Special functions:

```
restart;
rescaling := proc(f)
 global x;
 local f1, f2, i;
 description " Rescale r";
f1 := f;
 if f=x[1] then f1 := R_E + f;
elif f=x[2] or f=x[3] then f1 := f;
 elif f=H[1] or f=H[2] then f1 := f + R_E;
 elif type(f, atomic) then f1 := f;
 elif type( f, function) and op(0, op(0, f) ) ≠ symbol then f1 := f
elif  op(0, f) = Diff or op(0, f) = diff then f1 := Diff(rescaling(op(1, f) ), op(2, f) );
 elif op(0, f) = `=` then f1 := `=`(rescaling(op(1, f) ), rescaling(op(2, f) ) );
 elif  op(0, f) = `*` or op(0, f) = `+` then
f1 := op(0, f) (rescaling(op(1, f) ) );
 for i from 2 to nops(f) do
f1 := op(0, f) (f1, rescaling(op(i, f) ) );
 end do;
 else
f1 := f;
for i from 1 to nops(f) do
f1 := applyop(rescaling, i, f1);
 end do;
 end if;
f1;
 end proc:

Differentiate := proc(f, v)
 global x;
 local i, f2, f1, f3;
f=expand(f);
 if op(0, f) = `+` then
f1 := 0;
 for i from 1 to nops(f) do
f1 := f1 + Differentiate(op(i, f), v);
 end do;
 elif op(0, f) = `*` then
f3 := 1; f2 := 1;
 for i from 1 to nops(f) do
 if  diff(op(i, f), v) = 0 then f3 := f3·op(i, f);
 else f2 := f2·op(i, f);
```

```
 end if;
 end do;
if diff(f2, v) ≠ 0 then f1 := f3·Diff(f2, v); else f1 := 0; end if;
 elif diff(f, v) ≠ 0 then f1 := Diff(f, v);
 else f1 := 0; end if;
f1;
 end proc:


nondim := proc (f)
 global x;
 local f1, f2, i;
 description "Non Dimensionalization";
f1 := f;
 if f=x[1] then f1 := epsilon·R_E·f; elif f=x[2] or f=x[3] then f1 := f;
  elif op(0, op(0, f)) =BF or op(0, op(0, f)) =b then f1 := G·f;
  elif op(0, op(0, f)) =h then f1 := epsilon·R_E·f;
 elif type(f, atomic) then f1 := f;
 elif op(0, f) =u[1] then f1 := epsilon·sqrt(G·R_E)·f; elif op(0, f) =u[2] or op(0, f)
      =u[3] then f1 := sqrt(G·R_E)·f;
 elif op(0, f) =p[1, 1] or op(0, f) =p[2, 2] or op(0, f) =p[3, 3] or op(0, op(0, f))
      =Fr then f1 := epsilon·rho·G·R_E·f; elif op(0, f) =p[1, 2] or op(0, f) =p[1, 3]
     or op(0, f) =p[2, 3] then f1 := delta·epsilon·rho·G·R_E·f;
  elif op(0, f) =Diff or op(0, f) =diff then
 if op(2, f) =x[1] then f1 := 1/(epsilon·R_E); elif op(2, f) =t then f1 := 1/sqrt(R_E/G);
     else f1 := 1; end if;
 f2 := nondim(op(1, f)); f1 := Differentiate(f2, op(2, f))·f1;
  elif op(0, f) = `=` then f1 := `=`(nondim(op(1, f)), nondim(op(2, f)));
 elif op(0, f) = `*` or op(0, f) = `+` then
f1 := op(0, f)(nondim(op(1, f)));
 for i from 2 to nops(f) do
f1 := op(0, f)(f1, nondim(op(i, f)));
 end do;
 else f1 := f;
for i from 1 to nops(f) do
f1 := applyop(nondim, i, f1);
 end do;
 end if;
f1;
 end proc:
```

```
selection := proc(f)
global x;
 local f1, j, i;
 description "Selecting appropriate equations";
 j := 1;
 if op(0,f) = `=` then f1 := selection(op(1,f)) = selection(op(1,f))/op(1,f) · op(2,f);
    if diff(denom(op(2,f1)), epsilon) ≠ 0 then f1 := denom(op(2,f1))·f1; end if;
 elif op(0,f) = `*` then
 for i from 2 to nops(f) do if nops(op(i,f)) > nops(op(j,f)) then j := i; end if; end
    do;
f1 := op(j,f);
 else
f1 := f;
 end if;
f1;
 end proc:

integrate := proc(f)
 global x;
 local f1, i;
f1 := 0;
 if op(0,f) = `+` then
  for i from 1 to nops(f) do
f1 := f1 + lib_part(op(i,f));
 end do;
 elif op(0,f) = `=` then f1 := integrate(op(1,f)) = integrate(op(2,f));
 else
f1 := lib_part(f);
 end if;
f1;
 end proc:

lib_part := proc(f)
global x, H;
 local f2, f3, f4, f5, f6, i, j;
f2 := f; f3 := 1; f4 := 1 ;
 if op(0,f2) = `*` then
for i from 1 to nops(f2) do
if op(0, op(i,f2)) = `Diff` then
f4 := f4·op(i,f2);
 else
f3 := f3·op(i,f2);
 end if ;
```

```
end do ;
elif op(0, f2) = `Diff` then
f4 := f4·f2;
else
f3 := f3·f2;
end if;
if f4 ≠ 1 then
f5 := op(1, f4)·f3;
f6 := -op(1, f4)·`diff`(f3, op(2, f4));
if op(2, f4) = x[1] then
f5 := subs(x[1] = H[1], f5) − subs(x[1] = H[2], f5);
else
f5 := Differentiate(simplify(int(f5, x[1] = H[2]..H[1])), op(2, f4)) + `Diff`(H[2], op(2, f4))
    ·subs(x[1] = H[2], f5) - `Diff`(H[1], op(2, f4))·subs(x[1] = H[1], f5);
end if;
f6 := simplify(int(f6, x[1] = H[2]..H[1]));
else
f5 := 0; f6 := f3;
f6 := simplify(int(f6, x[1] = H[2]..H[1]));
end if;

f5 + f6;
end proc:

boundary := proc(f, i)
global x, H, n_K_BC, n_D_BC;
local f3, f4, f5;
f3 := subs(diff = Diff, n_K_BC);
f5 := subs(diff = Diff, n_D_BC);
f4 := simplify( expand(f), {f3[1], f3[2]}) ;
if i > 0 then
f4 := simplify(expand(f4), {f5[1][i]});
f4 := simplify(expand(f4), {f5[2][i]});
end if;
f4;
end proc:

Final_form := proc(f)
local f1;
if type(f, atomic) then if f = x[1] then f1 := (h/2) else f1 := f; end if
elif op(0, f) = `int` then f1 := h·Final_form(op(1, f));
elif op(0, f) = 'Diff' then f1 := Diff(Final_form(op(1, f)), op(2, f));
elif type(f, function) then
if diff(f, x[1]) ≠ 0 then
if op(0, op(0, f)) = `symbol` then f1 := conjugate( op(0, f) ) else f1 := conjugate( op(0, op(0,
    f)))[op( op(0, f) )];
end if;
elif op(0, op(0, f)) = Fr or op(0, op(0, f)) = h then f1 := op(0, f) else f1 := f;
end if;
else f1 := op(0, f) (seq(Final_form(op(i, f)), i = 1 ..nops(f)));
```

end if;
f1;
end proc:

Curvilinear Coordinates:

$with(LinearAlgebra) : x[1], x[2], x[3] := r, \text{theta}, \text{phi}; dim := 3;$

$$r, \theta, \phi$$
$$3 \tag{1}$$

Relation of curvilinear coordinates with cartesian coordinates:

$X := Vector(1..3) : X[1], X[2], X[3] := r \cdot \cos(x[3]) \cdot \sin(x[2]), r \cdot \sin(x[3]) \cdot \sin(x[2]), r \cdot \cos(x[2]);$

$$r \cos(\phi) \sin(\theta), r \sin(\phi) \sin(\theta), r \cos(\theta) \tag{2}$$

Defining covariant basis vectors:

$e := Matrix(1..3, 1..3) :$ **for** i **from** 1 **to** 3 **do for** j **from** 1 **to** 3 **do** $e[i, j] := diff(X[j], x[i]);$ **end do end do**;

Magnitudes of the basis vectors:

$\#G:=Vector(1..3) :$ **for** k **from** 1 **to** 3 **do** $G[k] := g[k](x[1], x[2]);$**end do;**
$g := Vector(1..3) :$ **for** i **from** 1 **to** 3 **do** $g[i] := simplify(DotProduct(e[i], e[i], conjugate = false))$
end do:
$\#assume(0 < r, 0 < \sin(\text{theta}));$

for i **from** 1 **to** 3 **do** $e[i] := simplify\left(\frac{e[i]}{\text{sqrt}(g[i])}, assume = [0 < r, 0 < \sin(\text{theta})]\right)$;**end do**;

$$\left[\cos(\phi) \sin(\theta) \quad \sin(\phi) \sin(\theta) \quad \cos(\theta)\right]$$
$$\left[\cos(\phi) \cos(\theta) \quad \sin(\phi) \cos(\theta) \quad -\sin(\theta)\right]$$
$$\left[-\sin(\phi) \quad \cos(\phi) \quad 0\right] \tag{3}$$

Defining christoffel symbols:

$R := Array(1..3, 1..3, 1..3) :$ **for** i **from** 1 **to** 3 **do**
for j **from** 1 **to** 3 **do**
for k **from** 1 **to** 3 **do**
$R[i, j, k] := simplify(DotProduct(diff\sim(e[i], x[j]), e[k], conjugate = false));$
end do;
end do;
end do;

$R1 := R :$

$U := Vector(1..3)$: **for** k **from** 1 **to** 3 **do** $U[k] := u[k](x[1], x[2], x[3], t)$;**end do**:

(4)

$div\_u := expand\left(simplify\left(add\left(\frac{1}{\mathrm{sqrt}(g[i])}\cdot(Diff(U[i], x[i]) + add(U[k]\cdot R1[k, i, i], k = 1..3)), i = 1..3\right), assume = [0 < r,\ 0 < \sin(\mathrm{theta})]\right)\right);$

$$\frac{\partial}{\partial r} u_1(r, \theta, \phi, t) + \frac{u_2(r, \theta, \phi, t)\cos(\theta)}{r\sin(\theta)} + \frac{\frac{\partial}{\partial \theta} u_2(r, \theta, \phi, t)}{r} + \frac{2\, u_1(r, \theta, \phi, t)}{r} + \frac{\frac{\partial}{\partial \phi} u_3(r, \theta, \phi, t)}{r\sin(\theta)} \tag{5}$$

for i **from** 1 **to** 3 **do** $div\_uu[i] := expand\left(simplify\left(add\left(\frac{1}{\mathrm{sqrt}(g[j])}(Diff(U[j]\cdot U[i], x[j]) + add(U[j]\cdot U[k]\cdot R1[k, j, i] + U[k]\cdot U[i]\cdot R1[k, j, j], k = 1..3)), j = 1..3\right), assume = [0 < r,\ 0 < \sin(\mathrm{theta})]\right)\right)$;**end do**;

$$\frac{\partial}{\partial r}\left(u_1(r, \theta, \phi, t)^2\right) - \frac{u_3(r, \theta, \phi, t)^2}{r} - \frac{u_2(r, \theta, \phi, t)^2}{r} + \frac{2\, u_1(r, \theta, \phi, t)^2}{r} + \frac{\cos(\theta)\, u_1(r, \theta, \phi, t)\, u_2(r, \theta, \phi, t)}{r\sin(\theta)} + \frac{\frac{\partial}{\partial \theta}\left(u_2(r, \theta, \phi, t)\, u_1(r, \theta, \phi, t)\right)}{r} + \frac{\frac{\partial}{\partial \phi}\left(u_3(r, \theta, \phi, t)\, u_1(r, \theta, \phi, t)\right)}{r\sin(\theta)}$$

$$-\frac{\cos(\theta)\, u_3(r, \theta, \phi, t)^2}{r\sin(\theta)} + \frac{\cos(\theta)\, u_2(r, \theta, \phi, t)^2}{r\sin(\theta)} + \frac{3\, u_2(r, \theta, \phi, t)\, u_1(r, \theta, \phi, t)}{r} + \frac{\partial}{\partial r}\left(u_2(r, \theta, \phi, t)\, u_1(r, \theta, \phi, t)\right) + \frac{\frac{\partial}{\partial \theta}\left(u_2(r, \theta, \phi, t)^2\right)}{r} + \frac{\frac{\partial}{\partial \phi}\left(u_3(r, \theta, \phi, t)\, u_2(r, \theta, \phi, t)\right)}{r\sin(\theta)}$$

$$\frac{2\cos(\theta)\, u_3(r, \theta, \phi, t)\, u_2(r, \theta, \phi, t)}{r\sin(\theta)} + \frac{3\, u_3(r, \theta, \phi, t)\, u_1(r, \theta, \phi, t)}{r} + \frac{\partial}{\partial r}\left(u_3(r, \theta, \phi, t)\, u_1(r, \theta, \phi, t)\right) + \frac{\frac{\partial}{\partial \theta}\left(u_3(r, \theta, \phi, t)\, u_2(r, \theta, \phi, t)\right)}{r} + \frac{\frac{\partial}{\partial \phi}\left(u_3(r, \theta, \phi, t)^2\right)}{r\sin(\theta)} \tag{6}$$

$P := Matrix(1..3, 1..3)$:**for** i **from** 1 **to** 3 **do for** j **from** 1 **to** 3 **do** $P[i,j] := p[i,j](x[1], x[2], x[3], t)$; **end do**; **end do**; $P := subs(\{P[2,1]=P[1,2], P[3,2]=P[2,3], P[3,1]=P[1,3]\}, P)$:

for i **from** 1 **to** 3 **do** $div\_P[i]$

$$:= expand\left(simplify\left(add\left(\frac{1}{\mathrm{sqrt}(g[j])}(Diff(P[j,i], x[j]) + add(P[j,k]\cdot R1[k,j,i] + P[k,i]\cdot R1[k,j,j], k=1..3)), j=1..3\right), assume=[0<r,\ 0<\sin(\theta)]\right)\right)$$;**end do**:

for k **from** 1 **to** 3 **do** $div\_P[k] := expand(simplify(subs(\{P[2,1]=P[1,2], P[3,2]=P[2,3], P[3,1]=P[1,3]\}, div\_P[k])))$ **end do**:

In prinicipal CS of the body:

$\Omega := Vector(1..3)$: **for** i **from** 1 **to** 3 **do** $\Omega[i] := \omega[i](t)$ **end do**:

In Body fixed curvilinear coodinates:

$B := Vector(1..3)$: **for** i **from** 1 **to** 3 **do** $B[i] := b[i](x[1], x[2], x[3])$; **end do**:

In principal CS:

$pos := Vector(1..3)$: $pos := X$:

$Cent := simplify(CrossProduct(\Omega, CrossProduct(\Omega, pos)))$:

In curvilinear coordinates:

$temp := Vector(1..3)$:**for** i **from** 1 **to** 3 **do** $temp[i] := simplify(DotProduct(Cent, e[i], conjugate=false))$ **end do**:

$B := B - temp$:
$B := simplify(subs(\{\omega[1](t)=0, \omega[2](t)=0\}, B), trig)$;

$$\begin{bmatrix} -\omega_3(t)^2\cos(\theta)^2 r + \omega_3(t)^2 r + b_1(r,\theta,\phi) \\ b_2(r,\theta,\phi) + r\,\omega_3(t)^2\sin(\theta)\cos(\theta) \\ b_3(r,\theta,\phi) \end{bmatrix} \tag{7}$$

In Principal CS:
$A := Vector(1..3)$: **for** i **from** 1 **to** 3 **do** $A[i] := \alpha[i](t)$; **end do**:
$A\_acc := CrossProduct(A, pos)$:
In CCS:
$temp[1..3] := A\_acc[1..3]$; **for** i **from** 1 **to** 3 **do** $A\_acc[i] := simplify(DotProduct(temp, e[i], conjugate=false))$ **end do**:

$$\begin{bmatrix} \alpha_2(t)\,r\cos(\theta) - \alpha_3(t)\,r\sin(\phi)\sin(\theta) \\ -\alpha_1(t)\,r\cos(\theta) + \alpha_3(t)\,r\cos(\phi)\sin(\theta) \\ \alpha_1(t)\,r\sin(\phi)\sin(\theta) - \alpha_2(t)\,r\cos(\phi)\sin(\theta) \end{bmatrix} \tag{8}$$

$A\_acc := simplify(subs(\{\mathrm{alpha}[1](t)=0, \mathrm{alpha}[2](t)=0\}, A\_acc));$

$$\begin{bmatrix} 0 \\ 0 \\ \alpha_3(t)\sin(\theta)\, r \end{bmatrix} \quad \textbf{(9)}$$

In BFCS:

$Omega\_B := Vector(1..3)$: **for** i **from** 1 **to** 3 **do** $Omega\_B[i] := simplify(DotProduct(\mathrm{Omega}, e[i], conjugate = false))$ **end do**: $corr := simplify(CrossProduct(Omega\_B, U))$:

$corr := simplify(subs(\{\mathrm{omega}[1](t)=0, \mathrm{omega}[2](t)=0\}, corr)); simplify(CrossProduct(e[1], e[2]))$:

$$\begin{bmatrix} -u_3(r,\theta,\phi,t)\,\omega_3(t)\sin(\theta) \\ -u_3(r,\theta,\phi,t)\,\omega_3(t)\cos(\theta) \\ \omega_3(t)\left(u_2(r,\theta,\phi,t)\cos(\theta)+u_1(r,\theta,\phi,t)\sin(\theta)\right) \end{bmatrix} \quad \textbf{(10)}$$

Acceleration of the core in PCS:
$A\_core := Vector(1..3)$:**for** i **from** 1 **to** 3 **do** $A\_core[i] := a[i](t)$ **end do**:
Acceleration of the core in CCS:
a_B:=Vector(1..3):
for i **from** 1 **to** 3 **do** $a\_B[i] := simplify(DotProduct(A\_core, e[i], conjugate = false))$ **end do**;

$$a_1(t)\cos(\phi)\sin(\theta)+a_2(t)\sin(\phi)\sin(\theta)+a_3(t)\cos(\theta)$$

$$a_1(t)\cos(\phi)\cos(\theta)+a_2(t)\sin(\phi)\cos(\theta)-a_3(t)\sin(\theta)$$

$$-a_1(t)\sin(\phi)+a_2(t)\cos(\phi) \quad \textbf{(11)}$$

Momentum balance:

$mom := Vector(1..3)$: **for** i **from** 1 **to** 3 **do** $mom[i] := \mathrm{rho}\cdot Diff(U[i], t) + \mathrm{rho}\cdot div\_uu[i] = -div\_P[i] + \mathrm{rho}\cdot BF[i](x[1], x[2], x[3], t)$;**end do**:*#BF also contains acceleration of the core*

for i **from** 1 **to** 3 **do** $mom[i] := collect(mom[i], \mathrm{rho})$ **end do**;

$$\left(\frac{\partial}{\partial t}u_1(r,\theta,\phi,t)+\frac{\partial}{\partial r}\left(u_1(r,\theta,\phi,t)^2\right)-\frac{u_3(r,\theta,\phi,t)^2}{r}-\frac{u_2(r,\theta,\phi,t)^2}{r}+\frac{2\,u_1(r,\theta,\phi,t)^2}{r}+\frac{\cos(\theta)\,u_1(r,\theta,\phi,t)\,u_2(r,\theta,\phi,t)}{r\sin(\theta)}+\frac{\frac{\partial}{\partial\theta}\left(u_2(r,\theta,\phi,t)\,u_1(r,\theta,\phi,t)\right)}{r}+\frac{\frac{\partial}{\partial\phi}\left(u_3(r,\theta,\phi,t)\,u_1(r,\theta,\phi,t)\right)}{r\sin(\theta)}\right)\rho =$$

$$-\left(\frac{\partial}{\partial r}p_{1,1}(r,\theta,\phi,t)\right)-\frac{\frac{\partial}{\partial\theta}p_{1,2}(r,\theta,\phi,t)}{r}-\frac{2\,p_{1,1}(r,\theta,\phi,t)}{r}+\frac{p_{2,2}(r,\theta,\phi,t)}{r}$$

$$+\frac{p_{3,3}(r,\theta,\phi,t)}{r}-\frac{p_{1,2}(r,\theta,\phi,t)\cos(\theta)}{r\sin(\theta)}-\frac{\frac{\partial}{\partial\phi}p_{1,3}(r,\theta,\phi,t)}{r\sin(\theta)}+\rho\,BF_1(r,\theta,\phi,t)$$

$$\left(\frac{\partial}{\partial t}u_2(r,\theta,\phi,t)-\frac{\cos(\theta)\,u_3(r,\theta,\phi,t)^2}{r\sin(\theta)}+\frac{\cos(\theta)\,u_2(r,\theta,\phi,t)^2}{r\sin(\theta)}\right.$$

$$+\frac{3\,u_2(r,\theta,\phi,t)\,u_1(r,\theta,\phi,t)}{r}+\frac{\partial}{\partial r}\left(u_2(r,\theta,\phi,t)\,u_1(r,\theta,\phi,t)\right)$$

$$\left.+\frac{\frac{\partial}{\partial\theta}\left(u_2(r,\theta,\phi,t)^2\right)}{r}+\frac{\frac{\partial}{\partial\phi}\left(u_3(r,\theta,\phi,t)\,u_2(r,\theta,\phi,t)\right)}{r\sin(\theta)}\right)\rho=-\left(\frac{\partial}{\partial r}p_{1,2}(r,\theta,\phi,t)\right)$$

$$-\frac{\frac{\partial}{\partial\theta}p_{2,2}(r,\theta,\phi,t)}{r}-\frac{3\,p_{1,2}(r,\theta,\phi,t)}{r}-\frac{p_{2,2}(r,\theta,\phi,t)\cos(\theta)}{r\sin(\theta)}$$

$$+\frac{p_{3,3}(r,\theta,\phi,t)\cos(\theta)}{r\sin(\theta)}-\frac{\frac{\partial}{\partial\phi}p_{2,3}(r,\theta,\phi,t)}{r\sin(\theta)}+\rho\,BF_2(r,\theta,\phi,t)$$

$$\left(\frac{\partial}{\partial t}u_3(r,\theta,\phi,t)+\frac{2\cos(\theta)\,u_3(r,\theta,\phi,t)\,u_2(r,\theta,\phi,t)}{r\sin(\theta)}+\frac{3\,u_3(r,\theta,\phi,t)\,u_1(r,\theta,\phi,t)}{r}\right. \tag{12}$$

$$+\frac{\partial}{\partial r}\left(u_3(r,\theta,\phi,t)\,u_1(r,\theta,\phi,t)\right)+\frac{\frac{\partial}{\partial\theta}\left(u_3(r,\theta,\phi,t)\,u_2(r,\theta,\phi,t)\right)}{r}$$

$$\left.+\frac{\frac{\partial}{\partial\phi}\left(u_3(r,\theta,\phi,t)^2\right)}{r\sin(\theta)}\right)\rho=-\left(\frac{\partial}{\partial r}p_{1,3}(r,\theta,\phi,t)\right)-\frac{\frac{\partial}{\partial\theta}p_{2,3}(r,\theta,\phi,t)}{r}$$

$$-\frac{3\,p_{1,3}(r,\theta,\phi,t)}{r}-\frac{2\,p_{2,3}(r,\theta,\phi,t)\cos(\theta)}{r\sin(\theta)}-\frac{\frac{\partial}{\partial\phi}p_{3,3}(r,\theta,\phi,t)}{r\sin(\theta)}+\rho\,BF_3(r,\theta,\phi,t)$$

Boundary conditions:

$H := Vector(1..2) : K\_BC := Vector(1..2) : N := Vector(1..2)$:**for** i **from** 1 **to** 2 **do** $H[i] := h[i](x[2], x[3], t); N[i] := convert\left(\left[\frac{diff(x[1]-H[i], x[1])}{\mathrm{sqrt}(g[1])}, \frac{diff(x[1]-H[i], x[2])}{\mathrm{sqrt}(g[2])}, \frac{diff(x[1]-H[i], x[3])}{\mathrm{sqrt}(g[3])}\right], Vector\right); K\_BC[i] := simplify(subs(x[1]=H[i], DotProduct(U, N[i], conjugate=false) = diff(H[i], t)), assume=[0 < H[i], \sin(x[2]) > 0]) :$
end do:
K_BC;

$$\left[\left[\frac{1}{h_1(\theta,\phi,t)\sin(\theta)}\left(u_1(h_1(\theta,\phi,t),\theta,\phi,t)\sin(\theta)\,h_1(\theta,\phi,t)-u_2(h_1(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\theta}h_1(\theta,\phi,t)\right)\sin(\theta)-u_3(h_1(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\phi}h_1(\theta,\phi,t)\right)\right)=\frac{\partial}{\partial t}h_1(\theta,\phi,t)\right],\right.$$
$$\left.\left[\frac{1}{h_2(\theta,\phi,t)\sin(\theta)}\left(u_1(h_2(\theta,\phi,t),\theta,\phi,t)\sin(\theta)\,h_2(\theta,\phi,t)-u_2(h_2(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\theta}h_2(\theta,\phi,t)\right)\sin(\theta)-u_3(h_2(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\phi}h_2(\theta,\phi,t)\right)\right)=\frac{\partial}{\partial t}h_2(\theta,\phi,t)\right]\right] \quad \textbf{(13)}$$

$n\_cont := rescaling(\,div\_u) : n\_cont := nondim(n\_cont) : n\_cont := selection(\,simplify(n\_cont, assume = [0 < R\_E, 0 < G, 0 < \sin(\text{theta})\,])\,);$

$$\left(\frac{\partial}{\partial r}u_1(r,\theta,\phi,t)\right)\sin(\theta)\,\varepsilon\,r+2\,\varepsilon\,u_1(r,\theta,\phi,t)\sin(\theta)+\left(\frac{\partial}{\partial r}u_1(r,\theta,\phi,t)\right)\sin(\theta)+\left(\frac{\partial}{\partial\theta}u_2(r,\theta,\phi,t)\right)\sin(\theta)+u_2(r,\theta,\phi,t)\cos(\theta)+\frac{\partial}{\partial\phi}u_3(r,\theta,\phi,t) \quad \textbf{(14)}$$

$n\_mom := Vector(1\,..3)$:**for** i **from** 1 **to** 3 **do** $n\_mom[i] := rescaling(\,mom[i]);\ \ n\_mom[i] := nondim(\,n\_mom[i]); n\_mom[i] := selection(simplify(n\_mom[i], assume = [0 < R\_E, 0 < G])\,);$
end do: n_mom;

$$\left[\left[2\,u_1(r,\theta,\phi,t)^2\sin(\theta)\,\varepsilon^2+\sin(\theta)\left(\frac{\partial}{\partial r}\left(u_1(r,\theta,\phi,t)^2\right)\right)\varepsilon^2 r+\sin(\theta)\left(\frac{\partial}{\partial t}u_1(r,\theta,\phi,t)\right)\varepsilon^2 r+u_1(r,\theta,\phi,t)\,u_2(r,\theta,\phi,t)\cos(\theta)\,\varepsilon-u_2(r,\theta,\phi,t)^2\sin(\theta)-u_3(r,\theta,\phi,t)^2\sin(\theta)+\sin(\theta)\left(\frac{\partial}{\partial r}\left(u_1(r,\theta,\phi,t)^2\right)\right)\varepsilon+\sin(\theta)\left(\frac{\partial}{\partial t}u_1(r,\theta,\phi,t)\right)\varepsilon+\sin(\theta)\left(\frac{\partial}{\partial\theta}\left(u_2(r,\theta,\phi,t)\,u_1(r,\theta,\phi,t)\right)\right)\varepsilon+\left(\frac{\partial}{\partial\phi}\left(u_3(r,\theta,\phi,t)\,u_1(r,\theta,\phi,t)\right)\right)\varepsilon=-\delta\,\varepsilon\,p_{1,2}(r,\theta,\phi,t)\cos(\theta)-\sin(\theta)\left(\frac{\partial}{\partial r}p_{1,1}(r,\theta,\phi,t)\right)\varepsilon\,r-\delta\,\varepsilon\left(\frac{\partial}{\partial\theta}p_{1,2}(r,\theta,\phi,t)\right)\sin(\theta)+\sin(\theta)\,BF_1(r,\theta,\phi,t)\,\varepsilon\,r-2\,\varepsilon\,p_{1,1}(r,\theta,\phi,t)\sin(\theta)+\varepsilon\,p_{2,2}(r,\theta,\phi,t)\sin(\theta)+\varepsilon\,p_{3,3}(r,\theta,\phi,t)\sin(\theta)-\delta\,\varepsilon\left(\frac{\partial}{\partial\phi}p_{1,3}(r,\theta,\phi,t)\right)-\left(\frac{\partial}{\partial r}p_{1,1}(r,\theta,\phi,t)\right)\sin(\theta)+BF_1(r,\theta,\phi,t)\sin(\theta)\right],\right.$$

$$\Big[3\, u_2(r,\theta,\phi,t)\, u_1(r,\theta,\phi,t)\sin(\theta)\,\varepsilon + \sin(\theta)\left(\frac{\partial}{\partial r}\left(u_2(r,\theta,\phi,t)\, u_1(r,\theta,\phi,t)\right)\right)\varepsilon\, r$$
$$+ \sin(\theta)\left(\frac{\partial}{\partial t} u_2(r,\theta,\phi,t)\right)\varepsilon\, r + u_2(r,\theta,\phi,t)^2\cos(\theta) - u_3(r,\theta,\phi,t)^2\cos(\theta)$$
$$+ \sin(\theta)\left(\frac{\partial}{\partial r}\left(u_2(r,\theta,\phi,t)\, u_1(r,\theta,\phi,t)\right)\right) + \sin(\theta)\left(\frac{\partial}{\partial t} u_2(r,\theta,\phi,t)\right)$$
$$+ \sin(\theta)\left(\frac{\partial}{\partial \theta}\left(u_2(r,\theta,\phi,t)^2\right)\right) + \frac{\partial}{\partial \phi}\left(u_3(r,\theta,\phi,t)\, u_2(r,\theta,\phi,t)\right) =$$
$$-\sin(\theta)\left(\frac{\partial}{\partial r} p_{1,2}(r,\theta,\phi,t)\right)\delta\,\varepsilon\, r + \sin(\theta)\, BF_2(r,\theta,\phi,t)\,\varepsilon\, r - 3\,\delta\,\varepsilon\, p_{1,2}(r,\theta,\phi,$$
$$t)\sin(\theta) - \varepsilon\, p_{2,2}(r,\theta,\phi,t)\cos(\theta) + \varepsilon\, p_{3,3}(r,\theta,\phi,t)\cos(\theta) - \delta\left(\frac{\partial}{\partial r} p_{1,2}(r,\theta,\phi,$$
$$t)\right)\sin(\theta) - \varepsilon\left(\frac{\partial}{\partial \theta} p_{2,2}(r,\theta,\phi,t)\right)\sin(\theta) - \delta\,\varepsilon\left(\frac{\partial}{\partial \phi} p_{2,3}(r,\theta,\phi,t)\right) + BF_2(r,\theta,\phi,$$
$$t)\sin(\theta)\Big],$$

$$\Big[3\, u_3(r,\theta,\phi,t)\, u_1(r,\theta,\phi,t)\sin(\theta)\,\varepsilon + \sin(\theta)\left(\frac{\partial}{\partial r}\left(u_3(r,\theta,\phi,t)\, u_1(r,\theta,\phi,t)\right)\right)\varepsilon\, r$$
$$+ \sin(\theta)\left(\frac{\partial}{\partial t} u_3(r,\theta,\phi,t)\right)\varepsilon\, r + 2\, u_3(r,\theta,\phi,t)\, u_2(r,\theta,\phi,t)\cos(\theta)$$
$$+ \sin(\theta)\left(\frac{\partial}{\partial r}\left(u_3(r,\theta,\phi,t)\, u_1(r,\theta,\phi,t)\right)\right) + \sin(\theta)\left(\frac{\partial}{\partial t} u_3(r,\theta,\phi,t)\right)$$
$$+ \sin(\theta)\left(\frac{\partial}{\partial \theta}\left(u_3(r,\theta,\phi,t)\, u_2(r,\theta,\phi,t)\right)\right) + \frac{\partial}{\partial \phi}\left(u_3(r,\theta,\phi,t)^2\right) =$$
$$-\sin(\theta)\left(\frac{\partial}{\partial r} p_{1,3}(r,\theta,\phi,t)\right)\delta\,\varepsilon\, r - 2\,\delta\,\varepsilon\, p_{2,3}(r,\theta,\phi,t)\cos(\theta) - \delta\,\varepsilon\left(\frac{\partial}{\partial \theta} p_{2,3}(r,\theta,\phi,$$
$$t)\right)\sin(\theta) - 3\,\delta\,\varepsilon\, p_{1,3}(r,\theta,\phi,t)\sin(\theta) + \sin(\theta)\, BF_3(r,\theta,\phi,t)\,\varepsilon\, r - \delta\left(\frac{\partial}{\partial r} p_{1,3}(r,\theta,\phi,$$
$$t)\right)\sin(\theta) + BF_3(r,\theta,\phi,t)\sin(\theta) - \varepsilon\left(\frac{\partial}{\partial \phi} p_{3,3}(r,\theta,\phi,t)\right)\Big]\Big]$$

(16)

$n\_K\_BC := Vector(1..2)$:**for** i **from** 1 **to** 2 **do** $n\_K\_BC[i] := rescaling(K\_BC[i]);\ n\_K\_BC[i] := simplify(nondim(n\_K\_BC[i])); n\_K\_BC[i] := simplify\left(\frac{selection(op(1, n\_K\_BC[i]))}{op(1, n\_K\_BC[i])} \cdot n\_K\_BC[i], assume = [0 < R\_E, 0 < G, 0 < \text{epsilon}]\right);$

end do:

$n\_K\_BC;$

$$\Big[\Big[u_1(h_1(\theta,\phi,t),\theta,\phi,t)\, h_1(\theta,\phi,t)\sin(\theta)\,\varepsilon - u_2(h_1(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial \theta} h_1(\theta,\phi,t)\right)\sin(\theta) \quad \textbf{(17)}$$

$$+u_1(h_1(\theta,\phi,t),\theta,\phi,t)\sin(\theta)-u_3(h_1(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\phi}h_1(\theta,\phi,t)\right)=(h_1(\theta,\phi,t)\,\varepsilon+1)\sin(\theta)\left(\frac{\partial}{\partial t}h_1(\theta,\phi,t)\right)\Big],$$

$$\Big[u_1(h_2(\theta,\phi,t),\theta,\phi,t)\,h_2(\theta,\phi,t)\sin(\theta)\,\varepsilon-u_2(h_2(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\theta}h_2(\theta,\phi,t)\right)\sin(\theta)+u_1(h_2(\theta,\phi,t),\theta,\phi,t)\sin(\theta)-u_3(h_2(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\phi}h_2(\theta,\phi,t)\right)=(h_2(\theta,\phi,t)\,\varepsilon+1)\sin(\theta)\left(\frac{\partial}{\partial t}h_2(\theta,\phi,t)\right)\Big]\Big]$$

```
D_BC := Vector(1..2) : D_BC[1] := simplify(P.N[1], assume = [0 < x[1], sin(x[2]) > 0]) :
 D_BC[2] := simplify(P.N[2], assume = [0 < x[1], sin(x[2]) > 0]) :
 for i from 1 to 3 do D_BC[1][i] := simplify(D_BC[1][i] = 0, assume = [0 < x[1], sin(x[2])
     > 0]); D_BC[2][i] := simplify(D_BC[2][i] = Fr[i](x[2], x[3], t), assume = [0 < x[1],
     sin(x[2]) > 0]); end do:
 n_D_BC := Vector(1..2) :for i from 1 to 2 do for j from 1 to 3 do n_D_BC[i][j]
     := rescaling( D_BC[i][j]); n_D_BC[i][j] := nondim( n_D_BC[i][j]); n_D_BC[i][j]
     := selection(simplify(n_D_BC[i][j], assume = [0 < R_E, 0 < G])) : n_D_BC[i][j]
     :=  convert(series(op(1, n_D_BC[i][j]), epsilon, 2), polynom) = convert(series(op(2,
     n_D_BC[i][j]), epsilon, 2), polynom) : n_D_BC[i][j] := selection(simplify(n_D_BC[i][j],
     assume = [0 < R_E, 0 < G])) :  n_D_BC[i][j] := expand(subs(x[1] = H[i], n_D_BC[i][j])) :
end do;end do:
 n_D_BC[1][1]; n_D_BC[1][2]; n_D_BC[1][3]; n_D_BC[2][1]; n_D_BC[2][2]; n_D_BC[2][3];
```

$$-\sin(\theta)\,p_{1,2}(h_1(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\theta}h_1(\theta,\phi,t)\right)\delta\varepsilon+p_{1,1}(h_1(\theta,\phi,t),\theta,\phi,t)\sin(\theta)\,\varepsilon\,h_1(\theta,\phi,t)-p_{1,3}(h_1(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\phi}h_1(\theta,\phi,t)\right)\delta\varepsilon+p_{1,1}(h_1(\theta,\phi,t),\theta,\phi,t)\sin(\theta)=0$$

$$\sin(\theta)\,p_{1,2}(h_1(\theta,\phi,t),\theta,\phi,t)\,\delta\varepsilon\,h_1(\theta,\phi,t)-\sin(\theta)\left(\frac{\partial}{\partial\theta}h_1(\theta,\phi,t)\right)p_{2,2}(h_1(\theta,\phi,t),\theta,\phi,t)\,\varepsilon-\left(\frac{\partial}{\partial\phi}h_1(\theta,\phi,t)\right)p_{2,3}(h_1(\theta,\phi,t),\theta,\phi,t)\,\delta\varepsilon+\delta p_{1,2}(h_1(\theta,\phi,t),\theta,\phi,t)\sin(\theta)=0$$

$$\sin(\theta)\left(\frac{\partial}{\partial\theta}h_1(\theta,\phi,t)\right)p_{2,3}(h_1(\theta,\phi,t),\theta,\phi,t)\,\delta\varepsilon-\sin(\theta)\,p_{1,3}(h_1(\theta,\phi,t),\theta,\phi,t)\,\delta\varepsilon\,h_1(\theta,\phi,t)+p_{3,3}(h_1(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\phi}h_1(\theta,\phi,t)\right)\varepsilon-\delta p_{1,3}(h_1(\theta,\phi,t),\theta,\phi,t)\sin(\theta)=0$$

$$\left(\frac{\partial}{\partial\theta}h_2(\theta,\phi,t)\right)\sin(\theta)\,p_{1,2}(h_2(\theta,\phi,t),\theta,\phi,t)\,\delta\varepsilon+\left(\frac{\partial}{\partial\phi}h_2(\theta,\phi,t)\right)p_{1,3}(h_2(\theta,\phi,t),\theta,\phi,t)\,\delta\varepsilon-p_{1,1}(h_2(\theta,\phi,t),\theta,\phi,t)\sin(\theta)\,\varepsilon\,h_2(\theta,\phi,t)-p_{1,1}(h_2(\theta,\phi,t),\theta,\phi,t)\sin(\theta)=$$

$$-\sin(\theta)\,Fr_1(\theta,\phi,t)\,h_2(\theta,\phi,t)\,\varepsilon-\sin(\theta)\,Fr_1(\theta,\phi,t)$$

$$-\sin(\theta)\,p_{1,2}(h_2(\theta,\phi,t),\theta,\phi,t)\,\delta\varepsilon\,h_2(\theta,\phi,t)+\left(\frac{\partial}{\partial\theta}h_2(\theta,\phi,t)\right)\sin(\theta)\,p_{2,2}(h_2(\theta,\phi,t),\theta,\phi,t)\,\varepsilon+\left(\frac{\partial}{\partial\phi}h_2(\theta,\phi,t)\right)p_{2,3}(h_2(\theta,\phi,t),\theta,\phi,t)\,\delta\varepsilon-\delta p_{1,2}(h_2(\theta,\phi,t),\theta,\phi,t)\sin(\theta)=-\sin(\theta)\,Fr_2(\theta,\phi,t)\,h_2(\theta,\phi,t)\,\varepsilon-\sin(\theta)\,Fr_2(\theta,\phi,t)$$

$$\left(\frac{\partial}{\partial\theta}h_2(\theta,\phi,t)\right)\sin(\theta)\,p_{2,3}(h_2(\theta,\phi,t),\theta,\phi,t)\,\delta\varepsilon-\sin(\theta)\,p_{1,3}(h_2(\theta,\phi,t),\theta,\phi,t)\,\delta\varepsilon\,h_2(\theta,\phi,t)+\left(\frac{\partial}{\partial\phi}h_2(\theta,\phi,t)\right)p_{3,3}(h_2(\theta,\phi,t),\theta,\phi,t)\,\varepsilon-\delta p_{1,3}(h_2(\theta,\phi,t),\theta,\phi,t)\sin(\theta)=-\sin(\theta)\,Fr_3(\theta,\phi,t)\,h_2(\theta,\phi,t)\,\varepsilon-\sin(\theta)\,Fr_3(\theta,\phi,t) \tag{18}$$

Depth Averaging:

DA_cont := *integrate*(*n_cont*) : *DA_cont* := *collect*(*boundary*(*DA_cont*, 0), epsilon) : *DA_cont* := *Final_form*(*DA_cont*);

$$\left(h_1\sin(\theta)\left(\frac{\partial}{\partial t}h_1\right)-\sin(\theta)\,h_2\left(\frac{\partial}{\partial t}h_2\right)+\sin(\theta)\,h\,\overline{u}_1\right)\varepsilon+\sin(\theta)\left(\frac{\partial}{\partial t}h_1\right)-\sin(\theta)\left(\frac{\partial}{\partial t}h_2\right)+\frac{\partial}{\partial\theta}\left(\sin(\theta)\,h\,\overline{u}_2\right)+\frac{\partial}{\partial\phi}\left(h\,\overline{u}_3\right) \tag{19}$$

DA_mom := *Vector*(1 ..3) :**for** *i* **from** 1 **to** 3 **do** *DA_mom*[*i*] := *integrate*(*n_mom*[*i*], *x*[1]=*H*[2] ..*H*[1]); *DA_mom*[*i*] := *boundary*(*DA_mom*[*i*], *i*) : *DA_mom*[*i*] := *Final_form*(*DA_mom*[*i*]);
end do:
DA_mom[1]; *DA_mom*[2]; *DA_mom*[3];

$$\left(h\,\overline{u}_1^2+\frac{\partial}{\partial t}\left(\frac{1}{2}h^2\,\overline{u}_1\right)\right)\varepsilon^2\sin(\theta)+\sin(\theta)\,\varepsilon\left(\frac{\partial}{\partial t}\left(\overline{u}_1\,h\right)\right)+\left(\frac{\partial}{\partial\phi}\left(h\,\overline{u}_3\,\overline{u}_1\right)+\frac{\partial}{\partial\theta}\left(\sin(\theta)\,h\,\overline{u}_2\,\overline{u}_1\right)\right)\varepsilon+\left(-h\,\overline{u}_2^2-h\,\overline{u}_3^2\right)\sin(\theta)=\left(-\left(\frac{\partial}{\partial\theta}\left(\sin(\theta)\,h\,\overline{p}_{1,2}\right)\right)-\left(\frac{\partial}{\partial\phi}\left(h\,\overline{p}_{1,3}\right)\right)\right)\delta\varepsilon+\left(-h\,\overline{p}_{1,1}+\frac{1}{2}h^2\,\overline{BF}_1+h\,\overline{p}_{2,2}+h\,\overline{p}_{3,3}\right)\sin(\theta)\,\varepsilon+\sin(\theta)\,\overline{BF}_1\,h+\sin(\theta)\,Fr_1\,h_2\,\varepsilon+\sin(\theta)\,Fr_1$$

$$\sin(\theta)\left(\frac{\partial}{\partial t}\left(h\,\overline{u}_2\right)\right)-\cos(\theta)\,h\,\overline{u}_3^2+\left(2\,h\,\overline{u}_2\,\overline{u}_1+\frac{\partial}{\partial t}\left(\frac{1}{2}h^2\,\overline{u}_2\right)\right)\sin(\theta)\,\varepsilon+\frac{\partial}{\partial\theta}\left(\sin(\theta)\,h\,\overline{u}_2^2\right)+\frac{\partial}{\partial\phi}\left(h\,\overline{u}_3\,\overline{u}_2\right)=-2\sin(\theta)\,\delta\varepsilon\,h\,\overline{p}_{1,2}-\delta\varepsilon\left(\frac{\partial}{\partial\phi}\left(h\,\overline{p}_{2,3}\right)\right)+\frac{1}{2}\sin(\theta)\,\varepsilon\,h^2\,\overline{BF}_2+\sin(\theta)\,\overline{BF}_2\,h+\left(\cos(\theta)\,h\,\overline{p}_{3,3}-\left(\frac{\partial}{\partial\theta}\left(\sin(\theta)\,h\,\overline{p}_{2,2}\right)\right)\right)\varepsilon+\sin(\theta)\,Fr_2\,h_2\,\varepsilon+\sin(\theta)\,Fr_2$$

$$\left(2 h \overline{u}_3 \overline{u}_1 + \frac{\partial}{\partial t}\left(\frac{1}{2} h^2 \overline{u}_3\right)\right)\sin(\theta)\,\varepsilon + \frac{\partial}{\partial\theta}\left(\sin(\theta)\, h \overline{u}_3 \overline{u}_2\right) + \frac{\partial}{\partial\phi}\left(h \overline{u_3^2}\right) + \sin(\theta)\left(\frac{\partial}{\partial t}\left(h \overline{u}_3\right)\right) + \cos(\theta)\, h \overline{u}_3 \overline{u}_2 = -2\sin(\theta)\,\delta\varepsilon\, h \overline{p}_{1,3} + \left(-\cos(\theta)\, h \overline{p}_{2,3} - \left(\frac{\partial}{\partial\theta}\left(\sin(\theta)\, h \overline{p}_{2,3}\right)\right)\right)\delta\varepsilon + \frac{1}{2}\sin(\theta)\,\varepsilon h^2 \overline{BF}_3 + \sin(\theta)\,\overline{BF}_3\, h - \varepsilon\left(\frac{\partial}{\partial\phi}\left(h \overline{p}_{3,3}\right)\right) + \sin(\theta)\, Fr_3 h_2 \varepsilon + \sin(\theta)\, Fr_3 \tag{20}$$

for i **from** 1 **to** 3 **do** $fr[i] := simplify(DotProduct(P.N[2], N[2], conjugate = false) N[2][i] - ub[i](x[2], x[3], t) \cdot DotProduct(P.N[2], N[2], conjugate = false) \cdot \text{delta}, assume = [0 < x[1], \sin(x[2]) > 0]); bf[i] := B[i] - 2 \cdot corr[i] - A\_acc[i];$ **end do**:

for j **from** 1 **to** 3 **do** $fr[j] := rescaling(fr[j]); fr[j] := simplify(nondim(fr[j])); fr[j] := simplify(convert(series(fr[j], \text{epsilon}, 3), polynom)); fr[j] := simplify\left(collect\left(\frac{subs(x[1] = H[2], fr[j])}{\text{epsilon} \cdot G \cdot R\_E \cdot \text{rho}}, \text{epsilon}\right)\right) : bf[j] := rescaling(bf[j]); bf[j] := simplify(nondim(bf[j])); bf[j] := simplify(convert(series(bf[j], \text{epsilon}, 3), polynom));$
$bf[j] := integrate(bf[j], x[1] = H[2]..H[1]) : bf[j] := \frac{Final\_form(bf[j])}{G} :$
end do: $fr[1]; fr[2]; fr[3]; bf[1]; bf[2]; bf[3];$

$$\frac{1}{\sin(\theta)}\Big(2\sin(\theta)\, ub_1(\theta,\phi,t)\, p_{1,2}(h_2(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\theta} h_2(\theta,\phi,t)\right)\delta^2\varepsilon + 2 p_{1,3}(h_2(\theta,\phi,t),\theta,\phi,t)\, ub_1(\theta,\phi,t)\left(\frac{\partial}{\partial\phi} h_2(\theta,\phi,t)\right)\delta^2\varepsilon - 2\left(\frac{\partial}{\partial\theta} h_2(\theta,\phi,t)\right)\sin(\theta)\, p_{1,2}(h_2(\theta,\phi,t),\theta,\phi,t)\,\delta\varepsilon - 2\left(\frac{\partial}{\partial\phi} h_2(\theta,\phi,t)\right) p_{1,3}(h_2(\theta,\phi,t),\theta,\phi,t)\,\delta\varepsilon - \sin(\theta)\, p_{1,1}(h_2(\theta,\phi,t),\theta,\phi,t)\, ub_1(\theta,\phi,t)\,\delta + p_{1,1}(h_2(\theta,\phi,t),\theta,\phi,t)\sin(\theta)\Big)$$

$$\frac{1}{\sin(\theta)}\Big(2\, ub_2(\theta,\phi,t)\, p_{1,2}(h_2(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\theta} h_2(\theta,\phi,t)\right)\sin(\theta)\,\delta^2\varepsilon + 2\, ub_2(\theta,\phi,t)\, p_{1,3}(h_2(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\phi} h_2(\theta,\phi,t)\right)\delta^2\varepsilon - ub_2(\theta,\phi,t)\, p_{1,1}(h_2(\theta,\phi,t),\theta,\phi,t)\sin(\theta)\,\delta - p_{1,1}(h_2(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\theta} h_2(\theta,\phi,t)\right)\sin(\theta)\,\varepsilon\Big)$$

$$\frac{1}{\sin(\theta)}\Big(2\sin(\theta)\, ub_3(\theta,\phi,t)\, p_{1,2}(h_2(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\theta} h_2(\theta,\phi,t)\right)\delta^2\varepsilon + 2 p_{1,3}(h_2(\theta,\phi,t),\theta,\phi,t)\, ub_3(\theta,\phi,t)\left(\frac{\partial}{\partial\phi} h_2(\theta,\phi,t)\right)\delta^2\varepsilon - \sin(\theta)\, ub_3(\theta,\phi,t)\, p_{1,1}(h_2(\theta,\phi,t),\theta,\phi,t)\,\delta - p_{1,1}(h_2(\theta,\phi,t),\theta,\phi,t)\left(\frac{\partial}{\partial\phi} h_2(\theta,\phi,t)\right)\varepsilon\Big)$$

$$\frac{1}{G}\Big(-\frac{1}{2}\,\omega_3(t)^2\cos(\theta)^2 R\_E\,\varepsilon\left(h_1^2 - h_2^2\right) - \omega_3(t)^2\cos(\theta)^2 R\_E\left(h_1 - h_2\right)$$

$$+\frac{1}{2}\,\omega_3(t)^2 R\_E\,\varepsilon\left(h_1^2-h_2^2\right)+2\sqrt{G\,R\_E}\,\omega_3(t)\sin(\theta)\,h\,\overline{u}_3+\omega_3(t)^2 R\_E\left(h_1-h_2\right)$$

$$+G\,h\,\overline{b}_1\Big)$$

$$\frac{1}{G}\left(\frac{1}{2}\,\omega_3(t)^2\cos(\theta)\sin(\theta)\,R\_E\,\varepsilon\left(h_1^2-h_2^2\right)+\omega_3(t)^2\cos(\theta)\sin(\theta)\,R\_E\left(h_1-h_2\right)\right.$$

$$\left.+2\sqrt{G\,R\_E}\,\omega_3(t)\cos(\theta)\,h\,\overline{u}_3+G\,h\,\overline{b}_2\right)$$

$$\frac{1}{G}\left(-2\,\omega_3(t)\sqrt{G\,R\_E}\sin(\theta)\,\varepsilon\,\overline{u}_1\,h-\frac{1}{2}\sin(\theta)\,\alpha_3(t)\,R\_E\,\varepsilon\left(h_1^2-h_2^2\right)\right. \tag{21}$$

$$\left.-2\sqrt{G\,R\_E}\,\omega_3(t)\cos(\theta)\,h\,\overline{u}_2-\sin(\theta)\,\alpha_3(t)\,R\_E\left(h_1-h_2\right)+G\,h\,\overline{b}_3\right)$$