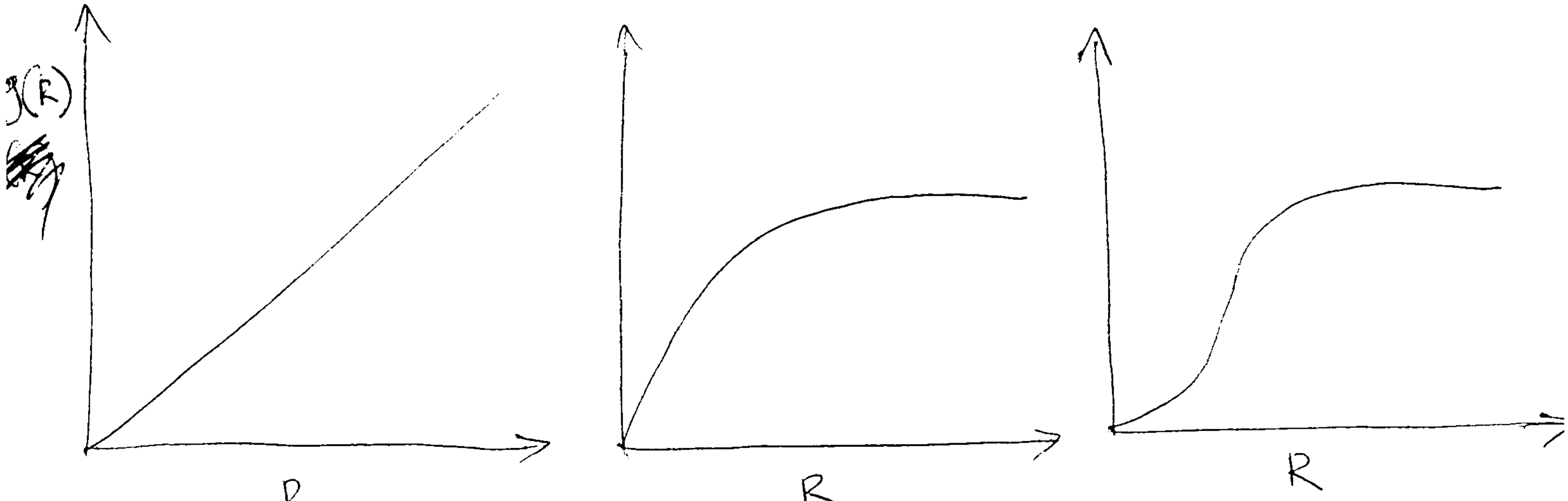
Types of functional responses

Type I — Type II — Type III



Type I

$$g(R) = bR$$

Per capita consumption rate ↑ linearly w/ resource density

Type II

$$g(R) = \frac{mR}{D + R}$$

Saturates at high resource densities

Type III

$$g(R) = \frac{mR^2}{D^2 + R^2}$$

accelerates at low resource densities and saturates at high resource densities.

Lotka - Volterra predator-prey model

$$\frac{dR}{dt} = aR - bRC$$

$$\frac{dC}{dt} = ebCR - dC$$

Quantities	Units
R, C	#
a, per capita growth rate of prey	$1/\text{Time}$
b, per head attack rate of predator	$1/\text{Time}(\#)$
d predator death rate	$1/\text{time}$
e conversion efficiency (scaling factor)	—
t	time

Assumptions :

(i)

(ii)

(iii)

Lotka - Volterra predator prey model

Equilibria : $(R^*, C^*) = (0, 0)$

$$= \left(\frac{d}{eb}, \frac{a}{b}\right)$$

Local stability analysis

$$\frac{dR}{dt} = aR - bRC = f_1$$

$$\frac{dC}{dt} = ebRC - dC = f_2$$

$$J_{11} = \frac{\partial f_1}{\partial R} = a - bC^*$$

$$J_{12} = \frac{\partial f_1}{\partial C} = -bR^*$$

$$J_{21} = \frac{\partial f_2}{\partial R} = ebC^*$$

$$J_{22} = \frac{\partial f_2}{\partial C} = ebR^* - d$$

Jacobian, $J = \begin{bmatrix} a - bC^* & -bR^* \\ ebC^* & ebR^* - d \end{bmatrix}$

Local stability of coexistence equilibrium :

$$J\Big|_{(R^*, C^*) > (0, 0)} = \begin{bmatrix} 0 & -\frac{d}{e} \\ ea & 0 \end{bmatrix}$$

Characteristic equation : $\lambda^2 + A_1\lambda + A_2 = 0$

$$A_1 = -\left(J_{11} + J_{22}\right) = 0$$

$$A_2 = J_{11}J_{22} - J_{12}J_{21} = 0 - (ea)\left(-\frac{d}{e}\right) = ad$$

$$\lambda_{1,2} = \frac{1}{2}\left(-A_1 \pm \sqrt{A_1^2 - 4A_2}\right)$$

$$= \frac{1}{2}\left(0 \pm \sqrt{-4ad}\right) = \frac{1}{2}\left(0 \pm 2\sqrt{-ad}\right.$$

$$\therefore \lambda_{1,2} = 0 \pm \sqrt{-ad}$$

* Eigenvalues have zero real parts.

Lotka-Volterra model with prey self limitation

$$\frac{dR}{dt} = aR\left(1 - \frac{R}{K}\right) - bRC = f_1$$

$$\frac{dC}{dt} = ebRC - dC = f_2$$

Equilibria : $(R^*, C^*) = (0, 0)$

$(K, 0)$

$\left(\frac{d}{eb}, \frac{a}{eb^2 K}(ebK - d)\right)$

Local stability analysis

$$J_{11} = \frac{\partial f_1}{\partial R} = a - \frac{2aR}{K} - bC$$

$$J_{12} = \frac{\partial f_1}{\partial C} = -bR$$

$$J_{21} = \frac{\partial f_2}{\partial R} = ebC$$

$$J_{22} = \frac{\partial f_2}{\partial C} = ebR - d$$

$$J = \begin{bmatrix} a - 2aR^* - bC & -bR^* \\ ebC^* & ebR^* - d \end{bmatrix}$$

Per cap. growth rate of resource $= \frac{dR}{dt} \cdot \frac{1}{R} = a - \frac{aR}{K} - bC$

At equilibrium, $\frac{dR}{dt} \cdot \frac{1}{R} = 0 \Rightarrow a - \frac{aR^*}{K} - bC^* = 0$

$$\Rightarrow J_{11} = \frac{-aR^*}{K}$$

$$J_{22} = eb \cdot \frac{d}{eb} = 0$$

$$J = \begin{bmatrix} -aR^* & -bR^* \\ ebC^* & 0 \end{bmatrix}$$

Characteristic equation: $\lambda^2 + A_1\lambda + A_2 = 0$

$$A_1 = -(J_{11} + J_{22}) = \frac{aR^*}{K}$$

$a, K > 0$

$\Rightarrow A_1 > 0$ if $R^* > 0$

$$A_2 = J_{11}J_{22} - J_{12}J_{21} = eb^2R^*C^*$$

$e, b > 0 \Rightarrow A_2 > 0$ if $R^*, C^* > 0$

* Coexistence equilibrium is locally stable as long as it is feasible.

Lotka - Volterra model with a Type II func. response for predator

$$\frac{dR}{dt} = aR - \frac{mRC}{H+R}$$

$$\frac{dC}{dt} = \frac{emRC}{H+R} - dC$$

(i) Solve for equilibria :

$$(R^*, C^*) = (0, 0)$$

$$= \left(\frac{dH}{em-d} , \frac{a}{m}\left(\frac{dH}{em-d}\right)\right)$$

(ii) Local stability analysis

$$\frac{dR}{dt} = aR - \frac{mRC}{H+R} = f_1$$

$$\frac{dC}{dt} = \frac{emRC}{H+R} - dC = f_2$$

Need quotient rule:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$$

$$J_{11} = \frac{\partial f_1}{\partial R} = a - \left[\frac{(H+R^*)mC^* - mR^*C^* \cdot 1}{(H+R^*)^2}\right]$$

$$= a - \frac{mHC^*}{(H+R^*)^2}$$

$$J_{12} = \frac{\partial f_1}{\partial C} = -\frac{mR^*}{H+R^*}$$

$$J_{21} = \frac{\partial f_2}{\partial R} = \left[\frac{(H+R^*)mC^* - mR^*C^* \cdot 1}{(H+R^*)^2}\right] = \frac{emHC^*}{(H+R^*)^2}$$

$$J_{22} = \frac{\partial f_2}{\partial C} = \frac{emR^*}{H+R^*} - d$$

No self limitation in predator. $\Rightarrow$ $J_{22} = 0$ (work this out)

* Equilibrium stable if $J_{11} < 0$, unstable otherwise.

$$J_{11} = a - \frac{mHC^*}{(H+R^*)^2}, \quad \text{Recall, } C^* = \frac{a}{m}(H+R^*)$$

⋮

$$J_{11} = \frac{aR^*}{H+R^*}, \quad R^* = \frac{dH}{em-d}, \quad em \neq d$$

$$\Rightarrow J_{11} > 0$$

Lotka-Volterra model with prey self limitation and a saturating (Type II) functional response for the predator

$$\frac{dR}{dt} = aR\left(1-\frac{R}{K}\right) - \frac{mRC}{H+R}$$

$$\frac{dC}{dt} = \frac{emRC}{H+R} - dC$$

Equilibria:

$$(R^*, C^*) = (0,0)$$

$$= (K, 0)$$

$$= \left(\frac{dH}{em-d}, \frac{aeH\left(K(em-d) - dH\right)}{K(em-d)^2}\right), \quad em-d \neq 0$$

Phase plane analysis

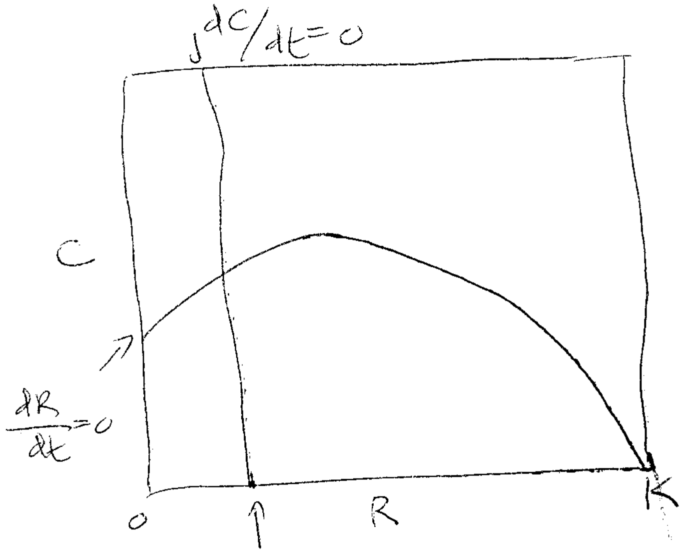
(i) Construct isoclines

At equilibrium $\frac{dR}{dt} = 0$,

$R \neq 0 \Rightarrow C = \frac{a}{mK}(K-R)(H+R)$ Prey zero isocline

$\frac{dC}{dt} = 0$, $C \neq 0 \Rightarrow R = \frac{dH}{em-d}$ predator zero isocline

(ii) Plot isoclines in phase space



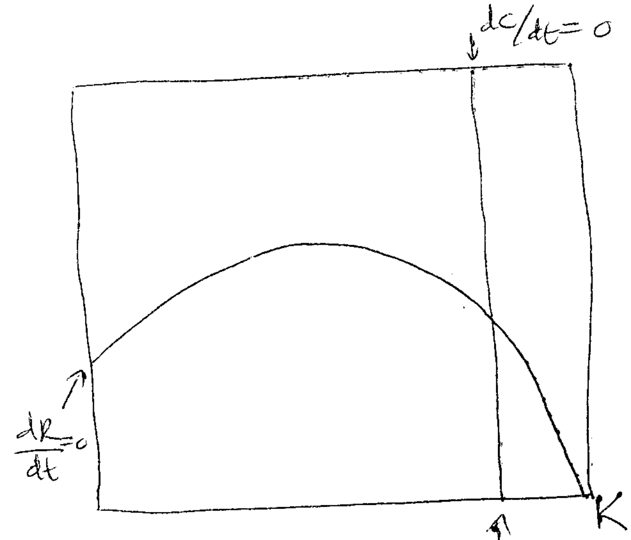
$R^* = \frac{dH}{em-d}$, prey density at which predator can just maintain itself

$R^* << K$

Strong consumer control relative to resource self limitation.

⇒ unstable equilibrium

⇒ ~~osc~~ ~~oscillations~~ persistent oscillations



$R^* = \frac{dH}{em-d}$

$R^* \to K$

Weak consumer control relative to resource self limitations

⇒ stable equilibrium

⇒ damped oscillations

* Paradox of enrichment (Rosenzweig 1971)

Transition from stability to instability

$$\lambda_{1,2} = \frac{1}{2}\left(-A_1 \pm \sqrt{A_1^2 - 4A_2}\right)$$

Zero real root

($A_1^2 > 4A_2$)

Transition from stability to instability involves

$\lambda = 0$ ($A_1, A_2 \neq 0$)

Result:

Qualitative change in the equilibrium itself (e.g., disappearance of equilibria, appearance of multiple equilibria)

e.g., saddle-node bifurcation

Complex root with zero real parts

($A_1^2 < 4A_2$)

Transition from stability to instability involves

$A_1 = 0 \Rightarrow$

$\lambda = 0 \pm i\sqrt{A_2}$

Result:

Transition from damped oscillations to sustained oscillations.

$A_1 = 0$ condition for oscillatory instability

e.g., Hopf bifurcation