3 Con't

Once we set the integral equal to 1, the denominator term 3 reduces to

$$\frac{\beta^{\alpha}\,\Gamma(\alpha+y)}{y!\,\Gamma(\alpha)\,(1+\beta)^{\alpha+y}}$$

Putting together 1, 2, and 3:

$$\frac{\dfrac{e^{-\lambda}\lambda^{y}}{y!}\cdot\dfrac{\beta^{\alpha}\lambda^{\alpha-1}e^{-\lambda\beta}}{\Gamma(\alpha)}}{\dfrac{\beta^{\alpha}\,\Gamma(\alpha+y)}{y!\,\Gamma(\alpha)\,(1+\beta)^{\alpha+y}}}$$

$$= \frac{e^{-\lambda}\lambda^{y}}{\cancel{y!}}\,\frac{\cancel{\beta^{\alpha}}\lambda^{\alpha-1}e^{-\lambda\beta}}{\cancel{\Gamma(\alpha)}}\cdot\frac{\cancel{y!}\,\cancel{\Gamma(\alpha)}\,(1+\beta)^{\alpha+y}}{\cancel{\beta^{\alpha}}\,\Gamma(\alpha+y)}$$

$$= \frac{(1+\beta)^{\alpha+y}\,\lambda^{\cancel{\alpha+y}\ \alpha+1}\,e^{-\lambda(\beta+1)}}{\Gamma(\alpha+y)}$$

which is the PDF of Gamma $(\lambda;\alpha+y,\beta+1)$