Stability, Accuracy and Convergence

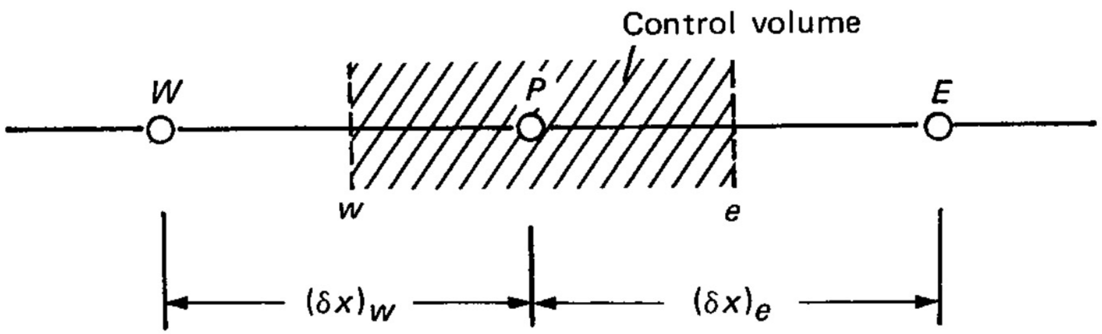
Exercise:

$$\frac{d}{dx}(\rho u) = 0$$

$$\Longrightarrow \rho u = \text{constant}$$

$$\frac{d}{dx}(\rho u \phi) = \frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right)$$

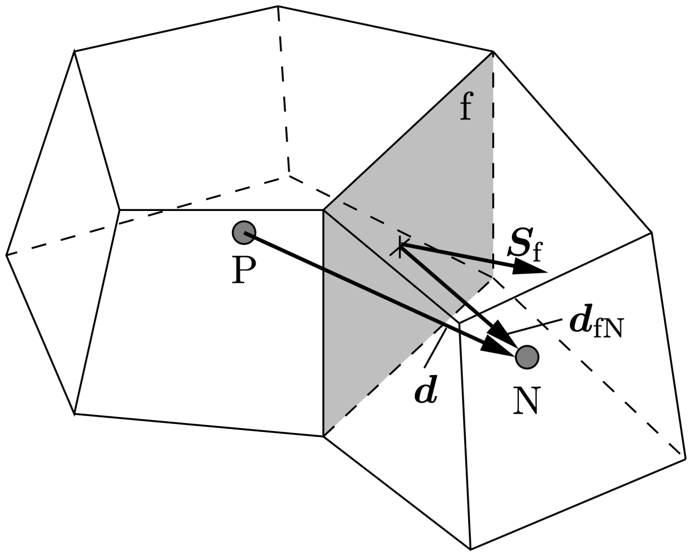
$$(\rho u \phi)_e - (\rho u \phi)_w = \left(\Gamma \frac{d\phi}{dx}\right)_e - \left(\Gamma \frac{d\phi}{dx}\right)_w$$



Central discretization: linear interpolation

$$\phi_e = \frac{\phi_E + \phi_P}{2}, \quad \phi_w = \frac{\phi_P + \phi_W}{2}$$

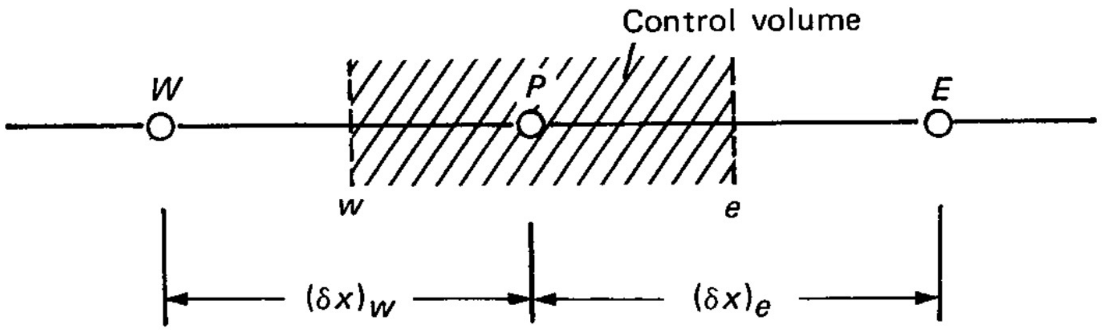
$$\frac{1}{2}(\rho u)_e(\phi_E + \phi_P) - \frac{1}{2}(\rho u)_w(\phi_P + \phi_W)$$
$$= \frac{\Gamma_e(\phi_E - \phi_P)}{(\delta x)_e} - \frac{\Gamma_e(\phi_P - \phi_W)}{(\delta x)_w}$$



Read chapter 5 in Patankar.

Stability, Accuracy and Convergence

$$\frac{1}{2}(\rho u)_e(\phi_E + \phi_P) - \frac{1}{2}(\rho u)_w(\phi_P + \phi_W) = \frac{\Gamma_e(\phi_E - \phi_P)}{(\delta x)_e} - \frac{\Gamma_e(\phi_P - \phi_W)}{(\delta x)_w}$$



$$a_P\phi_P = a_E\phi_E + a_W\phi_W$$

$$a_E = D_e - \frac{F_e}{2} \qquad a_w = D_w + \frac{F_e}{2}$$

$$F \equiv \rho u, \quad D \equiv \frac{\Gamma}{\delta x}$$

$$\begin{aligned} a_P &= D_e + \frac{F_e}{2} + D_w - \frac{F_w}{2} \\ &= a_E + a_w + (F_e - F_w) \\ &= a_E + a_w \end{aligned}$$

Continuity already satisfied: $F_e = F_w$

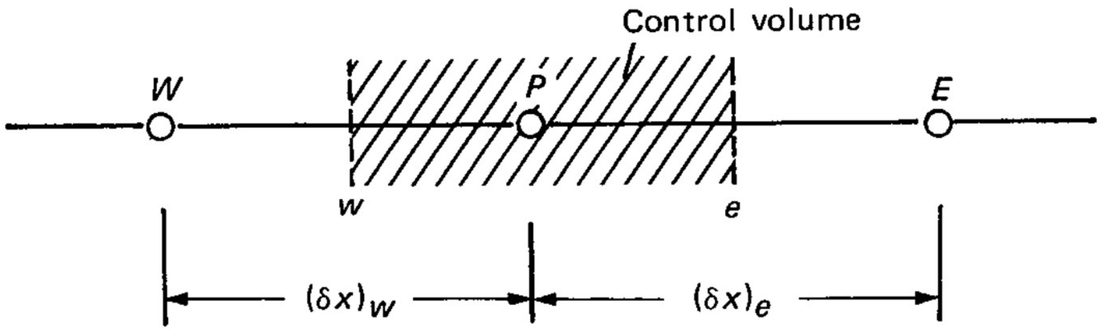
Let's pick $D_e = D_w = 1, \quad F_e = F_w = 4$

$$2\phi_P = -\phi_E + 3\phi_W$$

$$\phi_E = 200, \phi_W = 100 \implies \phi_P = 50!$$

$$\phi_E = 100, \phi_W = 200 \implies \phi_P = 250!$$

Stability, Accuracy and Convergence



$$D_e = D_w = 1, \quad F_e = F_w = 4$$

$$2\phi_P = -\phi_E + 3\phi_W$$

$$F \equiv \rho u, \quad D \equiv \frac{\Gamma}{\delta x}$$

Continuity already satisfied: $F_e = F_w$

$$a_P \phi_P = a_E \phi_E + a_W \phi_W$$

Instability occurs when $|F| > 2D$

Scarborough criterion for convergence:

$$a_P \phi_P = \sum_{f \in nb} a_{nb} \phi_{nb}$$

$$|a_P| \geq \sum |a_{nb}| \text{ at all nodes,}$$

$$|a_P| > \sum |a_{nb}| \text{ at atleast one nodes}$$

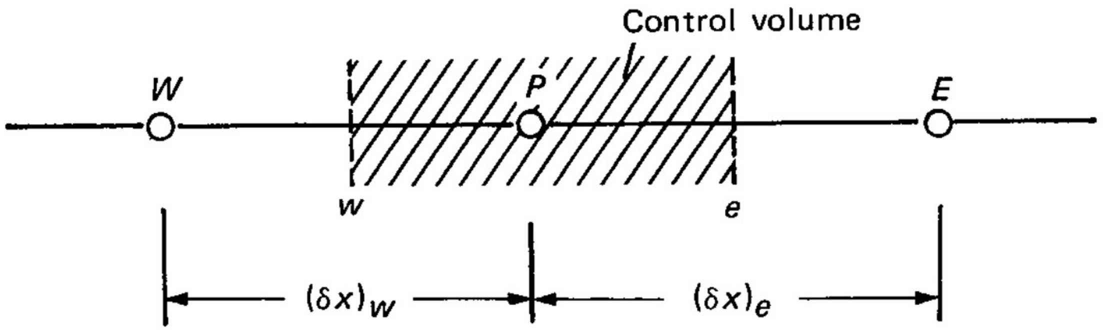
Solution: upwind scheme

$$\phi_f = \begin{cases} \phi_P & \text{if} \quad F_f > 0, \\ \phi_{nb} & \text{if} \quad F_f < 0 \end{cases}$$

- First order accurate
- Required at the point of flow reversal

Stability, Accuracy and Convergence

$$\frac{1}{2}(\rho u)_e(\phi_E+\phi_P)-\frac{1}{2}(\rho u)_w(\phi_P+\phi_W)$$
$$=\frac{\Gamma_e(\phi_E-\phi_P)}{(\delta x)_e}-\frac{\Gamma_e(\phi_P-\phi_W)}{(\delta x)_w}$$



Upwind:

$$(\rho u)_e\phi_e=\phi_P\max(F_e,0)-\phi_E\max(-F_e,0)$$
$$(\rho u)_w\phi_w=\phi_w\max(F_w,0)+\phi_P\max(-F_w,0)$$

$$F\equiv\rho u,\qquad D\equiv\frac{\Gamma}{\delta x}$$

$$a_P\phi_P=a_E\phi_E+a_W\phi_W$$

Continuity already satisfied: $F_e=F_w$

$$a_E=D_e+\max(-F_e,0)$$
$$a_W=D_w+\max(F_w,0)$$
$$a_P=D_e+\max(F_e,0)+D_w+\max(-F_w,0)$$
$$=a_E+a_W+(F_e-F_w)$$
$$=a_E+a_W$$

Let's pick $D_e=D_w=1,\quad F_e=F_w=4$

$$6\phi_P=\phi_E+5\phi_W$$

$$\phi_E=200,\phi_W=100\implies\phi_P=700/6$$
$$\phi_E=100,\phi_W=200\implies\phi_P=1100/6$$

Von Neumann stability analysis

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \qquad u = u_N + \epsilon$$

$$\left[\frac{\partial u_N}{\partial t} + c\frac{\partial u_N}{\partial x}\right] + \left[\frac{\partial \epsilon}{\partial t} + c\frac{\partial \epsilon}{\partial x}\right] = 0$$

$$\frac{\partial \epsilon}{\partial t} + c\frac{\partial \epsilon}{\partial x} = 0$$

FTCS $$\frac{\epsilon_i^{n+1} - \epsilon_i^n}{\Delta t} + c\frac{\epsilon_{i+1}^{n+1} - \epsilon_{i-1}^{n+1}}{2\Delta x} = 0$$

$$\epsilon(x,t) = \sum_{i=-\infty}^{\infty} \hat{\epsilon}(\kappa_i, t)e^{j\kappa_i x}, \quad j = \sqrt{-1}$$

For linear equations: $$\epsilon(x,t) = \hat{\epsilon}(\kappa_i, t)e^{j\kappa_i x}$$

$$\epsilon_i^n = \hat{\epsilon}_i^n e^{j(i\kappa\Delta x)} \qquad G(\kappa_i) = \frac{\hat{\epsilon}_i^{n+1}}{\hat{\epsilon}_i^n} = e^{j\omega\Delta t}$$

$$G - 1 + \left(\frac{c\Delta t}{\Delta x}\right)\frac{Ge^{jk\Delta x} - Ge^{-jk\Delta x}}{2} = 0$$

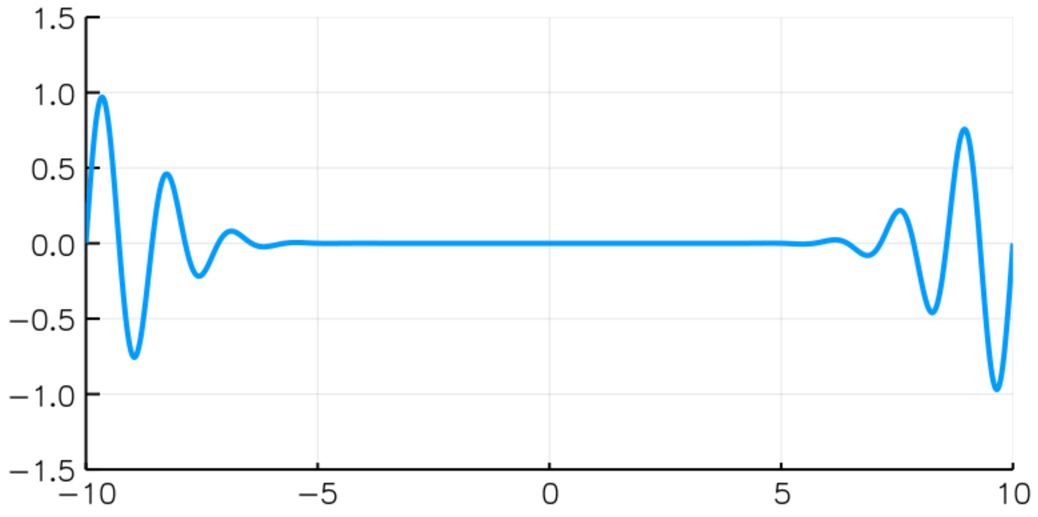
$$G = \frac{1}{1 - jN_c\sin(\kappa\Delta x)} = \frac{1 + jN_c\sin(\kappa\Delta x)}{1 + N_c^2\sin^2(\kappa\Delta x)}$$

$$|G| = \frac{1}{1 + N_c^2\sin^2(\kappa\Delta x)} \qquad \arg(G) = N_c\sin(\kappa\Delta x)$$

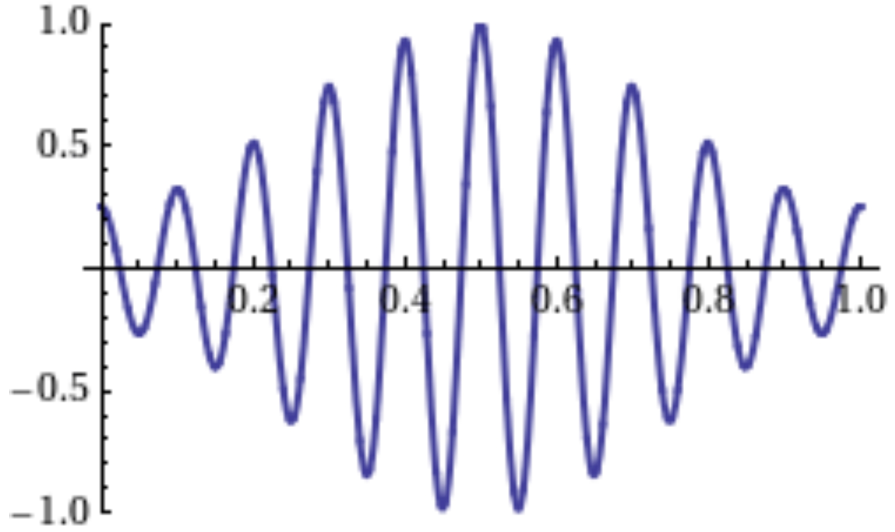
$$e^{j\omega\Delta t} = |G|e^{\arg(G)} \qquad \omega(k) = \frac{c}{\Delta x}\sin(\kappa\Delta x)$$

$$\frac{C_N}{c} = \frac{\omega(\kappa)}{\kappa c} = \frac{\sin(\kappa\Delta x)}{\kappa\Delta x} \qquad \frac{V_g}{c} = \frac{1}{c}\frac{d\omega(\kappa)}{d\kappa} = \cos(\kappa\Delta x)$$

Von Neumann stability analysis



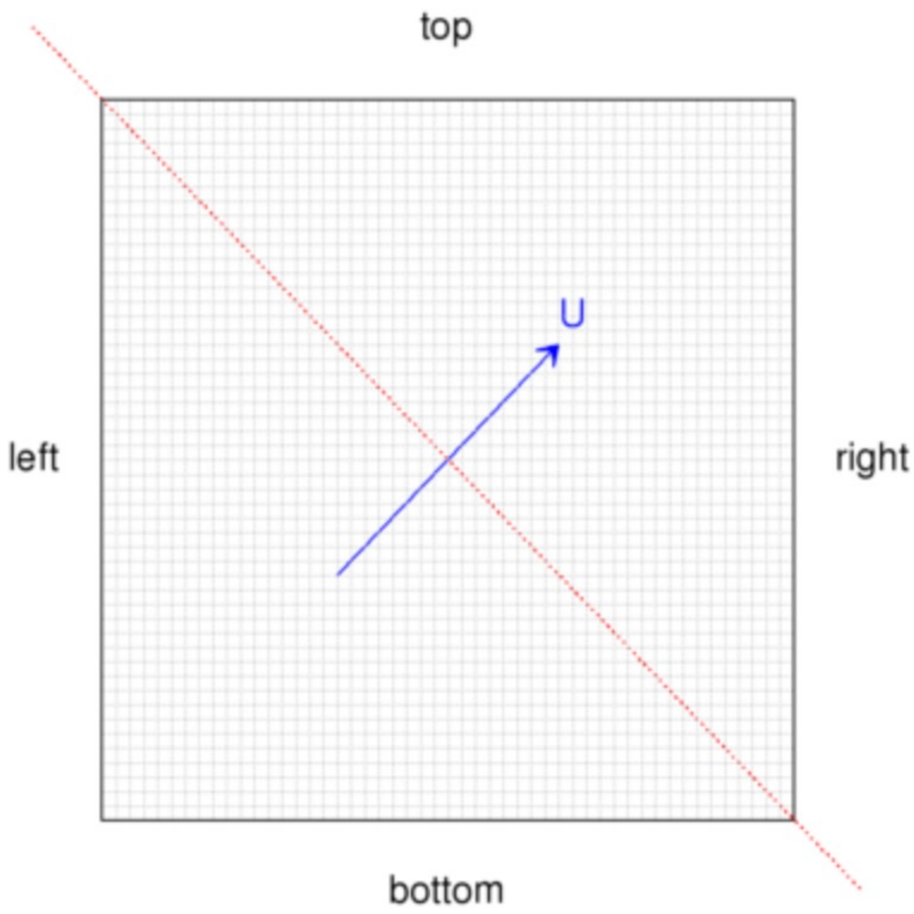
Propagation of a wave packet demonstrating a phase velocity greater than the group velocity without dispersion.

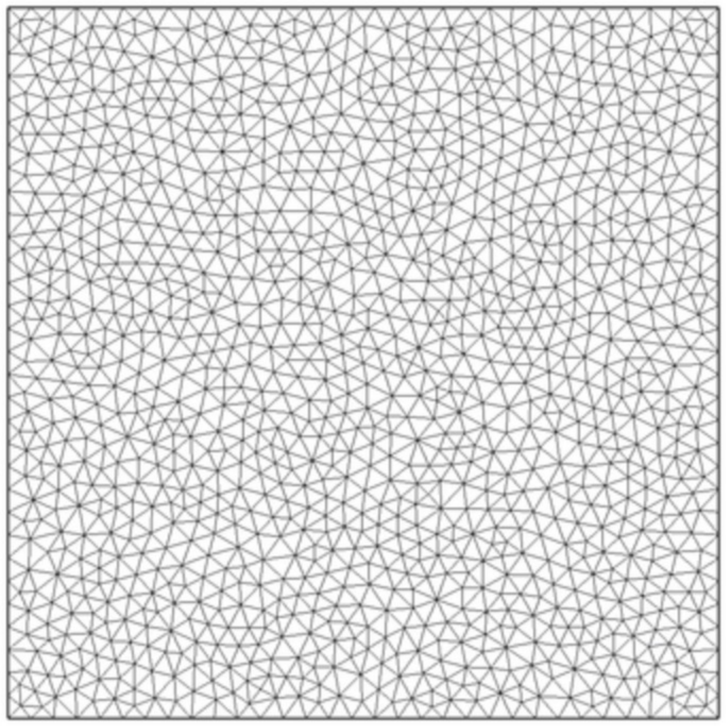


This shows a wave with the group velocity and phase velocity going in different directions. The group velocity is positive (i.e., the envelope of the wave moves rightward), while the phase velocity is negative (i.e., the peaks and troughs move leftward).

A 2-D square domain with side length 1m is employed, split into 50x50 uniform quad cells

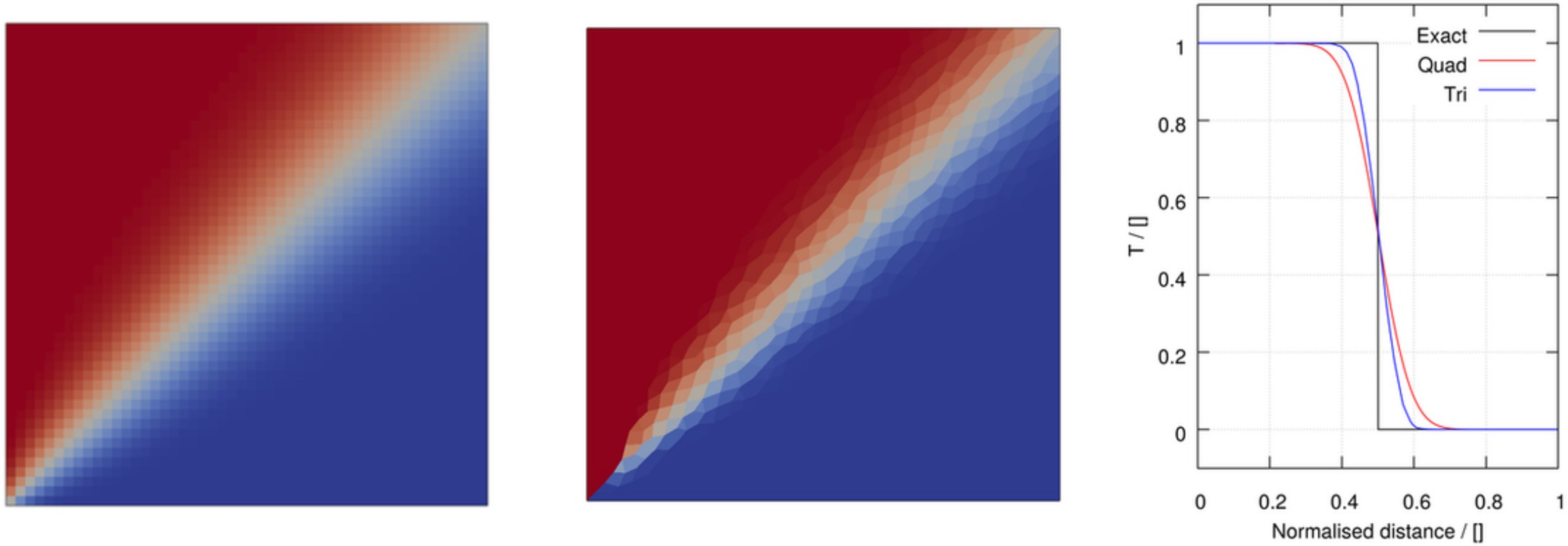
The velocity is set to a constant value of (1,1,0) m/s



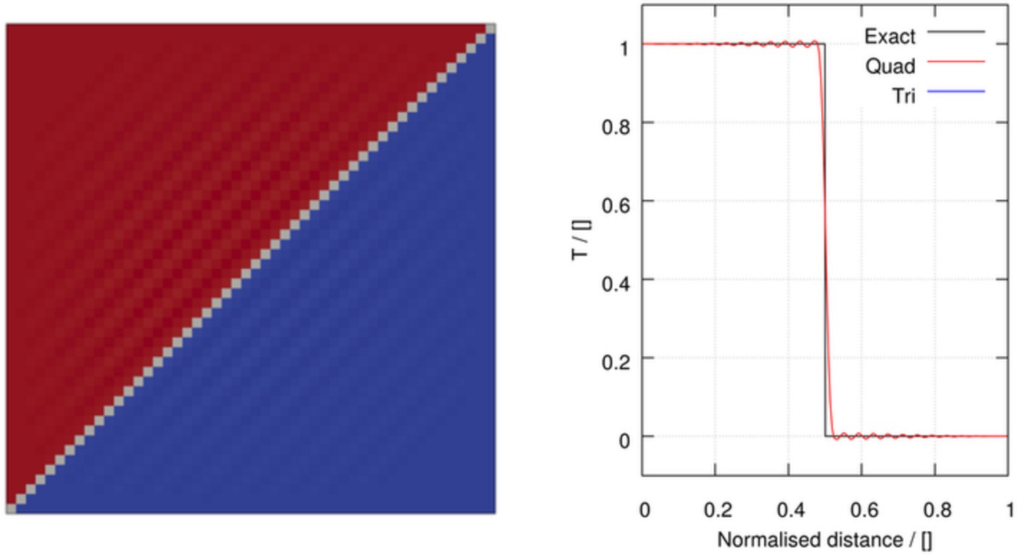


https://www.openfoam.com/documentation/guides/latest/doc/guide-schemes-divergence-example.html

Upwind:



Central:



$$u(c) = a_0 + \sum_{i=-\infty}^{\infty} a_i \cos(2\pi\kappa_i x) d\kappa + \sum_{i=-\infty}^{\infty} b_i \sin(2\pi\kappa_i x) d\kappa$$

https://www.openfoam.com/documentation/guides/latest/doc/guide-schemes-divergence-example.html