Getting the posterior for Poisson distributed dataset $Y = [y_1, y_2 \dots y_n]$ and a Gamma prior:

$$P(\lambda|Y) = \frac{{}^{①}P(Y|\lambda)\, P(\lambda)^{②}}{{}^{③}P(Y)}$$

① $$P(Y|\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{y_i}}{y_i!} = \frac{e^{-n\lambda}\lambda^{\sum y_i}}{\prod_{i=1}^{n} y_i!}$$

② $$P(\lambda) = \frac{\beta^{\alpha}\lambda^{\alpha-1}e^{-\lambda\beta}}{\Gamma(\alpha)} \quad \text{for } P(\lambda;\alpha,\beta)$$

③ $$\int_0^{\infty} \frac{\beta^{\alpha}\lambda^{\sum_{i=1}^{n} y_i+\alpha-1}\, e^{-\lambda(\beta+n)}}{\Gamma(\alpha)\prod_{i=1}^{n} y_i!}\, d\lambda$$

We multiply the integral by two convenience terms:

$$= \int_0^{\infty} \frac{\beta^{\alpha}\lambda^{\sum_{i=1}^{n} y_i+\alpha-1}\, e^{-\lambda(\beta+n)}}{\Gamma(\alpha)\prod_{i=1}^{n} y_i!} \cdot \frac{\Gamma(\alpha+\sum_i y_i)}{\Gamma(\alpha+\sum_i y_i)} \cdot \frac{(\beta+n)^{\alpha+\sum y_i}}{(\beta+n)^{\alpha+\sum y_i}}\, d\lambda$$

Factor out some terms that are constant wrt λ:

$$\frac{\beta^{\alpha}\cdot\Gamma(\alpha+\sum_i y_i)}{\Gamma(\alpha)\prod_{i=1}^{n} y_i!\cdot(\beta+n)^{\alpha+\sum y_i}} \int_0^{\infty} \frac{(\beta+n)^{\alpha+\sum y_i}\cdot\lambda^{\sum_{i=1}^{n} y_i+\alpha-1}\, e^{-\lambda(\beta+n)}}{\Gamma(\alpha+\sum_i y_i)}\, d\lambda$$

as the term inside integral is the PDF of $\text{Gamma}(\lambda;\alpha+\sum y_i, \beta+n)$, integrating over the whole range of (λ), ie $0 \to \infty$, will yield 1