Multi-channel Communications Fall 2022

Lecture 18
Introduction to OFDM

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Introduction

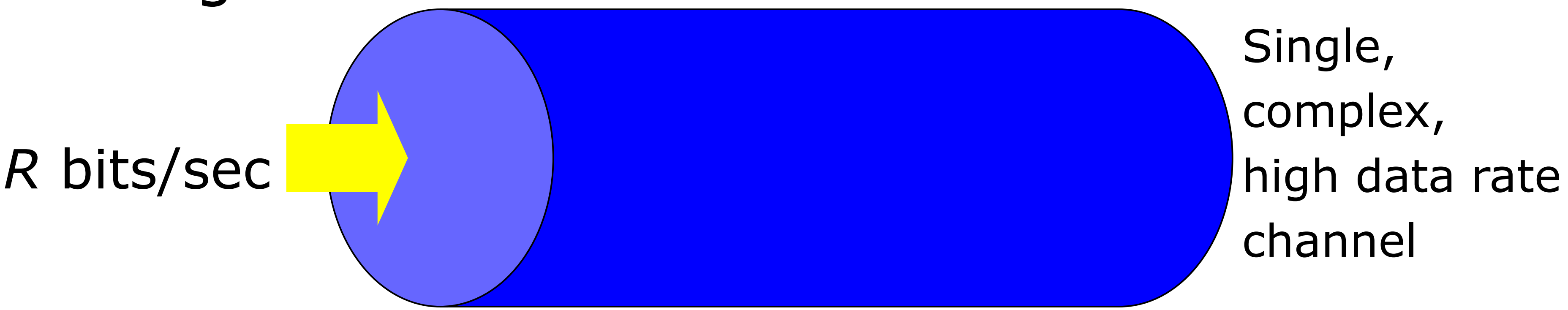
- Today we begin looking at a second multi-channel communications technique titled Orthogonal Frequency Domain Multiplexing or OFDM
- In OFDM multiple channels are created for a single link in the frequency domain
  - Frequency band is divided into *orthogonal* channels for transmission
- Key Concepts in OFDM
  - Use of Orthogonal carriers to obtain good bandwidth efficiency
  - The use of the FFT for efficient channel modulation
  - Individually modulating carriers and coding across carriers allows both efficient bandwidth utilization and good performance in frequency selective fading channels

What is OFDM?

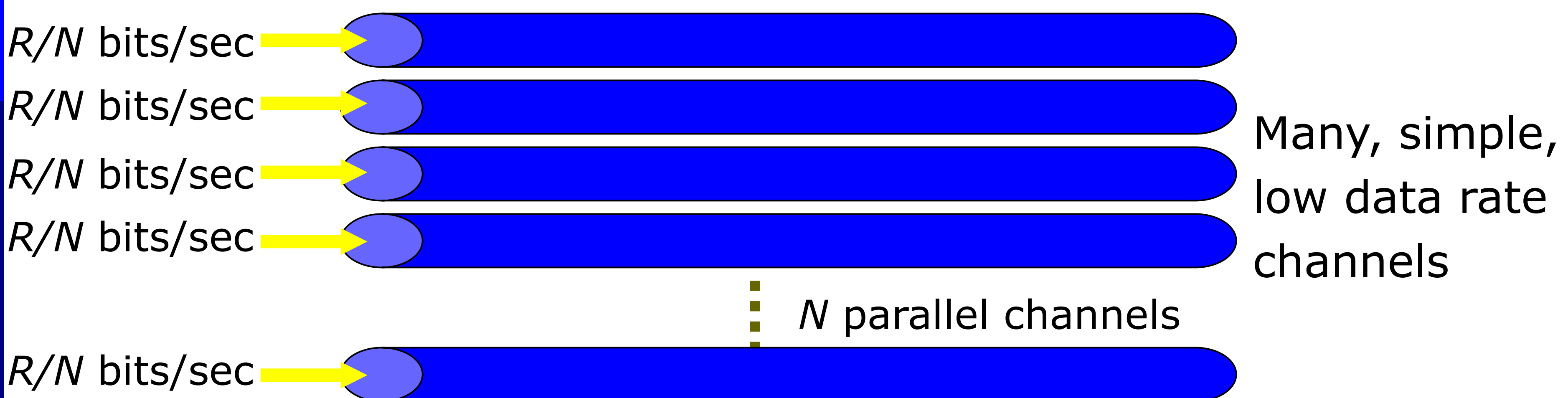
- OFDM – Orthogonal Frequency Division Multiplexing
  - Form of efficiently *multiplexing* a large number of parallel channels in the *frequency* domain using *orthogonal* carriers
- Not a modulation technique
  - Various modulation schemes can be used with OFDM – however only *linear* modulation schemes are used
- In *wired systems* was called Discrete Multitone (DMT) or Multi-Carrier Modulation

OFDM vs. Single Carrier

- Single Carrier



- OFDM

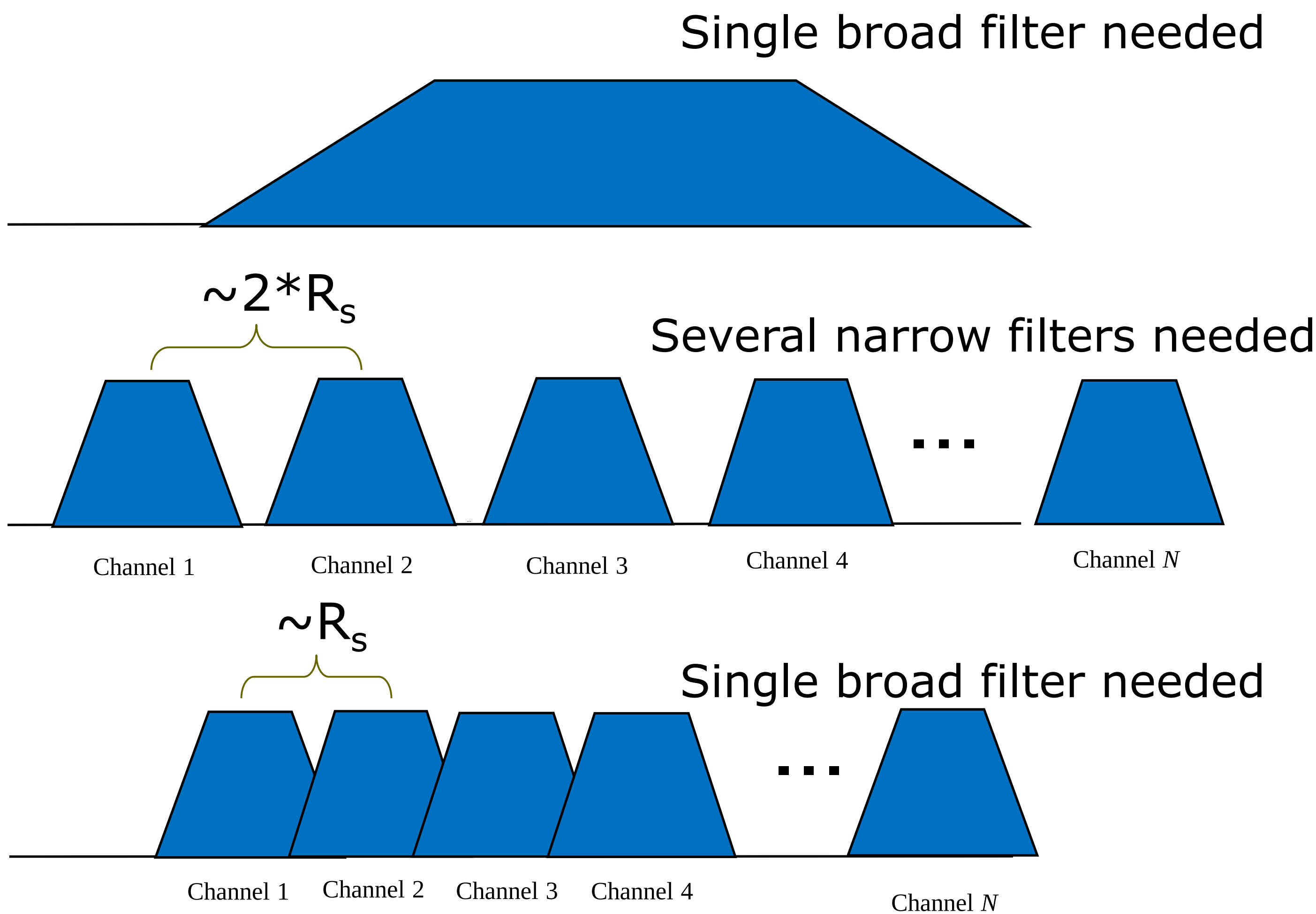


Key: channel spacing

Single Channel Transmission

Multi-carrier Parallel data Transmission

OFDM



Advantages of OFDM

- Simple equalization
  - High data rates are difficult with traditional single-carrier modulation due to multipath delay spread which requires complex equalization
  - Equalization in OFDM = one tap multiplication for each sub-carrier
- Bandwidth efficient
  - Closely spaced carriers allow large number of carriers in small bandwidth
  - Significantly more efficient than simple frequency division multiplexing
- Good performance in fading channels
  - Coding can provide frequency diversity

Advantages of OFDM (2)

- Robust against narrowband interference (impulse noise) owing to frequency diversity.
- Adaptive bit loading/ modulation, power distribution across sub-carriers is possible
  - Maximize capacity according to sub-channel responses – follows Information Theory concept for frequency selective channels
  - Water-filling in frequency
- Main advantage:
  - Allows much larger data rates without equalization than with single carrier modulation

Challenges with OFDM

- High Peak-to-Average Power Ratio (PAPR)
  - Being a sum of several sinusoids, the time-domain OFDM signal has a high PAPR which can cause nonlinear distortion of the signal at the transmit amplifier.
    - In-band and out-of-band distortion
    - Can be avoided with strictly linear amplifiers, but they are very inefficient
  - Having high PAPR is one of the main problems in OFDM.
- High Sensitivity to Frequency Offset Errors
  - Frequency offset destroys orthogonality among sub-carriers
  - High Doppler will also cause this problem

Key Concept: Orthogonal Carriers

- For any two linearly modulated carriers define

$$s_1(t) = d_1 \cos\left(2\pi f_1 t + \theta_1\right), \quad 0 \le t \le T$$
$$s_2(t) = d_2 \cos\left(2\pi f_2 t + \theta_2\right), \quad 0 \le t \le T$$

- If the two signals are orthogonal

$$\int_0^T s_1(t) s_2(t)\, dt = 0$$

Thus, using a matched filter for signal 1

$$\begin{aligned} Z &= 2\int_0^T r(t)\cos\left(2\pi f_1 t + \theta_1\right) dt \\ &= 2\int_0^T \left[d_1 \cos\left(2\pi f_1 t + \theta_1\right) + d_2 \cos\left(2\pi f_2 t + \theta_2\right) \right] \cos\left(2\pi f_1 t + \theta_1\right) dt \\ &= d_1 \end{aligned}$$

Orthogonality – Coherent Carriers

- For orthogonality

$$\int_0^T \sqrt{\frac{2E_s}{T}}\cos\left(2\pi f_1 t\right)\sqrt{\frac{2E_s}{T}}\cos\left(2\pi f_2 t\right)dt \quad \mathbf{= 0}$$

$$= \frac{E_s}{T}\int_0^T \cos\left[2\pi\left(f_1 - f_2\right)t\right] + \cos\left[2\pi\left(f_1 + f_2\right)t\right]dt$$

$$= \frac{E_s}{T}\left[\frac{\sin\left(2\pi\left(f_1 - f_2\right)t\right)}{2\pi\left(f_1 - f_2\right)} + \frac{\sin\left(2\pi\left(f_1 + f_2\right)t\right)}{2\pi\left(f_1 + f_2\right)}\right]\Bigg\|_0^T$$

$$= E_s\left[\text{sinc}\left(2\left(f_1 - f_2\right)T\right) + \text{sinc}\left(2\left(f_1 + f_2\right)T\right)\right]$$

$$\approx E_s\,\text{sinc}\left(2\left(f_1 - f_2\right)T\right) \qquad (\text{for } f_1 T >> 1)$$

Orthogonality (Continued)

- The first zero of sinc(x) is for x=1. Thus we set

$$2(f_1 - f_2)T = 1$$

- This gives the minimum frequency separation for orthogonality

$$f_1 - f_2 = \frac{1}{2T}$$

- However, we have assumed that the two frequencies were *coherently* aligned. With the presence of data modulation, this will not be the case – carriers will have different phases

Orthogonality – Noncoherent Carriers

- For both signals to be orthogonal over the symbol period T without coherent phase alignment they must both go through an integer number of cycles in a symbol period

$$2\pi f_1 T = 2\pi j, \quad f_1 = \frac{j}{T}$$

$$2\pi f_2 T = 2\pi k, \quad f_2 = \frac{k}{T}$$

$$f_1 - f_2 = \frac{j-k}{T} = \frac{m}{T}$$

Thus the requirement in OFDM for adjacent carriers is $\Delta f = \frac{1}{T}$

OFDM Subcarriers

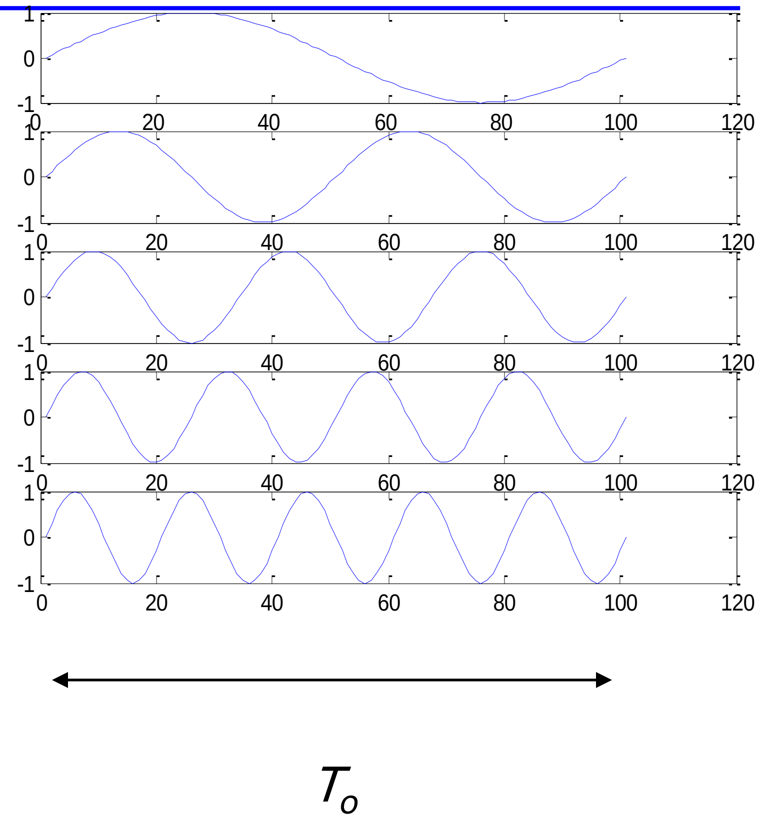
- Orthogonal waveforms are generated by using signals that have integer number of cycles in the OFDM symbol duration T_o
- The baseband equivalents of the orthogonal subcarriers satisfy the following relationship where *k* and *i* are subcarrier indices

$$\Delta f = 1/T_o$$

$$\int_0^{T_o} \cos(2\pi k \Delta f t)\cos(2\pi i \Delta f t)\,dt = \begin{cases} \frac{T_o}{2} & k = i \\ 0 & k \neq i \end{cases}$$

$$\int_0^{T_o} \sin(2\pi k \Delta f t)\sin(2\pi i \Delta f t)\,dt = \begin{cases} \frac{T_o}{2} & k = i \\ 0 & k \neq i \end{cases}$$

$$\int_0^{T_o} \sin(2\pi k \Delta f t)\cos(2\pi i \Delta f t)\,dt = 0$$



Subcarriers in OFDM

OFDM Subcarriers

- The base-band information in the *k*th subcarrier can be written as

$$\underbrace{\left(x_k + jy_k\right)}_{\text{data symbol}}\underbrace{\left\{\cos(2\pi k\Delta f t) + j\sin(2\pi k\Delta f t)\right\}}_{k\text{th carrier}}$$

$$\boxed{\Delta f = \frac{1}{T_o}}$$

- The OFDM signal is the sum of all the signals in each of its subcarriers which can be written as (usually implemented using IFFT)

$$s(t) = \sum_{k=0}^{N-1}\left(x_k + jy_k\right)\left\{\cos(2\pi k\Delta f t) + j\sin(2\pi k\Delta f t)\right\}$$

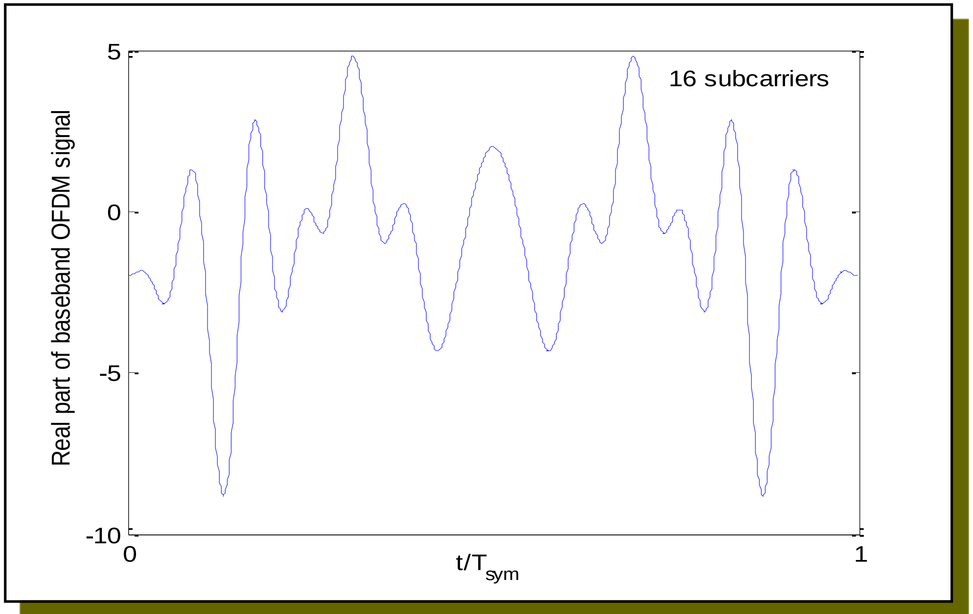
OFDM Sub-carriers (cont.)

The individual modulated symbols at the receiver are recovered using the Fourier Transform (Matched Filter). The *k*th output from the MF is:

$$
\begin{aligned}
Z_k &= \int_0^{T_o} s(t)\left\{\cos(2\pi k\Delta f t) - j\sin(2\pi k\Delta f t)\right\} dt \\
&= \sum_{n=0}^{N-1} \int_0^{T_o} (x_n + jy_n)\cos(2\pi n\Delta f t) + j\sin(2\pi n\Delta f t) \times \ldots \\
&\quad (\cos(2\pi k\Delta f t) + j\sin(2\pi k\Delta f t))\, dt \\
&= x_k + jy_k
\end{aligned}
$$

OFDM Symbol

- N serial input data elements separated by T_s = 1/f_s (f_s is input symbol rate)
- Symbol duration increased to $T_o = N\ T_s$ (helps in time dispersive channels)
- Subcarrier separation $\Delta f = 1/(N\ T_s) = 1/T_o$
- Sub-bands overlap but are still *orthogonal*



Example time domain waveform for one OFDM symbol

OFDM Spectrum

- Each symbol can be represented as

$$\underbrace{\left(x_k + jy_k\right)}_{\text{data symbol}}\underbrace{\left\{\cos(2\pi k\Delta ft) + j\sin(2\pi k\Delta ft)\right\}}_{k\text{th carrier}} \quad 0 \le t \le T_{ofdm}$$

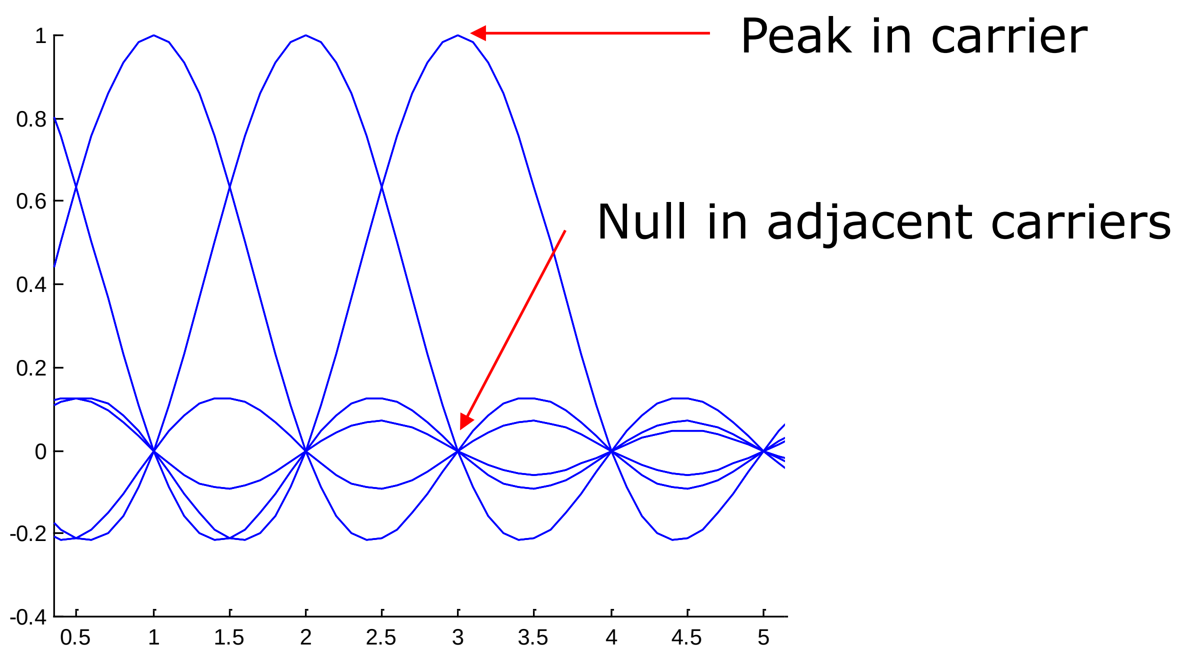
- The Fourier transform of a specific symbol is

$$\begin{aligned}
X(f) &= \mathcal{F}\left\{\underbrace{(x_k + jy_k)}_{\text{data symbol}}\ \underbrace{e^{j2\pi k\Delta ft}}_{\text{kth sub-carrier}}\ \underbrace{\Pi\left(\frac{t}{T_o}\right)}_{\text{square pulse}}\right\} \\
&= (x_k + jy_k)\mathcal{F}\left\{e^{j2\pi k\Delta ft}\right\} * \mathcal{F}\left\{\Pi\left(\frac{t}{T_o}\right)\right\} \\
&= (x_k + jy_k)\delta(f - k\Delta f) * T_o\mathrm{sinc}(T_o f) \\
&= (x_k + jy_k)T_o\mathrm{sinc}(T_o(f - k\Delta f))
\end{aligned}$$

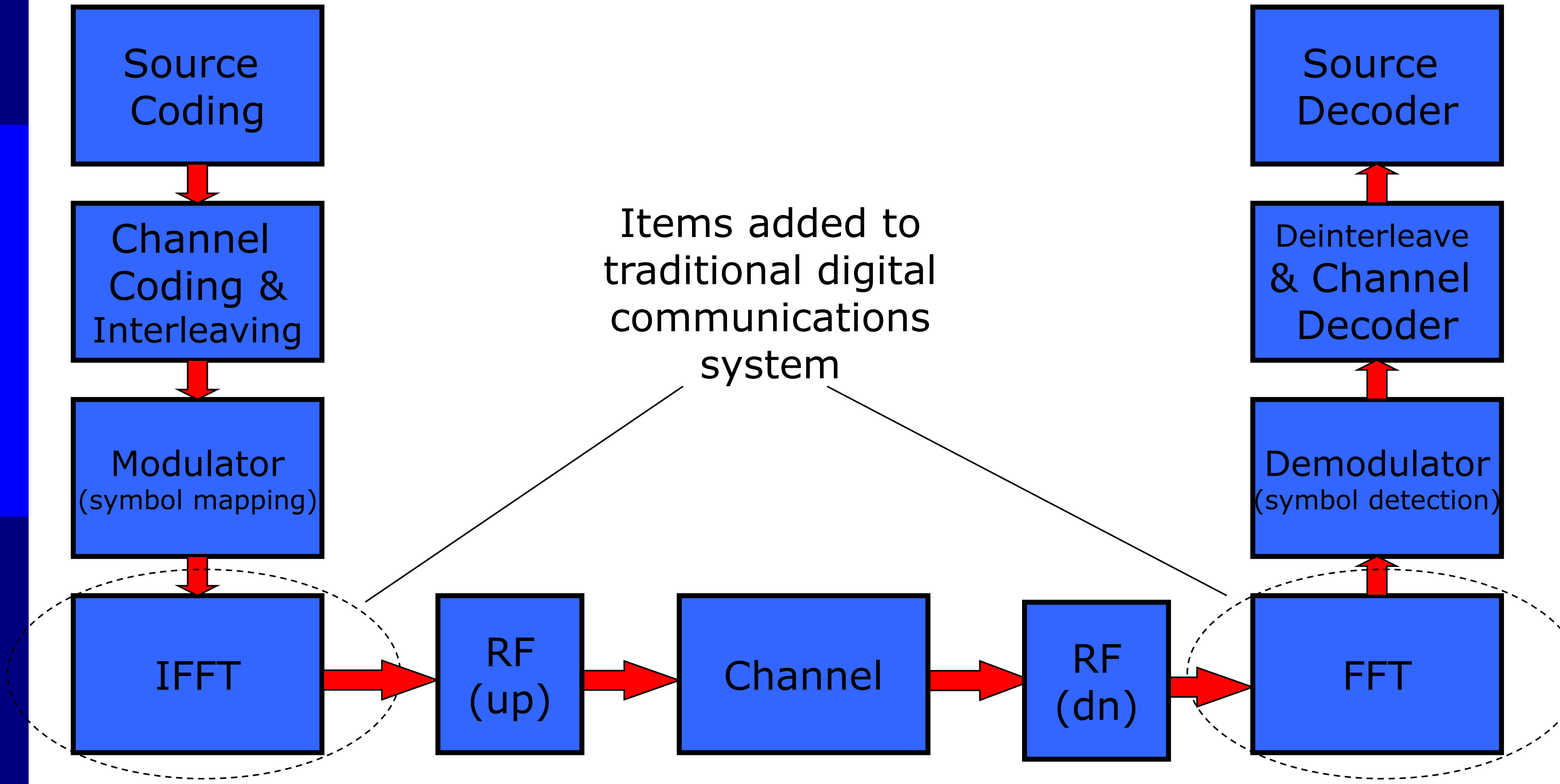
convolution

OFDM Spectrum

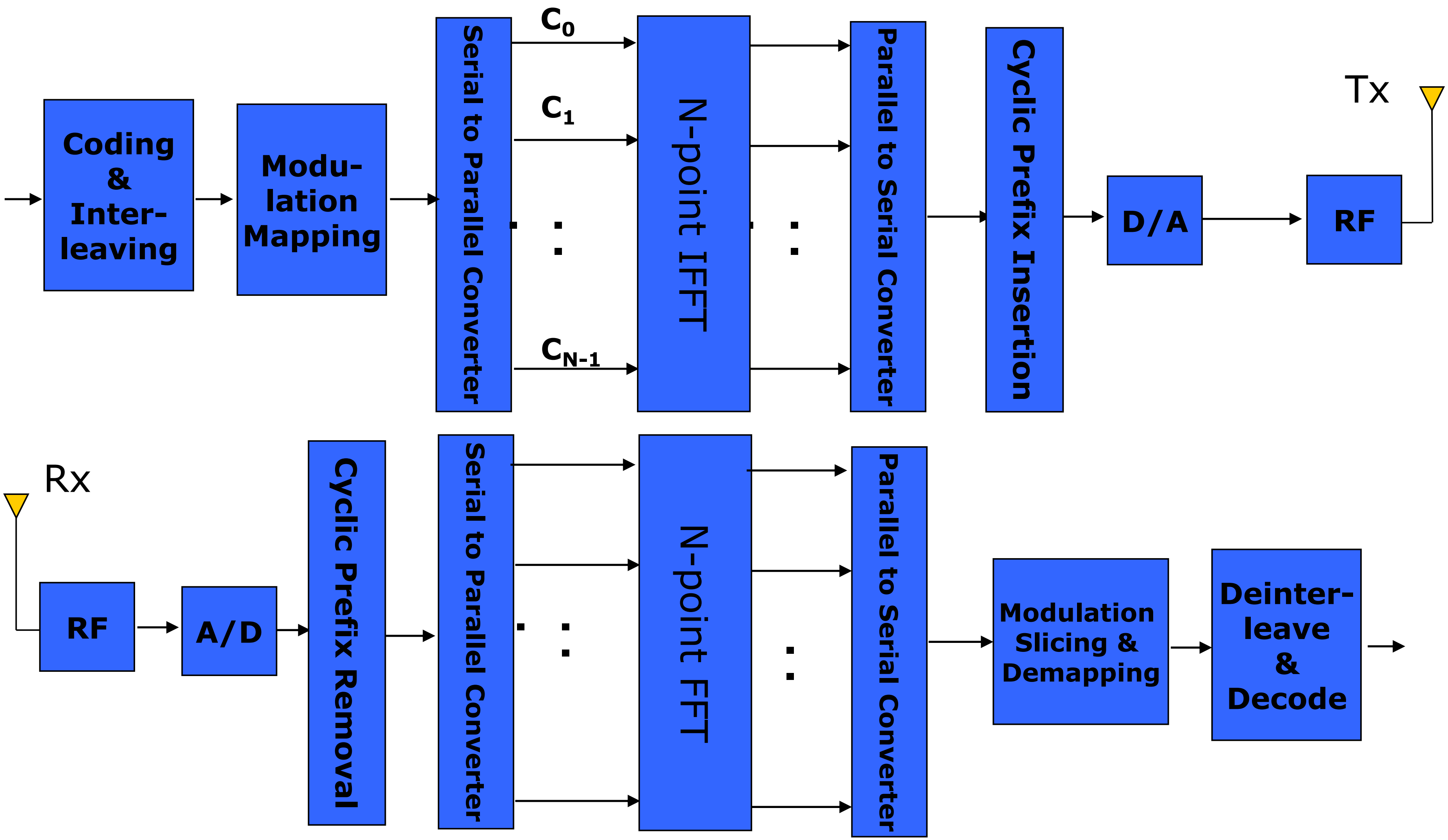
- The individual spectrum of the subcarriers are sinc functions
  - Time gated sinusoids
- Zero crossings occur at every integer multiple of fT_o and hence no Inter- Carrier Interference occurs
- For N sub-carriers, the smallest bandwidth of FDM is $2N/T_o$ while that of OFDM is $(N+1)/T_o$. By allowing the sub-carrier spectra to overlap, OFDM improves the spectral efficiency



Block Diagram

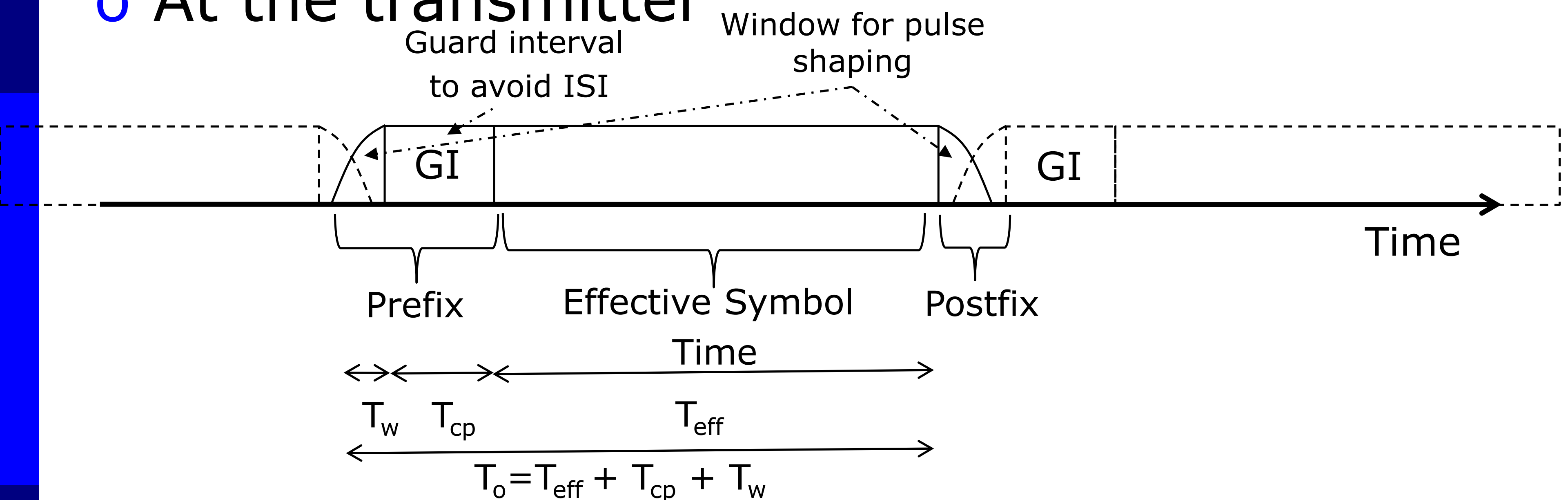


Detailed Diagram of OFDM

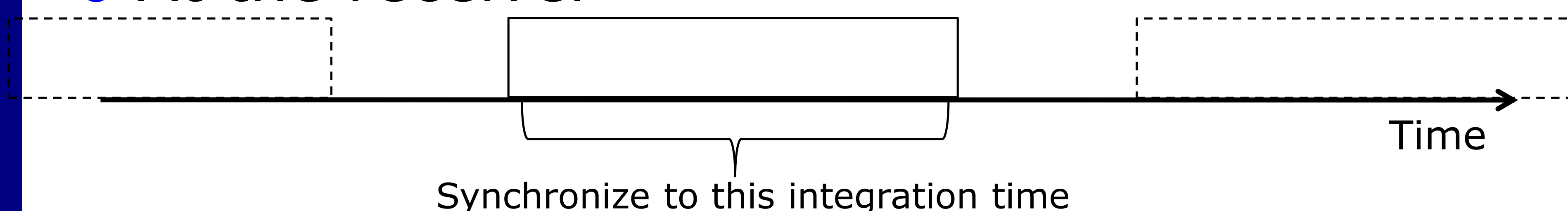


Symbol Timing and Windowing

- At the transmitter



- At the receiver



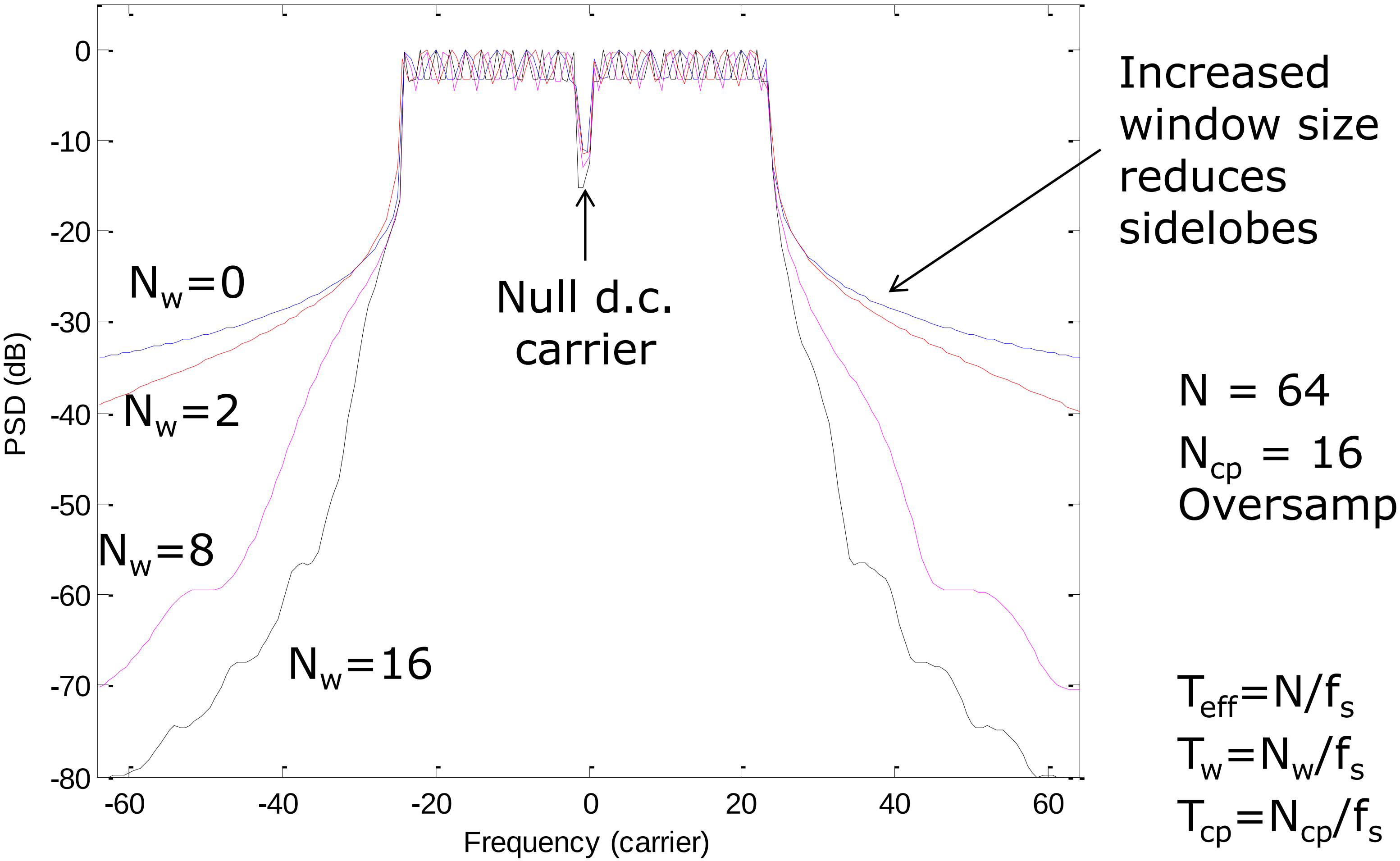
Window

- Common Window

$$
w(t) = \begin{cases} \frac{1}{2}\left[1-\cos\left(\frac{\pi(t+T_w+T_{cp})}{T_w}\right)\right] & -T_w - T_{cp} \le t \le -T_{cp} \\ 1 & -T_{cp} \le t \le T_{eff} \\ \frac{1}{2}\left[1-\cos\left(\frac{\pi(t-T_w-T_{eff})}{T_w}\right)\right] & T_{eff} \le t \le T_{eff} + T_w \end{cases}
$$

- Improves spectral properties
- Reduces overall energy efficiency since it requires unused symbol time T_w

Example Spectrum



$N = 64$

$N_{cp} = 16$

Oversampling = 2

$T_{eff}=N/f_s$

$T_w=N_w/f_s$

$T_{cp}=N_{cp}/f_s$

OFDM Signal Model

- Go to the board….

Discrete Time Analysis

- Define the DFT matrix

$$\mathbf{D}_N = \frac{1}{\sqrt{N}}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-j2\pi/N} & \cdots & e^{-j2\pi(N-1)/N} \\ : & : & : & : \\ 1 & e^{-j2\pi(N-1)/N} & \cdots & e^{-j2\pi(N-1)(N-1)/N} \end{bmatrix}$$

$$\left[\mathbf{D}\right]_{nk} = \frac{1}{\sqrt{N}} e^{-j2\pi kn/N}, 0 \le n \le N-1, 0 \le k \le N-1$$

Matrix Notation (cont.)

- Note that $\mathbf{D}_N$ is Unitary:

$$\mathbf{D}_N \mathbf{D}_N^H = \mathbf{I}$$

$$\mathbf{D}_N^{-1} = \mathbf{D}_N^H$$

- We can represent the DFT and IDFT using these matrices. For time samples **x** the frequency samples **X** are found as:

$$\mathbf{X} = \mathbf{D}_N \mathbf{x} \qquad \mathbf{x} = \mathbf{D}_N^{-1} \mathbf{X} = \mathbf{D}_N^H \mathbf{X}$$

Performance in AWGN

- Transmit signal (ignoring cyclic prefix):

$$\mathbf{x} = \mathbf{D}_N^H \mathbf{s}$$

Where **s** is the vector of transmit symbols (defined as PSK or QAM) each with energy E_s

- At the receiver we have

$$\mathbf{r} = \mathbf{x} + \mathbf{n} = \mathbf{D}_N^H \mathbf{s} + \mathbf{n}$$

Where **n** is a vector of complex WGN samples with variance $\sigma^2 = N_o/2$

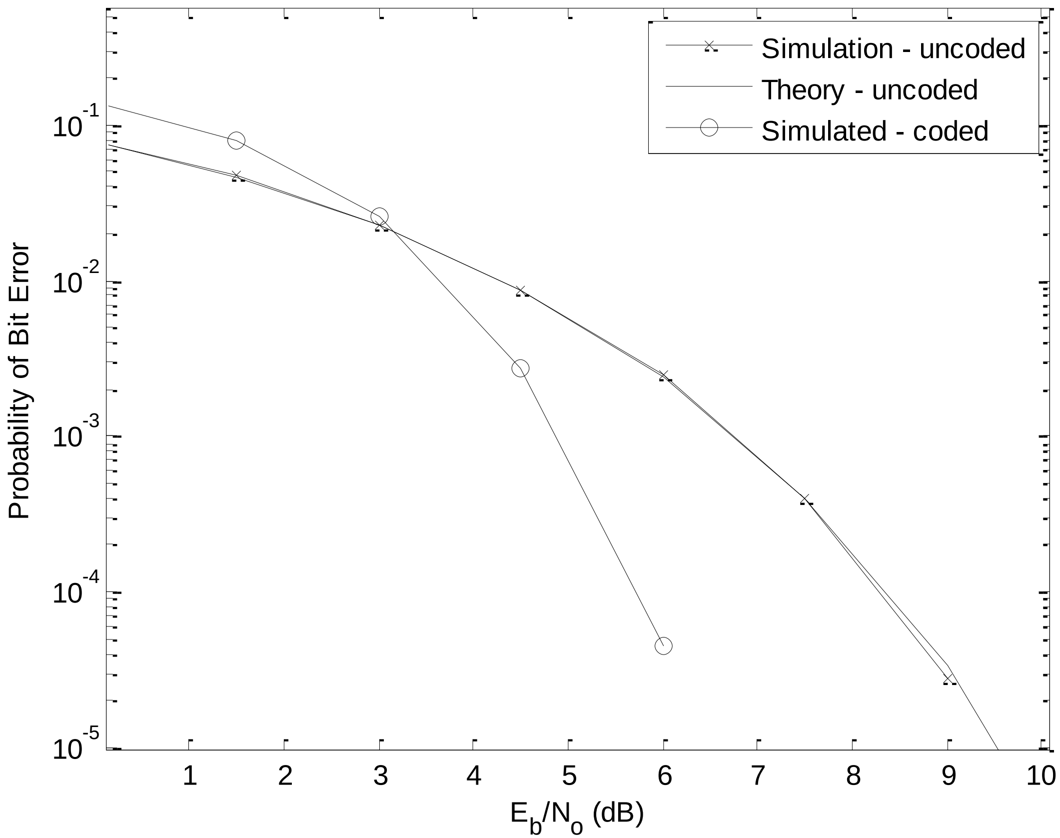
Performance (cont.)

- The matched filter receiver calculates:

$$\begin{aligned}\mathbf{z} &= \mathbf{D}_N \mathbf{r} \\ &= \mathbf{D}_N \left(\mathbf{D}_N^H \mathbf{s} + \mathbf{n} \right) \\ &= \mathbf{s} + \mathbf{D}_N \mathbf{n} = \mathbf{s} + \boldsymbol{\nu}\end{aligned}$$

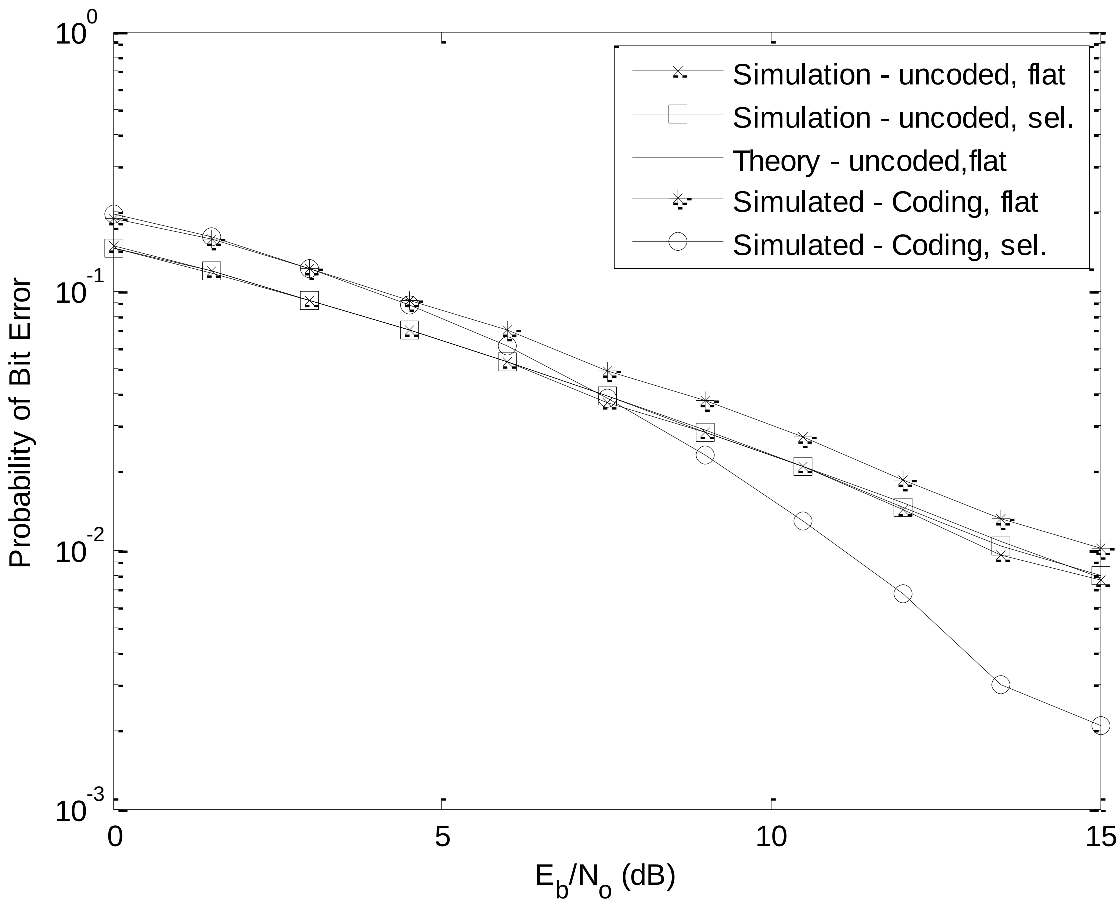
- Each element in **n** is a GRV with mean 0 and variance $\sigma^2 = N_o/2$
- Thus, the performance is identical to what we would achieve without OFDM in AWGN

AWGN



- BPSK
- BCH(63,36) code with hard decision decoding

Rayleigh Fading



- Perfect channel estimation
- Cyclic Prefix > Delay spread
- BCH(63,36) code with hard decision decoding (HDD)
- HDD in flat fading results in no gain over these E_b/N_o values (in fact even a loss)
- Frequency selective fading no different than flat fading in the absence of coding
- Frequency selective channel can provide DIVERSITY

Conclusions

- In this lecture we have introduced the concept of Orthogonal Frequency Division Multiplexing
  - OFDM is an efficient way to obtain high data rates in fading channels
- In the coming weeks we will examine many different aspects of the technique including
  - Performance in fading channels
    - Impact of Doppler and Multipath
  - Peak-to-Average Power reduction
  - Channel estimation
  - Bit loading / adaptive modulation
  - Standards