Homework 1

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Question 1

Explain what your measurements will be.
We will be exploring Gaurav's running speeds from his last ten runs.

Question 2

Deciding on some priors
We estimate a mean speed (μ_0) of 8 minutes and 10 seconds (i.e. 490 seconds) per mile. The basis for this is that on good days, I (Gaurav) generally try to run 8-minute (480 seconds) miles, and I know that over the past ~2 weeks I've been running slower due to a weird knee. We estimate a standard deviation (τ) of 30 seconds because I've done both road runs and treadmill runs, and I know that I run somewhat differently in the two conditions.

Question 3

Report the data and sample mean and variance (n-1) denominator.

```
runs <- read.csv("running_data.csv")
head(runs)
```

```
##   sample distance time_min time_sec speed_sec_per_mi
## 1      1      3.5       27     1620          462.857
## 2      2      3.1       24     1440          464.516
## 3      3      7.1       58     3480          490.141
## 4      4      4.0       35     2100          525.000
## 5      5      5.4       48     2880          533.333
## 6      6     13.5      130     7800          577.778
```

```
sample_mean <- mean(runs$speed_sec_per_mi)

# Calculate sample variance
sample_var <- sum((runs$speed_sec_per_mi-sample_mean)^2)/(nrow(runs)-1)
```

The mean of our sample is 513.719132; the variance of our sample is 2133.81829.

Question 4

Now specify the sampling standard deviation. Since we are doing a one parameter model, and since this value is usually not known, we need to do something because we are working with such a simple model.

We know that the speed estimates from the run tracking app are fairly accurate: the speeds from the app have closely matched my race speeds recorded independently. We don't think that the sampling standard deviation is higher than the sample σ of 46.193271, so we will proceed in the analysis assuming the sampling standard deviation is the same as the sd of the data .

Question 5

Calculate the posterior mean, variance, and SD.

Because we are working with a normal-normal model (mean is normally distributed with, in this case, a known sampling error), we can use the following formulae to collect posterior mean and variance:

$$\bar{\theta} = \frac{\frac{n}{\sigma^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} \bar{y} + \frac{\frac{1}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} \theta_0$$

$$\text{var}(\theta \mid \text{data}) = \left(\frac{n}{\sigma^2} + \frac{1}{\tau^2} \right)^{-1}$$

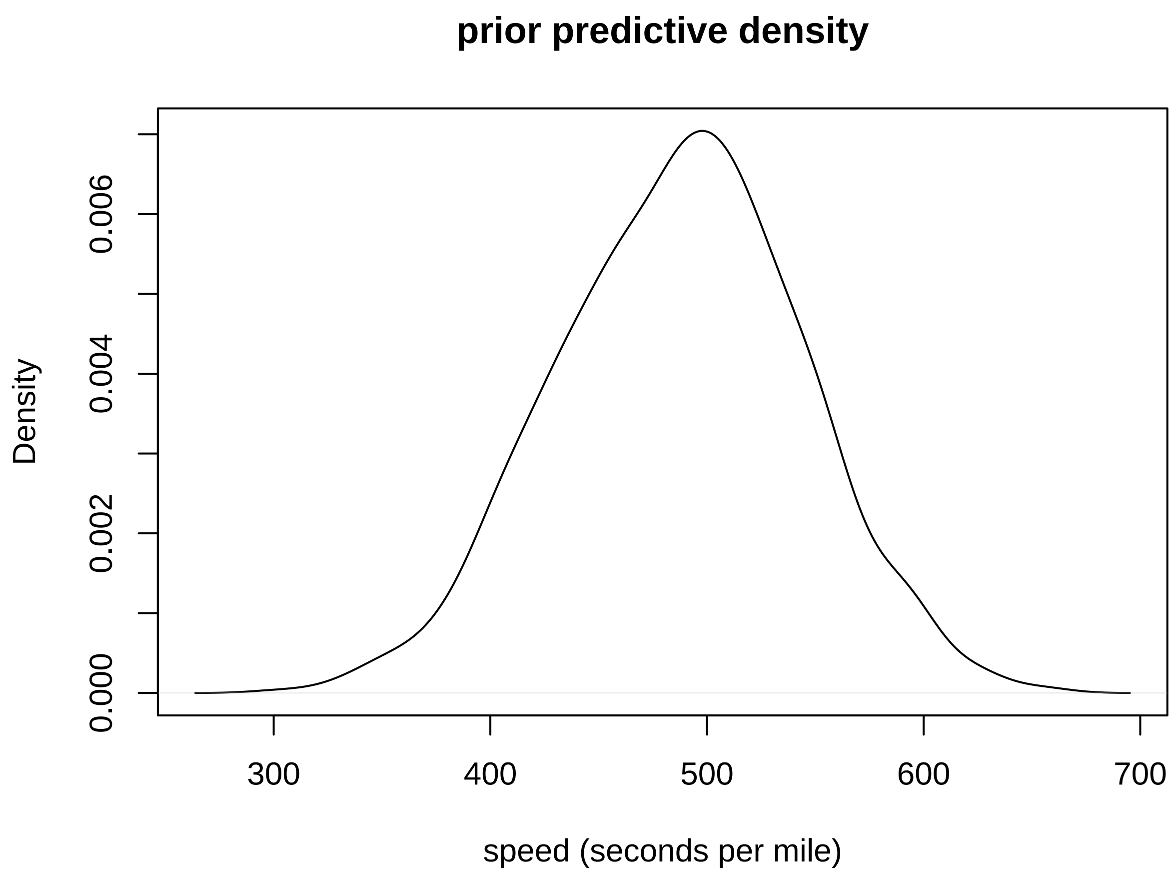
```
prior_mean = 490 #seconds
prior_sd   = 30 #seconds
sampling_sd = sqrt(sample_var)
n = nrow(runs)
ybar = sample_mean

# calculate the posterior mean using the formula
posterior_mean <- (n/(sampling_sd^2))/((n/sampling_sd^2)+(1/prior_sd^2)) * ybar +
  (1/prior_sd^2)/((n/sampling_sd^2)+(1/prior_sd^2)) * prior_mean

# calculate the posterior variance using the formula
posterior_var <- ((n/(sampling_sd^2))+(1/(prior_sd^2)))^-1
posterior_sd <- sqrt(posterior_var)
```

The posterior mean is $\bar{\mu} = 509.173313$;
The posterior var is $V = 172.48678$;
The posterior sd is $sd = 13.133422$.

Question 6

The prior predictive density is the density that you predict for a single observation before seeing any data.
It sure is!

```
# Sample from a normal distribution with the parameters above
plot(density(rnorm(1000, prior_mean, sqrt(prior_sd^2+sampling_sd^2))),
     main = "prior predictive density", xlab = "speed (seconds per mile)")
```



Question 7

Construct a table with means, sds and vars for the (i) posterior for mu, (ii) the prior for mu, (iii) the prior predictive for y, and (iv) the likelihood of mu.

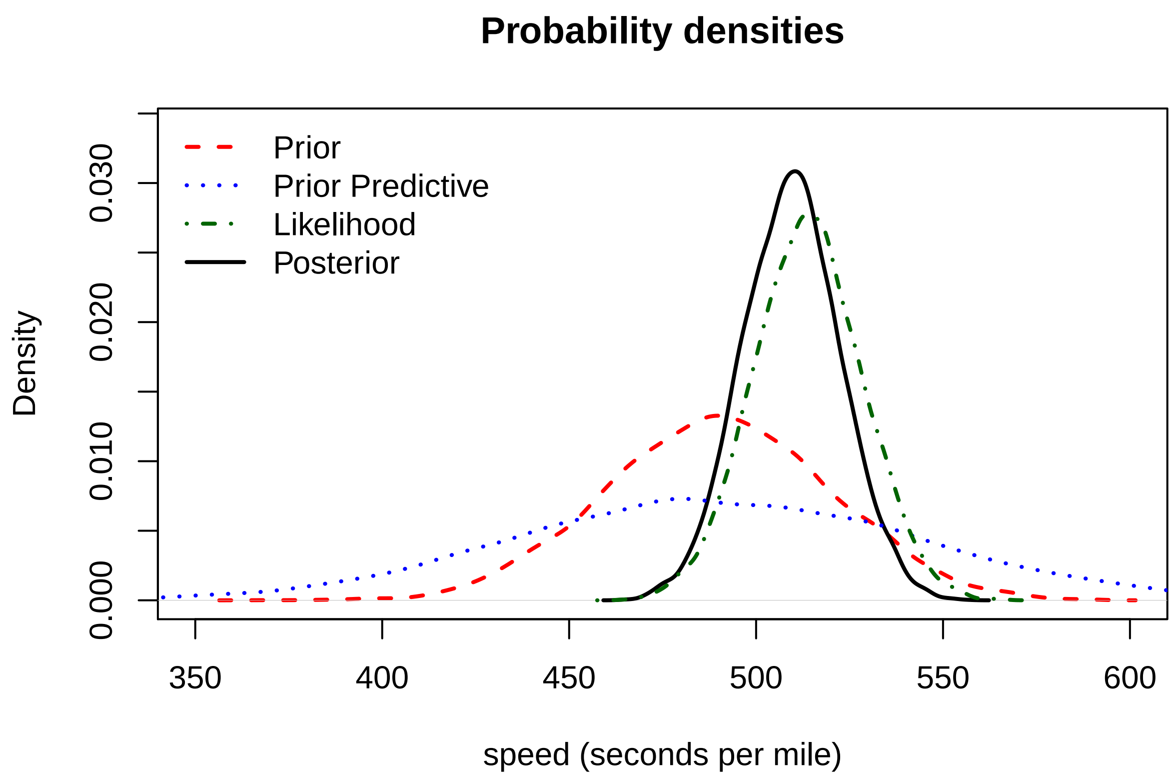
Table 1: Posterior, Prior, Prior Predictive, and Likelihood calculated from their definitions

	Mean	SD	Variance
Posterior	509.173	13.1334	172.487
Prior	490.000	30.0000	900.000
Prior Predictive	490.000	55.0801	3033.818
Likelihood	513.719	14.6076	213.382

note: code available in appendix

Question 8

Plot on a single plot the (i) posterior for mu, (ii) the prior for mu, (iii) the prior predictive for y, and (iv) the likelihood of mu (suitably normalized so it looks like a density, ie a normal with mean y-bar and variance sigma2 /n) all on the same graph. Interpret the plot.



Interpretation of plot: The posterior distribution of mean running speed is in between the prior mean and the likelihood estimate; in other words, the posterior is a compromise between the likelihood and the prior, which shrank the likelihood. The prior predictive has a mean equal to the prior mean but has a much wider distribution, which comes from our uncertainty in measurements as well as the prior estimate for variation in running speed. Our posterior distribution is quite similar to the likelihood distribution, which happens because the weight given to our prior mean is lower than the weight given to the sample mean in the posterior calculation. Practically, this means that I (Gaurav) have been running much slower than I thought over the past three weeks or so- my average speed is about twenty seconds a mile slower than I thought. Time to fix that!

note: code available in appendix

Question 9

Write R/WinBUGS programs to sample from the posterior of mu.

The following JAGS and R code sampels from the posterior of mu:

```
#sink("running_model.txt")
cat("model {
    for (i in 1:N) {
        x[i] ~ dnorm(mu, tau)
    }
    mu ~ dnorm(prior_mean, prior_tau)
    sigma <- sampling_sd
    tau <- 1/(sigma^2)             # tau equal to 1/sigma^2
}", fill = TRUE)
#sink()
```

```
# parameters
jags.params = c("mu", "sigma", "tau")

# data
x = runs$speed_sec_per_mi
N = length(runs$speed_sec_per_mi)
prior_tau = 1/(prior_sd^2)
jags.data = list("x", "N", "prior_mean", "sampling_sd", "prior_tau")

# initials
jags.inits = function(){
    list("mu" = 0) # part of algorithm!
}
```

Now that we have set up the model, data, and parameters, we can run the model:

```
#sampling from the posterior
hw1.sim = jags(jags.data, jags.inits, jags.params,
              model.file = "running_model.txt",
              n.chains = 3, n.iter = 11000, n.burnin = 1000)
```

```
## module glm loaded
```

We can summarize the output of the model:

```
to_print <- hw1.sim$BUGSoutput$summary[2:4,c("mean", "sd", "2.5%", "97.5%")]
knitr::kable(to_print, caption = "Summary of posterior from JAGS run")
```

Table 2: Summary of posterior from JAGS run

	mean	sd	2.5%	97.5%
mu	509.366910	13.2886	482.805128	534.961181
sigma	46.193271	0.0000	46.193271	46.193271
tau	0.000469	0.0000	0.000469	0.000469

Question 10

Adapt your BUGS program to sample from the prior and prior predictive. Do this by not loading your data, rather, in loading the initial values, move the data y over to the init list instead. There is an example at the end of Homework 2 for a Poisson-gamma likelihood/prior. [Helpful step: set keyword DIC=F in the call to bugs, as WinBUGS can not calculate DIC for prior predictions.]

Sampling from the prior:

```
### Sampling from the prior:

# data
prior_tau = 1/(prior_sd)^2 # same prior tau as before
# Note that the data object 'x' is not part of the data list
jags.data.prior = list("N", "prior_mean", "sampling_sd", "prior_tau")

# initials; note that x IS part of the inits
jags.inits_prior = function(){
    list("mu"=0, "x"=x) # part of algorithm!
}

hw1.sim2 = jags(jags.data.prior, jags.inits_prior, jags.params,
              model.file = "running_model.txt",
              n.chains = 3, n.iter = 11000, n.burnin = 1000, DIC=F)
```

And now, we **sample from the prior predictive**:

```
### Sampling from prior predictive

# data
prior_tau = 1/((prior_sd^2)+(sampling_sd^2)) # prior tau updated to be the prior predictive
# Note that the data object 'x' is not part of the data list
jags.data_prior_predictive = list("N", "prior_mean", "sampling_sd", "prior_tau")

# initials; note that x is part of the inits
jags.inits_prior_predictive = function(){
    list("mu"= 0, "x"=x) # part of algorithm!
}

hw1.sim3 = jags(jags.data_prior_predictive, jags.inits_prior_predictive, jags.params,
              model.file = "running_model.txt",
              n.chains = 3, n.iter = 11000, n.burnin = 1000,
              DIC=F)
```

Table 3: Summary of prior from JAGS run

	mean	sd	2.5%	97.5%
mu	490.748751	29.7999	433.195389	547.899215
sigma	46.193271	0.0000	46.193271	46.193271
tau	0.000469	0.0000	0.000469	0.000469

Table 4: Summary of prior predictive from JAGS run

	mean	sd	2.5%	97.5%
mu	489.873930	55.4974	381.412435	596.942520
sigma	46.193271	0.0000	46.193271	46.193271
tau	0.000469	0.0000	0.000469	0.000469

Question 11

Adapt your BUGS program to sample from the likelihood.

```
### Sampling from likelihood

# data
prior_tau = 1/(((prior_sd^2)/N))^2 # variance of the likelihood used here
# Note that the data object 'x' IS part of the data list:
jags.data_likelihood = list("x", "N", "prior_mean", "sampling_sd", "prior_tau")

# initials; note that X is NO LONGER part of inits
jags.inits = function(){
    list("mu"= 0)
}

hw1.sim4 = jags(jags.data_likelihood, jags.inits, jags.params,
              model.file = "running_model.txt",
              n.chains = 3, n.iter = 11000, n.burnin = 1000)


to_print_likelihood <- hw1.sim4$BUGSoutput$summary[,c("mean", "sd", "2.5%", "97.5%")]
knitr::kable(to_print_likelihood, caption = "Summary of likelihood from JAGS run")
```

Question 12

Report your WinBUGS models and R code, data, and inits. Use at least samples of size 10000.

R code, model, data, and inits are reported in questions 9-11.

Question 13

Construct a table with means, sds and vars for the (i) posterior for mu,(ii) the prior for mu, (iii) the prior predictive for y, and (iv) the likelihood of mu from the WinBUGS output.

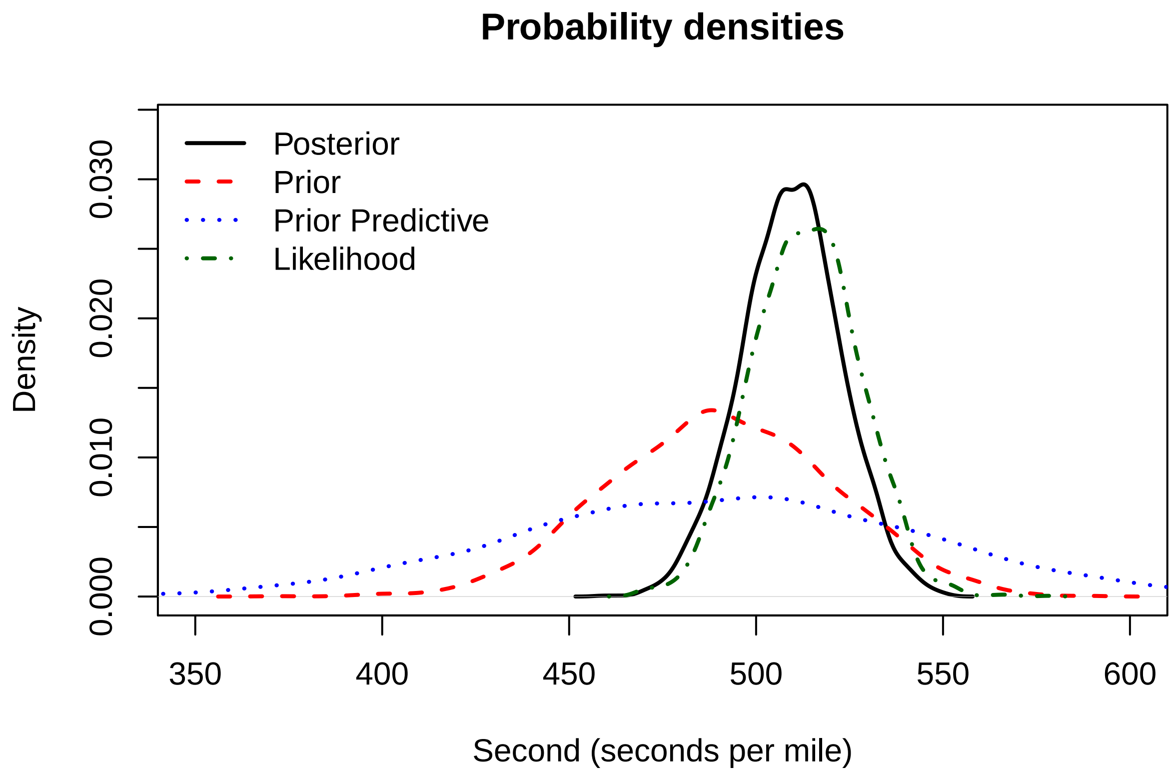
Table 5: Posterior, Prior, Prior Predictive, and Likelihood from JAGS simulations

	Mean	SD	Variance
Posterior	509.367	13.2886	176.587
Prior	490.749	29.7999	888.037
Prior predictive	489.874	55.4974	3079.967
Likelihood	513.275	14.3418	205.686

note: code available in appendix

Question 14

Plot on a single plot the (i) posterior for mu, (ii) the prior for mu (iii) the prior predictive for y, and (iv) the likelihood of mu (suitably normalized so it looks like a density, ie a normal with mean y-bar and variance sigma2/n) all on the same graph. All from the WinBUGS output. Interpret the plot.



Interpretation of the plot: As expected this plot is identical (more or less) to the plot generated from the same distributions in Question 8. As before, we see that the posterior distribution represents something of a compromise between the prior and likelihood distributions. Practically, this means that I (Gaurav) have been running much slower than I thought over the past three weeks or so- my average speed is about twenty seconds a mile slower than I thought. Time to fix that!

note: code available in appendix

Appendix

Question 7 code

```
posterior_row  <- c(posterior_mean, posterior_sd, posterior_var)
prior_row      <- c(prior_mean, prior_sd, prior_sd^2)
prior_pred_row <- c(prior_mean, sqrt(prior_sd^2+sampling_sd^2),
                    prior_sd^2+sampling_sd^2)
likelihood_row <- c(sample_mean, sqrt(sample_var/n), sample_var/n)

to_print <- rbind(posterior_row, prior_row, prior_pred_row, likelihood_row)
rownames(to_print) <- c("Posterior", "Prior", "Prior Predictive", "Likelihood")
colnames(to_print) <- c("Mean", "SD", "Variance")
knitr::kable(to_print, caption = "Posterior, Prior, Prior Predictive, and Likelihood calculated from the
```

Question 8 code

```
posterior_vec <- rnorm(10000, posterior_mean, posterior_sd)
prior_vec <- rnorm(10000, prior_mean, prior_sd)
prior_pred_vec <- rnorm(10000, prior_mean, sqrt(prior_sd^2+sampling_sd^2))
likelihood_vec <- rnorm(10000, sample_mean, sqrt(sample_var/n))

plot(density(prior_vec), lty = 2, col = "red", main = "Probability densities",
     ylim = c(0, 0.034), lwd = 2, xlim = c(350, 600),
     xlab = "seconds per mile")
lines(density(posterior_vec), lwd = 2)
lines(density(prior_pred_vec), col = "blue", lty = 3, lwd = 2)
lines(density(likelihood_vec), col = "darkgreen", lty = 4, lwd = 2)
legend("topleft", lty = c(2, 3, 4, 1), col = c("red", "blue", "darkgreen", "black"), lwd = 2,
       legend = c("Prior", "Prior Predictive", "Likelihood", "Posterior"), bty = "n")
```

Question 13 code

```
Posterior_mu   <- hw1.sim$BUGSoutput$summary[2, c("mean", "sd")]
Prior_mu       <- hw1.sim2$BUGSoutput$summary[1,c("mean", "sd")]
Prior_predictive_y <- hw1.sim3$BUGSoutput$summary[1,c("mean", "sd")]
Likelihood_mu <- hw1.sim4$BUGSoutput$summary[2,c("mean", "sd")]

to_print <- rbind(Posterior_mu, Prior_mu, Prior_predictive_y, Likelihood_mu)

var <- as.matrix((to_print[,2])^2)
results <- cbind(to_print, var)
rownames(results) <- c("Posterior", "Prior", "Prior predictive", "Likelihood")
colnames(results) <- c("Mean", "SD", "Variance")
knitr::kable(results, caption = "Posterior, Prior, Prior Predictive, and Likelihood from JAGS simulation
```

Question 14 code

```r
# Extract the posterior from the BUGS run
posterior = apply(hw1.sim$BUGSoutput$sims.array, 3, unlist)
posterior <- posterior[,2]
# Extract the prior from the BUGS run
prior = apply(hw1.sim2$BUGSoutput$sims.array, 3, unlist)
prior <- prior[,1]

# Extract the prior predictive from the BUGS run
prior_predictive = apply(hw1.sim3$BUGSoutput$sims.array, 3, unlist)
prior_predictive <- prior_predictive[,1]

# Extract the likelihood from the BUGS run
likelihood = apply(hw1.sim4$BUGSoutput$sims.array, 3, unlist)
likelihood <- likelihood[,2]

plot(density(posterior), main = "Probability densities", xlim= c(350, 600), ylim= c(0, 0.034), lwd = 2,
     xlab = "seconds per mile")
lines(density(prior), lty = 2, lwd = 2, col = "red")
lines(density(prior_predictive),lty = 3, lwd = 2, col = "blue")
lines(density(likelihood),lty = 4, lwd = 2, col = "darkgreen")
legend("topleft", lty = c(1, 2, 3, 4), col = c("black", "red", "blue", "darkgreen"), lwd = 2,
       legend = c("Posterior","Prior", "Prior Predictive", "Likelihood"), bty = "n")
```