Homework-5

1. Dual Formulation of SVM:-

SVM optimization problem for a linearly separable case with Lagrange multipliers α_i can be formulated as:

$$\max_{\alpha} \sum_{i=1}^{4} \alpha_i - \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} \alpha_i \alpha_j y_i y_j k(x_i, x_j)$$

Subject to the constraints:-

$$\sum_{i=1}^{4} \alpha_i y_i = 0$$

$\Rightarrow \alpha_i \geq 0$ for $i = 1,2,3,4$

Applying the Linear kernel

$k(x_i, x_j) = x_i^T x_j$, so we need to compute $x_i^T x_j$ for all i,j.

$\Rightarrow x_1 = (0,0)$ $\quad x_2 = (0,1)$ $\quad x_3 = (1,0)$ $\quad x_4 = (1,1)$

$x_1^T x_1 = 0$	$x_2^T x_1 = 0$	$x_3^T x_1 = 0$	$x_4^T x_1 = 0$
$x_1^T x_2 = 0$	$x_2^T x_2 = 1$	$x_3^T x_2 = 0$	$x_4^T x_2 = 1$
$x_1^T x_3 = 0$	$x_2^T x_3 = 0$	$x_3^T x_3 = 1$	$x_4^T x_3 = 1$
$x_1^T x_4 = 0$	$x_2^T x_4 = 1$	$x_3^T x_4 = 1$	$x_4^T x_4 = 2$

$\Rightarrow$ Substitute $k(x_i, x_j) = x_i^T x_j$ and $y_i y_j$ values into the dual objective function

$$\max_{\alpha} \sum_{i=1}^{4} \alpha_i - \frac{1}{2} \sum_{i=1}^{4} \sum_{j=1}^{4} \alpha_i \alpha_j y_i y_j x_i^T x_j$$

Using the values of y_i and $x_i^T x_j$

$$\Rightarrow \max_{\alpha} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - \frac{1}{2}(\alpha_2^2 + \alpha_3^2 + 2\alpha_4^2 + 2\alpha_2\alpha_4 + 2\alpha_3\alpha_4)$$

The sum of the weighted classes constraint:-

$$-\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$$

Non-negativity constraints:- $\alpha_i \geq 0$ for $i = 1,2,3,4$.

Final Dual Optimization problem

$$\max_{\alpha} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - \frac{1}{2}(\alpha_2^2 + \alpha_3^2 + 2\alpha_4^2 + 2\alpha_2\alpha_4 + 2\alpha_3\alpha_4)$$

Subject to :- $\quad -\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0$

$\alpha_i \geq 0$ for $i = 1,2,3,4$.

2A Dual Formulation Of SVM

$$\max_{\alpha} \sum_{i=1}^{4} \alpha_i - \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4} \alpha_i\alpha_j y_i y_j K(x_i, x_j)$$

Subject to the constraints

$$\sum_{i=1}^{4} \alpha_i y_i = 0$$

$$\alpha_i \geq 0 \text{ for } i = 1,2,3,4$$

$x_1 = (0,0)$ $x_2 = (0,1)$ $x_3 = (1,0)$ $x_4 = (1,1)$

$x_1^T x_1 = 0$ $x_2^T x_1 = 0$ $x_3^T x_1 = 0$ $x_4^T x_1 = 0$

$x_1^T x_2 = 0$ $x_2^T x_2 = 1$ $x_3^T x_2 = 0$ $x_4^T x_2 = 1$

$x_1^T x_3 = 0$ $x_2^T x_3 = 0$ $x_3^T x_3 = 1$ $x_4^T x_3 = 1$

$x_1^T x_4 = 0$ $x_2^T x_4 = 1$ $x_3^T x_4 = 1$ $x_4^T x_4 = 2$

Substitute $K(x_i, x_j)$ & $y_i y_j$ into the dual function

$$\Rightarrow \max_{\alpha} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4} \alpha_i \alpha_j y_i y_j x_i^T x_j$$

$$\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - \frac{1}{2}(\alpha_2^2 + \alpha_3^2 + 2\alpha_4^2 + 2\alpha_2\alpha_4 + 2\alpha_3\alpha_4).$$

We have $\alpha_1 = \alpha_2 + \alpha_3 + \alpha_4$

$$\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = (\alpha_2 + \alpha_3 + \alpha_4) + \alpha_2 + \alpha_3 + \alpha_4 = 2(\alpha_2 + \alpha_3 + \alpha_4)$$

$$\Rightarrow -\frac{1}{2}(\alpha_2^2 + \alpha_3^2 + 2\alpha_4^2 + 2\alpha_2\alpha_4 + 2\alpha_3\alpha_4) = -\frac{1}{2}((\alpha_2 + \alpha_4)^2 + (\alpha_3 + \alpha_4)^2$$

$\Rightarrow$ Final Dual Optimization

$$\max_{\alpha} 2(\alpha_2 + \alpha_3 + \alpha_4) - \frac{1}{2}((\alpha_2 + \alpha_4)^2 + (\alpha_3 + \alpha_4)^2)$$

subject to:-

$$\alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_3 \geq 0, \alpha_4 \geq 0, \alpha_1 = \alpha_2 + \alpha_3 + \alpha_4$$

This matches the form given in the question which is $\max_{\alpha} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - \frac{1}{2}((\alpha_2 + \alpha_4)^2 + (\alpha_3 + \alpha_4)^2)$.

Ba. In support vector Criteria the data ~~criteria~~ points with $\alpha_i > 0$, so.

1. $\alpha_1 = 4$
2. $\alpha_2 = 2$
3. $\alpha_3 = 2$

Bb. Recall the SVM decision function

$f(x) = w \cdot x + b$

we need to calculate w & b using

$\alpha_1 = 4, \alpha_2 = 2, \alpha_3 = 2, \alpha_4 = 0.$

$\Rightarrow w = \Sigma(\alpha i \cdot yi \cdot xi)$

$w = (2,2)$

$\Rightarrow b \Rightarrow$ Let's use $x_2 = (0,1)$

$1 = w \cdot x_2 + b$

$b = -1$

Now we can classify $x_5 = (-1,0)$

$f(x_5) = w \cdot x_5 + b$

$f(x_5) = -3.$

Rule is if $f(x_5) \geq 0$, classify as $+1$

if $f(x_5) < 0$, classify as -1

so, $f(x_5) = -3 < 0$, we classify $x_5 = (-1,0)$ as -1

computation $\Rightarrow f(x_5) = w \cdot x_5 + b = -3.$

so it is classified as $-1.$

Bc. Same as above calculated

$w = (2,2)$, $b = -1$

$x_5 = (-1,1) \Rightarrow f(x_5) = w \cdot x_5 + b \Rightarrow -1$

As per the rule $f(x_5) = -1 < 0$ we classify as -1

$\therefore$ computation of $x_5(-1,1) \Rightarrow f(x_5) = -1$

Bd. $w = (2,2)$, $b = -1.$

$x_5 = (1,1) \Rightarrow f(x_5) = w \cdot x_5 + b \Rightarrow 3.$

As per the rule $f(x_5) = 3 > 0$ we classify as 1

$\therefore$ computation of $x_5(-1,1)$

$\Rightarrow 3.$

$x_5 = (1,1)$ is classified as $+1$ by SVM.

3. $K=(U,V)=U^TV$

Dual Formulation

$$W(\alpha)=\sum_{i=1}^{4}\alpha_i-\frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}\alpha_i\alpha_j y_i y_j K(x_i,x_j)$$

$$\sum_{i=1}^{4}\alpha_i y_i=0,\ \alpha_i\geq 0 \text{ for all } i$$

$$K=(x_i,x_j)=x_i^T x_j$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

$$w(\alpha)=\alpha_1+\alpha_2+\alpha_3+\alpha_4-\frac{1}{2}(\alpha_2^2+\alpha_3^2+2\alpha_4^2+2\alpha_2\alpha_4+2\alpha_3\alpha_4]$$

subject to :-

$\alpha_1y_1+\alpha_2y_2+\alpha_3y_3+\alpha_4y_4=0$ which simplifies to

$-\alpha_1+\alpha_2+\alpha_3-\alpha_4=0$ & $\alpha_i\geq 0$ for $i=1,2,3,4$.

so, the dual problem

$$\Rightarrow w(\alpha)=\alpha_1+\alpha_2+\alpha_3+\alpha_4-\frac{1}{2}[\alpha_2^2+\alpha_3^2+2\alpha_4^2+2\alpha_2\alpha_4+2\alpha_3\alpha_4].$$

subject to :-

$-\alpha_1+\alpha_2+\alpha_3-\alpha_4=0$

$\alpha_i\geq 0$ for all i

4A Dual Form

$$\omega(\alpha)=\sum_{i=1}^{4}\alpha_i-\frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}\alpha_i\alpha_j y_i y_j K(x_i,x_j)$$

Subject to the constraints

- $\sum_{i=1}^{4}\alpha_i y_i=0$ & $\alpha_i\geq 0$ for all i

Expand the dual function after doing $K=(x_i,x_j)=x_i^Tx_j$

$$\omega(\alpha)=\alpha_1+\alpha_2+\alpha_3+\alpha_4-\frac{1}{2}(\alpha_2^2+\alpha_3^2+2\alpha_4^2-2\alpha_2\alpha_4-2\alpha_3\alpha_4)$$

$\sum_{i=1}^{4}\alpha_i y_i=0$ is $-\alpha_1+\alpha_2+\alpha_3-\alpha_4=0$.

so the dual form

Maximize $\omega(\alpha)=\alpha_1+\alpha_2+\alpha_3+\alpha_4-\frac{1}{2}((\alpha_2-\alpha_4)^2+(\alpha_3-\alpha_4)^2)$

$\alpha_1\geq 0,\ \alpha_2\geq 0,\ \alpha_3\geq 0,\ \alpha_4\geq 0$

$\alpha_1=\alpha_2+\alpha_3-\alpha_4$

4B. $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = k$, $k \geq 0$

maximize $4k$ is $4k$

$\alpha_1 = \alpha_2 + \alpha_3 - \alpha_4$

$\left.\begin{array}{l}\alpha_1 \geq 0, \alpha_2 \geq 0, \\ \alpha_4 \geq 0, \alpha_3 \geq 0\end{array}\right\} k \geq 0$

$k = k + k - k \Rightarrow k = k$

$w(k) = k + k + k + k - \frac{1}{2}[(k-k)^2 + (k-k)^2]$

$= 4k - \frac{1}{2}(0+0) \Rightarrow 4k.$

5A Dual form

$w(\alpha) = \sum_{i=1}^{5} \alpha_i - \frac{1}{2} \sum_{i=1}^{5} \sum_{j=1}^{5} \alpha_i \alpha_j y_i y_j k(x_i, x_j)$

Now $k(x_i, x_j) = x_i^T x_j$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 2 & 4 & 6 & 0 \\ 0 & 3 & 6 & 9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

constraints are :- $\alpha_i \geq 0$ for all i & $\sum_{i=1}^{5} \alpha_i y_i = 0$.

$\therefore -\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - \alpha_5$

$w(\alpha) = \sum_{i=1}^{5} \alpha_i - \frac{1}{2}(\alpha_2^2 + 4\alpha_3^2 + 9\alpha_4^2 + \alpha_5^2 + 2\alpha_2\alpha_3 + 3\alpha_2\alpha_4 + 0 + 0 + 0)$

$\alpha_1 = \alpha_2 + \alpha_3 + \alpha_4 - \alpha_5$

$w(\alpha) = \sum_{i=1}^{5} \alpha_i - \frac{1}{2}(\alpha_2^2 + 4\alpha_3^2 + 9\alpha_4^2 + \alpha_5^2 + 2\alpha_2\alpha_3 + 3\alpha_2 + \alpha_4 + 0 + 0 + 0)$

$\alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_3 \geq 0, \alpha_4 \geq 0, \alpha_5 \geq 0.$

5Ba. $\alpha_1 = 2$, $\alpha_2 = 2$, $\alpha_3 = 0$, $\alpha_4 = 0$, $\alpha_5 = 0$.

$x = (1,1)$

SVM model

$f(x) = \sum_{i=1}^{5} \alpha_i y_i k(x_i, x) + b$

$f(1,1) = \sum_{i=1}^{5} \alpha_i y_i k(x_i, (1,1)) + b$

$k(x_1, x) = k((0,0),(1,1)) = 0$

$k(x_2, x) = 1$

$k(x_3, x) = 2$
$k(x_4, x) = 3$
$k(x_5, x) = 1$
$f(1,1) = 2(-1)(0) + 2(1)(1) + 0 + 0 + 0 = 2 + b$
$f(x)$ depends on the sign of $2+b$

Bb. $x = (0,-2)$
$k(x_1, x) = 0$
$k(x_2, x) = 0$
$k(x_3, x) = 0$
$k(x_4, x) = 0$
$k(x_5, x) = -2$
$f(0,-2) = 2(-1)(0) + 2(1)(0) + (0)(4)(0) + (0)(1)(0) + (0)(-1)(-2) = 0 + b$

$\Rightarrow 0 + b \Rightarrow 0 + 1 = 1$

$f(0,-2) > 0$ classify as $+1$

Ca. soft margins L_1 norm
$C = 10$
$x = (1,1)$
$k(x_1, (1,1)) = 0$
$k(x_2, (1,1)) = 1$
$k(x_3, (1,1)) = 2$
$k(x_4, (1,1)) = 3$
$k(x_5, (1,1)) = 1$
soft margin SVM -
$f(x) = \sum_{i=1}^{n} \alpha_i y_i k(x_i, x) + b - C \sum_{i=1}^{m} y_i$
$\Rightarrow f(1,1) = \sum_{i=1}^{5} \alpha_i y_i k(x_i, (1,1)) + b - C \sum_{i=1}^{m} y_i$
$\Rightarrow 2 - 1 = 1 > 0$
$\therefore f(1,1) > 0$ classified on $y = 1$

Cb. $x = (0,-2)$
$f(0,-2) = \sum_{i=1}^{5} \alpha_i y_i k(x_i, (0,-2)) + b - C \sum_{i=1}^{5} y_i$
$\Rightarrow -1 < 0$
$\therefore f(0,-2) < 0 \Rightarrow$ classified on $y = -1$