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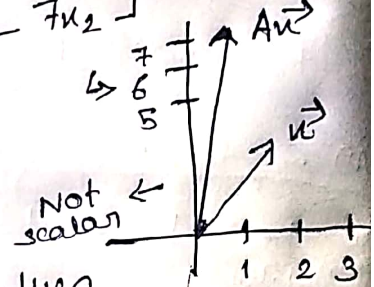
1. Describe eigen vectors of a matrix geometrically with a suitable example

Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$

then if we multiply the matrix A with vectors u_1 and u_2 then we end up with vectors.

$$A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2u_1 \\ 7u_2 \end{bmatrix}$$



The vector $2u_1 = \vec{u}$
vector $7u_2 = A\vec{u}$

For eigen vectors and eigen values we need a situation $A\vec{u} = \lambda\vec{u}$, i.e., for which $\vec{u}$ are $\vec{u}$ and $A\vec{u}$ are scalar mult

$$A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2u_1 \\ 7u_2 \end{bmatrix}$$

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \end{bmatrix}$$

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 7 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A\vec{u} = \lambda\vec{u}$$

∴ The 7 is eigenvalue for matrix A and the basis for eigen space $E_7 : \{ \begin{smallmatrix} 0 \\ 1 \end{smallmatrix} \}$ or any non zero multiple.

$$A \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2u_1 \\ 7u_2 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A\vec{u} = \lambda\vec{u}$$

∴ 2 is eigen value for matrix A and the basis for eigen space $E_2 : \{ \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \}$ or any non-zero multiple.

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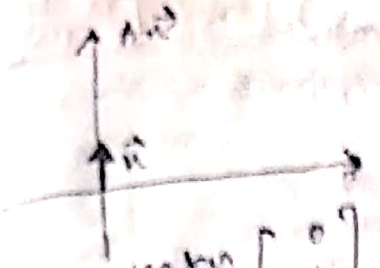

eigen vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ corresponding to eigen value 7.

eigen vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ corresponding to eigen value 2

Q. Find the eigen values, eigen vectors, algebric multiplicity and geometric multiplicity of the following matrix:

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

a) Eigen values:-

$|A - \lambda I| = 0$

$$\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$= \begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} = 0.$$

$(-2-\lambda)[-\lambda(1-\lambda)-12] - 2[-2\lambda-6] - 3[-4+1-\lambda] = 0$

$2\lambda - \cancel{2\lambda^2} + 24 + \lambda^2 - \lambda^3 + 12\lambda + \cancel{4\lambda} + 12 + 9 + \cancel{3\lambda} = 0.$

$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$

$\Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0.$

$\Rightarrow \lambda = 5$

$\therefore \lambda - 5 = 0.$

$\lambda^3 - 5\lambda^2 + 6\lambda^2 - 36\lambda + 9\lambda - 45$

$\lambda^2(\lambda-5)+6\lambda(\lambda-5)+9(\lambda-5)$

$\therefore (\lambda-5)(\lambda^2+6\lambda+9)=0$

$\therefore (\lambda-5)(\lambda+3)^2=0$.

$\therefore$ The eigen values are:-

$\lambda=5, -3, -3$.

$\boxed{\text{Eigen vectors}}$

b) $\underline{\lambda=5}$ $[A-\lambda I]X=0$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x_1}{\begin{vmatrix} -4 & -6 \\ -2 & -5 \end{vmatrix}}=\frac{x_2}{\begin{vmatrix} 2 & -6 \\ -1 & -5 \end{vmatrix}}=\frac{x_3}{\begin{vmatrix} 2 & -4 \\ -1 & -2 \end{vmatrix}}$$

$$\Rightarrow \frac{x_1}{20-12}=\frac{x_2}{-10-6}=\frac{x_3}{-4-4}$$

$$\Rightarrow \frac{x_1}{8}=\frac{x_2}{-16}=\frac{x_3}{-8}$$

$$\Rightarrow \frac{x_1}{1}=\frac{x_2}{-2}=\frac{x_3}{-1}.$$

$\therefore$ for $\lambda=5$, the eigen vectors are $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$

$\lambda=-3$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here we will get $\frac{x}{0}=\frac{-y}{0}=\frac{z}{0}$ which is not possible for eigen vectors.

This means there is only one independent equation in this system.

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Solving by Gaussian elimination.

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore x_1 + 2x_2 - 3x_3 = 0$ — (1)

$\therefore x_1 = -2x_2 + 3x_3$; $x_2 = x_2$; $x_3 = x_3$

$\therefore$ general solution :- $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$X = \begin{bmatrix} -2x_2 + 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \left[x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right]$$

$\therefore$ for $\lambda = -3$

The eigen vectors are :-

$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ eigen vector for the system are.

$\begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

(c) <u>Algebraic multiplicity</u>.

The algebraic multiplicity of an eigen values is the exponent on the apppropriate factor in characteristic equation,

$$(\lambda + 3)^2 (\lambda - 5) = 0.$$

Am of $\lambda = 5$ is 1.

Am of $\lambda = -3$ is 2.

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d) Geometric multiplicity

Dimension of the eigenspace
No. of basis factors in eigenspace.

$1 \le GM \le AM$

$\lambda = 5$

$[A - 5I]$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -11 \\ -1 & -2 & -5 \end{bmatrix}$$

$|D| = -7(20-32) - 2(-10-11) - 3(-4-9)$

$= 82$

$|D| \neq 0$

$\therefore$ Rank $= 3$.

$\therefore$ $GM = n - 3$

$= 3 - 3 = 0$.

4) For the following matrix A, find its eigen values a maximal orthogonal set s of eigen vectors and an orthogonal matrix P such that $D = P^{-1}AP$ is diagonal.

$$A = \begin{bmatrix} 2 & 2 & 4 \\ 2 & 5 & 8 \\ 4 & 8 & 17 \end{bmatrix}$$

Eigen values -

$|A - \lambda I| = 0$.

$$\begin{bmatrix} 2-\lambda & 2 & 4 \\ 2 & 5-\lambda & 8 \\ 4 & 8 & 17-\lambda \end{bmatrix} = 0.$$

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$\Rightarrow -\lambda^3 + 23\lambda^2 - 45\lambda + 22$

$\lambda = 1.$

$-\lambda^3 + \lambda^2 + 22\lambda^2 - 22\lambda - 22\lambda + 22$

$\Rightarrow -\lambda^2(\lambda - 1) + 22\lambda(\lambda - 1) - 22(\lambda - 1)$

$\Rightarrow (\lambda - 1)(-\lambda^2 + 23\lambda - 22)$

$= -(\lambda - 1)(\lambda^2 - 23\lambda + 22)$

$= -(\lambda - 1)(\lambda^2 - 22\lambda - \lambda + 22)$

$= -(\lambda - 1)(\lambda(\lambda - 22) - 1(\lambda - 22))$

$\therefore$ The eigen values are :-

$\lambda_1 = 1$

$\lambda_2 = 22.$

Eigen vectors.

$\lambda = 22$

$$\begin{bmatrix} -20 & 2 & 4 \\ 2 & -17 & 8 \\ 4 & 8 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\frac{x}{\begin{vmatrix} -17 & 8 \\ 8 & -5 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 2 & 8 \\ 4 & -5 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2 & -17 \\ 4 & 8 \end{vmatrix}}$$

$$\frac{x}{21} = \frac{-y}{-42} = \frac{z}{84}$$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{4}$$

$\therefore$ For $\lambda = 22$, eigen vectors are $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

$\lambda = 1$

for $\lambda = 1$,

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$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 6 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore x + 2y + 4z = 0$

$\Rightarrow x = -2y - 4z$

$y = y$

$z = z.$

$$x = \begin{bmatrix} -2x - 4y \\ y \\ z \end{bmatrix}$$

$$= \left[y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} \right]$$

for $\lambda = 1$, eigen vectors :

$$\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

The eigen vectors for the system,

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} 1 & -2 & -4 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

$$\boxed{P^{-1} = \frac{\text{adj}(P)}{|P|}}$$

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$|P| = 1(1-0) - (-2)(2-0) - 4(0-4)$

$= 1 + 4 + 16 = 21.$

adj (P) :-

1) $(-1)^{1+1}[1-0] = 1.$
2) $(-1)^{1+2}(2-0) = -2$
3) $(-1)^{1+3}(0-4) = -4.$
4) $(-1)^{2+1}(-2-0) = +2$
5) $(-1)^{2+2}(1-(-16)) = +17.$
6) $(-1)^{2+3}(0-(-8)) = -8$
7) $(-1)^{3+1}(0-(-4)) = +4$
8) $(-1)^{3+2}(0-(-8)) = -8$
9) $(-1)^{3+3}(1-(-4)) = 5.$

$\therefore \text{adj } P = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 17 & -8 \\ -4 & -8 & 5 \end{bmatrix}$

$\therefore P^{-1} = \frac{1}{21}\begin{bmatrix} 1 & 2 & 4 \\ -2 & 17 & -8 \\ -4 & -8 & 5 \end{bmatrix}$

Now,

$P^{-1}AP = \frac{1}{21}\begin{bmatrix} 1 & 2 & 4 \\ -2 & 17 & -8 \\ -4 & -8 & 5 \end{bmatrix}\begin{bmatrix} 2 & 2 & 4 \\ 2 & 5 & 8 \\ 4 & 8 & 17 \end{bmatrix}\begin{bmatrix} 1 & -2 & -4 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$

$= \frac{1}{21}\begin{bmatrix} 1 & 2 & 4 \\ -2 & 17 & -8 \\ -4 & -8 & 5 \end{bmatrix}\begin{bmatrix} 22 & -2 & -4 \\ 44 & 1 & 0 \\ 88 & 0 & 1 \end{bmatrix}$

$= \frac{1}{21}\begin{bmatrix} 462 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 21 \end{bmatrix}$

$= \frac{21}{21}\begin{bmatrix} 22 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 22 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= D.$

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$$D = P^{-1}AP = \begin{bmatrix} 22 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6) Factorize the following matrix using singular value Decomposition Algorithm.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}. \quad m = 2, \; n = 3$$

$$A^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Eigen value -

$$|AA^T - \lambda I| = 0.$$

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (2-\lambda)^2 - 1 = 0$$

$$\Rightarrow 4 - 4\lambda + \lambda^2 - 1 = 0.$$

$$\Rightarrow 3 - 4\lambda + \lambda^2 = 0.$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - \lambda + 3 = 0$$

$$\Rightarrow \lambda(\lambda - 3) - 1(\lambda - 3) = 0.$$

$$\therefore \lambda_1 = 3, \; \lambda_2 = 1.$$

Again $\lambda_1 = \sigma_1^2$

$\therefore \sigma_1 = \sqrt{3}, \; \sigma_2 = 1 \Rightarrow$ singular values

$\therefore \sigma_1 \geq \sigma_2 \geq \dots \sigma_n.$

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$\therefore U_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}, \quad U_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix}$

$\therefore U = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$

$\Sigma = \begin{bmatrix}\sqrt{3} & 0 & 0\\0 & 1 & 1\end{bmatrix}$

Now, $V_i = \frac{1}{\sqrt{\lambda_i}} A^T U_i$ for non zero singular value.

$V_1 = \frac{1}{\sigma_1} A^T U_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1 & 0\\1 & 1\\0 & 1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$

$= \frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\1\end{bmatrix}$

$V_2 = \frac{1}{\sigma_2} A^T U_2 = \frac{1}{1}\begin{bmatrix}1 & 0\\1 & 1\\0 & 1\end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}$

To find a vector orthogonal to V_1 & V_2.

$V_3 = V_1 \times V_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\1\end{bmatrix} \times \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}$

(cross product)

$= \frac{1}{\sqrt{12}}\begin{bmatrix}2\\-3\\2\end{bmatrix}$

Now to normalise,

$V_3' = \frac{1}{\sqrt{17}}\begin{bmatrix}2\\-3\\2\end{bmatrix}$

$V = \begin{bmatrix}V_1 & V_2 & V_3\end{bmatrix}$

According to SVD,

$A = U \Sigma V^T = \begin{bmatrix}U_1 & U_2\end{bmatrix}\begin{bmatrix}\sigma_1 & 0 & 0\\0 & \sigma_2 & 0\end{bmatrix}\begin{bmatrix}V_1 & V_2 & V_3\end{bmatrix}^T$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\sqrt{3} & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\left[\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}\frac{1}{\sqrt{12}}\right.$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\sqrt{3} & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\frac{1}{12}\begin{bmatrix}1 & -1 & 2\\ 2 & 0 & 3\\ 1 & 1 & 2\end{bmatrix}^T$$

$$\frac{1}{12\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}\sqrt{3} & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 2 & 1\\ -1 & 0 & 1\\ 2 & 3 & 2\end{bmatrix}$$

$$= \frac{1}{12\sqrt{2}}\begin{bmatrix}1.73 & -1 & 0\\ 1.73 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 2 & 1\\ -1 & 0 & 1\\ 2 & 3 & 2\end{bmatrix} = \frac{1}{12\sqrt{2}}\begin{bmatrix}2.73 & 3.46 & 0.7\\ 0.73 & 3.46 & 2.73\end{bmatrix}$$