Comparative Analysis of Portfolio Optimisation Models During The COVID-19 Pandemic

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***Abstract*—We implement and comparatively analyze the performance of portfolios constructed by three different optimization models i.e Monte-Carlo, Markowitz, and Black-Litterman models during the COVID-19 pandemic period. We compare the performance of the different optimal portfolios constructed by these models among themselves and against the benchmark market returns to demonstrate the comparative impact on the market and equity portfolios optimized by models under consideration.**

I. Introduction

A "Public Health Emergency of International Concern" recognized by the WHO on January 30, 2020 was later classified as a "Pandemic" as of March 11, 2020. The COVID-19 pandemic has taken the entire world by shock. Even countries like Italy and the USA with their superior healthcare systems were taken aback by the impact of the virus. In these uncertain times where even the most prudent governments struggle to contain the damage caused and mitigate future risks, the markets behave even more erratically. This is a huge concern for institutional investors and portfolio managers who invest a large number of funds in the markets hoping for superior returns as compared to less risky assets. In this article, we endeavor to analyze the impact on such portfolios during the COVID-19 period.

Institutional investors tackle a common problem — the Asset Allocation problem. This problem poses the question of constructing an optimal portfolio of assets from a given universe of assets. There are different types of investors due to which the definition of an 'optimal portfolio' differs depending on the goals and risk tolerance of the individual investor, but as a general objective all investors try to maximize their returns while minimizing the risk of said returns. In the context of this article, our asset universe consists of the thirty listed stocks that make up the Sensex index of the Bombay Stock Exchange (BSE) on the date of submission. The BSE is India's oldest stock exchange and the Sensex forms an important market index constituted by the most prominent listed stocks of the Indian market.

Portfolio optimization models are used to tackle the Asset allocation problem stated above. These mathematical models use historical inputs and given objectives to provide an optimal distribution of funds among the asset universe. To understand the impact of the pandemic on such portfolios, we implement important portfolio optimization models and then back-test their performance to understand the impact on their value. We compare the performance of these portfolio optimization models with the benchmark BSE Sensex index to understand the impact on them with respect to the general market.

Since the article aims to analyze the impact of market conditions on equity portfolios, it is assumed that there are no other alternative assets available for investment in our asset universe. As is prevalent in many institutional settings as well, the article limits itself to long-only equity portfolios. Throughout the project, we assume that there are 240 trading days in a financial year. For model components that need a risk-free rate input, the RBI 10-year treasury bond yield is used. For all implementations, the historical data begins from January 1, 2000, and it is assumed that the investment period under consideration is from January 1, 2019, to April 30, 2020.

The following sections deal with three important portfolio optimization models — section II uses Monte-Carlo methods of randomized simulations, section III uses the Markowitz model which models the asset allocation as a convex quadratic programming problem and section IV uses the Black-Litterman model which uses a Bayesian approach allowing the investor to factor in their subjective views. Within each section, we first outline the implementation undertaken and then provide the results of our experiments. Finally in section V we bring together all the results of the previous approaches and compare them among themselves and with the benchmark market returns. The complete code for this article is hosted online.[1]

II. Monte-Carlo Method

A. Overview

The Asset allocation problem as outlined in section I can be approached as an optimization problem — we need to optimize the way we allocate our funds among the assets which constitute our asset universe intending to gain a superior return on our investment. While many optimization problems are deterministic in nature, as a first approach we try to approach this problem at hand using the Monte-Carlo method. Monte-Carlo methods are a class of computation-intensive

[1] https://github.com/jmulherkar/BTP2020/

algorithms that leverage randomized sampling methods. They simulate a very large number of possible scenarios to arrive at an optimal estimate. For our problem, we use Monte-Carlo methods to analyze a large number of simulated portfolios to find the optimal choice.

B. Processing Historical Prices

To perform Monte-Carlo simulations on our asset universe, first we need to compute the parameters that will characterise each asset under consideration. To compute these parameters we use historical daily closing price data over a considerable period of analysis for each asset under consideration.

Let us model our asset universe as a set

$$X = \{x_1, x_2, \dots x_N\}$$

where N denotes the number of assets under consideration. Consider an asset $x \in X$ with daily closing prices $P_x = \{p_0, p_1, \dots p_t\}$ over the period of analysis. Using this historical data available to use, we can compute $r_k(x)$ - the daily return for asset $x \in X$ on a given day $i \in [1, t]$. [2] We can further use those values to compute $\overline{r_d}(x)$, the expected daily returns of asset x over the period of analysis.

$$\begin{aligned} r_k(x) &= \frac{p_k(x) - p_{k-1}(x)}{p_{k-1}(x)},\ k \in [1,t] \\ \overline{r_d}(x) &= \frac{\sum_{i=1}^{t} r_k(x)}{t} \end{aligned} \tag{1}$$

Since we conventionally work with per annum returns, the expected daily returns $\overline{r_d}(x)$ can be annualised by compounding $\overline{r_d}(x)$ over n_t, the average number of trading days in a financial year to compute $\overline{r}(x)$ the annualised expected daily returns of asset $x \in X$ over one financial year.

$$\overline{r}(x) = (1 + \overline{r_d}(x))^{n_t} - 1 \tag{2}$$

Carrying out the above computations for all assets in X allows us to construct $\overline{\mathbf{r}}$, the expected return matrix for our asset universe. Each matrix element $\overline{r}_i$ in $\overline{\mathbf{r}}$ represents the annualised expected return value of asset $x_i \in X$.

$$\overline{\mathbf{r}} = \begin{bmatrix} \overline{r}_1 \\ \overline{r}_2 \\ \vdots \\ \overline{r}_N \end{bmatrix} \tag{3}$$

The second parameter of consideration is the volatility of returns of assets in X. We need to compute not only the volatility of returns of a particular asset, but also the volatility of returns of each asset with respect to other assets which are a part of X. To do this, we use the sample co-variance matrix which has conventionally been used as an estimator of volatility. Thus, we use the daily historical data available to us to compute the sample co-variance matrix for daily returns across our X. The sample co-variance matrix is denoted by $\mathbf{S}$.

Consider the asset $x \in X$ with daily closing prices $P_x = \{p_0, p_1, \dots p_t\}$ over the period of analysis as before. We use the daily historical closing prices and values from equation (1) to construct the daily expected returns matrix $\overline{\mathbf{r_d}}$.

$$\overline{\mathbf{r_d}} = \begin{bmatrix} \overline{r}(x_1) & \overline{r}(x_2) & \dots & \overline{r}(x_n) \end{bmatrix}$$

To compute (S) we also construct a daily return matrix $\mathbf{r_d}$ where every element $r_a(x_b)$ denotes the daily returns on day $a \in [1, t]$ for asset $x_b \in X$.

$$\overline{\mathbf{r_d}} = \begin{bmatrix} r_1(x_1) & r_1(x_2) & \dots & r_1(x_n) \\ r_2(x_1) & r_2(x_2) & \dots & r_2(x_n) \\ \vdots & \ddots & \ddots & \vdots \\ r_t(x_1) & r_t(x_2) & \dots & r_t(x_n) \end{bmatrix}$$

We can now compute $\mathbf{S_d}$, the sample co-variance matrix of daily returns for our asset universe. Assume $\mathbf{1}$ to be a conformable matrix full of ones.

$$\mathbf{S} = \left(\frac{1}{t}\right) \sum_{i=1}^{t} (\mathbf{r_d} - \mathbf{1}\,\overline{\mathbf{r_d}})(\mathbf{r_d} - \mathbf{1}\,\overline{\mathbf{r_d}})^{\intercal} \tag{4}$$

C. Monte-Carlo Simulations

Once we have computed the annualised expected returns matrix $\overline{\mathbf{r}}$ and the sample co-variance matrix of returns $\mathbf{S}$ we can perform the required Monte-Carlo simulations. Since we are trying to find the optimal portfolio, we will randomise the relative weight-allocations to different assets that constitute our asset universe X. We repeat this operation for a large number of times and analyse each randomised portfolio created. As discussed in section I, we will find two different *optimal portfolios* for each approach — one with the maximum Sharpe ratio, and the other with the least volatility.

D. Optimum Portfolio: Minimum Volatility

Consider a randomised portfolio $\mathbf{w} = [\, w_1\ w_2\ \dots\ w_N \,]$ created using the Monte-Carlo approach as highlighted above where element w_i denotes the relative weight-allocation of funds to asset $x_i \in X$. Since we are dealing with relative weight-allocations, a randomised portfolio can only be considered feasible if the condition $\sum_{i=1}^{N} w_i = 1$ holds true.

The expected rate of return obtained by randomised portfolio allocation $\mathbf{w}$, can be computed as $\overline{r_p}(\mathbf{w}) = \mathbf{w}\,\overline{\mathbf{r}}$, where $\overline{\mathbf{r}}$ denotes the annualised expected returns matrix as computed in (3). The optimal portfolio we are finding minimises the volatility of returns $\sigma_p^2(\mathbf{w}) = \mathbf{w}\,\mathbf{S}\,\mathbf{w}^{\intercal}$ and thus the variance of the same.

Upon analysing all randomised portfolios generated by Monte-Carlo simulations, the one with the minimum volatility will be considered the optimal portfolio obtained by this approach.

[2] Throughout this paper $[x, y]$ denotes the integral range between x and y i.e $[x, y] = \mathbb{Z} \cap [x, y]$

E. Optimum Portfolio: Maximum Sharpe Ratio

The Sharpe ratio is a reward-to-risk ratio first introduced by Nobel-laureate William Sharpe to measure the performance of mutual funds [1]. It is a useful indicator of returns as it factors in not just the returns of the portfolio and their volatility, but also the rate of risk-free returns prevailing in the market. Consider a feasible randomised portfolio $\mathbf{w} = [\, w_1 \; w_2 \; \dots \; w_N \,]$ where $\sum_{i=1}^{N} w_i = 1$. Assume r_f denotes the rate of risk-free return while $\overline{r_p}(\mathbf{w})$ and $\sigma_p^2(\mathbf{w})$ denote respectively the expected return and their variance obtained through randomised portfolio allocation $\mathbf{w}$ as defined above.

$$\text{Sharpe Ratio} = \frac{\overline{r_p}(\mathbf{w}) - r_f}{\sqrt{\sigma_p^2(\mathbf{w})}}$$

Upon analysing all randomised portfolios generated by Monte-Carlo simulations, the one with the highest Sharpe ratio will be considered the optimal portfolio obtained by this approach.

F. Results

Using the historical price data over the period of analysis we compute equations (1) and (2) to construct the annualised expected returns matrix $\bar{\mathbf{r}}$ as shown in (3). Figure 1 is a graphical representation of $\bar{\mathbf{r}}$ with the expected returns expressed as percentages.

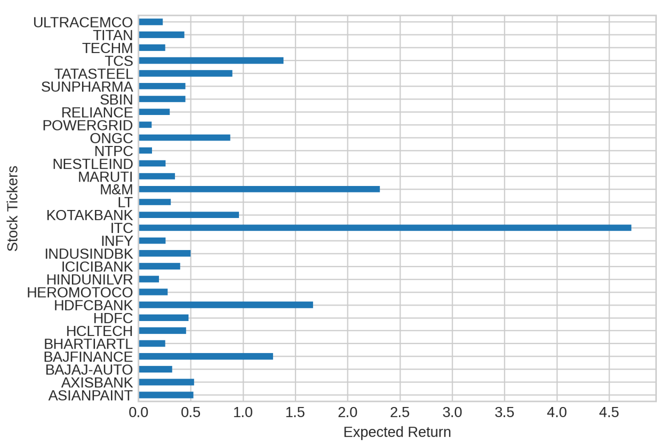


Fig. 1. Annualised expected returns of Sensex constituents

To estimate the volatility of returns of the assets under consideration over the period of analysis we compute the sample co-variance matrix $\mathbf{S}$ in accordance with equation (4). Figure 2 graphically represents $\mathbf{S}$ in the form of a heatmap.

Once we have estimated $\bar{\mathbf{r}}$ and $\mathbf{S}$ as shown above, we perform Monte-Carlo simulations as outlined in section II-C. For this articles, we compute and analyse 1,000,000 randomised portfolios. Figure 3 shows a risk-return scatter plot of the randomised plot and the locations of the optimal portfolios.

Table II lists the percentage composition of assets in both the optimal portfolios computed using Monte-Carlo Methods. Finally, Figure 4 plots the relative value of the optimal portfolios over the investment period and compares the same against the benchmark market index values over the same period.

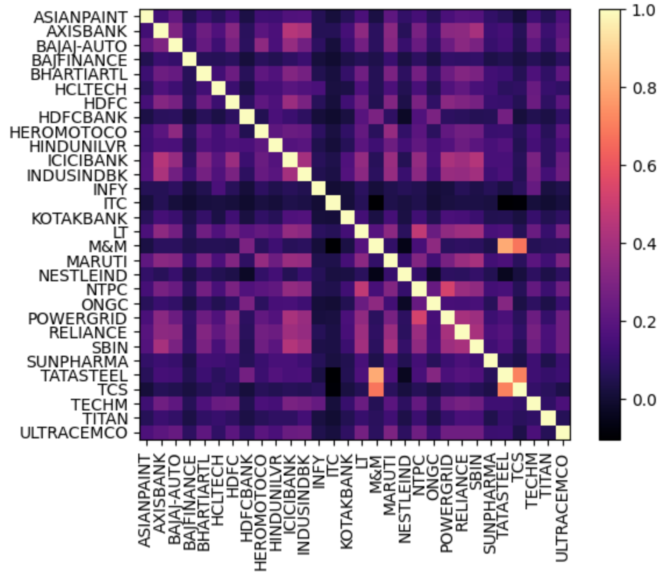


Fig. 2. Heatmap for the co-variance matrix of expected returns

III. Markowitz Model

A. Overview

The Markowitz model is a fundamental part of Modern Portfolio Theory, a field of portfolio optimization pioneered by Henry Markowitz with his seminal 1952 paper titled ‘Portfolio Selection’ [2]. The Markowitz model is also called the Mean-Variance model because these are most important parameters of an asset for the model. It analyses a risky asset based on the expected returns (a function of mean) and volatility of returns (a function of variance). The Markowitz model then

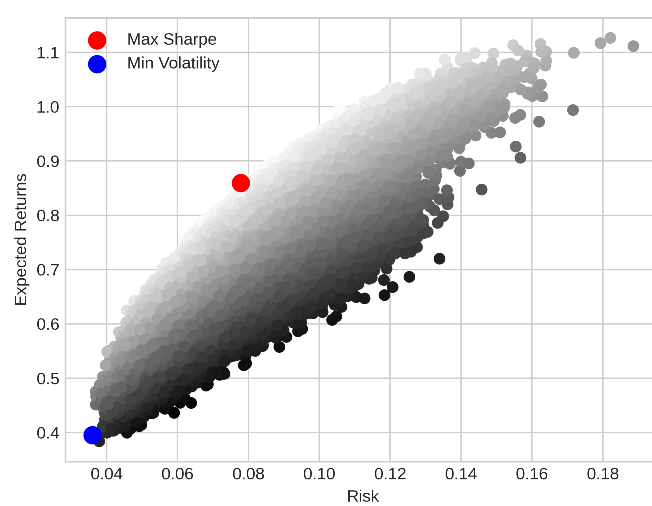


Fig. 3. Scatter plot for randomised samples and optimal portfolios

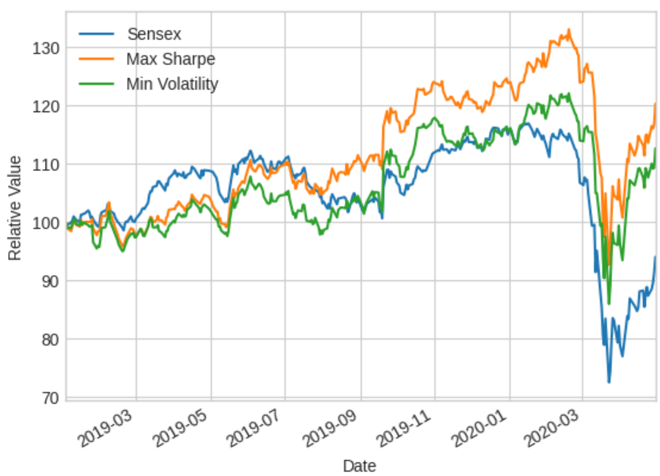


Fig. 4. Relative value of optimal Monte-Carlo simulated portfolios against benchmark market index

uses these parameters and the inter-relationship of these assets, characterized by the correlation between the returns of these risky assets to find a portfolio with an optimal amount of return and volatility for the investor.

The Efficient Set and Efficient Frontier form the heart of the Markowitz model. There are infinite portfolios that can constructed from a particular set of assets by varying the relative weights of each of these assets. The Markowitz model helps the investor by considering only a subset of these possibilities - the 'Efficient Set' of portfolios. Based on the assumptions of the Markowitz model, the Efficient Set of portfolio contains portfolios that satisfy the following conditions —

- Offer maximum expected return for varying levels of risk
- Offer minimum risk for varying levels of expected returns

When the the efficient set is plotted on a plot of expected return vs. volatility, it takes a convex shape which is called the

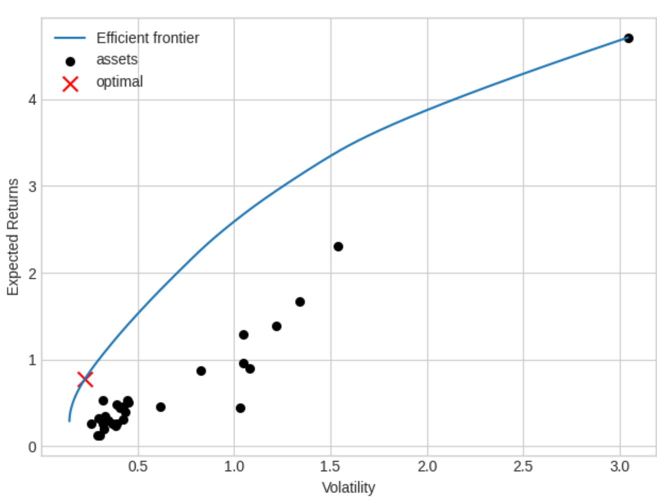


Fig. 5. Efficient Frontier for assets under consideration

'Efficient Frontier'. The optimal portfolio required necessarily lies on this efficient frontier. In the context of this article we consider two different optimal portfolios, one with the maximum Sharpe Ratio and the second with the minimum volatility. This helps us analyse the performance of the model even better, since the portfolio with the maximum Sharpe Ratio gives a portfolio which favors high returns and compared to a moderately risk-seeking investor, while one with minimum volatility can be compared to a risk-averse investor.

B. Processing historical prices

To implement the Markowitz model on our asset universe, we first need to compute parameters that will characterise each asset. To compute this we use historical daily closing price data for each asset in our asset universe. Let us model our asset universe as a set $X = \{x_1, x_2, \dots x_N\}$ as done in section II. We use this data to compute $\bar{\mathbf{r}}$ as in (3) and $\mathbf{S}$ as in (4) respectively.

The sample co-variance matrix is a conventional estimator of volatility, due to multiple factors including its unbiased nature and having maximum likelihood under normality. Notwithstanding, we avoid using this traditional estimator of volatility because it is known to be replete with estimation errors around the extremes, in particular when the number of historical observations are comparable to the number of assets under consideration. We will use a transformation of the sample co-variance matrix, constructed using the Ledoit-Wolf Shrinkage Estimator [3].

The Ledoit-Wolf Shrinkage Estimator involves the use of a Shrinkage constant $\delta \in [0, 1]$ and a structured estimator $\mathbf{F}$ (also known as the shrinkage target) to transform the sample co-variance matrix $\mathbf{S}$ closer to the true (or population) co-variance matrix. We use the *constant correlation* Ledoit-Wolf shrinkage model and accordingly, the shrinkage target is based on the estimation that all pairwise correlations are identical. This identical value is equal to the mean of all sample correlations. Our shrinkage target is the implied using this value and the previously obtained sample variances [4].

Let typical entries $\mathbf{S}$ be defined as σ_{ij}, $i, j \in X$. Then, the sample co-relation between two assets x_i and x_j ($x_i, x_j \in X$) is denoted by ρ_{ij}.

$$\rho_{ij} = \frac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}}$$

The average sample correlation across $\mathbf{S}$ is $\bar{\rho}$

$$\bar{\rho} = \frac{2}{N(N-1)} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \rho_{ij}$$

Correspondingly if the elements of the shrinkage target matrix $\mathbf{F}$ are given by f_{ij} then —

$$f_{ij} = \begin{cases} \bar{\rho}\sqrt{\sigma_{ii}\sigma_{jj}} & \text{if } i \neq j \\ \sigma_{ii} & \text{otherwise} \end{cases}$$

Finally we perform a convex linear combination of the constant co-relation estimator $\mathbf{F}$, the sample co-variance matrix

$\mathbf{S}$ and the optimal shrinkage constant δ^* to obtain our Ledoit-Wolf shrinkage estimation of the co-variance matrix $\mathbf{\Sigma}$ which will be used as an input for the Markowitz Model.

$$\mathbf{\Sigma} = \delta^* \mathbf{F} + (1 - \delta^*)\mathbf{S} \tag{5}$$

C. Min-Volatility

The *optimal portfolio* is a subjective term largely dependant on investor outlook and risk-tolerance. As stated earlier, the portfolio best-suited to a risk-averse investor would be the minimum volatility portfolio. Graphically, the minimum volatility portfolio would be represented by the point closest to the origin on the Efficient Frontier in a risk-return graph. The problem to find this allocation can be approached as a convex quadratic programming problem. From our asset universe X as above, we wish to find an optimal weight allocation matrix $\mathbf{w} = [\, w_1\, w_2\, \ldots\, w_N \,]$ where w_i is the proportion of total funds invested in asset x_i. A natural condition of an feasible solution to this problem is that $\sum_{i=1}^{s} w_i = 1$. Assuming $\mathbf{1}$ to be a conformable matrix of ones, the problem can be modelled as below.

$$\begin{aligned} &\underset{\mathbf{w}}{min}\ \frac{1}{2}\mathbf{w}^\intercal \Sigma \mathbf{w} \\ &s.t\ \mathbf{1}\,\mathbf{w} = 1 \end{aligned} \tag{6}$$

This is now a problem that is solvable by known convex quadratic programming methods.

D. Max-Sharpe

While the minimum volatility ratio caters to a risk-averse investor, other moderate risk-seeking investors may prefer the maximum Sharpe ratio as it tracks a risk-to-reward ratio. Graphically, the portfolio with the maximum Sharpe ratio would be represented by the most "north-west" point on the Efficient Frontier in a risk-return graph. The problem to find this allocation can also be approached as a quadratic-programming problem . As earlier, we wish to find an optimal weight allocation matrix $\mathbf{w} = [\, w_1\, w_2\, \ldots\, w_N \,]$ where w_i is the proportion of total funds invested in asset x_i. Like earlier, the condition $\sum_{i=1}^{N} w_i = 1$ must hold true. We assume that such an optimal portfolio allocation $\hat{\mathbf{w}}$, with $\bar{\mathbf{r}}^\intercal \hat{\mathbf{w}} > r_f$ exists, where r_f is the risk-free rate of return in the market. Considering this, we can now formulate a quadratic programming problem to find the portfolio with the maximum Sharpe ratio.

$$\begin{aligned} &\underset{\mathbf{w}}{max}\ \frac{\bar{\mathbf{r}}^\intercal \hat{\mathbf{w}} - r_f}{(\mathbf{w}^\intercal \mathbf{\Sigma} \mathbf{w})^{1/2}} \\ &s.t\ \mathbf{1}\,\mathbf{w} = 1 \end{aligned} \tag{7}$$

Although it possesses a polyhedral feasible region, this problem cannot be solved in its current form and must due to it's complex objective function and possible non-concavity [5]. Thus, we will transform this problem to an equivalent convex QP problem.

If χ denotes the set of feasible portfolios such that $\mathbf{1}\,\mathbf{w} = 1, \forall \mathbf{w} \in \chi$ and $\exists\ \bar{\mathbf{r}}^\intercal \hat{\mathbf{w}} > r_f$, we now show that the problem in (7) can be transformed to the following equivalent —

$$\begin{aligned} &\underset{\mathbf{y}}{min}\ \mathbf{y}^\intercal \mathbf{\Sigma} \mathbf{y} \\ &s.t\ (\bar{r} - r_f I)^\intercal \mathbf{y} = 1 \\ &(\mathbf{y}, \kappa) \in \chi^+ \end{aligned} \tag{8}$$

where $\chi^+ := \left\{ \mathbf{w} \in \mathbb{R}^n, \kappa \in \mathbb{R} \mid \kappa > 0, \frac{\mathbf{w}}{\kappa} \in \chi \right\} \cup\ (0,0)$

From our assumption $\exists\ \bar{\mathbf{r}}^\intercal \hat{\mathbf{w}} > r_f$ we can limit our search to only those $\mathbf{w}$ for which $(\bar{\mathbf{r}} - r_f \mathbf{I})^\intercal \mathbf{w} > 0$, where $\mathbf{I}$ is a conformable identity matrix. We can now make the following change of variables:

$$\kappa = \frac{1}{(\bar{\mathbf{r}} - r_f \mathbf{I})^\intercal \mathbf{w}}$$
$$\mathbf{y} = \kappa \mathbf{w}$$

Then $\sqrt{\mathbf{w}^\intercal \mathbf{\Sigma} \mathbf{w}} = (1/\kappa)\sqrt{\mathbf{y}^T \mathbf{\Sigma} \mathbf{y}}$ and the objective function of the original problem can be rewritten as $1/(\sqrt{\mathbf{y}^\intercal \Sigma \mathbf{y}})$ using the new variables. We can also see that,

$$(\bar{\mathbf{r}} - r_f \mathbf{I})^\intercal \mathbf{w} > 0, x \in \chi \iff \kappa > 0,\ \frac{\mathbf{y}}{\kappa} \in \chi$$

and,

$$\kappa = \frac{1}{(\bar{\mathbf{r}} - r_f \mathbf{I})^\intercal \mathbf{w}} \iff (\bar{\mathbf{r}} - r_f \mathbf{I})\mathbf{w} = 1$$

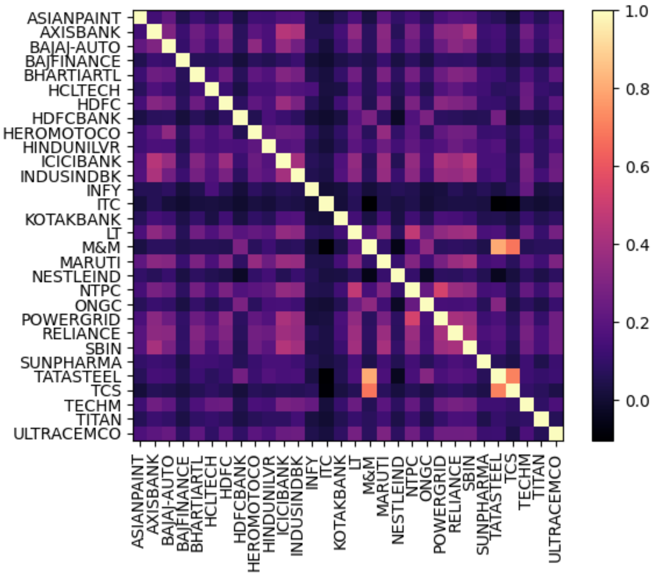
Since $(\bar{\mathbf{r}} - r_f \mathbf{I})\mathbf{w} = 1$ rules out $(0,0)$ as a solution, replacing $\kappa > 0, (y, \kappa) \in \chi$ with $(y, \kappa) \in \chi^+$ does not affect the solutions – it just makes the feasible set a closed set. Our original QP problem is now transformed to one that is solvable by known convex quadratic programming methods [5].

E. Results

To implement the Markowitz model to our asset universe we must first construct the annualised expected returns matrix $\bar{\mathbf{r}}$ using equations (1) and (2) identical to what was done in section II-F. As stated in section III-B we use the equation (5) to transform the sample co-variance $\mathbf{S}$ to its corresponding Ledoit-Wolf shrinkage estimation $\mathbf{\Sigma}$ which is used in further steps of the model. Figure 6 represents a heatmap representation of the Ledoit-Wolf shrinkage estimate for the assets under consideration.

Once we have computed our annualised expected returns matrix $\bar{\mathbf{r}}$ and co-variance matrix $\mathbf{\Sigma}$ we model the task of solving the task of portfolio optimisation in our hands as convex quadratic programming problems. To find the the optimal portfolio with the minimum volatility for our asset universe we solve the optimisation problem (6) laid out in section III-C. Similarly to find the optimal portfolio with the maximum Sharpe ratio for our asset universe we solve the optimisation problem (8) from section III-D. Finally, Figure 7 helps us understand the change in relative value of the optimal portfolios produced by the Markowitz model against the benchmark market index over the investment period.

Fig. 6. Heatmap for Ledoit-Wold shrinkage co-variance matrix



IV. Black-Litterman

A. Overview

The Markowitz model we previously analysed is a fundamental part of Modern Portfolio Theory, yet it has its own share of criticisms. It is very common for Markowitz model outputs to be highly concentrated in a few assets while assigning zero-allocations to all others. This issue is compounded by the fact that the Markowitz model is highly input-sensitive, a slight modification in the input parameters can lead to a huge change in model output. This makes the output of the model seem very uncertain and unintuitive in nature. Lastly, due to the model's high dependence on inputs, the extreme values are often prone to estimation error maximisation. These well-documented issues form an important set of reasons why many portfolio managers avoid leveraging the Markowitz model for optimizing their investments. This is where the Black-Litterman model by Fischer Black and Robert Litterman comes in. The Black-Litterman model aims to use a Bayesian approach to incorporate investor views surrounding asset returns in a prior return estimate to generate a mixed return of asset returns. The model also allows investors to input their confidence level for the subjective views being considered. The resulting return estimate is designed to return intuitive and sensible portfolio weights [6]. The rest of the section assumes the investor has N assets under consideration with a total of K subjective views.

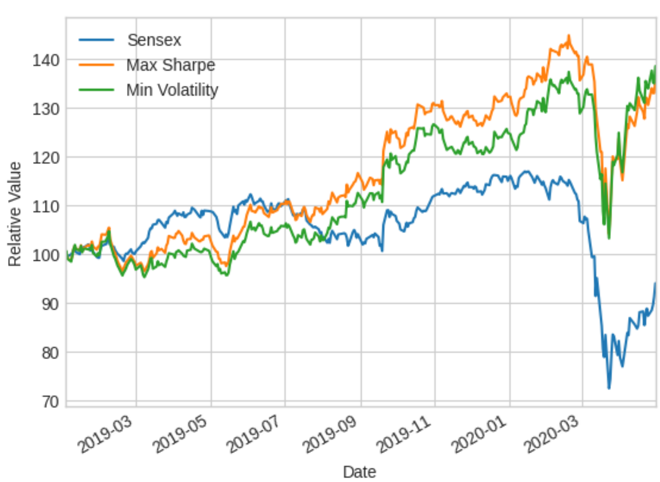


Fig. 7. Relative value of optimal Markowitz model portfolios against benchmark market index

B. Implied Equilibrium Return Vector

It has been discussed earlier that the expected returns vector forms the most important input in mean-variance optimisation methods. Due to input-sensitivity of these methods, a small change in the expected returns can result in a huge change in the optimal portfolio allocation [7]. To deal with this issue, previous literature explores several alternatives such as equal "mean" returns for all assets, risk-adjusted equal mean returns, etc. but all most possess the same issues of producing extreme portfolio allocations. To tackle this issue the Black-Litterman model uses a neutral starting point of "equilibrium" returns — those returns that just clear the market. Reverse optimization methods involving the market capitalization weights of assets $\mathbf{w_{mkt}}$, the co-variance matrix $\mathbf{\Sigma}$ and the risk aversion coefficient λ. The risk-aversion factor is a scaling factor that characterizes the rate at which an investor will forego expected return for less variance.

$$\lambda = \frac{\overline{r_m} - r_f}{\sigma^2} \tag{9}$$

where $\overline{r_m}$ is the expected benchmark returns, r_f is the risk-free rate of investment and σ^2 represents the sample variance of benchmark returns.

Finally the implied equilibrium return vector Π can be computed

$$\mathbf{\Pi} = \lambda \mathbf{\Sigma}\, \mathbf{w_{mkt}} \tag{10}$$

C. Investor Views

One of the novel features of the Black-Litterman model is its ability to incorporate investor views with prior estimates. This provides investment managers with a logical and intuitive way to influence the implied excess return vector (Π) with their own subjective views. These views may be of two kinds —

- Absolute views [eg. asset x will provide a return of 7% (with confidence = 30%)]
- Relative views [eg. asset y will outperform asset z by 2% (with confidence = 60%)]

These investor inputs are included in the model via a views matrix $\mathbf{Q}$ of dimensions $K \times 1$, where K denotes the total number of investor views. Since the views matrix $\mathbf{Q}$ contains subjective inputs, the uncertainty gives rise to a random, normally-distributed Error Term Matrix ε with $\overline{\varepsilon} = 0$

and co-variance matrix $\mathbf{\Omega}$. Unless the model is input with 100% confidence for views input, all error terms are non-zero quantities. Instead of the error terms ε the variance of each error term ω is directly involved in the Black-Litterman formula through the co-variance matrix $\mathbf{\Omega}$. The off-diagonal elements of $k \times k$ dimensional matrix $\mathbf{\Omega}$ are zero, to enforce the assumption that all views are independent of one another while the variance terms express the uncertainty of the views.

$$\mathbf{Q} + \varepsilon = \begin{bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_K \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_K \end{bmatrix} \tag{11}$$

$$\mathbf{\Omega} = \begin{bmatrix} \omega_1 & 0 & \dots & 0 \\ 0 & \omega_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \omega_K \end{bmatrix} \tag{12}$$

Now that we have successfully expressed our value-based views and certainty in each one, there needs to be a way to link these views to the respective asset returns, so that a mixed estimate can be computed. This is done with the help of matrix $\mathbf{P}$. With N assets under consideration and K views, each view contributes to a $1 \times N$ row in the matrix P of dimension $K \times N$.

$$\mathbf{P} = \begin{bmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,N} \\ p_{2,1} & p_{2,2} & \cdots & p_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ p_{K,1} & p_{K,2} & \cdots & p_{K,N} \end{bmatrix} \tag{13}$$

There are different methods of populating $\mathbf{P}$ such as those based on assign percentages to assets [8], an equal weightage system [9], etc. In this article however, we avoid these methods in favor of one based on the market capitalization of assets [6]. Accordingly, the relative weightage of an asset in a view is proportional to the market capitalization of the particular asset with respect to the aggregate market capitalization of assets falling in the same category (i.e total market cap of either out-performing or under-performing assets). After populating $\mathbf{P}$ one can compute the variance of a particular view as $\mathbf{p_i}\Sigma\mathbf{p_i}^\intercal$, where $\mathbf{p_i}$ is the $1 \times N$ size matrix which is a part of the i^{th} subjective investor view.

D. Posterior Expected Returns

We already have everything required to compute the posterior expected returns $\overline{\mathbf{R}}$, except the scalar τ. While the value of τ should be more or less inversely proportional to the relative weight given to Π, there is a dearth of guidance about the estimation of τ across Black-Litterman literature. Accordingly there are several different methods of estimating the value scalar τ [10]. For th purposes of this article, we approximate $\tau = 0.05$, a sensible and conventional value.

$$\overline{\mathbf{R}} = [(\tau\mathbf{\Sigma})^{-1} + \mathbf{P}^\intercal\mathbf{\Omega}^{-1}\mathbf{P}]^{-1}\ [(\tau\mathbf{\Sigma})^{-1}\mathbf{\Pi} + \mathbf{P}^\intercal\mathbf{\Omega}^{-1}\mathbf{Q}] \tag{14}$$

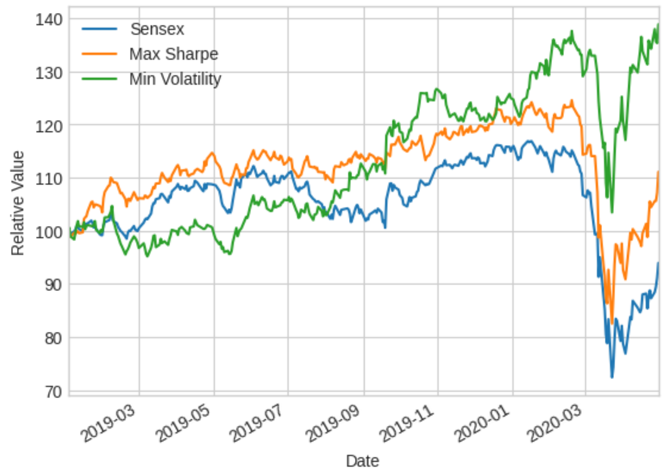


Fig. 8. Relative value of optimal Black-Litterman portfolios against benchmark market index

E. Results

To implement the Black-Littrman model on our asset universe we first need to calculate the portfolio's risk aversion factor using equation (9). To estimate $\overline{r_m}$ we apply equations (1) and (2) to the historical daily closing levels of Sensex over the same period of analysis as our asset universe. The co-variance matrix $\mathbf{\Sigma}$ represents the Ledoit-Wolf shrinkage estimation of the co-variance matrix computed using the equation (5). Finally, we use historical data sourced from the official BSE website to find the FF market capitalization of each asset under consideration. We are now ready to compute $\mathbf{\Pi}$, the implied equilibrium return vector using equation (10).

The nature of investors view contribute significantly to the performance of the Black-Litterman model. Since views are subjective to the investor, we have based our views upon the market outlooks as per the date of investment of funds (i.e Jan 1, 2019). We supply the model with the following views —

- NBFCs will fall by 5% [50% confidence]
- RELIANCE will surge by 10% [70% confidence]
- Auto sector will fall by 5% [30% confidence]
- IT sector will surge by 10% [60% confidence]

Based on these views and the details outlined in section IV-C, we can now construct our views vector $\mathbf{Q}$ as in (11), the picking matrix $\mathbf{P}$ as in (13 and the confidence matrix $\mathbf{\Omega}$ as in (12).

At this point we have all the components to estimate the posterior expected returns $\overline{R}$ as given by the Black-Litterman model equation (14). The effect of the investor views is demonstrated in Table IV which contrasts the prior returns of $\mathbf{\Pi}$ with posterior returns $\overline{\mathbf{R}}$ obtained using the Black Litterman model.

To obtain the optimal portfolio compositions, we solve the optimisation problems (6) and (8) just as done with the Markowitz model, but we use the expected returns $\mathbf{R}$ and $\mathbf{\Sigma}$ as defined by the Black-Litterman model instead. (6 helps us find the Black-Litterman minimum volatility portfolio composition while (8) helps us find the portfolio composition for the Black-Litterman portfolio with maximum Sharpe ratio. Table V lists the composition of these Black-Litterman optimal portfolios.

Figure 8 helps us measure the change in value of the optimal Black-Litterman portfolios over the investment period. It also compares these changes against change in benchmark market levels, as depicted by the BSE Sensex.

V. Conclusion

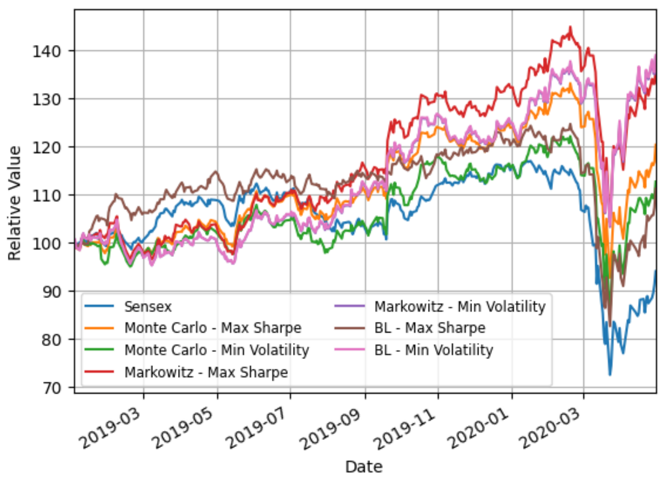


Fig. 9. Relative value of all optimal portfolios against benchmark market index

Over the course of the article we construct six different *optimal portfolios* based on three different portfolio optimisation approaches in a bid to tackle the Asset Allocation problem. Figure 9 consolidates the performance of all these *optimal portfolios* in a single graph. Table I gives the Compound Annual Growth Rate (CAGR) and Maximum Drawdown (MDD) of all portfolios under consideration over the investment period.

We can analyse the consolidated performance graph to infer the impact of the COVID-19 pandemic on typical portfolios constructed using these techniques. An important observation in support of general resiliency of the portfolio optimisation models is the fact that all portfolios end in a profit while the benchmark Sensex falls into the red. Among other factors, the Sensex depends on free-float market capitalization of its components, hence negative Sensex returns against net positive portfolio returns suggest that the portfolio optimization models are successful in diversifying risk across the portfolio to protect it from losses during tumultuous times such as these.

Another notable observation we can infer from empirical analysis of Figure 9 shows that all portfolios take losses as the markets succumbed under pandemic conditions, but (a) these losses were softened by the risk management measures and (b) the models surge significantly during normal market cycles, which allow them to profitable in the long run even after incurring short periods of significant losses. This observation is supported by the MDD values in Table I, while MDD values vary only by 5-6 percentage points indicating that all portfolios have incurred comparable amounts of losses, the CAGR values show that non-benchmark portfolios still manage to book profits owing to the gains early on before the pandemic.

In section IV-C we said that the investor views can significantly affect the portfolio performance. The returns as seen in Table I fortify this view — since we adjusted our investor views according to the general market outlook (which could never predict the pandemic situation) at the time of investment, the Black-Litterman Max Sharpe portfolio, which was tilted in favor of our views gives us very little return as compared to the Black-Litterman Min Volatility portfolio geared towards minimising risk through proper diversification, even if that required foregoing some returns. In a pandemic situation, the Max Sharpe portfolio performs poorly as the views were not adjusted to the pandemic, but the portfolio that minimised volatility was highly resilient towards any black swan events in the future.

TABLE I
CAGR and MDD of Portfolios Under Consideration

Portfolio	CAGR	MDD
Sensex (Benchmark)	-4.584%	38.070%
Monte Carlo - Max Sharpe	14.851%	30.402%
Monte Carlo - Min Volatility	9.308%	29.645%
Markowitz - Max Sharpe	25.735%	28.137%
Markowitz - Min Volatility	27.692%	24.863%
Black Litterman - Max Sharpe	8.224%	33.775%
Black Litterman - Min Volatility	27.975%	24.831%

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TABLE II
COMPOSITION OF OPTIMAL MONTE-CARLO SIMULATED PORTFOLIOS

Ticker	Max Sharpe	Min Volatility
ASIANPAINT	7.34%	1.97%
AXISBANK	0.86%	7.92%
BAJAJ-AUTO	3.81%	4.73%
BAJFINANCE	6.12%	0.61%
BHARTIARTL	0.46%	6.39%
HCLTECH	2.09%	2.02%
HDFC	3.61%	2.1%
HDFCBANK	4.04%	0.31%
HEROMOTOCO	6.36%	6.42%
HINDUNILVR	0.14%	8.36%
ICICIBANK	4.36%	2.45%
INDUSINDBK	0.45%	0.24%
INFY	8.04%	3.5%
ITC	5.37%	0.53%
KOTAKBANK	2.49%	1.52%
LT	0.53%	2.61%
M&M	2.68%	0.78%
MARUTI	0.22%	4.59%
NESTLEIND	6.08%	5.2%
NTPC	1.52%	4.61%
ONGC	5.44%	1.68%
POWERGRID	3.26%	4.83%
RELIANCE	1.56%	6.8%
SBIN	5.65%	1.73%
SUNPHARMA	4.9%	6.42%
TATASTEEL	0.71%	1.1%
TCS	6.35%	0.88%
TECHM	2.48%	2.58%
TITAN	0.17%	0.04%
ULTRACEMCO	2.92%	7.1%

TABLE III
COMPOSITION OF OPTIMAL MARKOWITZ PORTFOLIOS

Ticker	Max Sharpe	Min Volatility
ASIANPAINT	25.82%	10.34%
AXISBANK	3.05%	0.0%
BAJAJ-AUTO	0.0%	6.87%
BAJFINANCE	6.99%	0.26%
BHARTIARTL	0.0%	3.78%
HCLTECH	0.67%	0.0%
HDFC	8.12%	1.36%
HDFCBANK	4.34%	0.0%
HEROMOTOCO	0.0%	6.45%
HINDUNILVR	0.0%	8.03%
ICICIBANK	0.0%	0.0%
INDUSINDBK	0.0%	0.0%
INFY	14.56%	25.73%
ITC	4.18%	0.0%
KOTAKBANK	3.91%	0.0%
LT	0.0%	0.0%
M&M	4.78%	0.0%
MARUTI	0.0%	1.04%
NESTLEIND	9.68%	13.49%
NTPC	0.0%	4.79%
ONGC	1.9%	0.37%
POWERGRID	0.0%	9.92%
RELIANCE	0.0%	0.0%
SBIN	1.21%	0.0%
SUNPHARMA	8.08%	4.92%
TATASTEEL	0.0%	0.0%
TCS	2.73%	0.04%
TECHM	0.0%	0.0%
TITAN	0.0%	0.02%
ULTRACEMCO	0.0%	2.58%

TABLE IV
COMPARISON OF PRIOR AND POSTERIOR EXPECTED ASSET RETURNS

Ticker	Prior Return	Posterior Return
ASIANPAINT	8.83%	6.82%
AXISBANK	12.57%	8.22%
BAJAJ-AUTO	9.91%	3.94%
BAJFINANCE	12.89%	2.16%
BHARTIARTL	10.95%	7.83%
HCLTECH	12.28%	9.85%
HDFC	11.47%	7.9%
HDFCBANK	30.55%	24.93%
HEROMOTOCO	9.74%	3.42%
HINDUNILVR	10.8%	8.37%
ICICIBANK	13.22%	8.4%
INDUSINDBK	12.52%	7.96%
INFY	7.85%	8.89%
ITC	46.0%	47.44%
KOTAKBANK	16.33%	12.12%
LT	13.12%	9.0%
M&M	36.22%	16.0%
MARUTI	11.26%	4.55%
NESTLEIND	7.79%	6.85%
NTPC	10.98%	8.3%
ONGC	15.52%	11.01%
POWERGRID	10.38%	7.88%
RELIANCE	12.65%	9.49%
SBIN	12.22%	7.75%
SUNPHARMA	9.71%	8.1%
TATASTEEL	29.66%	14.64%
TCS	30.12%	15.34%
TECHM	10.76%	9.36%
TITAN	13.96%	10.03%
ULTRACEMCO	10.44%	7.1%

TABLE V
COMPOSITION OF OPTIMAL BLACK-LITTERMAN PORTFOLIOS

Ticker	Max Sharpe	Min Volatility
ASIANPAINT	0.0%	10.37%
AXISBANK	0.0%	0.0%
BAJAJ-AUTO	0.0%	6.76%
BAJFINANCE	0.0%	0.32%
BHARTIARTL	0.0%	3.78%
HCLTECH	2.17%	0.0%
HDFC	0.0%	1.36%
HDFCBANK	12.9%	0.0%
HEROMOTOCO	0.0%	6.43%
HINDUNILVR	3.62%	8.0%
ICICIBANK	0.0%	0.0%
INDUSINDBK	0.0%	0.0%
INFY	37.65%	25.82%
ITC	6.12%	0.04%
KOTAKBANK	4.49%	0.0%
LT	0.0%	0.0%
M&M	0.0%	0.0%
MARUTI	0.0%	0.86%
NESTLEIND	0.0%	13.58%
NTPC	0.0%	4.58%
ONGC	0.01%	0.45%
POWERGRID	0.0%	9.91%
RELIANCE	16.85%	0.0%
SBIN	0.0%	0.0%
SUNPHARMA	0.96%	4.98%
TATASTEEL	0.0%	0.0%
TCS	7.62%	0.1%
TECHM	6.25%	0.0%
TITAN	1.34%	0.08%
ULTRACEMCO	0.0%	2.58%