Multi-channel Communications Fall 2022

Lecture 9

Combatting Multipath with Antennas – Spatial Diversity

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Introduction

- Multipath fading is the dominant impediment to wireless communications
- Diversity is the primary means of combating fading
- Diversity = receiving multiple copies of the data
  - If the copies are independent there is a good chance that one will have good fidelity
- Diversity can be accomplished in *time, frequency, or space.*
  - We focus on spatial diversity but much of the analysis applies to any form of diversity

Impact of Fading

- The average probability of error depends on the distribution of the signal-to-noise ratio (SNR)

$$P_b = \int_0^\infty \underbrace{P_b(x)}_{\text{Probability of error at a specific SNR}} \overbrace{f_\gamma(x)}^{\text{Distribution of SNR}} dx$$

- Thus, the primary goal of diversity is to "improve" the distribution of SNR
  - "Improve" means shifting the distribution towards higher values of SNR and reducing the probability of low values of SNR

Rayleigh Fading

- The most common model for multipath fading is the Rayleigh fading model
- The received signal (in complex baseband) is modeled as

$$r(t) \quad = \quad \alpha(t)e^{j\phi(t)}x(t) + n(t)$$

  - α(t) – Rayleigh distributed channel amplitude
  - ϕ(t) - uniformly distributed channel phase
  - x(t) – transmitted message signal
  - n(t) – complex Gaussian noise

Rayleigh Fading – cont.

- The amplitude is distributed as

$$f_\alpha(x) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \quad x \geq 0$$

σ² = variance of underlying Gaussian r.v.

- Moments:

$$\begin{aligned} \mathrm{E}\{x\} &= \sqrt{\frac{\pi\sigma^2}{2}} \\ \mathrm{E}\{x^2\} &= 2\sigma^2 \\ \sigma_\alpha^2 &= \frac{\sigma^2(4-\pi)}{2} \end{aligned}$$

SNR Distribution

- For a Rayleigh distributed amplitude, the channel power (and SNR for constant noise power) follows an exponential distribution:

$$f_\gamma(x) = \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}} \quad x \geq 0$$

- where $\bar{\gamma}$ is the average channel SNR

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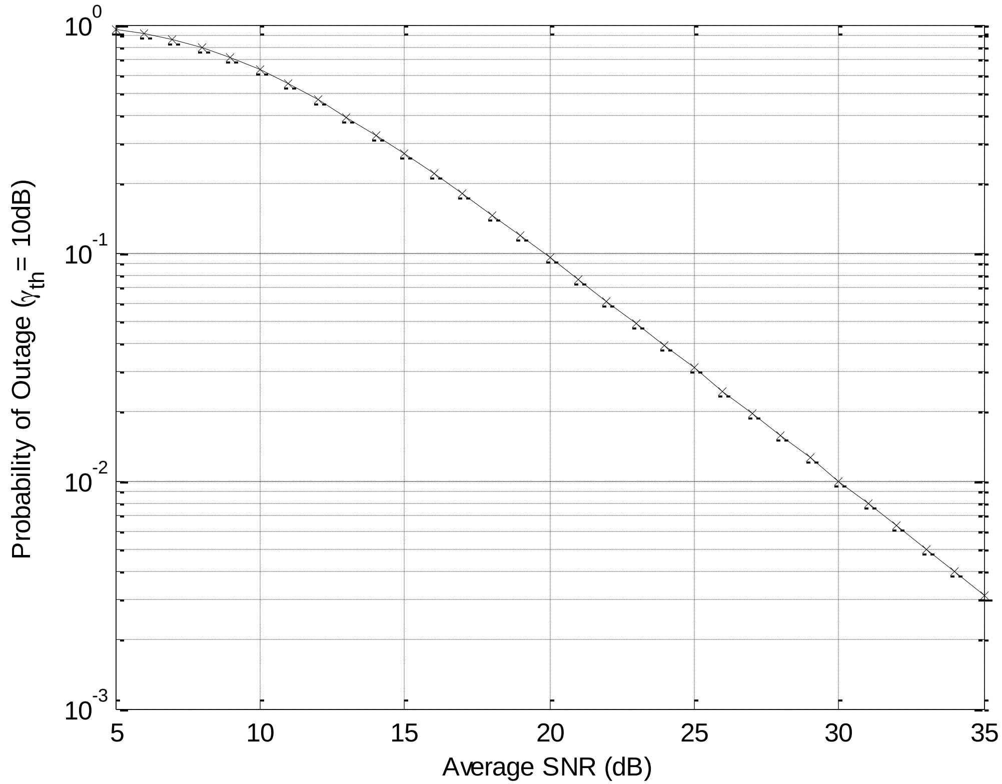
Outage Probability

- The probability that the SNR falls below a threshold γ_{th} is found from the CDF:

$$
\begin{aligned}
P_{out} &= \Pr\left\{x \le \gamma_{th}\right\} \\
&= F_{\gamma}\left(\gamma_{th}\right) \\
&= 1 - e^{-\left(\gamma_{th}/\bar{\gamma}\right)}
\end{aligned}
$$

Outage Probability – cont.



- In order to keep the SNR above 10dB (γ_{th}=10) 90% of the time (P_{out} = 10%), the average SNR must be 20dB
- For P_{out} = 1%, the average SNR must be 30dB

BER for BPSK

- BER for a fixed channel SNR (per bit)

$$P_e = Q\left(\sqrt{2\gamma}\right)$$

- Averaging over the SNR distribution:

$$\begin{aligned} P_b &= \int_0^\infty f_\gamma(x) Q\left(\sqrt{2x}\right) dx \\ &= \int_0^\infty \frac{1}{\bar{\gamma}} e^{-\frac{x}{\bar{\gamma}}} Q\left(\sqrt{2x}\right) dx \\ &= \frac{1}{2}\left(1 - \sqrt{\frac{\bar{\gamma}}{1+\bar{\gamma}}}\right) \\ &\approx \frac{1}{4\bar{\gamma}} \end{aligned}$$

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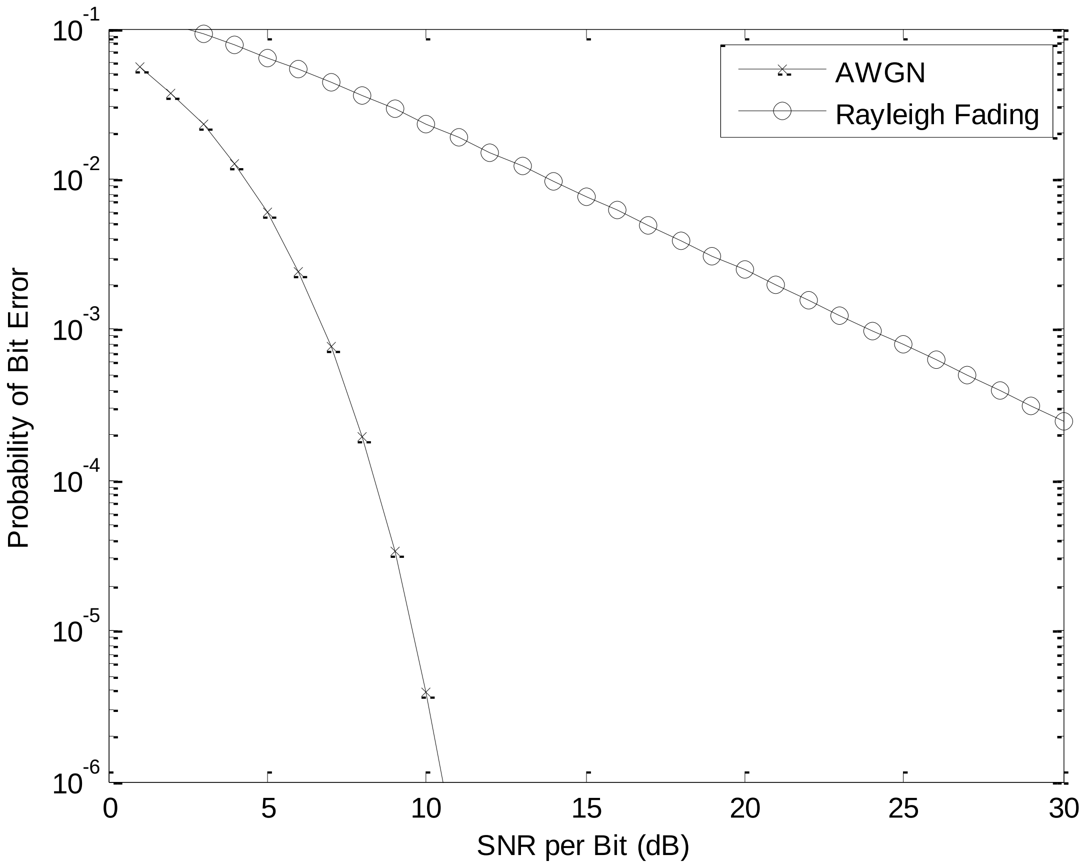
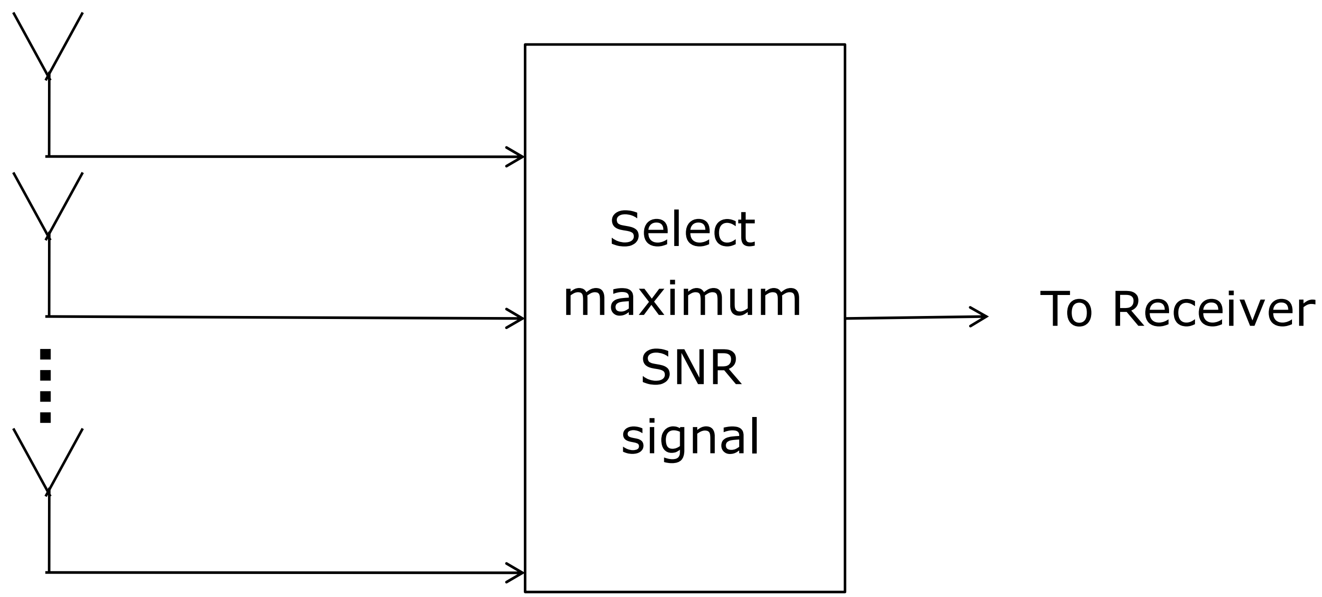
BER for BPSK



- Fading causes extremely large degradation in performance

Selection Diversity

- Simplest form of diversity
- Receiver has multiple antennas available
- Circuit in front end selects signal with the largest SNR and sends it to the receiver



Selection - SNR Distribution

- By choosing the signal with maximum SNR from L independent signals, we can improve SNR distribution

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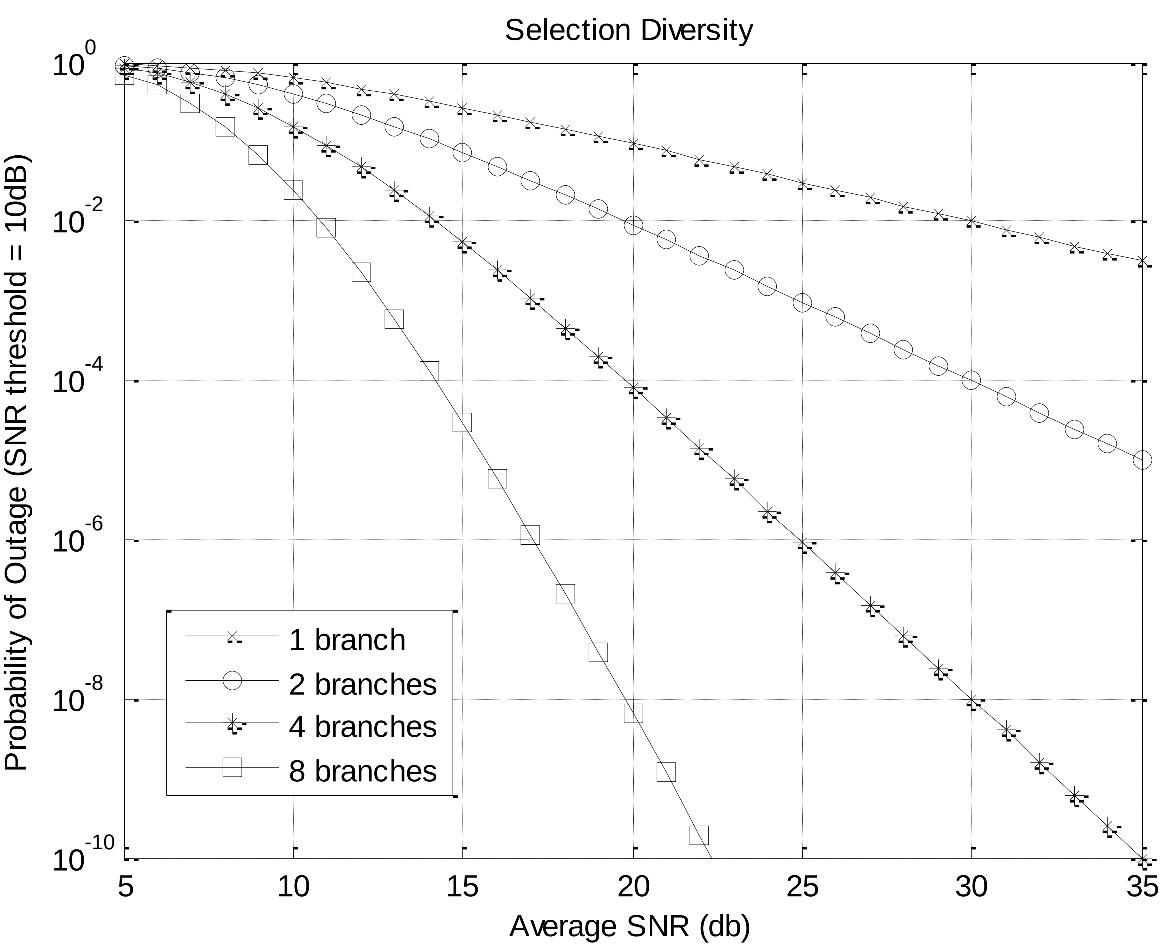
Selection Diversity



- P_{out} = 1%
- γ_{th} = 10dB

L	$\bar{\gamma}$
1	$30dB$
2	$20dB$
4	$14dB$
8	$11dB$

Bit Error Rate Performance (BPSK)

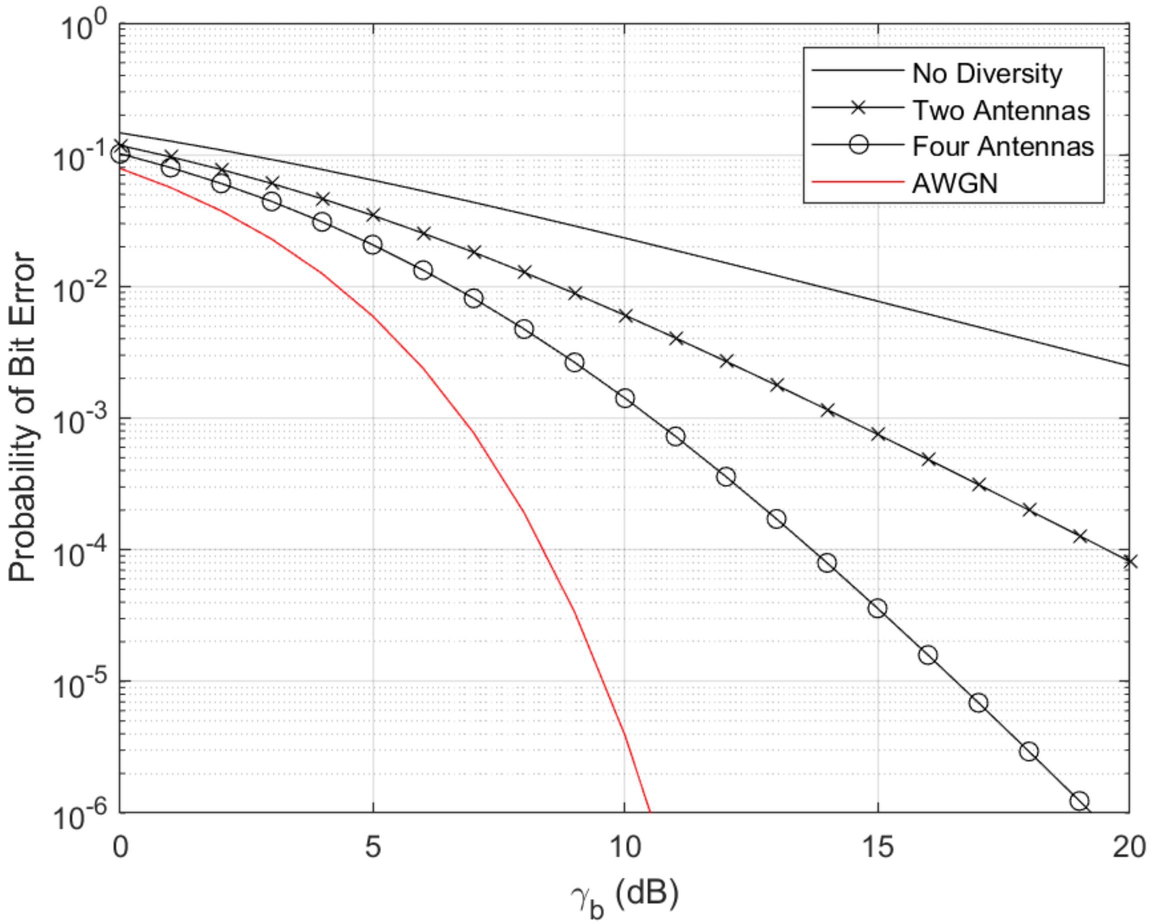
- We can show that the BER performance for BPSK in Rayleigh fading with *L*-fold selection diversity is

$$P_b = \frac{1}{2}\sum_{k=1}^{L}\left(\begin{array}{c} L \\ k \end{array}\right)(-1)^{k-1}\left(1-\sqrt{\frac{\overline{\gamma}_b}{k+\overline{\gamma}_b}}\right)$$

$$\overline{\gamma}_b = \frac{\overline{\gamma}_c}{\sum_{k=1}^{L}\frac{1}{k}}$$

Average SNR Gain

Bit Error Rate Performance (BPSK)



- Slope = diversity gain
- Diversity gain increases with L
- Infinite diversity approaches AWGN

Equal Gain Combining – SNR Dist.

- Instead of selecting one of *L* diversity branches, a better approach is to keep all signals and combine them coherently
- Let the signal on the *i*th antenna be written as

$$r_i(t) \quad = \quad \alpha_i(t)e^{j\phi_i(t)}x(t) + n_i(t)$$

- The combined signal is written as

$$\begin{aligned} r(t) &= \sum_{i=1}^{L} e^{-j\phi_i} r_i(t) \\ &= \sum_{i=1}^{L} \alpha_i(t)x(t) + \sum_{i=1}^{L} \hat{n}_i(t) \end{aligned}$$

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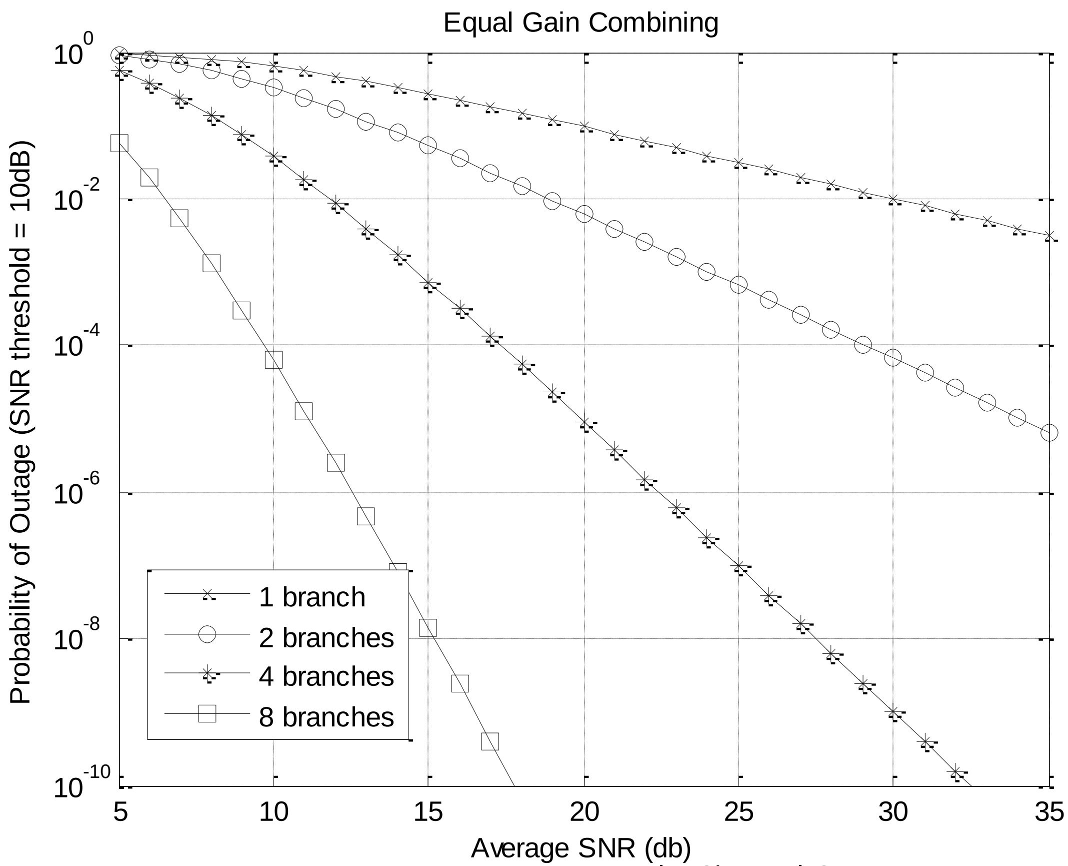
Equal Gain Combining



- $P_{out} = 1\%$
- $\gamma_{th} = 10\text{dB}$

L	$\bar{\gamma}$
1	$30dB$
2	$19dB$
4	$12dB$
8	$7dB$

Optimal Combining

- In general
$$r(t) = \sum_{i=1}^{L} w_i e^{-j\phi} r_i(t)$$
- Selection
$$w_i = \begin{cases} 1 & i = \arg\max\{\alpha_i(t)\} \\ 0 & else \end{cases}$$
- Equal Gain Combining $w_i = 1$
  - Better than selection since all signal energy captured
  - Sub-optimal since low SNR branches are given equal weight with high SNR branches
- What are the optimal weights?

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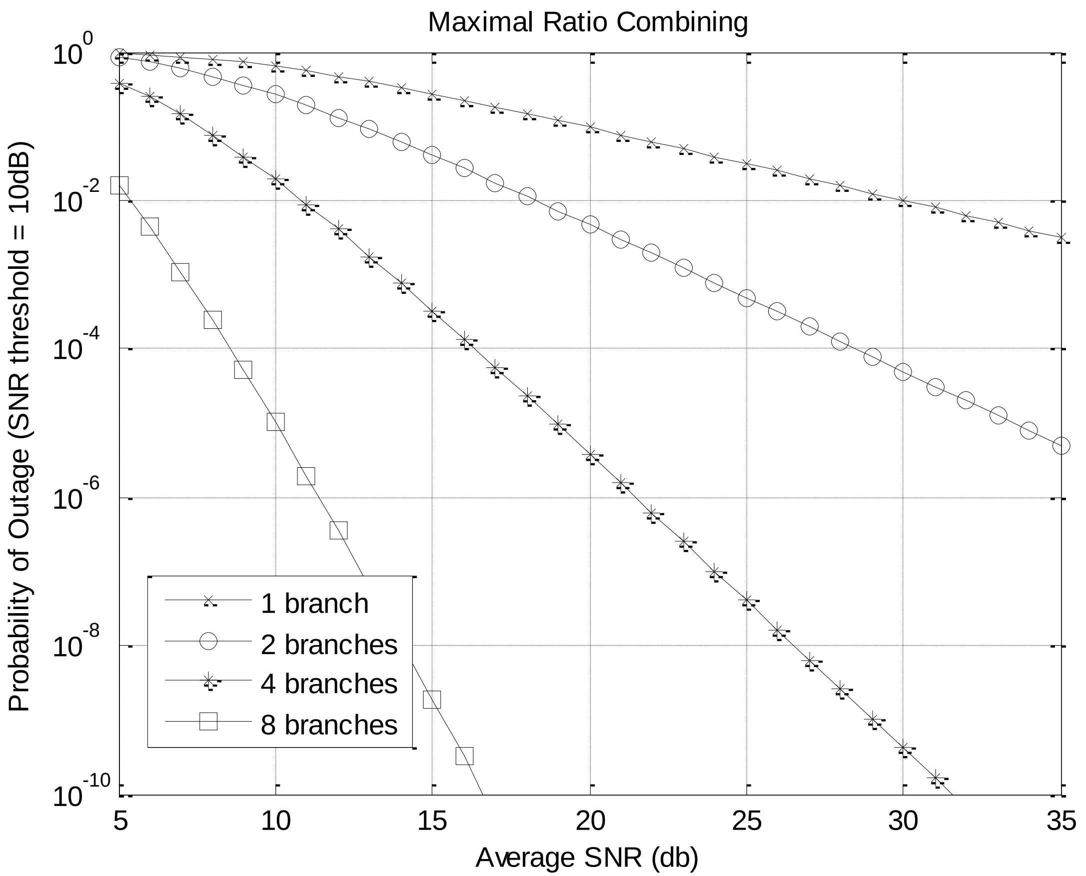
Maximal Ratio Combining



- $P_{out} = 1\%$
- $\gamma_{th} = 10dB$

L	$\overline{\gamma}$
1	$30dB$
2	$18dB$
4	$11dB$
8	$5.5dB$

Bit Error Rate Performance (BPSK)

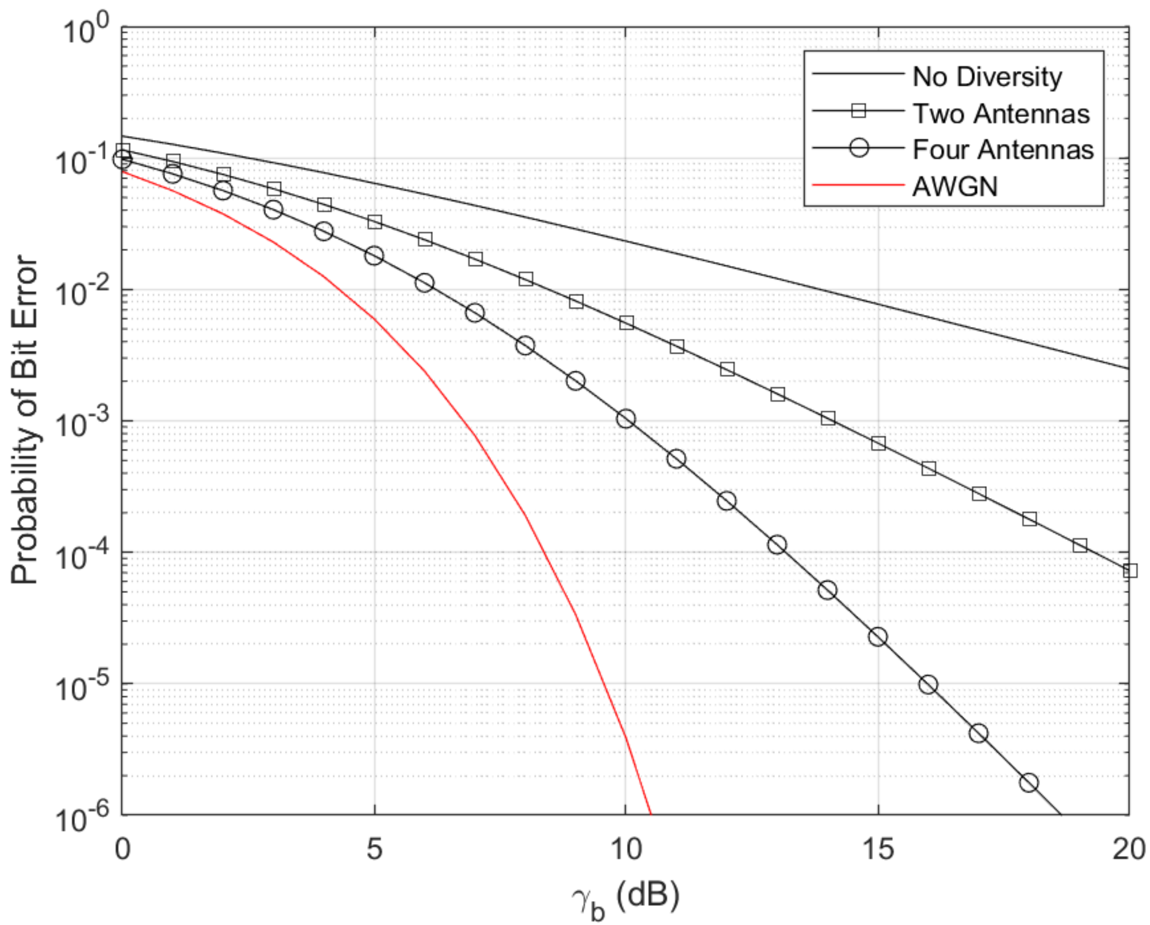
- We can show that the BER performance for BPSK in Rayleigh fading with *L*-fold Maximal Ratio Combining is

$$P_b = \left[\frac{1}{2}\left(1-\sqrt{\frac{\overline{\gamma}_b}{1+\overline{\gamma}_b}}\right)\right]^L \sum_{k=0}^{L-1} \binom{L-1+k}{k} \left[\frac{1}{2}\left(1+\sqrt{\frac{\overline{\gamma}_b}{1+\overline{\gamma}_b}}\right)\right]^k$$

$$\overline{\gamma}_b = \frac{\overline{\gamma}_c}{L}$$

Average SNR Gain

Bit Error Rate Performance (BPSK)



- Slope = diversity gain
- Diversity gain increases with *L*
- Infinite diversity approaches AWGN

Types of Gain

- Diversity Gain
  - Slope of the log-log outage probability curve
  - L for all schemes

- Average SNR Gain
  - Improvement in average SNR
  - Varies depending on combining scheme

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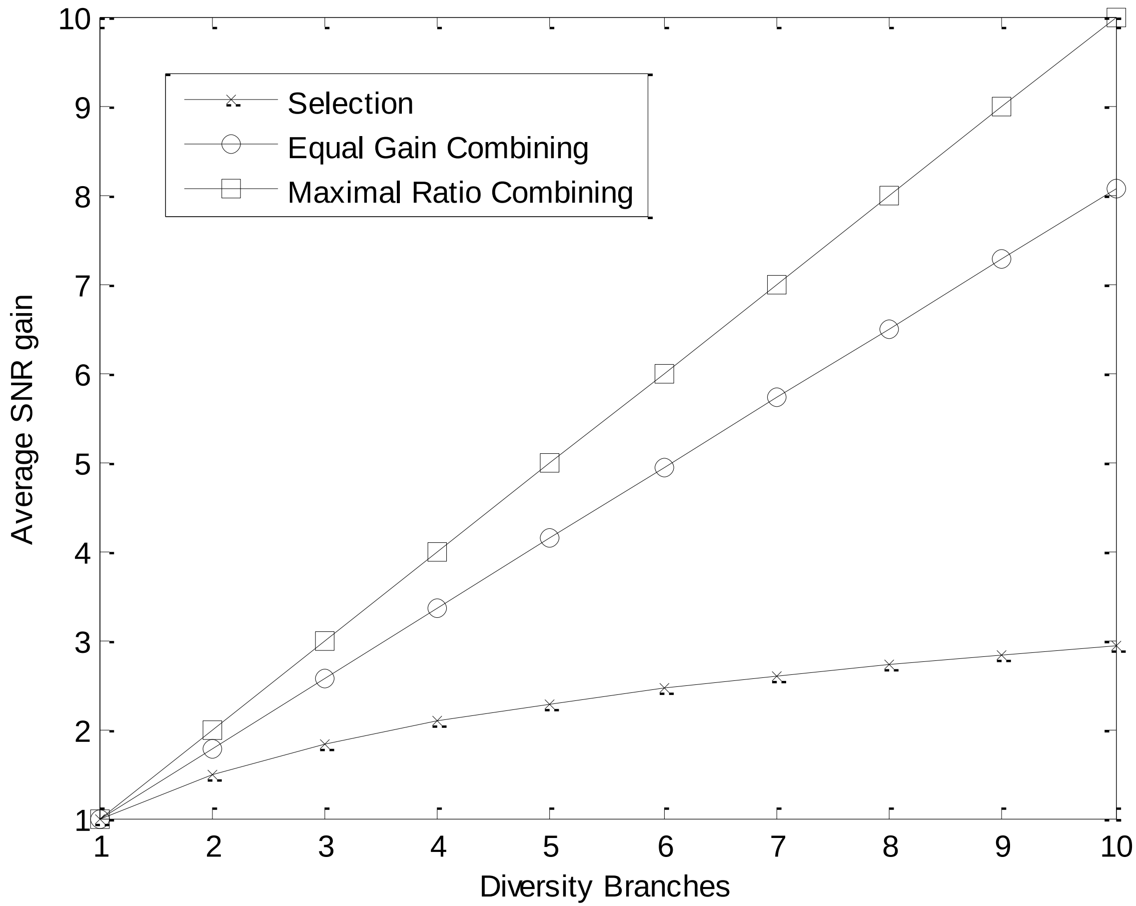
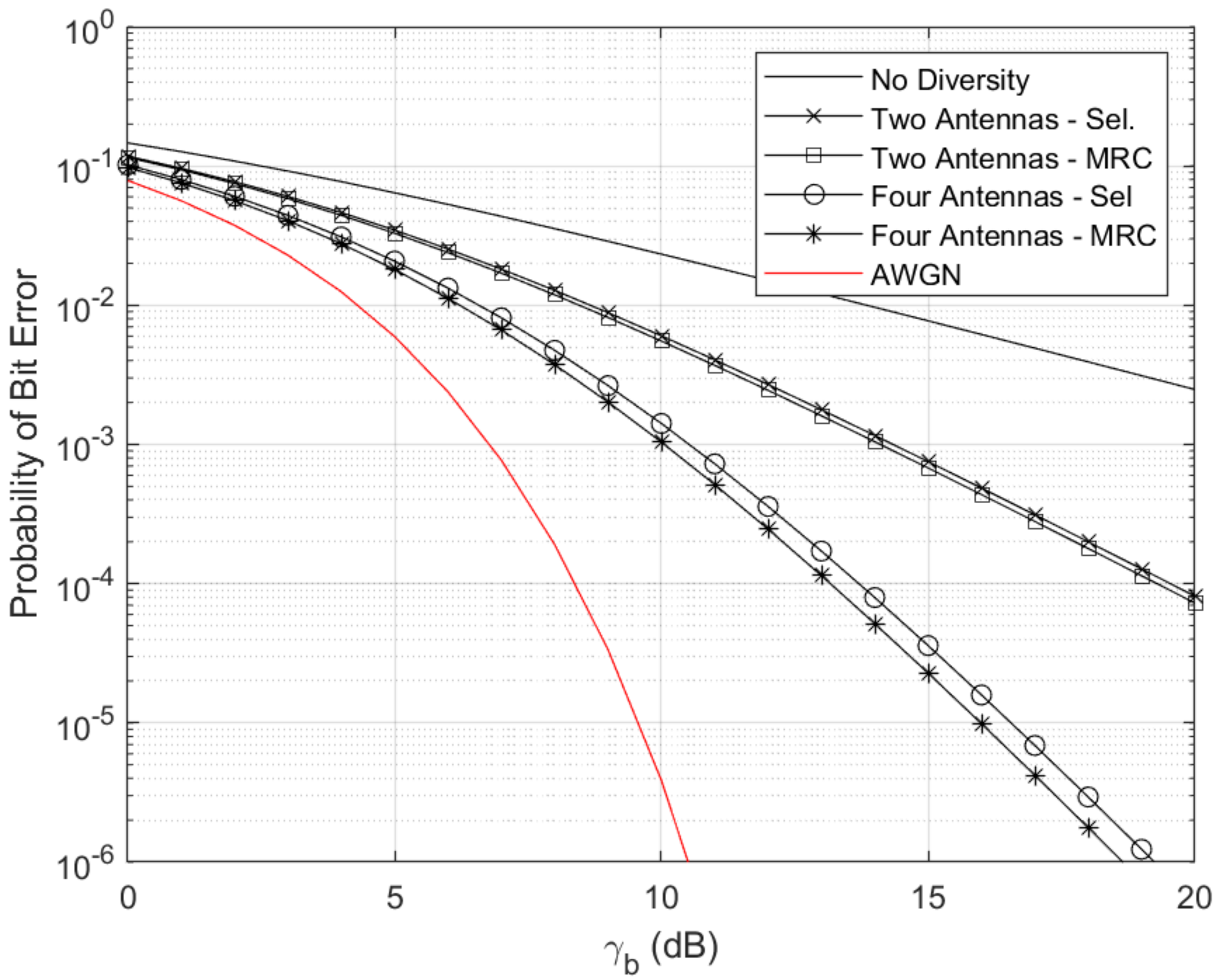
Average SNR Gain



BER Comparison



- MRC provides slight improvement (in terms of SNR per bit)
- Note that on a log-log plot, the diversity gain (i.e., slope) is what we would expect:

$$\log(P_e) = \log\begin{pmatrix} 2L-1 \\ L \end{pmatrix} - L\log(4\overline{\gamma})$$

Impact of Average SNR per Branch

- All diversity schemes perform the best with equal power per branch
- We will examine the impact of unequal powers on MRC, but all degrade similarly

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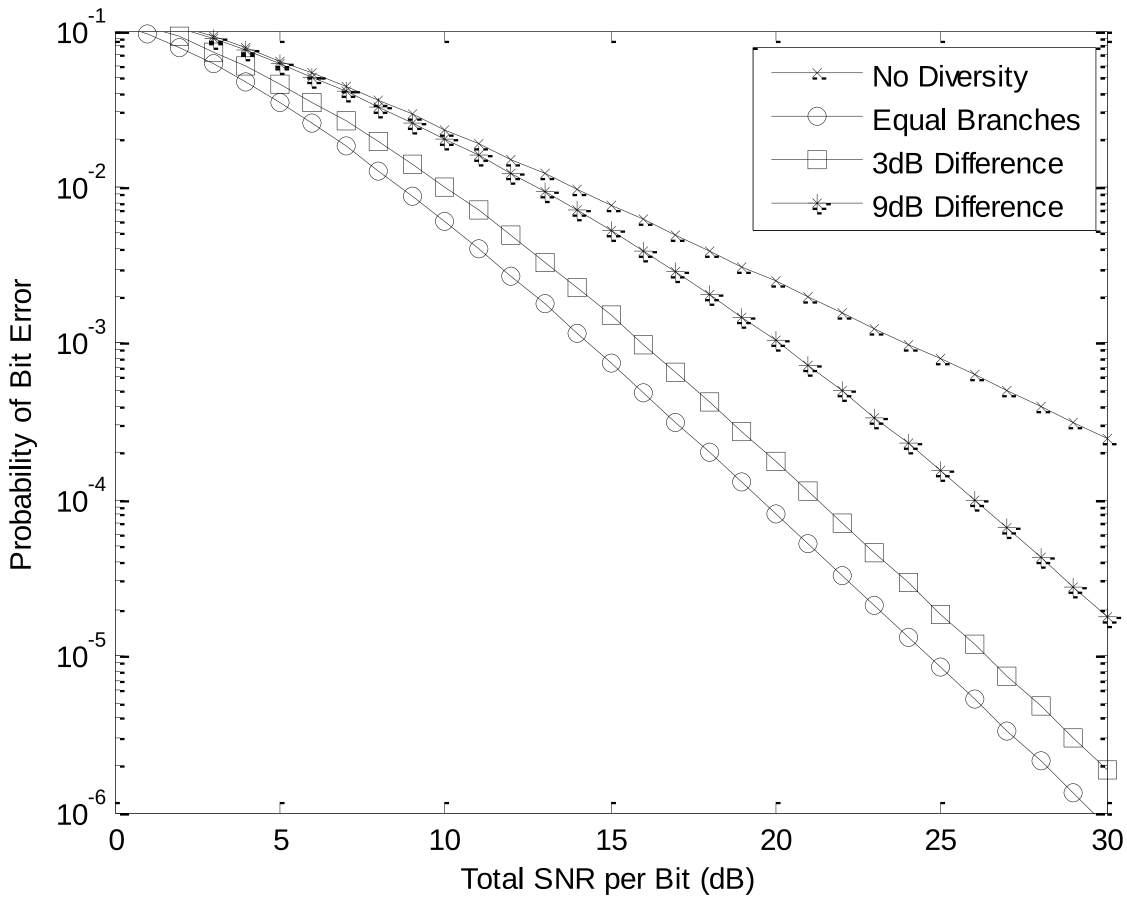
Unequal Paths



- Same total E_b/N_o
- BPSK modulation
- Two branches
- Diversity gain not affected, but average SNR gain reduced

Conclusions

- In this lecture we looked at one of the oldest and most dominant uses of multiple antennas – diversity
- Diversity changes the distribution of the SNR which improves BER
- MRC provides best performance of all combining techniques, but all provide substantial improvement
- Gains experience diminishing returns versus number of antennas