02/08/2024

Tutorial - 1

Q1. What is the shortest possible code length, in bits per average symbol, that could be acheived for a six letter alphabet whose symbols have the following probability distribution.

$$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{32}\right\}$$

Sol:- Entropy, $H(x) = \sum_{i=1}^{6} P(x_i) \cdot \log_2 \frac{1}{P(x_i)}$

$$= \frac{1}{2}(1) + \frac{1}{4}(2) + \frac{1}{8}(3) + \frac{1}{16}(4) + \frac{1}{32}(5) + \frac{1}{32}(5)$$

$$= \frac{16+16+12+8+10}{32}$$

$= \boxed{\cancel{1.625}}$ 1.9375 bits/message.

Q2. An image uses 512x512 picture elements. Each picture element can take any of the 8 different intensity levels. Compute the maximum entropy of this image.

Sol:- $n = 8$, $N = 512 \times 512$.

$H(x) = \log_2 n = \log_2 8 = 3$ bits/picture element.

$\therefore H(x_n) = N \cdot H(x) = 512 \times 512 \times 3$ bits/image.

$= 2^{18} \times 3$ bits/image.

Q4. A binary source has symbol probabilities 0.7, 0.15 and 0.15. If extension coding (blocks of 4 symbols) is used, then compute lower and upper bounds on average code word length.

Sol:- $H(x) = \sum_{i=1}^{3} p(x_i) \log_2 \frac{1}{P(x_i)}$

$$= 0.7 \log_2\left(\frac{10}{7}\right) + 0.15 \cdot \log_2\left(\frac{20}{3}\right) + 0.15 \cdot \log_2\left(\frac{20}{3}\right)$$

$= 0.36 + 0.82$

$= 1.18$ bits/symbol.

$H(x_4) = N \cdot H(x)$

$= 4 (1.18)$

$= 4.72$ bits/block.

Upper Bound :- $4 \times \log_2 3 = 1.53 \times 4 = 6.12$ bits / coding block.

Lower Bound :- $4 \times 0 = 0$.

Q5 A source generates three symbols with probabilities of 0.25, 0.25 and 0.50 at a rate of 3000 symbols per second. Assuming the independent generation of symbols. Calculate the average bit rate of the most efficient source encoder.

Sol:- $H(x) = 0.25(2) + 0.25(2) + 0.5(1)$

$= 1.5$ bits/symbol.

average bit rate $=$ H(x) $\times$ Symbol rate

$= 1.5 \times 3000 = 4500$ bits/second.

Q6. A vedio transmission system transmits 625 picture frames per second. Each frame consists of a 200×200 pixel grid with 64 intensity levels per pixel. Compute the data rate of the system.

Sol:- No. of required bits / pixel $= \log_2 64 = 6$ bits/pixel.

No. of required bits/picture frame $= 6 \times 200 \times 200 = 2,40,000$ bits/frame.

Data rate $= 625 \times 2,40,000$ bits/sec.

$= 15 \times 10^6$ bits/sec.

Q3. Consider a discrete memoryless source with alphabet $S = \{s_1, s_2, s_3, \ldots\}$ and their respective probabilities occurance $P = \{1/2, 1/4, 1/8, 1/16, 1/32, \ldots\}$. Compute the entropy of source in bits.

Sol:- No. of bits required/alphabet $= \sum_{i=1}^{n} P(x_i) \cdot \log \frac{1}{P(x_i)}$

(Let) $K = \frac{1}{2}(1) + \frac{1}{4}(2) + \frac{1}{8}(3) + \ldots$ —①

$\cancel{K = \sum_{i=1}^{n} \frac{i}{2^i}}$

$K/2 = \frac{1}{4}(1) + \frac{1}{8}(2) + \ldots$ —②

① − ② $\Rightarrow K/2 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

$= \frac{1/2}{1-1/2} = 1$

$\therefore \boxed{K=2}$

$\therefore$ No. of bits required / alphabet = 8 bits/alphabet
$\hookrightarrow$ Source Entropy

09/08/2024

Tutorial-2

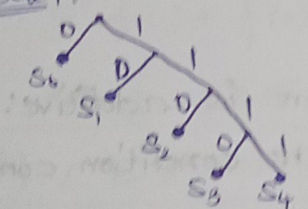
Q. Consider the following four different source codes listed below.

Symbol	Code 1	Code 2	Code 3	Code 4
S_0	0	0	0	00
S_1	10	01	01	01
S_2	110	001	011	10
S_3	1110	000	110	110
S_4	1111	0011	111	111

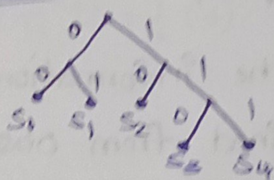
1. Two of these four codes are prefix-free codes. Identify them and construct their individual decision tree.

Sol:- Code 1 & Code 4 are prefix-free codewords.

code 1:



code 4:



2. Apply kraft-Mcmillan inequalities to codes 1, 2, 3 and 4. Discuss your results in light of those obtained in part (1).

Sol:- code 1 : $\sum_{i=1}^{n} 2^{-l_i} = 2^{-1}+2^{-2}+2^{-3}+2^{-4}+2^{-4}$

$= \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16} = 1 \rightarrow$ Satisfy

Code 2 : $\sum_{i=1}^{n} 2^{-l_i} = 2^{-1}+2^{-2}+2^{-3}+2^{-4}+2^{-4} = 1 \longrightarrow$ Satisfy

code 3 : $\sum_{i=1}^{n} 2^{-l_i} = 2^{-1}+2^{-2}+2^{-3}+2^{-3}+2^{-3} = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$

$= \frac{9}{8} > 1 \rightarrow$ Not Satisfied

code 4 : $\sum_{i=1}^{n} 2^{-l_i} = 2^{-2}+2^{-2}+2^{-2}+2^{-3}+2^{-3}$

$= \frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8} = 1 \longrightarrow$ Satisfy

* Codes which are ~~satisfy~~ prefix-free code words will satisfy kraft McMillan inequality but converse is not true.

Q2. Consider a discrete memoryless source whose alphabet consists of K equiprobable. symbol.

1. Explain why using a fixed-length code to represent such a source is about as efficient as any code can be.

Sol:- $H(x) = \log_2 K$

No. of min. bits required for codeword $(R) = \log_2 K$

efficiency $= \frac{H(x)}{R} = 100\%$ if 'K' is power of 2.

efficiency $= \frac{H(x)}{R} < 100\%$ if 'K' is not a power of 2.

2. What condition has to be satisfied by K and the code-word length for the coding efficiency to be 100%.

Sol:- Efficiency $= \frac{H(x)}{R} = 100\%$

'K' should be very large. for 100% efficiency or 'K'. Should be power of '2'.

Q3. Prove that the information measure is additive: that the information gained from observing the condition combination of N independent events, whose probabilities are P_i for $i = 1, 2, \ldots N$. is the sum of information gained from observing each one of these events seperately and in any order.

Sol:- $P = P_1 . P_2 . P_3 \ldots P_N$.

$$I = \log_2 \frac{1}{P}$$

$$= \log_2 \frac{1}{\prod_{n=1}^{N} P_n}$$

$$= \log_2 \frac{1}{P_1} + \log_2 \frac{1}{P_2} + \log_2 \frac{1}{P_3} + \ldots \log_2 \frac{1}{P_N} \qquad [\log ab = \log a + \log b]$$

$$\boxed{I = I_1 + I_2 + \ldots + I_N}$$

Q4. Construct an efficient, uniquely decodable binary code, having the prefix property and having the shortest possible arrange code length per symbol, for an alphabet whose five letters appear with these probabilities.

Symbol	A	B	C	D	E
Probability	1/2	1/4	1/8	1/16	1/16

How do you know that your code has the shortest possible average code length per symbol?

Sol:-

Symbol	Probability	Code
A	1/2	0
B	1/4	10
C	1/8	110
D	1/16	1110
E	1/16	1111

Avg. Length, $L = \sum_{i=1}^{5} P_i l_i$

$= \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{1}{4} + \frac{1}{4}$

$= \frac{15}{8} = 1.875$

$H(x) = \sum_{i=1}^{5} P_i \cdot \log_2 \frac{1}{P_i}$

$= \frac{1}{2}(1) + \frac{1}{4}(2) + \frac{1}{8}(3) + \frac{1}{16}(4) + \frac{1}{16}(4)$

$= \frac{15}{8} = 1.875$

$\boxed{\therefore H(x) \leq L \leq H(x) + 1}$

Hence, we can say that this code is shortest possible code length per symbol.

Q5. Cosider a telegraph source having two symbols, dot and dash. The dot duration is 0.2 s. The dash duration is 3 times the dot duration. The probability of the dots occuring is twice that of the dash, and the time between symbols is 0.2 s. Calculate the information rate of the telegraph source.

Sol:- probability of dash = 1/3.

probability of dot = 2/3.

$H(x) = \frac{1}{3} \cdot \log_2 3 + \frac{2}{3} \log_2 (3/2)$

$= 0.5283 + 0.3899$

$= 0.918$ bits/symbol.

Avg. time duration $= 0.2 \times \frac{2}{3} + 0.6 \times \frac{1}{3} + 0.2$

$= 0.53$ sec/symbol.

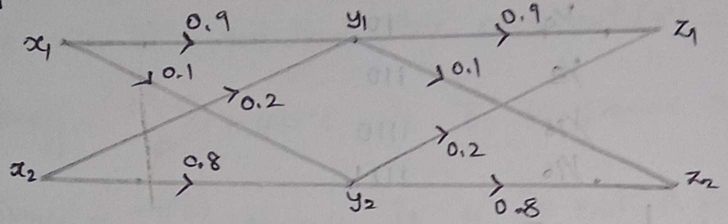
Rate of Telegraph Source $= 0.918 \times \frac{1}{0.533}$

$= 1.72$ bits/second.

Tutorial-3

30/08/2024

Q1. Two binary channels are connected in a cascade, as shown in figure.



1. Find the overall channel matrix of the resultant channel and draw the resultant equivalent channel diagram.
2. Find $P(z_1)$ and $P(z_2)$ when $P(x_1) = P(x_2) = 0.5$

Sol:- (2) $P(y_1) = P(x_1).P(y_1/x_1) + P(x_2).P(y_1/x_2)$

$= (0.5)(0.9) + (0.5)(0.2)$

$= 0.55$

$P(y_2) = P(x_1).P(y_2/x_1) + P(x_2).P(y_2/x_2)$

$= (0.5)(0.1) + (0.5)(0.8)$

$= 0.45$

$P(z_1) = P(y_1).P(z_1/y_1) + P(y_2).P(z_1/y_2)$

$= (0.55)(0.9) + (0.45)(0.2)$

$\boxed{P(z_1) = 0.585}$

$P(z_2) = P(y_1).P(z_2/y_1) + P(y_2).P(z_2/y_2)$

$= (0.55)(0.1) + (0.45)(0.8)$

$\boxed{P(z_2) = 0.415}$

(1) $\cancel{P(z_1/x_1)}$

$$P(z/y) = \begin{bmatrix} P(z_1/y_1) = 0.9 & P(z_2/y_1) = 0.1 \\ P(z_1/y_2) = 0.2 & P(z_2/y_2) = 0.8 \end{bmatrix}$$

$$P(y/x) = \begin{bmatrix} P(y_1/x_1) = 0.9 & P(y_2/x_1) = 0.1 \\ P(y_1/x_2) = 0.2 & P(y_2/x_2) = 0.8 \end{bmatrix}$$

$$P(Z/x) = \cancel{P(Z/y)} \cdot P(Y/x) \cdot P(Z/y)$$

$$= \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

$$P(Z/x) = \begin{bmatrix} 0.83 & 0.17 \\ 0.34 & 0.66 \end{bmatrix}$$

Q2. Proof that mutual Information is always non-negative.

$$I(x:Y) \geq 0.$$

Sol:-

$$I(x:Y) = \sum_i \sum_j P(x_i, y_j) \cdot \log \frac{P(x_i, y_j)}{P(x_i) \cdot P(y_j)}$$

$$-I(x:y) = \sum_i \sum_j P(x_i, y_j) \cdot \log \frac{P(x_i) \cdot P(y_j)}{P(x_i, y_j)}$$

$$-I(x:Y) \leq \sum_i \sum_j P(x_i, y_j) \cdot \left[\frac{P(x_i) \cdot P(y_j)}{P(x_i, y_j)} - 1 \right] \quad \left[\because \log x \leq x - 1\right]$$

$$-I(x:y) \leq \sum_i \sum_j P(x_i) \cdot P(y_j) - \sum_i \sum_j P(x_i, y_j)$$

$$-I(x:y) \leq 1 - 1 \Rightarrow -I(x:y) \leq 0$$

$$\therefore \boxed{I(x:y) \geq 0}$$

Q3. First, show that $H(Y,X) \leq H(X) + H(Y)$ and also that $H(Y,X) \leq H(X) + H(Y)$ with equality if and only if X and Y are independent.

Sol:-

$$I(X:Y) \geq 0$$

$$H(X) - H(\cancel{X/Y}) \geq 0$$

$$\sum_i P(x_i) \cdot \log \frac{1}{P(x_i)} - \sum_i \sum_j P(x_i, y_j) \cdot \log \frac{1}{P(x_i/y_j)} \geq 0.$$

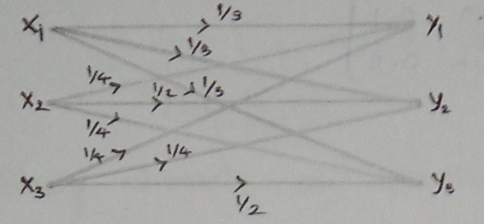
$$\sum_i P(x_i) \cdot \log \frac{1}{P(x_i)} - \sum_i \sum_j P(x_i, y_j) \log \frac{P(y_j)}{P(x_i, y_j)} \geq 0. \quad \left[\because P(x_i/y_j) = \frac{P(x_i, y_j)}{P(y_j)}\right]$$

$$\sum_i P(x_i) \cdot \log \frac{1}{P(x_i)} - \sum_j P(y_j) \cdot \log P(y_j) + \sum_i \sum_j P(x_i, y_j) \cdot \log P(x_i, y_j) \geq 0.$$

$$H(X) + H(Y) - H(Y,X) \geq 0$$

$$\therefore \boxed{H(Y/X) \leq H(X) + H(Y)}$$

Q4. Consider the DMC shown in figure.



1. Find the output probabilities if $p(x_1)=\frac{1}{2}$ and $p(x_2)=p(x_3)=\frac{1}{4}$.
2. Find the output Entropy $H(Y)$.

Sol:- (1) $P(y_1) = P(x_1)\cdot p(y_1/x_1) + P(x_2)\cdot P(y_1/x_2) + P(x_3)\cdot P(y_1/x_3)$

$$= \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4}$$

$$= 7/24 = 0.292$$

$P(y_2) = P(x_1)\cdot P(y_2/x_1) + P(x_2)\cdot P(y_2/x_2) + P(x_3)\cdot P(y_2/x_3)$

$$= \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{1}{4}$$

$$= 17/48 = 0.354$$

$P(y_3) = P(x_1)\cdot P(y_3/x_1) + P(x_2)\cdot P(y_3/x_2) + P(x_3)\cdot P(y_3/x_3)$

$$= \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{2}$$

$$= 17/48 = 0.354$$

(2) $H(Y) = \sum_{i=1}^{3} P(y_i)\cdot \log_2 \frac{1}{P(y_i)}$

$$= P(y_1)\cdot\log_2\frac{1}{P(y_1)} + P(y_2)\cdot\log_2\frac{1}{P(y_2)} + P(y_3)\log_2\frac{1}{P(y_3)}$$

$$= \frac{7}{24}\cdot\log_2\left(\frac{24}{7}\right) + \frac{17}{48}\cdot\log_2\left(\frac{48}{17}\right) + \frac{17}{48}\cdot\log_2\left(\frac{48}{17}\right)$$

$$= 0.5184 + 0.5303 + 0.5303$$

$$\boxed{H(Y) = 1.579}$$

Q5. Consider two sources A and B with entropies $H(A) = 1$ bit and $H(B) = 2$ bits. If the joint entropy $H(A,B) = 2.5$ bits, calculate the mutual information $I(A:B)$. What does the mutual information tell you about the relationship between A and B?

Sol:- $I(A:B) = H(A) + H(B) - H(A,B)$

$$= 1 + 2 - 2.5 = 0.5 \text{ bits.}$$

06/09/2024

Tutorial - 4

Q1. Consider the following binary sequence.

111010011000101101000......

Encode this sequence using the Lempel-Ziv algorithm. Assume that the binary symbols 0 and 1 are already in the codebook

Sol:- Phrases: 0, 1, 11, 10, 100, 110, 001, 101, 1010, 00

Locations: 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000,

Codewords:- 00000, 00001, 00101, 00100, 01000, 00110, 00010, 01001, 10000, 00010

(above the codewords: 0, 1, 11, 10, 100, 110, 00, 101, 1010)

repeat.

Q3. Calculate the amount of information needed to open a lock whose combination consists of three integers, each ranging from 00 to 99.

Sol:- $P = \frac{1}{10^6}$, $I = \log_2(10^6) = 6(\log_2 10)$

$= 19.93$ bits.

Q2. Calculate $H(X)$, $H(Y)$, $H(X/Y)$ and $I(X:Y)$ using the provided binary channel matrix and the probabilities $P(X_1) = \frac{1}{3}$ & $P(X_2) = \frac{2}{3}$

Sol:-

$$P(x_1) = \frac{1}{3} \text{ and } P(x_2) = 2/3.$$

$$H(X) = \frac{1}{3}\log_2 3 + \frac{2}{3}\log_2(3/2)$$

$$= 0.5283 + 0.3899 = 0.918$$

$$\boxed{H(X) = 0.9182}$$

Given, $P\left(\frac{y}{x}\right) = \begin{bmatrix} 2/3 & 1/3 \\ 1/10 & 9/10 \end{bmatrix}$

$$P(y_1) = P(x_1) \cdot P(y_1/x_1) + P(x_2) \cdot P(y_1/x_2)$$

$$= \frac{1}{3} \cdot \frac{2}{3} + \frac{2}{3} \cdot \frac{1}{10} = \frac{2}{9} + \frac{1}{15} = \frac{10+3}{45} = \frac{13}{45}.$$

$$P(y_2) = P(x_1) \cdot P(y_2/x_1) + P(x_2) \cdot P(y_2/x_2)$$

$$= \frac{1}{3}\left(\frac{1}{3}\right) + \frac{2}{3}\left(\frac{9}{10}\right) = \frac{1}{9} + \frac{3}{5} = \frac{5+27}{45} = \frac{32}{45}$$

$$H(Y) = \frac{13}{45}\cdot\log_2\left(\frac{45}{13}\right) + \frac{32}{45}\log_2\left(\frac{45}{32}\right).$$

$$= 0.5175 + 0.3497 = 0.8672.$$

$$\boxed{H(Y) = 0.8672}$$

$$P(x,y) = P(x) * P(y/x) = \cancel{\begin{bmatrix} \frac{1}{8} & \frac{2}{3} \\ \frac{1}{10} & \frac{9}{10} \end{bmatrix}} = \begin{bmatrix} \frac{2}{9} & \frac{1}{9} \\ \frac{2}{30} & \frac{18}{30} \end{bmatrix}$$

$$H(x,y) = \frac{2}{9}\log_2(9/2) + \frac{1}{9}\log_2(9) + \frac{2}{30}\log_2(15) + \frac{18}{30}\log_2\left(\frac{5}{3}\right).$$

$$= 0.4822 + 0.3522 + 0.2604 + 0.4421$$

$$\Rightarrow \boxed{1.5369 = H(x,y)}$$

$$\cancel{P(x,y) = P(y) * P(x/y) = \begin{bmatrix} \frac{2}{9} & \frac{1}{9} \\ \frac{2}{30} & \frac{18}{30} \end{bmatrix}}$$

$$[P(x,y)]^T = P(y,x) = \begin{bmatrix} 2/9 & 2/30 \\ 1/9 & 18/30 \end{bmatrix}$$

$$P(y) * P(x/y)$$

$$\Rightarrow P(x/y) = \begin{bmatrix} 10/13 & 5/32 \\ 3/13 & 27/32 \end{bmatrix} \quad \begin{array}{c|cc} & x_1 & x_2 \\ \hline y_1 & 10/13 & 3/13 \\ y_2 & 5/32 & 27/32 \end{array}$$

$$H(X/Y) = \sum_i \sum_j P(x_i/y_j)\cdot\log_2\frac{1}{P(x_i/y_j)}$$

$$= \frac{10}{13}\cdot\log_2\left(\frac{13}{10}\right) + \frac{3}{13}\log_2\left(\frac{13}{3}\right) + \frac{5}{32}\log_2\left(\frac{32}{5}\right) + \frac{27}{32}\log_2\left(\frac{32}{27}\right)$$

$$= 0.2911 + \overset{0.4881}{\cancel{0.0873}} + 0.4184 + 0.2068$$

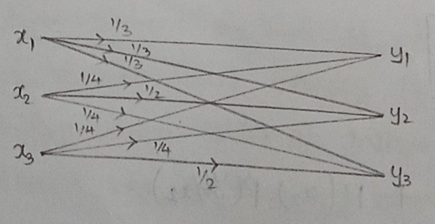
$$\boxed{H(X/Y) = 1.4044}$$

WKT, $I(X:Y) = H(X) + H(Y) - H(X,Y)$

$$= 0.9182 + 0.8672 - 1.5369$$

$$\boxed{I(X:Y) = 0.2425 \text{ bit}}$$

Q5. Consider the DMC shown in figure.



1. Find the output probabilities if $P(x_1) = 1/2$ and $P(x_2) = P(x_3) = 1/4$
2. Find the output entropy $H(Y)$.

Sol:- (i) Given $P(x_1) = 1/2$; $P(x_2) = P(x_3) = 1/4$

$$P(Y/X) = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$P(Y) = P(X) . P(Y/X) = \begin{bmatrix} 1/2 & 1/4 & 1/4 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$P(Y) = \begin{bmatrix} 7/24 & 17/48 & 17/48 \end{bmatrix}$$

(ii) $H(Y) = \sum_i P(y_i) \log_2 \frac{1}{P(y_i)}$

$$= \frac{7}{24} \log_2\left(\frac{24}{7}\right) + \frac{17}{48} \log_2\left(\frac{48}{17}\right) + \frac{17}{48} \log_2\left(\frac{48}{17}\right)$$

$$= 0.5184 + 0.5303 + 0.5303$$

$$\boxed{H(Y) = 1.5791}$$

Q4. The Joint entropy of a discrete system is defined as

$$H(X,Y) = \sum_{x,y} P(x_i, y_j) \log \frac{1}{P(x_i, y_j)} .$$

Show that $H(x,y) = H(Y) + H(X/Y)$.

Sol:-

$$H(X,Y) = \sum_i \sum_j P(x_i, y_j) . \log_2 \frac{1}{P(x_i, y_j)}$$

$$= \sum_i \sum_j P(x_i, y_j) \log \frac{1}{P(y_j) . P(x_i/y_j)}$$

$$= \sum_i \sum_j P(x_i, y_j) \log \frac{1}{P(y_j)} + \sum_i \sum_j P(x_i, y_j) . \log \frac{1}{P(x_i/y_j)}$$

$$= \sum_j P(y_j) . \log \frac{1}{P(y_j)} + \sum_i \sum_j P(x_i, y_j) \log \frac{1}{P(x_i/y_j)}$$

$$\therefore \boxed{H(X,Y) = H(Y) + H(X/Y)}$$