Special functions:

```
restart;
```

For changing r=R_E+r in the equations wherever it is not appearing in the differentiation.

```
rescaling := proc(f)
 global x;
 local f1, f2, i;
 description " Rescale r";
f1 := f;
 if f=x[1] then f1 := R_E + f;
elif f=x[2] or f=x[3] then f1 := f;
 elif f=H[1] or f=H[2] then f1 := f + R_E;
 elif type(f, atomic) then f1 := f;
 elif type( f, function) and op(0, op(0, f) ) ≠ symbol then f1 := f
elif  op(0, f) = Diff or op(0, f) = diff then f1 := Diff(rescaling(op(1,f) ), op(2,f) );
 elif op(0, f) = `=` then f1 := `=`(rescaling(op(1,f) ), rescaling(op(2,f) ) );
 elif op(0, f) = `*` or op(0, f) = `+` then
f1 := op(0, f) (rescaling(op(1,f) ) );
 for i from 2 to nops(f) do
f1 := op(0, f) (f1, rescaling(op(i,f) ) );
 end do;
 else
f1 := f;
for i from 1 to nops(f) do
f1 := applyop(rescaling, i, f1);
 end do;
 end if;
f1;
 end proc:
```

///

Taking out all constants from Diff

```
Differentiate :=proc(f, v)
 global x;
 local i, f2, f1, f3;
 f=expand(f);
```

```
if op(0, f) = `+` then
f1 := 0;
for i from 1 to nops(f) do
f1 := f1 + Differentiate(op(i, f), v);
end do;
elif op(0, f) = `*` then
f3 := 1; f2 := 1;
for i from 1 to nops(f) do
if diff(op(i, f), x[1]) = 0 and diff(op(i, f), x[2]) = 0 and diff(op(i, f), x[3]) = 0
    then f3 := f3·op(i, f);
else f2 := f2·op(i, f);
end if;
end do;
if diff(f2, v) ≠ 0 then f1 := f3·Diff(f2, v); else f1 := 0; end if;
elif diff(f, v) ≠ 0 then f1 := Diff(f, v);
else f1 := 0;
end if;
f1;
end proc:
```

//

Non-dimensionalization

```
nondim := proc (f)
global x;
local f1, f2, i;
description "Non Dimensionalization";
f1 := f;
if f=x[1] then f1 := epsilon·R_E·f;
 elif op(0, op(0, f) ) =BF or op(0, op(0, f) ) =b then f1 := G·f;
 elif op(0, op(0, f) ) =h then f1 := epsilon·R_E·f;
elif type(f, atomic) then f1 := f;
elif op(0, f) =u[1] then f1 := epsilon·sqrt(G·R_E)·f; elif op(0, f) =u[2] or op(0, f)
    =u[3] then f1 := sqrt(G·R_E)·f;
elif op(0, op(0, f) ) =omega then f1 := sqrt(G/R_E)·f;
 elif op(0, f) =p[1, 1] or op(0, f) =p[2, 2] or op(0, f) =p[3, 3] or op(0, op(0, f) )
    =Pb then f1 := epsilon·rho·G·R_E·f; elif op(0, f) =p[1, 2] or op(0, f) =p[1, 3]
    or op(0, f) =p[2, 3] or op(0, op(0, f) ) =Fr then f1 := delta·epsilon·rho·G·R_E·f;
```

#
elif $op(0, f) = Diff$ **or** $op(0, f) = diff$ **then**
if $op(2, f) = x[1]$ **then** $f1 := \frac{1}{\text{epsilon} \cdot R\_E}$; **elif** $op(2, f) = t$ **then** $f1 := \frac{1}{\text{sqrt}\left(\frac{R\_E}{G}\right)}$;
else $f1 := 1$;**end if**;
$f2 := simplify(nondim(op(1, f)))$;$f1 := Differentiate(f2, op(2, f)) \cdot f1$;
elif $op(0, f) = \text{`=`}$ **or** $op(0, f) = \text{`^`}$ **then** $f1 := op(0, f)\,(nondim(op(1, f)),$
$nondim(op(2, f)))$;
elif $op(0, f) = \text{`*`}$ **or** $op(0, f) = \text{`+`}$ **then**
$f1 := op(0, f)\,(nondim(op(1, f)))$;
for i **from** 2 **to** $nops(f)$ **do**
$f1 := op(0, f)\,(f1, nondim(op(i, f)))$;
end do;
else
for i **from** 1 **to** $nops(f)$ **do**
$f1 := applyop(nondim, i, f1)$;
end do;
end if;
$f1$;
end proc:

//

Removing constant factors

$selection :=$ **proc**(f)
global x;
local $f1, j, i$;
description "Selecting appropriate equations";
$j := 1$;
if $op(0, f) = \text{`=`}$ **then** $f1 := simplify\left(selection(op(1, f)) = \frac{selection(op(1, f))}{op(1, f)} \cdot op(2, f)\right)$;**if** $diff(denom(op(2, f1)), \text{epsilon}) \neq 0$ **then** $f1 := denom(op(2, f1)) \cdot f1$; **end if**;
elif $op(0, f) = \text{`*`}$ **then**
for i **from** 2 **to** $nops(f)$ **do** **if** $nops(op(i, f)) > nops(op(j, f))$ **then** $j := i$; **end if**; **end do**;
$f1 := op(j, f)$;
else

```
f1 := f;
 end if;
f1;
 end proc:
```

//
//
//

Integration by parts used in the depth averaging

```
integrate :=proc(f)
 global x;
 local f1, i;
f1 := 0;
 if op(0, f) = `+` then
  for i from 1 to nops(f) do
 f1 := f1 + lib_part(op(i, f) );
 end do;
 elif op(0, f) = `=` then f1 := integrate(op(1, f) ) = integrate(op(2, f) );
 else
 f1 := lib_part(f);
 end if;
 f1;
 end proc:

lib_part :=proc(f)
global x, H;
 local f2, f3, f4, f5, f6, i, j;
 f2 := f; f3 := 1; f4 := 1 ;
 if  op(0, f2) = `*` then
for i from 1 to nops(f2) do
if op(0, op(i, f2) ) = `Diff` then
f4 := f4·op(i, f2);
 else
 f3 := f3·op(i, f2);
 end if ;
 end do  ;
 elif op(0, f2) = `Diff` then
 f4 := f4·f2;
 else
  f3 := f3·f2;
  end if;
 if  f4 ≠ 1 then
 f5 := op(1, f4)·f3;
 f6 := -op(1, f4)·`diff`(f3, op(2, f4));
```

```
 if op(2, f4) = x[1] then
 f5 := subs(x[1] = H[1], f5) - subs(x[1] = H[2], f5);
 else
 f5 := Differentiate(simplify(int(f5, x[1] = H[2]..H[1])), op(2, f4)) + `Diff`(H[2], op(2, f4))
       ·subs(x[1] = H[2], f5) - `Diff`(H[1], op(2, f4))·subs(x[1] = H[1], f5);
 end if;
f6 := simplify(int(f6, x[1] = H[2]..H[1]));
 else
 f5 := 0; f6 := f3;
 f6 := simplify(int(f6, x[1] = H[2]..H[1]));
 end if;

 f5 + f6;
 end proc:
```

//
//
///

Substituting the Bundary conditions

```
boundary := proc(f, i)
 global x, H, n_K_BC, n_D_BC;
 local f3, f4, f5;
f3 := subs(diff = Diff, n_K_BC);
 f5 := subs(diff = Diff, n_D_BC);
 f4 := simplify( expand(f), {f3[1], f3[2]}) ;
 if i > 0 then
 f4 := simplify(expand(f4), {f5[1][i]});
 f4 := simplify(expand(f4), {f5[2][i]});
 end if;
 f4;
 end proc:
```

//
//
//

Final form removing irrelevant terms

```
Final_form := proc(f)
local f1;
 global x, H;
  if diff(f, x[2]) = 0 and diff(f, x[3]) = 0 then f1 := simplify( 1/(h[1] - h[2]) int(f, x[1] = h[2]
       .. h[1]), assume = [0 < (h[1] - h[2])] );
 elif op(0, f) = `int` then f1 := simplify((h[1] - h[2])·Final_form(op(1, f)), assume = [0 < (h[1]
       - h[2])]);
 elif op(0, f) = 'Diff' then f1 := Diff(Final_form(op(1, f)), op(2, f));
 elif type(f, function) then
```

```
if diff(f, x[1]) ≠ 0 then
if op(0, op(0,f)) = `symbol` then f1 := conjugate( op(0,f)) else f1 := conjugate( op(0, op(0,
   f))) [op( op(0,f))];
end if;
elif op(0, op(0,f)) = Fr or op(0, op(0,f)) = h or op(0, op(0,f)) = Pb or op(0,f) = h then f1
   := op(0,f) else f1 := f;
end if;
 elif op(0,f) = `^` then f1 := op(0,f) ( Final_form(op(1,f)), op(2,f));
else f1 := op(0,f) (seq(Final_form(op(i,f)), i = 1 ..nops(f)));
end if;
f1;
end proc:
```

//

```
Linearize :=proc(f)
global x;
local f1, deg, i, f2, f3, f4;
description "Linearization";
f2 := f;
f2 := expand(f2);
if op(0, f2) = `=` then f1 := op(0, f2)( Linearize(op(1, f2)), Linearize(op(2, f2)));
elif op(0, f2) = `Diff` then f1 := op(0, f2) (Linearize(op(1, f2)), op(2, f2)) ;
elif op(0, f2) = `+` then
f1 := 0;
for i from 1 to nops(f2) do
f1 := op(0, f2) (f1, Linearize(op(i, f2)));
end do;
elif op(0, f2) = `*` then
deg := 0;
f3 := 1; f4 := 1;
for i from 1 to nops(f2) do
if op(0, op(i, f2)) ≠ `Diff` then deg := deg + degree(op(i, f2), epsilon); f4 := f4·op(i,
   f2); else f3 := simplify(Diff(Linearize(op(1, op(i, f2))), op(2, op(i, f2))));
end if;
end do;
if deg > 1 then f1 := 0 else f1 := f4·f3; end if;
elif degree(f, epsilon) > 1 then f1 := 0
else f1 := f;
end if;
f1;
end proc:
```

//

Curvilinear Coordinates:

$with(LinearAlgebra) : x[1], x[2], x[3] := r, \text{theta}, \text{phi}; dim := 3;$

$$r, \theta, \phi$$

$$3 \tag{1}$$

Relation of curvilinear coordinates with cartesian coordinates:

$X := Vector(1..3) : X[1], X[2], X[3] := r \cdot \cos(x[3]) \cdot \sin(x[2]), r \cdot \sin(x[3]) \cdot \sin(x[2]), r \cdot \cos(x[2]);$

$$r\cos(\phi)\sin(\theta), r\sin(\phi)\sin(\theta), r\cos(\theta) \tag{2}$$

Defining covariant basis vectors:

$e := Matrix(1..3, 1..3) :$ **for** i **from** 1 **to** 3 **do for** j **from** 1 **to** 3 **do** $e[i, j] := diff(X[j], x[i]);$ **end do end do**;

$e;$

$$\begin{bmatrix} \cos(\phi)\sin(\theta) & \sin(\phi)\sin(\theta) & \cos(\theta) \\ r\cos(\phi)\cos(\theta) & r\sin(\phi)\cos(\theta) & -r\sin(\theta) \\ -r\sin(\phi)\sin(\theta) & r\cos(\phi)\sin(\theta) & 0 \end{bmatrix} \tag{3}$$

Metric tensor:

(4)

$M := Matrix(1..3, 1..3) :$ **for** i **from** 1 **to** 3 **do for** j **from** 1 **to** 3 **do** $M[i, j] := simplify(DotProduct(e[i], e[j], conjugate = false), assume = [0 < \text{theta}]);$ **end do**; **end do**; $M;$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & -(\cos(\theta)^2 - 1)r^2 \end{bmatrix} \tag{5}$$

$det\_M := Determinant(M);$

$$-r^4(\cos(\theta)^2 - 1) \tag{6}$$

(7)

$U := Vector(1..3) :$ **for** k **from** 1 **to** 3 **do** $U[k] := u[k](x[1], x[2], x[3], t);$**end do**:

$div\_u := expand\left(simplify\left(add\left(\frac{1}{\mathrm{sqrt}(det\_M)} \cdot \left(Diff\left(\frac{U[i] \cdot \mathrm{sqrt}(det\_M)}{\mathrm{sqrt}(M[i,i])}, x[i]\right)\right), i = 1..3\right), assume = [0 < r,\ 0 < \sin(\mathrm{theta})]\right)\right);$

$$\frac{\frac{\partial}{\partial r}\left(u_1(r,\theta,\phi,t)\, r^2 \sin(\theta)\right)}{r^2 \sin(\theta)} + \frac{\frac{\partial}{\partial \theta}\left(u_2(r,\theta,\phi,t)\, r \sin(\theta)\right)}{r^2 \sin(\theta)} + \frac{\frac{\partial}{\partial \phi}\left(u_3(r,\theta,\phi,t)\, r\right)}{r^2 \sin(\theta)} \tag{8}$$

Defining christoffel symbols only for orthogonal systems:

$R := Array(1..3, 1..3, 1..3) :$

for i **from** 1 **to** 3 **do**

  for j **from** 1 **to** 3 **do**

    for k **from** 1 **to** 3 **do**

$R[i,j,k] := simplify\left(DotProduct\left(diff\sim(e[i], x[j]), \frac{e[k]}{M[k,k]}, conjugate = false\right)\right);$

end do;

end do;

end do;

for i **from** 1 **to** 3 **do** $div\_uu[i] := simplify\left(simplify\left(add\left(\frac{1}{\mathrm{sqrt}(det\_M) \cdot \mathrm{sqrt}(M[i,i])} \cdot \left(Diff\left(\frac{U[k] \cdot U[i] \cdot \mathrm{sqrt}(det\_M) \cdot \mathrm{sqrt}(M[i,i])}{\mathrm{sqrt}(M[k,k])}, x[k]\right)\right), k = 1..3\right), assume = [0 < r,\ 0 < \sin(\mathrm{theta})]\right) + simplify\left(add\left(add\left(\frac{U[k] \cdot U[j]}{\mathrm{sqrt}(M[k,k]) \cdot \mathrm{sqrt}(M[j,j])} \cdot R[j,k,i] \cdot \mathrm{sqrt}(M[i,i]), j = 1..3\right) + \frac{2 \cdot U[k] \cdot U[i] \cdot \mathrm{sqrt}(M[i,i])}{\mathrm{sqrt}(M[k,k])} \cdot diff\left(\frac{1}{\mathrm{sqrt}(M[i,i])}, x[k]\right), k = 1..3\right), assume = [0 < r,\ 0 < \sin(\mathrm{theta})]\right)\right);$ **end do**;

$$-\frac{1}{r^2 \sin(\theta)}\left(u_3(r,\theta,\phi,t)^2\, r \sin(\theta) + u_2(r,\theta,\phi,t)^2\, r \sin(\theta) - \left(\frac{\partial}{\partial r}\left(u_1(r,\theta,\phi,t)^2\, r^2 \sin(\theta)\right)\right) - \left(\frac{\partial}{\partial \theta}\left(u_2(r,\theta,\phi,t)\, u_1(r,\theta,\phi,t)\, r \sin(\theta)\right)\right) - \left(\frac{\partial}{\partial \phi}\left(u_3(r,\theta,\phi,t)\, u_1(r,\theta,\phi,t)\, r\right)\right)\right)$$

$$-\frac{1}{r^3 \sin(\theta)}\left(u_3(r,\theta,\phi,t)^2 \cos(\theta)\, r^2 - \left(\frac{\partial}{\partial r}\left(u_1(r,\theta,\phi,t)\, u_2(r,\theta,\phi,t)\, r^3 \sin(\theta)\right)\right) - \left(\frac{\partial}{\partial \theta}\left(u_2(r,\theta,\phi,t)^2\, r^2 \sin(\theta)\right)\right) - \left(\frac{\partial}{\partial \phi}\left(u_3(r,\theta,\phi,t)\, u_2(r,\theta,\phi,t)\, r^2\right)\right)\right)$$

$$\frac{1}{r^3 \sin(\theta)^2}\left(\frac{\partial}{\partial r}\left(u_1(r,\theta,\phi,t)\, u_3(r,\theta,\phi,t)\, r^3 \sin(\theta)^2\right) + \frac{\partial}{\partial \theta}\left(u_2(r,\theta,\phi,t)\, u_3(r,\theta,\phi,t)\, r^2 \sin(\theta)^2\right) + \frac{\partial}{\partial \phi}\left(u_3(r,\theta,\phi,t)^2\, r^2 \sin(\theta)\right)\right) \tag{9}$$

$P := Matrix(1..3, 1..3)$:**for** i **from** 1 **to** 3 **do for** j **from** 1 **to** 3 **do** $P[i,j] := p[i,j](x[1], x[2], x[3], t)$; **end do**; **end do**; $P := subs(\{P[2,1] = P[1,2], P[3,2] = P[2,3], P[3,1] = P[1,3]\}, P)$:

for i **from** 1 **to** 3 **do** $div\_P[i] := simplify\left(simplify\left(add\left(\frac{1}{\mathrm{sqrt}(det\_M)\cdot \mathrm{sqrt}(M[i,i])} \cdot \left(Diff\left(\frac{P[k,i]\cdot \mathrm{sqrt}(det\_M)\cdot \mathrm{sqrt}(M[i,i])}{\mathrm{sqrt}(M[k,k])}, x[k]\right)\right), k = 1..3\right), assume = [0 < r,\ 0 < \sin(\mathrm{theta})]\right) + simplify\left(add\left(add\left(\frac{P[k,j]}{\mathrm{sqrt}(M[k,k])\cdot \mathrm{sqrt}(M[j,j])}\cdot R[j,k,i]\cdot \mathrm{sqrt}(M[i,i]), j = 1..3\right) + \frac{2\cdot P[k,i]\cdot \mathrm{sqrt}(M[i,i])}{\mathrm{sqrt}(M[k,k])}\cdot diff\left(\frac{1}{\mathrm{sqrt}(M[i,i])}, x[k]\right), k = 1..3\right), assume = [0 < r,\ 0 < \sin(\mathrm{theta})]\right)\right)$;**end do**;

$$-\frac{1}{r^2 \sin(\theta)}\left(r \sin(\theta)\, p_{2,2}(r,\theta,\phi,t) + r\sin(\theta)\, p_{3,3}(r,\theta,\phi,t) - \left(\frac{\partial}{\partial r}\left(p_{1,1}(r,\theta,\phi,t)\, r^2 \sin(\theta)\right)\right) - \left(\frac{\partial}{\partial \theta}\left(p_{1,2}(r,\theta,\phi,t)\, r\sin(\theta)\right)\right) - \left(\frac{\partial}{\partial \phi}\left(p_{1,3}(r,\theta,\phi,t)\, r\right)\right)\right)$$

$$-\frac{1}{r^3 \sin(\theta)}\left(p_{3,3}(r,\theta,\phi,t)\cos(\theta)\, r^2 - \left(\frac{\partial}{\partial r}\left(p_{1,2}(r,\theta,\phi,t)\, r^3 \sin(\theta)\right)\right) - \left(\frac{\partial}{\partial \theta}\left(p_{2,2}(r,\theta,\phi,t)\, r^2 \sin(\theta)\right)\right) - \left(\frac{\partial}{\partial \phi}\left(p_{2,3}(r,\theta,\phi,t)\, r^2\right)\right)\right)$$

$$\frac{1}{r^3 \sin(\theta)^2}\left(\frac{\partial}{\partial r}\left(p_{1,3}(r,\theta,\phi,t)\, r^3 \sin(\theta)^2\right) + \frac{\partial}{\partial \theta}\left(p_{2,3}(r,\theta,\phi,t)\, r^2 \sin(\theta)^2\right) + \frac{\partial}{\partial \phi}\left(p_{3,3}(r,\theta,\phi,t)\, r^2 \sin(\theta)\right)\right) \tag{10}$$

for k **from** 1 **to** 3 **do** $div\_P[k] := expand(simplify(subs(\{P[2,1] = P[1,2], P[3,2] = P[2,3], P[3,1] = P[1,3]\}, div\_P[k])))$ **end do**:

In prinicipal CS of the body:

$\mathrm{Omega} := Vector(1..3)$: **for** i **from** 1 **to** 3 **do** $\mathrm{Omega}[i] := \mathrm{omega}[i](t)$ **end do**:

In Body fixed curvilinear coodinates:

$B := Vector(1..3)$: **for** i **from** 1 **to** 3 **do** $B[i] := b[i](x[1], x[2], x[3])$; **end do**:

In principal CS:

$pos := Vector(1..3) : pos := X$:

$Cent := simplify(CrossProduct(\mathrm{Omega}, CrossProduct(\mathrm{Omega}, pos)), assume = [0 < x[1], \sin(x[2]) > 0]) :$

In curvilinear coordinates:

$temp := Vector(1..3)$:**for** i **from** 1 **to** 3 **do** $temp[i] := simplify\left(DotProduct\left(Cent, \frac{e[i]}{\mathrm{sqrt}(M[i,i])}, conjugate = false\right), assume = [0 < x[1], \sin(x[2]) > 0]\right)$ **end do**:

$B := B - temp :$
$B := simplify(subs(\{\mathrm{omega}[1](t) = 0, \mathrm{omega}[2](t) = 0\}, B), assume = [0 < x[1], \sin(x[2]) > 0]);$

$$\begin{bmatrix} -\omega_3(t)^2 \cos(\theta)^2 r + \omega_3(t)^2 r + b_1(r, \theta, \phi) \\ b_2(r, \theta, \phi) + r\,\omega_3(t)^2 \sin(\theta) \cos(\theta) \\ b_3(r, \theta, \phi) \end{bmatrix} \quad \textbf{(11)}$$

In Principal CS:
$A := Vector(1..3)$: **for** i **from** 1 **to** 3 **do** $A[i] := \mathrm{alpha}[i](t)$; **end do**:
$A\_acc := CrossProduct(A, pos)$:
In CCS:
$temp[1..3] := A\_acc[1..3]$: **for** i **from** 1 **to** 3 **do** $A\_acc[i] := simplify\left(DotProduct\left(temp, \frac{e[i]}{\mathrm{sqrt}(M[i,i])}, conjugate = false\right), assume = [0 < x[1], \sin(x[2]) > 0]\right)$ **end do**:
$A\_acc := simplify(subs(\{\mathrm{alpha}[1](t) = 0, \mathrm{alpha}[2](t) = 0\}, A\_acc), assume = [0 < x[1], \sin(x[2]) > 0]);$

$$\begin{bmatrix} 0 \\ 0 \\ \sin(\theta)\,\alpha_3(t)\,r \end{bmatrix} \quad \textbf{(12)}$$

In BFCS:

$Omega\_B := Vector(1..3)$: **for** i **from** 1 **to** 3 **do** $Omega\_B[i] := simplify\left(DotProduct\left(\mathrm{Omega}, \frac{e[i]}{\mathrm{sqrt}(M[i,i])}, conjugate = false\right), assume = [0 < x[1], \sin(x[2]) > 0]\right)$ **end do**: $corr := simplify(CrossProduct(Omega\_B, U), assume = [0 < x[1], \sin(x[2]) > 0]) :$

$corr := simplify(subs(\{\mathrm{omega}[1](t) = 0, \mathrm{omega}[2](t) = 0\}, corr));$

$$\begin{bmatrix} -\omega_3(t) \sin(\theta)\, u_3(r, \theta, \phi, t) \\ -\omega_3(t) \cos(\theta)\, u_3(r, \theta, \phi, t) \\ \omega_3(t) \left(\sin(\theta)\, u_1(r, \theta, \phi, t) + \cos(\theta)\, u_2(r, \theta, \phi, t)\right) \end{bmatrix} \quad \textbf{(13)}$$

Acceleration of the core in PCS:

$A\_core := Vector(1..3)$:**for** i **from** 1 **to** 3 **do** $A\_core[i] := a[i](t)$ **end do**:
Acceleration of the core in CCS:
$a\_B := Vector(1..3)$:
for i **from** 1 **to** 3 **do** $a\_B[i] := simplify\left(DotProduct\left(A\_core, \frac{e[i]}{\mathrm{sqrt}(M[i,i])}, conjugate = false\right),\right.$
$\left. assume = [0 < x[1], \sin(x[2]) > 0]\right)$ **end do**;

$$a_1(t)\cos(\phi)\sin(\theta) + a_2(t)\sin(\phi)\sin(\theta) + a_3(t)\cos(\theta)$$
$$\cos(\phi)\cos(\theta)\,a_1(t) + \sin(\phi)\cos(\theta)\,a_2(t) - \sin(\theta)\,a_3(t)$$
$$\cos(\phi)\,a_2(t) - \sin(\phi)\,a_1(t) \tag{14}$$

Momentum balance:

$mom := Vector(1..3)$: **for** i **from** 1 **to** 3 **do** $mom[i] := \mathrm{rho}\cdot Diff(U[i]\cdot denom(div\_uu[i]), t) + \mathrm{rho}\cdot numer(div\_uu[i]) = -numer(div\_P[i]) + \mathrm{rho}\cdot BF[i](x[1], x[2], x[3], t)\cdot denom(div\_uu[i])$;**end do**:*#BF also contains acceleration of the core*
$cont := numer(div\_u)$;

$$\frac{\partial}{\partial r}\left(u_1(r,\theta,\phi,t)\,r^2\sin(\theta)\right) + \frac{\partial}{\partial \theta}\left(u_2(r,\theta,\phi,t)\,r\sin(\theta)\right) + \frac{\partial}{\partial \phi}\left(u_3(r,\theta,\phi,t)\,r\right) \tag{15}$$

for i **from** 1 **to** 3 **do** $mom[i] := collect(mom[i], \mathrm{rho})$ **end do**;

$$\left(-u_2(r,\theta,\phi,t)^2 r\sin(\theta) - u_3(r,\theta,\phi,t)^2 r\sin(\theta) + \frac{\partial}{\partial t}\left(u_1(r,\theta,\phi,t)\,r^2\sin(\theta)\right) + \frac{\partial}{\partial r}\left(u_1(r,\theta,\phi,t)^2 r^2\sin(\theta)\right) + \frac{\partial}{\partial \theta}\left(u_2(r,\theta,\phi,t)\,u_1(r,\theta,\phi,t)\,r\sin(\theta)\right) + \frac{\partial}{\partial \phi}\left(u_3(r,\theta,\phi,t)\,u_1(r,\theta,\phi,t)\,r\right)\right)\rho = r\sin(\theta)\,p_{2,2}(r,\theta,\phi,t) + r\sin(\theta)\,p_{3,3}(r,\theta,\phi,t) - \left(\frac{\partial}{\partial r}\left(p_{1,1}(r,\theta,\phi,t)\,r^2\sin(\theta)\right)\right) - \left(\frac{\partial}{\partial \theta}\left(p_{1,2}(r,\theta,\phi,t)\,r\sin(\theta)\right)\right) - \left(\frac{\partial}{\partial \phi}\left(p_{1,3}(r,\theta,\phi,t)\,r\right)\right) + \rho\,BF_1(r,\theta,\phi,t)\,r^2\sin(\theta)$$

$$\left(-u_3(r,\theta,\phi,t)^2\cos(\theta)\,r^2 + \frac{\partial}{\partial t}\left(u_2(r,\theta,\phi,t)\,r^3\sin(\theta)\right) + \frac{\partial}{\partial r}\left(u_1(r,\theta,\phi,t)\,u_2(r,\theta,\phi,t)\,r^3\sin(\theta)\right) + \frac{\partial}{\partial \theta}\left(u_2(r,\theta,\phi,t)^2 r^2\sin(\theta)\right) + \frac{\partial}{\partial \phi}\left(u_3(r,\theta,\phi,t)\,u_2(r,\theta,\phi,t)\,r^2\right)\right)\rho = p_{3,3}(r,\theta,\phi,t)\cos(\theta)\,r^2 - \left(\frac{\partial}{\partial r}\left(p_{1,2}(r,\theta,\phi,t)\,r^3\sin(\theta)\right)\right) - \left(\frac{\partial}{\partial \theta}\left(p_{2,2}(r,\theta,\phi,t)\,r^2\sin(\theta)\right)\right) - \left(\frac{\partial}{\partial \phi}\left(p_{2,3}(r,\theta,\phi,t)\,r^2\right)\right) + \rho\,BF_2(r,\theta,\phi,t)\,r^3\sin(\theta)$$

$$\left(\frac{\partial}{\partial t}\left(u_3(r,\theta,\phi,t)\,r^3\sin(\theta)^2\right) + \frac{\partial}{\partial r}\left(u_1(r,\theta,\phi,t)\,u_3(r,\theta,\phi,t)\,r^3\sin(\theta)^2\right) + \frac{\partial}{\partial \theta}\left(u_2(r,\theta,\phi,t)\,u_3(r,\theta,\phi,t)\,r^2\sin(\theta)^2\right) + \frac{\partial}{\partial \phi}\left(u_3(r,\theta,\phi,t)^2 r^2\sin(\theta)\right)\right)\rho = -\left(\frac{\partial}{\partial r}\left(p_{1,3}(r,\theta,\phi,\right.\right. \tag{16}$$

$t)\ r^3 \sin(\theta)^2\big)\Big) - \Big(\frac{\partial}{\partial\theta}\big(p_{2,3}(r,\theta,\phi,t)\ r^2 \sin(\theta)^2\big)\Big) - \Big(\frac{\partial}{\partial\phi}\big(p_{3,3}(r,\theta,\phi,$
$t)\ r^2 \sin(\theta)\big)\Big) + \rho\, BF_3(r,\theta,\phi,t)\ r^3 \sin(\theta)^2$

Boundary conditions:

$H := Vector(1..2) : K\_BC := Vector(1..2) : N := Vector(1..2)$:**for** i **from** 1 **to** 2 **do** $H[i] := h[i](x[2], x[3], t); N[i] := subs\left(x[1] = H[i], convert\left(\left[\frac{diff(x[1] - H[i], x[1])}{\mathrm{sqrt}(M[1,1])}, \frac{diff(x[1] - H[i], x[2])}{\mathrm{sqrt}(M[2,2])}, \frac{diff(x[1] - H[i], x[3])}{\mathrm{sqrt}(M[3,3])}\right], Vector\right)\right); K\_BC[i] := simplify(subs(x[1] = H[i], DotProduct(U, N[i], conjugate = false) = diff(H[i], t)), assume = [0 < H[i], \sin(x[2]) > 0]) :$
end do:
$K\_BC;$

$$\left[\left[\frac{1}{h_1(\theta,\phi,t)\sin(\theta)}\left(u_1(h_1(\theta,\phi,t),\theta,\phi,t)\sin(\theta)\,h_1(\theta,\phi,t) - u_2(h_1(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\theta}h_1(\theta,\phi,t)\right)\sin(\theta) - u_3(h_1(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\phi}h_1(\theta,\phi,t)\right)\right) = \frac{\partial}{\partial t}h_1(\theta,\phi,t)\right],\right.$$
$$\left.\left[\frac{1}{h_2(\theta,\phi,t)\sin(\theta)}\left(u_1(h_2(\theta,\phi,t),\theta,\phi,t)\sin(\theta)\,h_2(\theta,\phi,t) - u_2(h_2(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\theta}h_2(\theta,\phi,t)\right)\sin(\theta) - u_3(h_2(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\phi}h_2(\theta,\phi,t)\right)\right) = \frac{\partial}{\partial t}h_2(\theta,\phi,t)\right]\right] \quad \textbf{(17)}$$

$n\_cont := simplify(rescaling(\ cont)) : n\_cont := nondim(n\_cont) : n\_cont := selection(\ simplify(n\_cont, assume = [0 < R\_E, 0 < G, 0 < \sin(\mathrm{theta})]));$

$$\frac{\partial}{\partial r}\left(u_1(r,\theta,\phi,t)\,(\varepsilon r + 1)^2 \sin(\theta)\right) + \frac{\partial}{\partial\theta}\left(u_2(r,\theta,\phi,t)\,(\varepsilon r + 1)\sin(\theta)\right) + \frac{\partial}{\partial\phi}\left(u_3(r,\theta,\phi,t)\,(\varepsilon r + 1)\right) \quad \textbf{(18)}$$

$n\_mom := Vector(1..3)$:**for** i **from** 1 **to** 3 **do** $n\_mom[i] := rescaling(\ mom[i]);\ \ n\_mom[i] := nondim(\ n\_mom[i]); n\_mom[i] := selection(simplify(n\_mom[i], assume = [0 < R\_E, 0 < G])\ );$
end do:

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$n\_K\_BC := Vector(1..2)$:**for** i **from** 1 **to** 2 **do** $n\_K\_BC[i] := rescaling(\ K\_BC[i]);\ \ n\_K\_BC[i] := simplify(\ nondim(\ n\_K\_BC[i])); n\_K\_BC[i] := simplify(selection(\ n\_K\_BC[i]), assume = [0 < R\_E, 0 < G, 0 < \mathrm{epsilon}]);$

end do:
n_K_BC;

$$\left[\left[\sin(\theta)\, u_1(h_1(\theta,\phi,t),\theta,\phi,t)\, h_1(\theta,\phi,t)\,\varepsilon - \sin(\theta)\, u_2(h_1(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\theta} h_1(\theta,\phi,t)\right) + u_1(h_1(\theta,\phi,t),\theta,\phi,t)\sin(\theta) - u_3(h_1(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\phi} h_1(\theta,\phi,t)\right) = (h_1(\theta,\phi,t)\,\varepsilon+1)\sin(\theta)\left(\frac{\partial}{\partial t} h_1(\theta,\phi,t)\right)\right], \left[\sin(\theta)\, h_2(\theta,\phi,t)\, u_1(h_2(\theta,\phi,t),\theta,\phi,t)\,\varepsilon - \sin(\theta)\, u_2(h_2(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\theta} h_2(\theta,\phi,t)\right) + u_1(h_2(\theta,\phi,t),\theta,\phi,t)\sin(\theta) - u_3(h_2(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\phi} h_2(\theta,\phi,t)\right) = (h_2(\theta,\phi,t)\,\varepsilon+1)\sin(\theta)\left(\frac{\partial}{\partial t} h_2(\theta,\phi,t)\right)\right]\right] \quad \textbf{(20)}$$

$D\_BC := Vector(1..2) : D\_BC[1] := subs\left(x[1]=H[1], simplify\left(\frac{P.N[1]}{VectorCalculus[Norm](N[1])}, assume=[0<x[1], \sin(x[2])>0]\right)\right) :$
$D\_BC[2] := subs\left(x[1]=H[2], simplify\left(\frac{P.N[2]}{VectorCalculus[Norm](N[2])}, assume=[0<x[1], \sin(x[2])>0]\right)\right) :$
for i **from** 1 **to** 3 **do** $D\_BC[1][i] := simplify(D\_BC[1][i]=0, assume=[0<x[1], \sin(x[2])>0]); D\_BC[2][i] := simplify(D\_BC[2][i]=Pb[i](x[2],x[3],t)+Fr[i](x[2],x[3],t), assume=[0<x[1], \sin(x[2])>0]);$ **end do**:
$n\_D\_BC := Vector(1..2)$:**for** i **from** 1 **to** 2 **do for** j **from** 1 **to** 3 **do** $n\_D\_BC[i][j] := rescaling(D\_BC[i][j]); n\_D\_BC[i][j] := nondim(n\_D\_BC[i][j]); n\_D\_BC[i][j] := selection(simplify(n\_D\_BC[i][j], assume=[0<R\_E, 0<G, \sin(x[2])>0, H[i]>0, \text{epsilon}>0])) : n\_D\_BC[i][j] := convert(series(op(1, n\_D\_BC[i][j]), \text{epsilon}, 2), polynom) = convert(series(op(2, n\_D\_BC[i][j]), \text{epsilon}, 2), polynom) : n\_D\_BC[i][j] := selection(simplify(n\_D\_BC[i][j], assume=[0<R\_E, 0<G, \sin(x[2])>0])) : n\_D\_BC[i][j] := expand(subs(x[1]=H[i], n\_D\_BC[i][j])) :$
end do:**end do**:
$n\_D\_BC[1][1]; n\_D\_BC[1][2]; n\_D\_BC[1][3]; n\_D\_BC[2][1]; n\_D\_BC[2][2]; n\_D\_BC[2][3];$

$$-\left(\frac{\partial}{\partial\theta} h_1(\theta,\phi,t)\right) p_{1,2}(h_1(\theta,\phi,t),\theta,\phi,t)\sin(\theta)\,\delta\varepsilon + h_1(\theta,\phi,t)\sin(\theta)\, p_{1,1}(h_1(\theta,\phi,t),\theta,\phi,t)\,\varepsilon - \left(\frac{\partial}{\partial\phi} h_1(\theta,\phi,t)\right) p_{1,3}(h_1(\theta,\phi,t),\theta,\phi,t)\,\delta\varepsilon + p_{1,1}(h_1(\theta,\phi,t),\theta,\phi,t)\sin(\theta) = 0$$

$$h_1(\theta,\phi,t)\sin(\theta)\, p_{1,2}(h_1(\theta,\phi,t),\theta,\phi,t)\,\delta\varepsilon - p_{2,2}(h_1(\theta,\phi,t),\theta,\phi,t)\sin(\theta)\left(\frac{\partial}{\partial\theta} h_1(\theta,\phi,t)\right)\varepsilon - p_{2,3}(h_1(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\phi} h_1(\theta,\phi,t)\right)\delta\varepsilon + \delta p_{1,2}(h_1(\theta,\phi,t),\theta,\phi,t)\sin(\theta) = 0$$

$$p_{2,3}(h_1(\theta,\phi,t),\theta,\phi,t)\sin(\theta)\left(\frac{\partial}{\partial\theta}h_1(\theta,\phi,t)\right)\delta\varepsilon-\sin(\theta)p_{1,3}(h_1(\theta,\phi,t),\theta,\phi,t)h_1(\theta,\phi,t)\delta\varepsilon-\delta p_{1,3}(h_1(\theta,\phi,t),\theta,\phi,t)\sin(\theta)+p_{3,3}(h_1(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\phi}h_1(\theta,\phi,t)\right)\varepsilon=0$$

$$p_{1,2}(h_2(\theta,\phi,t),\theta,\phi,t)\sin(\theta)\left(\frac{\partial}{\partial\theta}h_2(\theta,\phi,t)\right)\delta\varepsilon-\sin(\theta)p_{1,1}(h_2(\theta,\phi,t),\theta,\phi,t)h_2(\theta,\phi,t)\varepsilon+p_{1,3}(h_2(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\phi}h_2(\theta,\phi,t)\right)\delta\varepsilon-p_{1,1}(h_2(\theta,\phi,t),\theta,\phi,t)\sin(\theta)=-\sin(\theta)h_2(\theta,\phi,t)Fr_1(\theta,\phi,t)\delta\varepsilon-\sin(\theta)h_2(\theta,\phi,t)Pb_1(\theta,\phi,t)\varepsilon-\sin(\theta)Fr_1(\theta,\phi,t)\delta-\sin(\theta)Pb_1(\theta,\phi,t)$$

$$h_2(\theta,\phi,t)p_{1,2}(h_2(\theta,\phi,t),\theta,\phi,t)\sin(\theta)\delta\varepsilon-p_{2,3}(h_2(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\phi}h_2(\theta,\phi,t)\right)\delta\varepsilon-p_{2,2}(h_2(\theta,\phi,t),\theta,\phi,t)\sin(\theta)\left(\frac{\partial}{\partial\theta}h_2(\theta,\phi,t)\right)\varepsilon+\delta p_{1,2}(h_2(\theta,\phi,t),\theta,\phi,t)\sin(\theta)=\sin(\theta)Fr_2(\theta,\phi,t)h_2(\theta,\phi,t)\delta\varepsilon+\sin(\theta)Pb_2(\theta,\phi,t)h_2(\theta,\phi,t)\varepsilon+\sin(\theta)Fr_2(\theta,\phi,t)\delta+\sin(\theta)Pb_2(\theta,\phi,t)$$

$$h_2(\theta,\phi,t)\sin(\theta)p_{1,3}(h_2(\theta,\phi,t),\theta,\phi,t)\delta\varepsilon-p_{2,3}(h_2(\theta,\phi,t),\theta,\phi,t)\sin(\theta)\left(\frac{\partial}{\partial\theta}h_2(\theta,\phi,t)\right)\delta\varepsilon+\delta p_{1,3}(h_2(\theta,\phi,t),\theta,\phi,t)\sin(\theta)-p_{3,3}(h_2(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\phi}h_2(\theta,\phi,t)\right)\varepsilon=\sin(\theta)Fr_3(\theta,\phi,t)h_2(\theta,\phi,t)\delta\varepsilon+\sin(\theta)Pb_3(\theta,\phi,t)h_2(\theta,\phi,t)\varepsilon+\sin(\theta)Fr_3(\theta,\phi,t)\delta+\sin(\theta)Pb_3(\theta,\phi,t) \quad \textbf{(21)}$$

Depth Averaging:

$DA\_cont := integrate(\ n\_cont) : DA\_cont := collect(boundary(DA\_cont, 0), \text{epsilon}) : DA\_cont := Linearize(collect(algsubs(h[1]-h[2]=h, simplify(Final\_form(DA\_cont), assume=[(H[1]-H[2])>0])), \text{epsilon}));$

$$2\varepsilon h_2\sin(\theta)\left(\frac{\partial}{\partial t}(h_2+h)\right)-2\left(\frac{\partial}{\partial t}h_2\right)\sin(\theta)\varepsilon h_2+2\left(\frac{\partial}{\partial t}(h_2+h)\right)\sin(\theta)\varepsilon h+\sin(\theta)\left(\frac{\partial}{\partial t}(h_2+h)\right)-\sin(\theta)\left(\frac{\partial}{\partial t}h_2\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{2}\varepsilon h^2\overline{u}_3+\varepsilon h h_2\overline{u}_3+\overline{u}_3 h\right)+\frac{\partial}{\partial\theta}\left(\frac{1}{2}\sin(\theta)\overline{u}_2 h^2\varepsilon+\sin(\theta)\overline{u}_2 h\varepsilon h_2+\sin(\theta)\overline{u}_2 h\right) \quad \textbf{(22)}$$

$DA\_mom := Vector(1..3)$:**for** i **from** 1 **to** 3 **do** $DA\_mom[i] := integrate(\ n\_mom[i], x[1]=H[2]..H[1]) : DA\_mom[i] := boundary(DA\_mom[i], i) : DA\_mom[i] := collect(algsubs(h[1]-h[2]=h, simplify(Final\_form(DA\_mom[i]), assume=[(h[1]-h[2])>0])), \text{epsilon}) : DA\_mom[i] := Linearize(expand(DA\_mom[i])) :$
end do:

$DA\_mom[2];$

$$-\cos(\theta)\,\varepsilon\,\overline{u}_3^2 h^2 - 2\,\varepsilon\cos(\theta)\,\overline{u}_3^2 h\,h_2 - \cos(\theta)\,\overline{u}_3^2 h + \frac{\partial}{\partial\theta}\left(\sin(\theta)\,\varepsilon\,\overline{u}_2^2 h^2 + 2\,\varepsilon\,h_2\sin(\theta)\,\overline{u}_2^2 h + \sin(\theta)\,\overline{u}_2^2 h\right) + \frac{\partial}{\partial\phi}\left(\varepsilon\,h^2\,\overline{u}_2\,\overline{u}_3 + 2\,\varepsilon\,h\,h_2\,\overline{u}_2\,\overline{u}_3 + h\,\overline{u}_2\,\overline{u}_3\right) + \frac{\partial}{\partial t}\left(\frac{3}{2}\sin(\theta)\,\overline{u}_2 h^2\varepsilon + 3\sin(\theta)\,\overline{u}_2 h\,\varepsilon\,h_2 + \sin(\theta)\,\overline{u}_2 h\right) = -\varepsilon\,\delta\left(\frac{\partial}{\partial\phi}\left(\varepsilon\,h^2\,\overline{p}_{2,3} + 2\,\varepsilon\,h\,h_2\,\overline{p}_{2,3} + h\,\overline{p}_{2,3}\right)\right) + \sin(\theta)\,\overline{BF}_2 h + \sin(\theta)\,Pb_2 + \sin(\theta)\,Fr_2\,\delta - \varepsilon\left(\frac{\partial}{\partial\theta}\left(\sin(\theta)\,\overline{p}_{2,2} h\left(\varepsilon\,h + 2\,\varepsilon\,h_2 + 1\right)\right)\right) + \frac{3}{2}\sin(\theta)\,\varepsilon\,\overline{BF}_2 h^2 + \cos(\theta)\,\overline{p}_{3,3}\,\varepsilon\,h + 3\,\varepsilon\,h_2\sin(\theta)\,\overline{BF}_2 h + 3\sin(\theta)\,Fr_2\,h_2\,\delta\,\varepsilon + 3\sin(\theta)\,Pb_2\,h_2\,\varepsilon \qquad \textbf{(23)}$$

for i **from** 1 **to** 3 **do** $Fr[i] := simplify\left(-\left(U[i] - \frac{DotProduct\left(U, \frac{N[2]}{VectorCalculus[Norm](N[2])}, conjugate = false\right)\cdot N[2][i]}{VectorCalculus[Norm](N[2])}\right) \middle/ \left(VectorCalculus[Norm]\left(U - \frac{DotProduct\left(U, \frac{N[2]}{VectorCalculus[Norm](N[2])}, conjugate = false\right)\cdot N[2]}{VectorCalculus[Norm](N[2])}\right)\right) \cdot \frac{DotProduct(P.N[2], N[2], conjugate = false)}{VectorCalculus[Norm](N[2])^2}\cdot \text{delta}, assume = [0 < x[1], \sin(x[2]) > 0, H[2] > 0, H[1] > 0]\right); Pb[i] := simplify\left(\frac{DotProduct(P.N[2], N[2], conjugate = false) N[2][i]}{VectorCalculus[Norm](N[2])^3}, assume = [0 < x[1], \sin(x[2]) > 0, H[2] > 0, H[1] > 0]\right); BF[i] := B[i] - 2\cdot corr[i] - A\_acc[i];$**end do**:

for j **from** 1 **to** 3 **do** $Fr[j] := rescaling(\ Fr[j]); Fr[j] := simplify\left(\frac{nondim(\ Fr[j])}{\text{delta}\cdot\text{epsilon}\cdot G\cdot R\_E\cdot\text{rho}}\right); Fr[j] := simplify(convert(series(Fr[j], \text{epsilon}, 2), polynom)); Fr[j] := simplify(collect(subs(x[1] = H[2], Fr[j]), \text{epsilon}), assume = [0 < R\_E, 0 < G, \sin(x[2]) > 0]) : Pb[j] := rescaling(\ Pb[j]); Pb[j] := simplify\left(\frac{nondim(\ Pb[j])}{\text{epsilon}\cdot G\cdot R\_E\cdot\text{rho}}\right); Pb[j] := simplify(convert(series(Pb[j], \text{epsilon}, 2), polynom)); Pb[j] := simplify(collect(subs(x[1] = H[2], Pb[j]), \text{epsilon}), assume = [0 < R\_E, 0 < G, \sin(x[2]) > 0]) : BF[j] := rescaling(\ BF[j]); BF[j] := simplify(\ nondim(\ BF[j])); BF[j] := simplify(convert(series(BF[j], \text{epsilon}, 2), polynom)); BF[j] := integrate(\ BF[j], x[1] = H[2]..H[1]\ ) : BF[j] := algsubs\left(h[1] - h[2] = h\ , simplify\left(\frac{Final\_form(BF[j])}{G},\right.\right.$

$assume[H[1]-H[2] > 0, R\_E > 0, G > 0]\Big)\Big) : BF[j] := simplify(BF[j], assume = [0 < R\_E, G > 0]);$

end do:

$BF[1]; BF[2]; BF[3]; Fr[1]; Fr[2]; Fr[3];$

$$-\frac{1}{2}\, h \left(\cos(\theta)^2 \omega_3(t)^2 \varepsilon h + 2\cos(\theta)^2 \omega_3(t)^2 \varepsilon h_2 + 2\,\omega_3(t)^2 \cos(\theta)^2 - \omega_3(t)^2 \varepsilon h - 2\,\omega_3(t)^2 \varepsilon h_2 - 4\,\overline{u}_3\, \omega_3(t) \sin(\theta) - 2\,\omega_3(t)^2 - 2\,\overline{b}_1\right)$$

$$\frac{1}{2}\, h \left(\cos(\theta)\, \omega_3(t)^2 \sin(\theta)\, \varepsilon h + 2\cos(\theta)\, \omega_3(t)^2 \sin(\theta)\, \varepsilon h_2 + 2\cos(\theta)\, \omega_3(t)^2 \sin(\theta) + 4\,\overline{u}_3 \cos(\theta)\, \omega_3(t) + 2\,\overline{b}_2\right)$$

$$-\frac{1}{2}\,\frac{1}{G}\Big(h\,\big(4\,G \sin(\theta)\, \omega_3(t)\, \varepsilon\, \overline{u}_1 + \sin(\theta)\, \alpha_3(t)\, R\_E\, \varepsilon h + 2\sin(\theta)\, \alpha_3(t)\, R\_E\, \varepsilon h_2 + 4\,G\, \omega_3(t) \cos(\theta)\, \overline{u}_2 + 2\sin(\theta)\, \alpha_3(t)\, R\_E - 2\,G\,\overline{b}_3\big)\Big)$$

$$-\left(\varepsilon \left(\sin(\theta)\, u_2(h_2(\theta,\phi,t),\theta,\phi,t) \left(\frac{\partial}{\partial\theta} h_2(\theta,\phi,t)\right) + u_3(h_2(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\phi} h_2(\theta,\phi,t)\right)\right) p_{1,1}(h_2(\theta,\phi,t),\theta,\phi,t)\right) \Big/ \left(\sqrt{u_2(h_2(\theta,\phi,t),\theta,\phi,t)^2 + u_3(h_2(\theta,\phi,t),\theta,\phi,t)^2}\,\sin(\theta)\right)$$

$$\left(u_2(h_2(\theta,\phi,t),\theta,\phi,t)\left(2\,p_{1,2}(h_2(\theta,\phi,t),\theta,\phi,t)\sin(\theta)\left(\frac{\partial}{\partial\theta} h_2(\theta,\phi,t)\right)\delta\varepsilon + 2\,p_{1,3}(h_2(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\phi} h_2(\theta,\phi,t)\right)\delta\varepsilon - p_{1,1}(h_2(\theta,\phi,t),\theta,\phi,t)\sin(\theta)\right)\right) \Big/ \left(\sqrt{u_2(h_2(\theta,\phi,t),\theta,\phi,t)^2 + u_3(h_2(\theta,\phi,t),\theta,\phi,t)^2}\,\sin(\theta)\right)$$

$$\left(u_3(h_2(\theta,\phi,t),\theta,\phi,t)\left(2\,p_{1,2}(h_2(\theta,\phi,t),\theta,\phi,t)\sin(\theta)\left(\frac{\partial}{\partial\theta} h_2(\theta,\phi,t)\right)\delta\varepsilon + 2\,p_{1,3}(h_2(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\phi} h_2(\theta,\phi,t)\right)\delta\varepsilon - p_{1,1}(h_2(\theta,\phi,t),\theta,\phi,t)\sin(\theta)\right)\right) \Big/ \left(\sqrt{u_2(h_2(\theta,\phi,t),\theta,\phi,t)^2 + u_3(h_2(\theta,\phi,t),\theta,\phi,t)^2}\,\sin(\theta)\right) \quad \textbf{(24)}$$

$nondim(\text{omega}[1](t));$

$$\sqrt{\frac{G}{R\_E}}\ \omega_1(t) \tag{25}$$

$Diff(r^2, R\_E + r)$

$$Diff(r^2, R\_E + r) \tag{26}$$

$op(op(op(op(\ 0, \text{sqrt}(r)))))$

$$builtin, 221 \tag{27}$$

$(23);$

$$23 \tag{28}$$

(29)

$test := op(2, div\_u);$

$$\frac{\frac{\partial}{\partial \theta}\left(u_2(r, \theta, \phi, t)\ r \sin(\theta)\right)}{r^2 \sin(\theta)} \tag{30}$$

$rescaling(div\_u)$

$$\frac{\frac{\partial}{\partial r}\left(u_1(r, \theta, \phi, t)\ (R\_E + r)^2 \sin(\theta)\right)}{(R\_E + r)^2 \sin(\theta)} + \frac{\frac{\partial}{\partial \theta}\left(u_2(r, \theta, \phi, t)\ (R\_E + r) \sin(\theta)\right)}{(R\_E + r)^2 \sin(\theta)} + \frac{\frac{\partial}{\partial \phi}\left(u_3(r, \theta, \phi, t)\ (R\_E + r)\right)}{(R\_E + r)^2 \sin(\theta)} \tag{31}$$

$op(\ test)$

$$\frac{1}{r^2}, \frac{1}{\sin(\theta)}, \frac{\partial}{\partial \theta}\left(u_2(r, \theta, \phi, t)\ r \sin(\theta)\right) \tag{32}$$

`` `+`(2, 1); ``

$$3 \tag{33}$$