{-1/2}}_{|AB|=|A||B|} \\
&= |\mathbf{I}+2s\mathbf{R}|^{-1/2}
\end{aligned}
\tag{6.130}
$$

Now, it can be shown that the final determinant is a perfect square so that

$$
|\mathbf{I}+2s\mathbf{R}|^{-1/2} = |\mathbf{I}+s\mathbf{\Psi}|^{-1}
\tag{6.131}
$$

where

$$
\mathbf{\Psi} = \begin{bmatrix}
\overline{P}_1 & \rho_{12}-j\beta_{12} & \cdots & \cdots & \rho_{1L}-j\beta_{1L} \\
\rho_{12}+j\beta_{12} & \overline{P}_2 & & & \vdots \\
\vdots & & & \vdots & \\
\rho_{1L}+j\beta_{1L} & \cdots & \cdots & \cdots & \overline{P}_L
\end{bmatrix}
\tag{6.132}
$$

Thus, we can write

$$
\begin{aligned}
\Phi(s) &= |\mathbf{I}+s\mathbf{\Psi}|^{-1} \\
&= \frac{1}{\prod_{i=1}^L (1+s\lambda_i)}
\end{aligned}
$$

where λ_i is the ith eigenvalue of Ψ. If all eigenvalues are distinct, we can write (using partial fraction expansion)

$$
\frac{1}{\prod_{i=1}^L (1+s\lambda_i)} = \sum_{i=1}^L \frac{\eta_i}{1+s\lambda_i}
\tag{6.133}
$$

where

$$
\eta_i = \frac{\lambda_i^{L-1}}{\prod_{k\neq i}(\lambda_i - \lambda_k)}
\tag{6.134}
$$

Thus, making this substitution

$$\Phi(s) = \sum_{i=1}^{L} \frac{\lambda_i^{L-1}}{\prod_{k \neq i}(\lambda_i - \lambda_k)} \frac{1}{1 + s\lambda_i} \tag{6.135}$$

$$= \sum_{i=1}^{L} \frac{\lambda_i^{L-2}}{\prod_{k \neq i}(\lambda_i - \lambda_k)} \frac{1}{s + 1/\lambda_i} \tag{6.136}$$

and taking the inverse Laplace transform we have

$$f_{P_t}(p) = \sum_{i=1}^{L} \frac{\lambda_i^{L-2}}{\prod_{k \neq i}(\lambda_i - \lambda_k)} e^{-p/\lambda_i} \tag{6.137}$$

Now, we can find the distribution of the combined SNR γ_{MRC} assuming that the noise power is constant (our usual assumption). Additionally, we assume that the average received signal power on each antenna is the same: $\overline{P}_1 = \overline{P}_2 \ldots = \overline{P}_L$. In other words $\overline{\gamma}_1 = \overline{gamma}_2 \ldots = \overline{\gamma}$. Let's define a matrix $\boldsymbol{\Lambda}$

$$\boldsymbol{\Lambda} = \frac{1}{\overline{P}_1} \boldsymbol{\Psi} \tag{6.138}$$

with eigenvalues $\tilde{\lambda}_i$ Then, the distribution of SNR can be easily shown to be

$$\boxed{f_{\gamma_{mrc}}(\gamma) = \sum_{i=1}^{L} \frac{\tilde{\lambda}_i^{L-2}}{\prod_{k \neq i}(\tilde{\lambda}_i - \tilde{\lambda}_k)} \frac{1}{\overline{\gamma}} e^{-\frac{\gamma}{\overline{\gamma}\tilde{\lambda}_i}}} \tag{6.139}$$

Note that the distribution is identical to the case of independent, but non-identical SNRs. Thus, correlated branches will behave like independent SNRs with unequal branch SNRs. Example bit error rates are plotted in Figure 6.17 for $L = M_r = 4$. Note that independent channels and no diversity cases are also plotted for reference. For the correlated cases, we assume that

$$\Lambda = \begin{bmatrix} 1 & \rho & \rho & \rho \\ \rho & 1 & \rho & \rho \\ \rho & \rho & 1 & \rho \\ \rho & \rho & \rho & 1 \end{bmatrix} \tag{6.140}$$

We can see that the effect of correlation is to shift the curves to the right (analogous to an average SNR loss). The slope is not changed, confirming that there is no loss in diversity order.

6.13 Transmit Diversity - Concept

The previous sections examine the use of multiple antennas at the receiver to provide diversity gain in fading channels. Multiple antennas can also be used at the transmitter to achieve diversity, although it is somewhat different, especially in the absence of channel knowledge at the transmitter. In other words, transmit diversity is analgous to receive diversity when either (a) a time division duplex (TDD) system is employed with a slow fading channel (meaning that received channel information can be used on transmit) or (b) feedback is employed. We will not specifically discuss TDD systems, but the application to TDD should be obvious after reading the chapter. Instead, we will assume FDD and first examine open loop techniques (i.e., techniques which function in
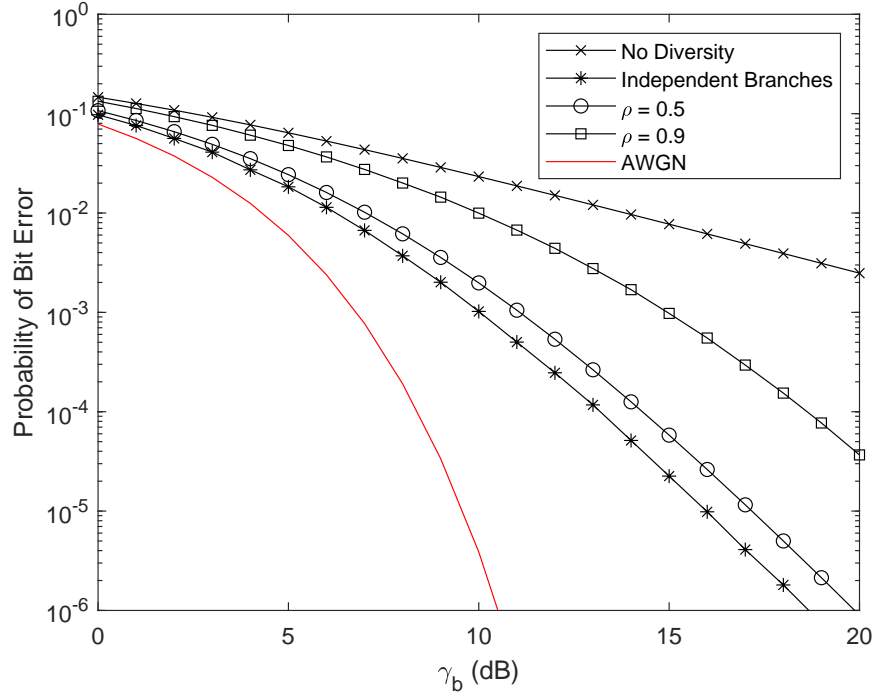
Figure 6.17: Bit Error Rate of BPSK in Correlated Rayleigh Fading for $L = M_r = 4$

the absence of channel knowledge at the transmitter) and then move on to closed-loop techniques (i.e., those techniques which feedback channel information to the transmitter).

The first question that might be asked is when not simply transmit from multiple widely-spaced antennas? To see why this isn't helpful, assume multiple transmit antennas with a single receive antenna and let $\tilde{\alpha}_i(t)$ be the complex channel from the ith antenna which is assumed to be a complex Gaussian random process (Rayleigh fading). Note that $\mathrm{E}\left\{|\tilde{\alpha}_i(t)|^2\right\} = 1$ and that $\mathrm{E}\left\{|\Re\left\{\tilde{\alpha}_i(t)\right\}|^2\right\} = \mathrm{E}\left\{|\Im\left\{\tilde{\alpha}_i(t)\right\}|^2\right\} = 1/2$. Let the transmit signal $x(t) = \sum_{i=-\infty}^{\infty} s_i p(t - iT)$. If we transmit the same symbols from all antennas, but with $1/N_t$ of the power (keeping the total power the same as the single-antenna case), and assuming a narrowband channel, at the receiver we have

$$
\begin{aligned}
r(t) &= \sqrt{P_r} \sum_{i=1}^{N_t} \sqrt{\frac{1}{N_t}} x(t)\tilde{\alpha}_i(t) + n(t) \\
&= \sqrt{P_r} \left(\underbrace{\sqrt{\frac{1}{N_t}} \sum_{i=1}^{N_t} \tilde{\alpha}_i(t)}_{\tilde{\alpha}_T(t)} \right) x(t) + n(t)
\end{aligned}
$$

where $\tilde{\alpha}_T(t)$ is the sum of complex Gaussian random processes. It is thus another complex Gaus-

sian random process. Further, $\mathrm{E}\left\{|\tilde{\alpha}_T(t)|^2\right\} = 1$ since the channels are independent with zero mean. Thus, the channel is still a Rayleigh fading channel with unit power. No improvement is obtained! Thus, we cannot simply transmit from multiple antennas and hope to obtain diversity benefit. However, there are various open-loop and closed-loop techniques to obtain diversity benefit with multiple transmit antennas, as we will discuss below.

6.14 Open Loop Transmit Diversity Techniques

6.14.1 Antenna Hopping

Let's assume slow fading such that $\tilde{\alpha}_i(t) = \tilde{\alpha}_i$. The received signal in this case is

$$r(t) = \sum_{i=-\infty}^{\infty} \sqrt{P_r} s_i p(t - iT) \tilde{\alpha}_{\lfloor i/N_t \rfloor} + n(t) \tag{6.141}$$

where P_r is the average received power (assumed to be the same on both channels) and $\mathrm{E}\left\{|\tilde{\alpha}_i(t)|^2\right\} = 1$. Let's further assume that all channels are independent. Without any coding, the decision metric (after matched filtering) can be written as

$$z[\nu] = \sqrt{P_r} s_\nu |\tilde{\alpha}_{\lfloor i/N_t \rfloor}|^2 + n[\nu]\tilde{\alpha}^*_{\lfloor i/N_t \rfloor} \tag{6.142}$$

. The bit error rate for a specific instantiation of $\tilde{\alpha}$ is

$$P_b = \mathrm{Q}\left(\sqrt{2\frac{\mathrm{E_b}}{\mathrm{N_o}}|\tilde{\alpha}|^2}\right) \tag{6.143}$$

Since each symbol has a potentially different channel, the average BER is the same as in single-channel Rayleigh fading. If BPSK is used:

$$P_b \approx \frac{1}{4E_b/N_o} \tag{6.144}$$

where E_b/N_o is the average energy per bit divided by the noise power spectral density. Now, let us try using a simple rate $1/N_t$ repeat code. In this case, the decision variable (coherent combination of N_t consecutive received symbols) is

$$
\begin{aligned}
z\left[\lfloor \nu/N_t \rfloor\right] &= \sum_{i=1}^{N_t} \left(\sqrt{P_r/N_t}\tilde{\alpha}_i s_{\lfloor \nu/N_t \rfloor} + n[\lfloor \nu/N_t \rfloor + i]\tilde{\alpha}_i^*\right) \\
&= \sqrt{\frac{P_r}{N_t}} s_{\lfloor \nu/N_t \rfloor} \sum_{i=1}^{N_t} |\tilde{\alpha}_i|^2 + \sum_{i=1}^{N_t} \tilde{\alpha}_i^* n[\lfloor \nu/N_t \rfloor + i]
\end{aligned}
\tag{6.145}
$$

$$P_b = \mathrm{Q}\left(\sqrt{\frac{2}{\mathrm{N_t}}\frac{\mathrm{E_b}}{\mathrm{N_o}}\sum_{i=1}^{N_t}|\tilde{\alpha}_i|^2}\right) \tag{6.146}$$

To determine the average error rate we recognize that

$$\eta = \frac{1}{N_t}\sum_{i=1}^{N_t}|\tilde{\alpha}_i|^2 \tag{6.147}$$

is a Chi Square R.V. with N_t degrees of freedom and average power $N_t/2$ in each of the underlying Gaussian random variables. Thus, we can write the BER as

$$P_b = \left[\left(1 - \sqrt{\frac{\overline{\gamma}_c}{1 + \overline{\gamma}_c}}\right)\right]^{N_t} \sum_{k=0}^{N_t-1} \left(\begin{array}{c} N_t - 1 + k \\ k \end{array}\right) \left[\frac{1}{2}\left(1 + \sqrt{\frac{\overline{\gamma}_c}{1 + \overline{\gamma}_c}}\right)\right]^k \tag{6.148}$$

where $\overline{\gamma}_c = \frac{1}{N_t}\frac{E_b}{N_o}$ and $\frac{E_b}{N_o}$ is the average energy per bit divided by the noise power spectral density. This can be approximated at high average $\frac{E_b}{N_o}$ as

$$P_b \approx \left(\frac{N_t}{4E_b/N_o}\right)^{N_t} \left(\begin{array}{c} 2N_t - 1 \\ N_t \end{array}\right) \tag{6.149}$$

This is identical to the probability of MRC with N_t receive antennas but without the N_t average SNR improvement.

6.14.2   Space Time Transmit Diversity

This is also known as a 2-dimensional Space-Time Block Code (STBC) or the Alamouti Code. During time slot #1, at the output of a matched filter we observe:

$$r_1 = \tilde{\alpha}_1\sqrt{E_s/2}s_1 + \tilde{\alpha}_2\sqrt{E_s/2}s_2 + n_1 \tag{6.150}$$

where n_1 is a GRV. During time slot #2

$$r_2 = -\tilde{\alpha}_1\sqrt{E_s/2}s_2^* + \tilde{\alpha}_2\sqrt{E_s/2}s_1^* + n_2 \tag{6.151}$$

where n_2 is a GRV. The decision metric for symbol one is created as

$$\begin{aligned} Z_1 &= \tilde{\alpha}_1^* r_1 + \tilde{\alpha}_2 r_2^* \\ &= \tilde{\alpha}_1^*\left(\tilde{\alpha}_1\sqrt{E_s/2}s_1 + \tilde{\alpha}_2\sqrt{E_s/2}s_2 + n_1\right) + \ldots \\ &\quad \tilde{\alpha}^2\left(-\tilde{\alpha}_1\sqrt{E_s/2}s_2^* + \tilde{\alpha}_2\sqrt{E_s/2}s_1^* + n_2\right)^* \\ &= \sqrt{E_s/2}\left(|\tilde{\alpha}_1|^2 + |\tilde{\alpha}_2|^2\right)s_1 \\ &\quad \sqrt{E_s/2}\left(\tilde{\alpha}_1^*\tilde{\alpha}_2 - \tilde{\alpha}_1^*\tilde{\alpha}_2\right)s_2 \\ &\quad \tilde{\alpha}_1^* n_1 + \tilde{\alpha}_2 n_2^* \\ &= \underbrace{\sqrt{E_s/2}\left(|\tilde{\alpha}_1|^2 + |\tilde{\alpha}_2|^2\right)s_1}_{\text{desired energy}} + \underbrace{\tilde{n}}_{\text{GRV with variance } \left(|\tilde{\alpha}_1|^2 + |\tilde{\alpha}_2|^2\right)\sigma^2} \tag{6.152} \end{aligned}$$

The per symbol SNR is

$$\gamma = \frac{E_s}{N_o}\frac{\left(|\tilde{\alpha}_1|^2 + |\tilde{\alpha}_2|^2\right)}{2} \tag{6.153}$$

And thus for BPSK (or QPSK) the BER is

$$Q\left(\sqrt{2\frac{E_b}{N_o}\frac{\left(|\tilde{\alpha}_1|^2 + |\tilde{\alpha}_2|^2\right)}{2}}\right) \tag{6.154}$$

Averaged over Rayleigh fading we obtain:

$$
\begin{aligned}
P_b &= \left[\left(1 - \sqrt{\frac{\overline{\gamma}_c}{1 + \overline{\gamma}_c}}\right)\right]^2 \sum_{k=0}^{1} \binom{2 - 1 + k}{k} \left[\frac{1}{2}\left(1 + \sqrt{\frac{\overline{\gamma}_c}{1 + \overline{\gamma}_c}}\right)\right]^k \\
&= \left[\left(1 - \sqrt{\frac{\overline{\gamma}_c}{1 + \overline{\gamma}_c}}\right)\right]^2 \left(1 + \sqrt{\frac{\overline{\gamma}_c}{1 + \overline{\gamma}_c}}\right) \tag{6.155}
\end{aligned}
$$

where $\overline{\gamma}_c = \frac{1}{2}\frac{E_s}{N_o}$. This is the same BER that we would obtain from two-fold receive diversity with MRC. Note that it is easily shown (and we will verify this in the next chapter) that a similar decision metric and bit error rate are obtained for the second symbol.

Now, what if there are multiple antennas? Let's define $\tilde{\alpha}_{1i}$ as the channel between transmit antenna 1 and receive antenna i. Similarly, let r_{1i} and r_{2i} be the matched filter outputs on the ith antenna during time slots 1 and 2 repectively. The decision metric for the first symbol is found as

$$
Z_1 = \sum_{i=1}^{M_r} (\tilde{\alpha}_{1i}^* r_{1i} + \tilde{\alpha}_{2i} r_{2i}^*) \tag{6.156}
$$

The BER would be identical to that of $2M_r$-fold diversity with MRC combining.

6.15 Closed-Loop Transmit Diversity Techniques

6.15.1 Antenna Selection

Assuming that all antennas are monitored (via pilot channels) and the best channel is fed back to the transmitter, the performance is equivalent to selection diversity. Specifically, for BPSK modulation and coherent detection the BER is

$$
P_b = \frac{1}{2}\left[1 - \sum_{k=1}^{N_t}\binom{N_t}{k}(-1)^{k-1}\sqrt{\frac{\overline{\gamma}}{\overline{\gamma} + k}}\right] \tag{6.157}
$$

where $\overline{\gamma} = \frac{E_b}{N_o}$. When bit errors occur, the performance can be approximated that in Rayleigh fading with no diversity[1].

$$
P_{b|e} = \frac{1}{2}\left(1 - \sqrt{\frac{\overline{\gamma}}{\overline{\gamma} + 1}}\right) \tag{6.158}
$$

Thus, the overall probability of bit error is

$$
P_{b,t} = P_b(1 - p) + P_{b|e}p \tag{6.159}
$$

where p is the probability of feedback error.

6.15.2 Generalized Beamforming

Recall that in transmit beamforming, we weight the signals sent from multiple antennas so that they arrive at the mobile with the same phases and thus add coherently. When the channel can be approximated as plane wave propagation, we simply need to know the array geometry and

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[1]Due to the fact that the channel is NOT the maximum SNR channel, the performance would be slightly worse than that in standard Rayleigh fading.

the desired angle of departure to accomplish this. However, in a fading channel with non-zero angle spread, this is not true. We can still, however, co-phase the signals to cause them to sum coherently at the mobile. However, we requite mobile feedback to accomplish this.

Let the channel from the ith antenna be $\tilde{\alpha}_i = \sqrt{P_r}\alpha_i e^{j\phi_i}$ where P_r is the average receive power in the channel. Assume that the mobile can feedback to the base station the measured values ϕ_i. The transmit signal from the ith antenna is

$$x_i(t) = e^{-j\phi_i}\sqrt{1/N_t}s(t) \tag{6.160}$$

where $s(t) = \sum_{i=-\infty}^{\infty} s_i p(t - iT)$. Thus, the received signal at the receiver is

$$r(t) = \sum_{i=1}^{N_t} \tilde{\alpha}_i x_i(t) + n(t) \tag{6.161}$$

$$= \sum_{i=1}^{N_t} \sqrt{P_r}\alpha_i e^{j\phi_i} e^{-j\phi_i}\sqrt{1/N_t}s(t) + n(t) \tag{6.162}$$

$$= \sqrt{P_r/N_t}\left(\sum_{i=1}^{N_t} \alpha_i\right)s(t) + n(t) \tag{6.163}$$

The instantaneous SNR is then

$$SNR = \frac{P_r\left(\sum_{i=1}^{N_t} \alpha_i\right)^2}{N_t\sigma_n^2} \tag{6.164}$$

where σ_n^2 is the variance of the noise signal $n(t)$. It is straightforward to show that the decision variable has the same statistics as equal gain combining with N_t receive antennas. To see this, consider the received signal on the ith receive antenna when there is one transmit antenna and M_r receive antennas:

$$r_i(t) = \sqrt{P_r}\alpha_i e^{j\phi_i}x(t) + n_i(t) \tag{6.165}$$

Now with equal gain combining we have:

$$r_T(t) = \sum_i e^{-j\phi_i}r_i(t) \tag{6.166}$$

$$= \sqrt{P_r}\left(\sum_i \alpha_i\right)x(t) + \sum_i e^{-j\phi}n_i(t) \tag{6.167}$$

$$= \sqrt{P_r}\left(\sum_i \alpha_i\right)x(t) + n_T(t) \tag{6.168}$$

Now, the SNR of this combined signal is then

$$SNR = \frac{P\left(\sum_i \alpha_i\right)^2}{M_r\sigma_n^2} \tag{6.169}$$

If instead of sending back only phase values, we instead send back normalized channel values we could create transmit weights

$$w_i = \frac{\tilde{\alpha}_i^*}{\sqrt{\sum_i |\tilde{\alpha}_i|^2}} \tag{6.170}$$

If we use these weights at the transmitter instead of phase only, we can show that the performance

is equivalent to maximal ratio combining with N_t receive antennas.

Note that although we have termed this "generalized beamforming", there is no beampattern in this case, due to the fact that the channel is not coherent across the antenna array.

6.15.3 Maximal Ratio Transmission (MRT)

Maximal Ratio Transmission is a generalization of the previous idea when there are multiple transmit and receive antennas. Specifically, let's assume that we are transmitting a scalar signal $s(t)$ from N_t antennas and receiving that signal on M_r receive antennas. We wish to find a transmit weight vector $\mathbf{v}$ to premulitply by the transmit signal so that the transmit vector is $\mathbf{s}(t) = \mathbf{v}s(t)$ and a receive vector $\mathbf{w}$ to use for combining the received signals $\mathbf{r}(t)$ on M_r receive antennas. Specifically, ignoring time we have a receive vector each symbol time

$$\mathbf{r} = \mathbf{H}\mathbf{s} + \mathbf{n} \tag{6.171}$$

where $\mathbf{H}$ is the $M_r \times N_t$ channel matrix and $\mathbf{n}$ is the $M_r \times 1$ noise vector. Our decision variable is

$$
\begin{aligned}
z &= \mathbf{w}^H \mathbf{r} & (6.172)\\
&= \mathbf{w}^H (\mathbf{H}\mathbf{s} + \mathbf{n}) & (6.173)\\
&= \mathbf{w}^H \mathbf{H}\mathbf{v}s + \mathbf{w}^H \mathbf{n} & (6.174)
\end{aligned}
$$

Now, we wish to choose $\mathbf{w}$ and $\mathbf{v}$ to maximize SNR. Now, the SNR is

$$
\begin{aligned}
SNR &= \frac{(\mathbf{w^H H v})^2 E_s}{\sum_i |w_i|^2 \sigma_n^2} & (6.175)\\
&= \frac{\mathbf{w}^H \mathbf{H}\mathbf{v}\mathbf{v}^H \mathbf{H}^H \mathbf{w}}{\mathbf{w}^H \mathbf{w}} \frac{E_s}{\sigma_n^2} & (6.176)
\end{aligned}
$$

Note that $\frac{E_s}{\sigma_n^2}$ is the average SNR on a single antenna. Now, by definition MRT assumes that $\mathbf{v}$ is a linear transformation of the channel matrix $\mathbf{H}$:

$$\mathbf{v} = \frac{(\mathbf{g}\mathbf{H})^H}{a} \tag{6.177}$$

where $\mathbf{g}$ is a $1 \times M_r$ vector and $a = |\mathbf{g}\mathbf{H}|$ is a normalizing constant. Thus,

$$
\begin{aligned}
\mathbf{r} &= \mathbf{H}\mathbf{H}^H \mathbf{g}^H \frac{s}{a} + \mathbf{n} & (6.178)\\
z &= \mathbf{w}^H \mathbf{r} & (6.179)\\
&= \mathbf{w}^H \mathbf{H}\mathbf{H}^H \mathbf{g}^H \frac{s}{a} + \mathbf{w}^H \mathbf{n} & (6.180)
\end{aligned}
$$

Now, for a given value of $\mathbf{g}$, the optimal value of $\mathbf{w}$ is $\mathbf{w} = \mathbf{g}^H$. Thus,

$$z = \mathbf{g}\mathbf{H}\mathbf{H}^H \mathbf{g}^H \frac{s}{a} + \mathbf{g}\mathbf{n} \tag{6.181}$$

The resulting SNR is

$$SNR \quad = \quad \frac{(\mathbf{g}\mathbf{H}\mathbf{H}^H\mathbf{g}^H)^2/a^2}{\mathbf{g}\mathbf{g}^H} \frac{E_s}{\sigma_n^2} \tag{6.182}$$

$$= \quad \frac{\mathbf{g}\mathbf{H}\mathbf{H}^H\mathbf{g}^H}{\mathbf{g}\mathbf{g}^H} \frac{E_s}{\sigma_n^2} \tag{6.183}$$

This is known as a Rayleigh quotient. It is well-known that the ratio is maximinzed when $\mathbf{g}$ is chosen as the maximum eigenvector of $\mathbf{H}\mathbf{H}^H$. Further, the maximum SNR achieved is

$$SNR_{max} = \lambda_{max} \frac{E_s}{\sigma_n^2} \tag{6.184}$$

where λ_{max} is the maximum eigenvalue of $\mathbf{H}\mathbf{H}^H$.