Vector function :- If for each value of a scalar variable t, there corresponds a value of a vector $\vec{r}$, then $\vec{r}$ is known as a vector function and defined as follows

$$\vec{r} = \vec{r}(t) \quad \text{or} \quad \vec{r} = \vec{f}(t)$$

In 3 Dimension, Every vector can be uniquely expressed as a linear combination of three fixed non-coplaner vector $\vec{F}(t) = f_1(t)\,\hat{i} + f_2(t)\,\hat{j} + f_3(t)\,\hat{k}$

where $\hat{i}$ is unit vector along x-axis
$\hat{j}$ is unit vector along y-axis
$\hat{k}$ is unit vector along z-axis.

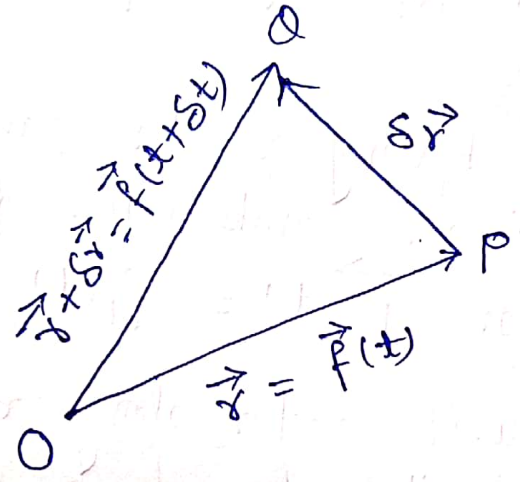
and $f_1(t)$ is component of vector $\vec{F}(t)$ along x-axis.
$f_2(t)$ is component of vector $\vec{F}(t)$ along y-axis.
$f_3(t)$ is component of vector $\vec{F}(t)$ along z-axis.

Ex- $\vec{F}(t) = t^2\,\hat{i} + 2t\,\hat{j} + t^3\,\hat{k}$

$$\underbrace{t^2}_{f_1(t)}\qquad \underbrace{2t}_{f_2(t)}\qquad \underbrace{t^3}_{f_3(t)}$$

Derivative of a vector function w.r.t a scalar

Let $\vec{r} = \vec{f}(t)$ be the vector function of the scalar variable t, and δt be the small increment of t and $\delta\vec{r}$ corresponding to $\vec{r}$.



$$\vec{PQ} = \vec{OQ} - \vec{OP} = \delta\vec{r}$$

Here $\vec{OQ} = \vec{PQ} + \vec{OP}$

$$\vec{f}(t+\delta t) = \vec{r} + \delta\vec{r}$$

$$\delta\vec{r} = \vec{f}(t+\delta t) - \vec{r}$$

$$= \vec{f}(t+\delta t) - \vec{f}(t)$$

$$\frac{\delta\vec{r}}{\delta t} = \frac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t}$$

$$\lim_{\delta t\to 0}\left(\frac{\delta\vec{r}}{\delta t}\right) = \lim_{\delta t\to 0}\left(\frac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t}\right)$$

If this limit exists then it is equal to $\frac{d\vec{r}}{dt}$

Unit - IV

$$\boxed{\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}}$$ - del operator

Gradient :- $\text{grad}(f) = \nabla f$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f$$

$$= \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

Note :- Gradient can be determined for scalar function $f(x,y,z)$.

Ex. ① If $\phi = x^2yz + 3x^2y - y^3$, to find $\nabla\phi$ at $(1,-1,2)$

Soln $\text{grad}\,\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2yz + 3x^2y - y^3)$

$$= \hat{i}\frac{\partial}{\partial x}(x^2yz+3x^2y-y^3) + \hat{j}\frac{\partial}{\partial y}(x^2yz+3x^2y-y^3) + \hat{k}\frac{\partial}{\partial z}(x^2yz+3x^2y-y^3)$$

$$= \hat{i}\,(2xyz + 6xy) + \hat{j}\,(x^2z + 3x^2 - 3y^2) + \hat{k}\,(x^2y)$$

$$\text{grad}\,\phi \Big]_{at\ (1,-1,2)} = \hat{i}\left(2(1)(-1)(2) + 6(1)(-1)\right) + \hat{j}\left((1)^2(2) + 3(1)^2 - 3(-1)^2\right) + \hat{k}\left(1^2(-1)\right)$$

$$\boxed{\text{grad}\,\phi = -10\hat{i} + 2\hat{j} - \hat{k}}$$ //

Q.2 If $u = x+y+z$, $v = x^2+y^2+z^2$, $w = xy+yz+zx$ then prove that

$$(\text{grad}\,u)\cdot((\text{grad}\,v)\times(\text{grad}\,w)) = 0$$

Note. $[\vec{a}\,\vec{b}\,\vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c})$

& $[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{a}\,\vec{b}]$

& $\vec{A}\times\vec{B} = -(\vec{B}\times\vec{A})$

Sol^n $\text{grad}\,u = \nabla u$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) u$$

$$= \hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}$$

$$= \hat{i}(1) + \hat{j}(1) + \hat{k}(1)$$

$$\boxed{\text{grad}\,u = \hat{i} + \hat{j} + \hat{k}}$$

$$\text{grad}\,v = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\text{grad}\,w = \hat{i}(y+z) + \hat{j}(x+z) + \hat{k}(x+y)$$

$$L.H.S = (\text{grad}\,u)\cdot[\text{grad}\,v \times \text{grad}\,w]$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x+y+z & x+y+z & x+y+z \end{vmatrix} \quad R_3 \to R_3 + R_2 \quad = 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 0$$

Note :- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

then $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c})$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Q. 3 Prove that $\nabla r^{-3} = -3r^{-5}\vec{r}$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Sol: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\vec{r}| = |x\hat{i} + y\hat{j} + z\hat{k}|$

$\boxed{r = \sqrt{x^2+y^2+z^2}}$ $\quad (\because |\vec{r}| = r)$ ↑ scalar

$\Rightarrow r^2 = x^2 + y^2 + z^2$

$2r\frac{\partial r}{\partial x} = 2x \quad , 2r\frac{\partial r}{\partial y} = 2y, \quad 2r\frac{\partial r}{\partial z} = 2z$

$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \quad , \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = z/r$

Now $\quad$ L.H.S $= \nabla r^{-3}$

$= \hat{i}\frac{\partial}{\partial x} r^{-3} + \hat{j}\frac{\partial}{\partial y} r^{-3} + \hat{k}\frac{\partial}{\partial z} r^{-3}$

$= \hat{i}\left(-3r^{-4}\frac{\partial r}{\partial x}\right) + \hat{j}\left(-3r^{-4}\frac{\partial r}{\partial y}\right) + \hat{k}\left(-3r^{-4}\frac{\partial r}{\partial z}\right.$

$$\nabla r^{-3} = -3r^{-4}\left[\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right)$$

$$= -3r^{-4}\left[\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right)$$

$$= -3r^{-5}\left[x\hat{i} + y\hat{j} + z\hat{k}\right]$$

$$\boxed{\nabla r^{-3} = -3r^{-5}.\vec{r}}$$ //

Q.4 Prove that $\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$

Solⁿ:- let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$|\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\Rightarrow r^2 = x^2+y^2+z^2$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

Now $\nabla\left(\frac{1}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\frac{1}{r}$

$$= \hat{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \hat{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) + \hat{j}\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right) + \hat{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$$

$$= -\frac{1}{r^2}\left(\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right)$$

$$\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2}\left(\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right)$$

$$= -\frac{1}{r^3}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= -\frac{1}{r^3}\vec{r}$$

$$\Rightarrow \boxed{\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}}$$

Question 5 Find the gradient and unit normal vector to the surface $x^2+y^2-z^2=2$ at the point $(1,-1,2)$

Sol^n:- Let $f = x^2+y^2-z^2-2 = 0$

$\longrightarrow$ scalar function.

$$\text{grad } f = \nabla f$$

$$= \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \hat{i}(2x) + \hat{j}(2y) + \hat{k}(-2z)$$

$$\boxed{\text{grad } f\Big|_{\text{at }(1,-1,2)} = 2\hat{i} - 2\hat{j} - 4\hat{k}}$$

Formula.

The unit normal vector to the surface is given by

$$= \frac{\text{grad } f}{|\text{grad } f|}$$

Now

unit normal vector to the surface $= \dfrac{2\hat{i}-2\hat{j}-4\hat{k}}{\sqrt{4+4+16}}$

$$= \frac{2\hat{i}-2\hat{j}-4\hat{k}}{\sqrt{24}}$$

$$= \frac{\cancel{2}(\hat{i}-\hat{j}-2\hat{k})}{\cancel{2}\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}}(\hat{i}-\hat{j}-2\hat{k})$$

Ans.

Note :- Gradient of scalar field f is normal to the surface.

Directional Derivative The directional derivative of surface f in direction $\vec{A}$ is given by

$$= (\text{grad } f)\cdot\hat{A}$$

where $\hat{A}$ = unit vector in direction $\vec{A}$.

$$= \frac{\vec{A}}{|\vec{A}|}$$

Ex. 1 Find directional derivative of $f = x^2yz + 4xz^2$ at point $(1,-2,-1)$ in the direction of vector $2\hat{i}-\hat{j}-2\hat{k}$.

Sol^n: Here $f = x^2yz + 4xz^2$

$\vec{A} = 2\hat{i}-\hat{j}-2\hat{k}$

Now $\text{grad } f = \nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f$

$$= \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \hat{i}(2xyz + 4z^2) + \hat{j}(x^2z) + \hat{k}(x^2y + 8xz)$$

Now $\text{grad } f \Big|_{at\ (1,-2,-1)} = \hat{i}\left(2(1)(-2)(-1) + 4(-1)^2\right) + \hat{j}\left((1)^2(-1)\right) + \hat{k}\left((1)^2(-2) + 8(1)(-1)\right]$

$$\boxed{\nabla f_{(1,-2,-1)} = 8\hat{i} + (-\hat{j}) + (-10)\hat{k}}$$

Now $\vec{A} = 2\hat{i} - \hat{j} - 2\hat{k}$

$$\hat{A} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{4+1+4}} = \frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})$$

So directional derivative at $(1,-2,-1)$ in direction of $2\hat{i} - \hat{j} - 2\hat{k}$ $= (\text{grad } f)\cdot\hat{A}$

$$= (8\hat{i} - \hat{j} - 10\hat{k})\cdot\frac{1}{3}(2\hat{i} - \hat{j} - 2\hat{k})$$

$$= \frac{1}{3}(16 + 1 + 20)$$

$$= \frac{37}{3}$$

Ans

Ex. 2 Find direction derivative of $f = x^2 - 2y^2 + 4z^2$ at $(1,1,-1)$ in $2\hat{i} + \hat{j} - \hat{k}$. Also find direction of maximum directional derivative at $(1,1,-1)$ & its maximum value?

Sol

$$\text{grad} f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \hat{i}(2x) + \hat{j}(-4y) + \hat{k}(8z)$$

$$\text{grad} f\Big|_{(1,1,-1)} = \hat{i}(2(1)) + \hat{j}(-4(1)) + \hat{k}(8(-1))$$

$$= \underline{2\hat{i} - 4\hat{j} - 8\hat{k}}$$

$\therefore$ directional derivative in direction of $\vec{A} = \text{grad} f \cdot \hat{A}$

$$= (2\hat{i} - 4\hat{j} - 8\hat{k}) \cdot \frac{(2\hat{i} + \hat{j} - \hat{k})}{\sqrt{4+1+1}}$$

$$= \frac{4-4+8}{\sqrt{6}} = \frac{8}{\sqrt{6}}$$

Note:- Direction of maximum directional derivatives is in direction of grad f.

Hence direction of maximum directional derivative is here $2\hat{i} - 4\hat{j} - 8\hat{k}$

Now Max directional derivative $= |\text{grad} f| = \sqrt{4+16+64}$

$$= \sqrt{84}$$

Note :- 1 Maximum directional derivative

$$= \text{grad} f \cdot \widehat{\text{grad}} f$$

$$= \text{grad} f \cdot \frac{\text{grad} f}{|\text{grad} f|}$$

$$= \frac{|\text{grad} f|^2}{|\text{grad} f|}$$

$$\boxed{M.D.D = |\text{grad} f|}$$

Note 2 Angle b/w two surfaces f and g is given by

$$\cancel{\cos\theta = \frac{(\text{grad} f)\cdot(\text{grad} g)}{|\text{grad} f|\,|\text{grad} g|}}$$

$$\boxed{\cos\theta = \frac{(\text{grad} f)\cdot(\text{grad} g)}{|\text{grad} f|\,|\text{grad} g|} = \frac{\nabla f \cdot \nabla g}{|\nabla f|\,|\nabla g|}}$$

Ex. ① Find angle b/w two surfaces $x^2+y^2+z^2=9$ & $z=x^2+y^2-3$ at $(2,-1,2)$

Sol^n Let $f = x^2+y^2+z^2-9$

$g = x^2+y^2-z-3$

$$\text{grad} f = \nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\text{grad} f\Big|_{(2,-1,2)} = 4\hat{i} - 2\hat{j} + 4\hat{k}$$

$$\text{grad} g = \nabla g = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\text{grad} g\Big|_{(2,-1,2)} = 4\hat{i} - 2\hat{j} - \hat{k}$$

Now

$$\cos\theta = \frac{(\text{grad} f)\,(\text{grad} g)}{|\text{grad} f|\;|\text{grad} g|}$$

$$= \frac{(4\hat{i}-2\hat{j}+4\hat{k})\cdot(4\hat{i}-2\hat{j}-\hat{k})}{\sqrt{16+4+16}\;\sqrt{16+4+1}}$$

$$= \frac{16+\cancel{4}-\cancel{4}}{\sqrt{36}\;\sqrt{21}} = \frac{16}{6\sqrt{21}}$$

$$\cos\theta = \frac{8}{3\sqrt{21}}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)}$$

Q.2 find the value of constant a and b so that surface $ax^2 - bxy = (a+2)x$ will be orthogonal to surface $4x^2y + z^3 = 4$ at $(1,-1,2)$?

Sol^n- Let $f = 4x^2y + z^3 - 4$

$g = ax^2 - bxy - (a+2)x$

$$\text{grad } f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \hat{i}(8xy) + \hat{j}(4x^2) + \hat{k}(3z^2)$$

$$\text{grad } f \Big|_{(1,-1,2)} = -8\hat{i} + 4\hat{j} + 12\hat{k}$$

$$\text{grad } g = \hat{i}\frac{\partial g}{\partial x} + \hat{j}\frac{\partial g}{\partial y} + \hat{k}\frac{\partial g}{\partial z}$$

$$= \hat{i}(2ax - by - (a+2)) + \hat{j}(-bx) + \hat{k}(0)$$

$$\text{grad } g \Big|_{(1,-1,2)} = i(2a + b - a - 2) + j(-b(1))$$

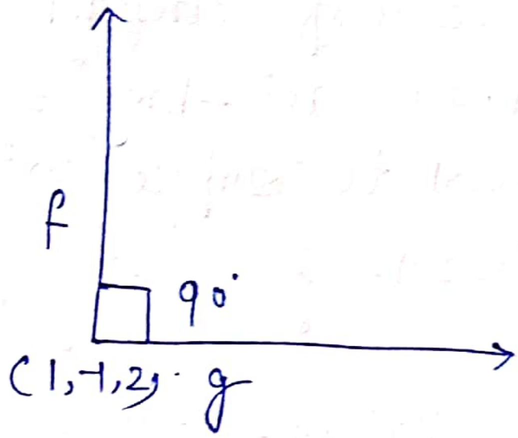
$$= (a + b - 2)\hat{i} - b\hat{j} + 0\hat{k}$$

Now $\theta = 90^\circ$

$$\Rightarrow \cos 90^\circ = \frac{(\text{grad } f)\cdot(\text{grad } g)}{|\text{grad } f||\text{grad } g|}$$

$$\Rightarrow 0 = \frac{-8(a+b-2) - 4b}{|\text{grad } f|\,|\text{grad } g|} \Rightarrow -8a - 8b + 16 - 4b = 0$$

$$\Rightarrow \boxed{-8a - 12b + 16 = 0}$$



Since g touching (1, -1, 2) so it will satisfy the equation $g = 0$

$$g = ax^2 - bxy - (a+2)x = 0$$

at (1, -1, 2)

$$a - b(1)(-1) - (a+2)(1) = 0$$

$$\cancel{a} + b - \cancel{a} - 2 = 0$$

$$\boxed{b = 2}$$

Now put $b = 2$ in equation $-8a - 12b + 16 = 0$

$$\Rightarrow -8a - 12(2) + 16 = 0$$

$$\Rightarrow -8a - 24 + 16 = 0$$

$$\Rightarrow -8a - 8 = 0$$

$$\Rightarrow 8a = -8$$

$$\Rightarrow \boxed{a = -1}$$

Hence $\boxed{a = -1, \ b = 2}$ Answer

Ques 3 find the maximum values for the directional derivative of the function

$f = 3x^2y^2 - 5x^3z + 3xy^2z^2$ at the point $(1,-1,-1)$

Sol^n $\text{grad } f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$

$= \hat{i}(6xy^2 - 15x^2z + 3y^2z^2)$
$+ \hat{j}(6x^2y + 6xyz^2) + \hat{k}(-5x^3 + 6xy^2z)$

$\text{grad } f\Big|_{(1,-1,-1)} = \hat{i}(6(1)(-1)^2 - 15(1)^2(-1) + 3(-1)^2(-1)^2)$
$+ \hat{j}(6(1)^2(-1) + 6(1)(-1)(-1)^2]$
$+ \hat{k}(-5(1)^3 + 6(1)(-1)^2(-1)]$

$$\boxed{\text{grad } f\Big|_{(1,-1,-1)} = 24\hat{i} - 12\hat{j} - 11\hat{k}}$$

The maximum value of directional derivative at point $(1,-1,-1)$ $= |\text{grad } f_{(1,-1,-1)}|$

$= |24\hat{i} - 12\hat{j} - 11\hat{k}|$

$= \sqrt{(24)^2 + (12)^2 + (11)^2}$

$= \sqrt{576 + 144 + 121}$

$= \sqrt{841}$

$= 29$ Ans.

Note ① Let $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

& $\vec{r}$ = position vector

$= x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

then equation of tangent plane is given by

$$\boxed{[\vec{R} - \vec{r}] \cdot \text{grad} f = 0}$$

$$[x\hat{i} + y\hat{j} + z\hat{k} - x_1\hat{i} - y_1\hat{j} - z_1\hat{k}] \cdot \text{grad} f = 0$$

$$\boxed{[(x - x_1)\hat{i} + (y - y_1)\hat{j} - (z - z_1)\hat{k}] \cdot \text{grad} f = 0}$$

this is required equation of tangent plane at point (x_1, y_1, z_1).

Note ② The equation of normal plane at point (x_1, y_1, z_1) is given by

$$\boxed{\frac{x - x_1}{\left.\frac{\partial f}{\partial x}\right|_{(x_1, y_1, z_1)}} = \frac{y - y_1}{\left.\frac{\partial f}{\partial y}\right|_{(x_1, y_1, z_1)}} = \frac{z - z_1}{\left.\frac{\partial f}{\partial z}\right|_{(x_1, y_1, z_1)}}}$$

Q.1 Find the equation of tangent and normal plane of surface $x^2+y^2+z^2=5$ at point $(1,-1,2)$

Sol^n Let $f = x^2+y^2+z^2-5$

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial z} = 2z$$

then $\text{grad } f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$$\boxed{\text{grad } f\Big|_{(1,-1,2)} = 2\hat{i} - 2\hat{j} + 4\hat{k}}$$

equation of tangent plane at point $(1,-1,2)$

$$\uparrow\ \uparrow\ \uparrow$$
$$x_1\ y_1\ z_1$$

$$\Rightarrow (\vec{R}-\vec{r})\cdot \text{grad } f = 0$$

$$\Rightarrow \left((X-1)\hat{i} + (Y+1)\hat{j} + (Z-2)\hat{k}\right)\cdot(2\hat{i}-2\hat{j}+4\hat{k}) = 0$$

$$\Rightarrow 2(X-1) - 2(Y+1) + 4(Z-2) = 0$$

$$\Rightarrow 2X-2-2Y-2+4Z-8=0$$

$$\Rightarrow \boxed{2X-2Y+4Z-12=0}$$

Ans.

& Equation of normal plane at $(1,-1,2)$ is

$$\frac{X-x_1}{\left.\frac{\partial f}{\partial x}\right|_{(x_1,y_1,z_1)}} = \frac{Y-y_1}{\left.\frac{\partial f}{\partial y}\right|_{(x_1,y_1,z_1)}} = \frac{Z-z_1}{\left.\frac{\partial f}{\partial z}\right|_{(x_1,y_1,z_1)}}$$

$$\Rightarrow \frac{X-1}{2} = \frac{Y+1}{-2} = \frac{Z-2}{4} \Rightarrow \boxed{\frac{X-1}{2} = \frac{Y+1}{-2} = \frac{Z-2}{4}}$$

Ans

Divergence of vector point function

Let $\vec{V}$ be the vector point function. Then the divergence of $\vec{V}$ can be given by

$$\text{div}\,\vec{V} = \nabla\cdot\vec{V}$$
$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\vec{V}$$
$$= \hat{i}\cdot\frac{\partial\vec{V}}{\partial x} + \hat{j}\cdot\frac{\partial\vec{V}}{\partial y} + \hat{k}\cdot\frac{\partial\vec{V}}{\partial z}$$
$$= \sum \hat{i}\cdot\frac{\partial\vec{V}}{\partial x}$$

or

$$\sum \hat{j}\cdot\frac{\partial\vec{V}}{\partial y}$$

or

$$\sum \hat{k}\cdot\frac{\partial\vec{V}}{\partial z}$$

Solenoidal Vector — A vector $\vec{V}$ is said to be solenoidal vector if $\boxed{\text{div}\,\vec{V} = 0}$"

Note :- ① div of any vector $\vec{V}$ is always comes as scalar quantity.

Note ② Gradient of a scalar function is always a vector quantity.

Curl of vector $\vec{V}$

$$\boxed{\text{Curl } \vec{V} = \nabla \times \vec{V}}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \vec{V}$$

Note (3) Curl$(\vec{V})$ is always a vector quantity.

Ex. (1) find div $\vec{f}$ at point $(2,1,4)$ where

$$\vec{f} = 2x^2y^3\,\hat{i} - 3y^2z\,\hat{j} + 3xy^2\,\hat{k}$$

Note $\hat{i}\cdot\hat{i} = 1$ $\hat{j}\cdot\hat{j} = 1$ $\hat{k}\cdot\hat{k} = 1$ $\hat{i}\cdot\hat{j} = 0 = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i}$

Now $$\nabla\cdot\vec{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(2x^2y^3\hat{i} - 3y^2z\hat{j} + 3xy^2\hat{k}\right)$$

$$= 4xy^3 - 6yz + 0$$

$$\left(\text{div }\vec{f}\right)_{at(2,1,4)} = 4(2)(1)^3 - 6(1)(4)$$

$$= 8 - 24$$

$$\boxed{\text{div }\vec{f}_{at(2,1,4)} = -16}$$ Ans.

Scalar quantity