Getting the posterior for Poisson dataset y and a Gamma prior:

$$P(\lambda|y) = \frac{\overset{①}{P(y|\lambda)}\ \overset{②}{P(\lambda)}}{③\ P(y)}$$

① $$P(y|\lambda) = \frac{e^{-\lambda}\lambda^{y}}{y!}$$

② $$P(\lambda) = \frac{\beta^{\alpha}\lambda^{\alpha-1}e^{-\lambda(\beta)}}{\Gamma(\alpha)}$$ for Gamma$(\lambda;\alpha,\beta)$

③ $$P(y) = \int_0^\infty p(y|\lambda)\ p(\lambda)\ d\lambda$$

$$= \int_0^\infty \frac{\beta^{\alpha}\lambda^{\alpha-1}e^{-\lambda\beta}}{\Gamma(\alpha)} \cdot \frac{e^{-\lambda}\lambda^{y}}{y!}\ d\lambda = \int_0^\infty \frac{\beta^{\alpha}\lambda^{\alpha+y-1}e^{-\lambda(\beta+1)}}{y!\,\Gamma(\alpha)}\ d\lambda$$

We multiply the integral by two convenience terms:

$$= \int_0^\infty \frac{\beta^{\alpha}\lambda^{\alpha+y-1}e^{-\lambda(\beta+1)}}{y!\,\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+y)}{\Gamma(\alpha+y)} \cdot \frac{(1+\beta)^{\alpha+y}}{(1+\beta)^{\alpha+y}}\ d\lambda$$

and we factor out some terms that are constant WRT λ:

$$\frac{\beta^{\alpha}\ \Gamma(\alpha+y)}{y!\ \Gamma(\alpha)\ (1+\beta)^{\alpha+y}} \int_0^\infty \frac{(\beta+1)^{\alpha+y}\ \lambda^{\alpha+y-1}\ e^{-\lambda(\beta+1)}}{\Gamma(\alpha+y)}\ d\lambda$$

As the term inside the integral is the PDF of Gamma $(\lambda;\ \alpha+y,\ \beta+1)$, integrating across the range of $0 \to \infty$ yields 1.