③, con't

Once we set the integral to 1, the denominator term ③ is:

$$\frac{\beta^{\alpha}\,\Gamma(\alpha+\sum_i y_i)}{\Gamma(\alpha)\cdot\prod_{i=1}^{n} y_i!\cdot(\beta+n)^{\sum_i y_i+\alpha}}$$

Putting together ①, ②, and ③:

$$\frac{\overset{①}{\dfrac{e^{-n\lambda}\,\lambda^{\sum y_i}}{\prod_{i=1}^{n} y_i!}}\cdot\overset{②}{\dfrac{\beta^{\alpha}\,\lambda^{\alpha-1}\,e^{-\lambda\beta}}{\Gamma(\alpha)}}}{\underset{③}{\dfrac{\beta^{\alpha}\,\Gamma(\alpha+\sum_i y_i)}{\Gamma(\alpha)\cdot\prod_{i=1}^{n} y_i!\,(\beta+n)^{\sum_i y_i+\alpha}}}}$$

$$=\frac{\dfrac{\beta^{\alpha}\,\lambda^{\sum y_i+\alpha-1}\,e^{-\lambda(\beta+n)}}{\prod_{i=1}^{n} y_i!\;\Gamma(\alpha)}}{\dfrac{\beta^{\alpha}\,\Gamma(\alpha+\sum y_i)}{\Gamma(\alpha)\cdot\prod_{i=1}^{n} y_i!\;(\beta+n)^{\sum y_i+\alpha}}}$$

$$=\frac{\cancel{\beta^{\alpha}}\,\lambda^{\sum y_i+\alpha-1}\,e^{-\lambda(\beta+n)}}{\cancel{\prod_{i=1}^{n} y_i!}\;\cancel{\Gamma(\alpha)}}\cdot\frac{\cancel{\Gamma(\alpha)}\cdot\cancel{\prod_{i=1}^{n} y_i!}\;(\beta+n)^{(\sum y_i+\alpha)}}{\cancel{\beta^{\alpha}}\;\Gamma(\alpha+\sum y_i)}$$

$$=\frac{(\beta+n)^{\sum y_i+\alpha}\,\lambda^{\sum y_i+\alpha-1}\,e^{-\lambda(\beta+n)}}{\Gamma(\alpha+\sum_i y_i)}$$

which is the pdf of $\text{Gamma}\left(\lambda;\ \alpha+\sum_{i=1}^{n} y_i,\ \beta+n\right)$