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Group Decision Making and Social Learning

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***Abstract*—Social learning or learning from actions of others is a key focus of microeconomics; it studies how individuals aggregate information in social networks. Following the seminal work of Aumann, a large literature studies the strategic interaction of agents in a social network, where they receive private information and act based upon that information while also observing the actions of each other. These observations are in turn informative about other agents' private signals; information that can be then used in making future decisions. By the same token, agents engage in group discussions to benefit from private information of others and come up with better decisions that aggregate every body's information as efficiently as possible.**

We begin by considering the decision problems of a Bayesian agent in a social learning scenario. As the Bayesian agent attempts to infer the true state of the world from her sequence of private signals and observations of actions of others, her decision problems at every epoch can be cast recursively; curbing some of the complexities of the decision scenario, but only to a limited extent. In a group decision scenario, the initial private signals of the agents constitute a state space and the ultimate goal of the agents is get informed about the private signals of each other. The Bayesian agent is initially informed of only her own signal; however, as her history of interactions with other group members becomes enriched, her knowledge of the possible private signals that others may have observed also gets refined; thus enabling her to make better decisions.

Bayesian calculations in the social learning setting are notoriously difficult. Successive applications of Bayes rule to the entire history of past observations lead to forebodingly complex inferences: due to lack of knowledge about the global network structure, and unavailability of private observations, as well as third-party interactions that precede every decision. This has given rise to a large literature on non-Bayesian social learning that suggest the use of decision-making heuristics, not only for mathematical tractability but also on the grounds that they are better descriptors of the bounded rational behaviors that are observed in reality: people rely on some simplifications of their environment to be able to analyze it faster and more reliably. These heuristics may be fitted for simple one-shot decision making but they are often not suited for handling the complexities of a public discussion or for making group decisions. As a result, many inefficiencies arise in the outcome of such discussions and their roots can be traced to the underlying heuristic decision-making mechanisms. These and other issues relating to heuristic decision making constitute the main focus of this paper.

***Index Terms*—Opinion Dynamics, Social Learning, Bayesian Learning, Rational Learning, Observational Learning, Statistical Learning, Distributed Learning, Distributed Hypothesis Testing, Distributed Detection**

I. Introduction

Individuals exchange opinions with their peers in order to learn from their knowledge and experiences; these interactions occur through a variety of media which we collectively refer to as social networks. Often, signals that people receive about the true state of the world are tainted with noise. Therefore, they must decide how to handle and aggregate the information that they receive in order to make decisions about voting, adopting new technologies, purchasing new products, and so on. It follows that understanding which learning paradigms best predicts group behavior is important both for understanding whether information transmission is efficient, and for thinking through policy designs that rely on information dissemination. Indeed, an enhanced understanding of learning in social networks will have many important and direct implications for social and organizational policy.

James Surowiecki in his popular science book on wisdom of crowds [1], provides well-known cases for information aggregation in social networks, and argues how under the right circumstances (diversity of opinion, independence, decentralization and aggregation) groups outperform even their smartest or best informed members; see for example the essentially perfect performance of the middlemost estimate at the weight-judging competition of the 1906 West of England Fat Stock and Poultry Exhibition studied by Francis Galton in his 1907 Nature article [2], entitled "Vox Populi" (The Wisdom of Crowds), or the study of market reaction to the 1986 challenger disaster in [3], where its is pointed out that the main responsible company's (Morton Thiokol) stock was hit hardest of all, even months before the cause of the accident could be officially determined.

On the other hand, several studies point out that the evolution of people's opinions and decisions in social networks is subject to various kind of biases and inefficiencies [4], [5], [6], [7], [8]. Daniel Kahneman in his highly acclaimed work, "*Thinking, Fast and Slow*", points out that the proper way to elicit information from a group is not through a public discussion but rather confidentially collecting each person's judgment [9, Chapter 23]. Indeed, decision making among groups of individuals exhibit many singularities and important inefficiencies that lead to Kahneman's noted advice. As a team converges on a decision expressing doubts about the wisdom of the consensus choice is suppressed; subsequently teams of decision makers are afflicted with *groupthink* as they appear to reach a consensus.[1] The mechanisms of uncritical optimism, overconfidence, and the illusions of validity in group interactions also lead to *group polarization*, making the individuals more amenable toward extreme opinions [11].

Such deviations from the efficient and/or rational outcome are often attributed to the structural effects that arise in networked interactions; in particular, the predominant influence

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[1] Gar Klein proposes a famous method of *project premortem* to overcome the groupthink through an exercise: imagining that the the planned decision was failed in implementation and writing a brief report of the failure [10].

of more central agents in shaping the group decision, in spite of the fact that such influential agents do not necessarily enjoy a high quality of observations or superior knowledge; cf. persuasion bias in [12], obstructions to wisdom of crowds in [13], and data incest in [14]. Subsequently, a better understanding of social learning can also help us analyze the effect of such biases and use our insights and conclusions to improve policy designs that are aimed at implementing desirable social norms or eradicating undesirable ones, or even to come up with more efficient procedures for aggregating individual beliefs, and to understand how media sources, prominent agents, government and politicians are able to manipulate public opinion and influence spread of beliefs in society [15].

Following the seminal work of [16], a large literature studies strategic interaction of agents in a social network, where they receive private information and act based upon that information while also observing each other's actions [17], [18], [19]. These observations are in turn informative about other agents' private signals; information that can be then used in making future decisions. Such a model is a good descriptor for online reputation and polling systems such as Yelp® and TripAdvisor®, where individuals' recommendations are based on their private observations and recommendations of their friends [20, Chapter 5]. The analysis of such systems is important not only because they play a significant role in generating revenues for the businesses that are being ranked [21], but also for the purposes of designing fair rankings and accurate recommendation systems. In this line of work, two critical questions of interest are: (i) effectiveness of information transmission and information sharing/exchange through observed actions and (ii) effectiveness of decision-making using the available information. In fact, these research issues are inter-related, as the quality of decision-making depends on the quality of of information exchanged among the the agents.

The network externalities that arise in above settings are purely informational. People are therefore interacting with each other, only to learn from one another, and to improve the quality of their decisions (for example, in jury deliberations); this lack of strategic externalities is an important characteristic of the kind of human interactions that we consider in this paper [19]. In the following sections, we study two main frameworks for modeling decision making in such scenarios: Bayesian and heuristic decision making.

II. Distributed Information Structure

Consider a set of n agents that are labeled by $[n]$ and interact according to a digraph $\mathcal{G} = ([n], \mathcal{E})$.[1] The neighborhood of agent i is the set of all agents whom she observes including herself, and it is denoted by $\mathcal{N}_i = \{j \in [n]; (j,i) \in \mathcal{E}\} \cup \{i\}$; every node having a self-loop: $(i,i) \in \mathcal{E}$ for all i. We refer to the cardinality of $\mathcal{N}_i$ as the degree of node i and denote it by $deg(i)$. There is a state $\theta \in \Theta$ that is unknown to the agents and belongs to a measurable space Θ.[2] Associated with each agent i, $\mathcal{S}_i$ is a measurable set called the signal space of i, and given θ, $\mathcal{L}_i(\cdot \mid \theta)$ is a probability measure on $\mathcal{S}_i$, which is referred to as the *signal structure* of agent i. We use $\mathcal{L}(\cdot \mid \theta)$ to denote the joint distribution of the private signals of all agents at every decisions epoch, signals being independent over time and across the agents.

An agents' belief about the unknown allows her to make decisions even as the outcome is dependent on the unknown value θ. These beliefs about the unknown state are probability distributions over Θ. Even before any observations are made, every agent $i \in [n]$ holds a prior belief $\mathcal{V}_i(\cdot) \in \Delta\Theta$; this represents her subjective biases about the possible values of θ and the way they are realized. For each time instant t, let $\boldsymbol{\mathcal{M}}_{i,t}(\cdot)$ be the (random) probability distribution over Θ, representing the *opinion* or *belief* at time t of agent i about the realized value of θ. Moreover, let the associated expectation operator be $\mathbb{E}_{i,t}\{\cdot\}$, representing integration with respect to $\boldsymbol{\mathcal{M}}_{i,t}(d\theta)$. We assume that all agents share the common knowledge of signal structures $\ell_i(\cdot|\hat{\theta}), \forall \hat{\theta} \in \Theta$, their priors $\mathcal{V}_i(\cdot)$, and their corresponding sample spaces $\mathcal{S}_i$ and Θ for all $i \in [n]$.[3] In dealing with measurable spaces (Θ or $\mathcal{S}$) we equip them with sigma-finite measures, denoted by $\mathcal{G}_\theta$ and $\mathcal{G}_s$ for Θ and $\mathcal{S}$, respectively. They are needed as we consider Radon-Nikodym derivatives of beliefs and signal structures when doing Bayes updates. If a measurable space is countable, then we can take its σ-finite measure to be the counting measure, denoted by $\mathcal{K}$, and if the space has positive Lebesgue measure, then we can take its σ-finite measure to be the Lebesgue measure, denoted by Λ_k. We denote the Radon-Nikodym derivatives of beliefs and signal structures by $\boldsymbol{\mu}_{i,t}(\cdot)$ and $\ell_i(\cdot|\theta)$, respectively.

III. Bayesian Decision Making

The rational approach advocates application of Bayes rule to the entire sequence of observations successively at every step. However, such repeated applications of Bayes rule in networks become very complex, especially if the agents are unaware of the global network structure; and as they use their local data to make inferences about all possible contingencies that can lead to their observations. While some analytical properties of rational learning is deduced and studies in the literature [18],

[1]Some notations: Throughout the paper, $\mathbb{R}$ is the set of real numbers, $\mathbb{N}$ denotes the set of all natural numbers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$ a fixed integer the set of integers $\{1, 2, \ldots, n\}$ is denoted by $[n]$, while any other set is represented by a capital Greek or calligraphic letter. For a measurable set $\mathcal{X}$ we use $\Delta\mathcal{X}$ to denote the set of all probability distributions over the set $\mathcal{X}$. Furthermore, any random variable is denoted in boldface letter, vectors are represented in lowercase letters and with a bar over them, measures are denoted by upper case Greek or calligraphic Latin letters, and matrices are denoted in upper case Latin letters. For a matrix A, its spectral radius $\rho(A)$ is the largest magnitude of all its eigenvalues.

[2]For instance, Θ could be a finite set, and its finitely many elements represent the possible alternatives that are available to the members of a group, as they engage in a public discussion to deliberate on their best option.

[3]The signal structures $\mathcal{L}_i(\cdot|\hat{\theta}), \forall \hat{\theta} \in \Theta$ and the priors $\mathcal{V}_i(\cdot)$, as well as the corresponding sample spaces $\mathcal{S}_i$ and Θ are common knowledge amongst the agents for all $i \in [n]$. The assumption of common knowledge in the case of fully rational (Bayesian) agents implies that given the same observations of one another's beliefs or private signals distinct agents would make identical inferences; in the sense that starting form the same belief about the unknown θ, their updated beliefs given the same observations would be the same; in Aumann's words, rational agents cannot agree to disagree [16].

[22], [23], [24], [25], their tractable modeling and analysis remains an important problem in network economics and continues to attract attention.

Some of the earliest results addressing the problem of social learning are due to Banerjee (1992) [26], and Bikhchandani et al. (1998) [27] who consider a complete graph structure where the agent's observations are public information and also ordered in time, such that each agent has access to the observations of all the past agents. These assumptions help analyze and explain the interplay between public and private information leading to fashion, fads, herds etc. Later results by Gale and Kariv [18] relax some of these assumptions by considering the agents that make simultaneous observations of only their neighbors rather than the whole network, but the computational complexities limit the analysis to networks with only two or three agents. In more recent results, Mueller-Frank [23] provides a framework of rational learning that is analytically amenable. Mossel, Sly and Tamuz [24] analyze the problem of estimating a binary state of the world from a single initial private signal that is independent and identically distributed among the agents conditioned on the true state; and they show that by repeatedly observing each other's best estimates of the unknown as the size of the network increases with high probability Bayesian agents asymptotically learn the true state. Hence, the agents are able to combine their initial private observations and learn the truth.

In this Section we assume that the signal spaces are finite sets because as we shall see some of the calculations of each agent rely critically on her enumeration of all possible private signals that the other network agents may have observed. The ultimate goal of a Bayesian agent can be described as learning all private signals of all other agents in the network based on his local observations of her neighbors; this however, can be extremely complex, if not impossible. We begin by considering the agents in a binary world in Subsection III-A, where actions are chosen from one of the two possibilities that are thought to explain the observations. The rewards are allocated according as to whether the actions coincide with the true case or not. Hence, the Bayesian agent at each epoch aims to maximize her probability of correctly guessing the true case, given all her observations thus far.

A. Decisions of a Single Agent in a Binary World

For clarity of exposition we consider a single agent, i, who observes binary signals $\mathbf{s}_{i,t} \in \{\pm 1\}$ and takes actions $\mathbf{a}_{i,t} \in \{\pm 1\}$ at every instant $t \in \mathbb{N}_0$. She lives in a binary world were the truth θ can take one of the two values ± 1 with equal probability (cf. Fig. 1); and her actions are rewarded by $+1$ if $\mathbf{a}_{i,t} = \theta$ and are penalized by -1 otherwise. Suppose further that her probability of receiving the signal θ is $p > 0.5$; so that

$$\mathcal{P}_\theta\{\mathbf{s}_{i,t} = \theta\} = 1 - \mathcal{P}_\theta\{\mathbf{s}_{i,t} = -\theta\} = p > 0.5.$$

We assume that the agent is myopic so that at every time t she is only concerned about her immediate reward at that decision epoch. Consider the decision problems of agent i at every time

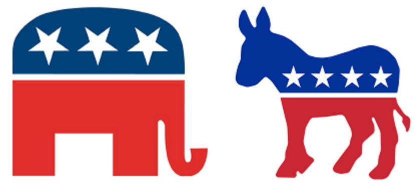

Fig. 1: Bipartisanship is an example of a binary state space.

instant $t \in \mathbb{N}_0$; to model them as Markov decision processes we define her state as the collection of all private signals that she would ever observe: $\bar{s}_\infty := (\mathbf{s}_{i,0}, \ldots, \mathbf{s}_{i,t}, \ldots) \in \mathcal{S}_i^{\mathbb{N}_0}$. Subsequently, at any time t she has only partial knowledge of her state. Let $\bar{\mathbf{s}}_t := (\mathbf{s}_{i,0}, \ldots, \mathbf{s}_{i,t})$ be the collection of all private signals that is revealed to her up until time t. Her expected reward from taking an action $\mathbf{a}_{i,t}$ is then given by

$$\begin{aligned} r(\mathbf{a}_{i,t}, \bar{\mathbf{s}}_t) &= \mathcal{P}_\theta\{\mathbf{a}_{i,t} = \theta \mid \bar{\mathbf{s}}_t\} - \mathcal{P}_\theta\{\mathbf{a}_{i,t} \neq \theta \mid \bar{\mathbf{s}}_t\} \\ &= 2\mathcal{P}_\theta\{\mathbf{a}_{i,t} = \theta \mid \bar{\mathbf{s}}_t\} - 1. \end{aligned}$$

Since both $\{\theta = +1\}$ and $\{\theta = -1\}$ are initially equally likely, the agents' optimal decisions is given by:

$$\mathbf{a}^\star_{i,t} = \begin{cases} +1, & \text{if } \mathcal{P}_{+1}\{\bar{\mathbf{s}}_t\} \geq \mathcal{P}_{-1}\{\bar{\mathbf{s}}_t\}, \\ -1, & \text{otherwise.} \end{cases} \tag{1}$$

Using the likelihood ratio statistics

$$\frac{\mathcal{P}_{+1}\{\bar{\mathbf{s}}_t\}}{\mathcal{P}_{-1}\{\bar{\mathbf{s}}_t\}} = \left(\frac{p}{1-p}\right)^{\sum_{\tau=0}^{t} \mathbf{s}_{i,\tau}},$$

we can rewrite (1) as a threshold rule in terms of the sufficient statistic $\mathbf{S}_{i,t} = \sum_{\tau=0}^{t} \mathbf{s}_{i,\tau}$ that is the running total of the observed signals:

$$\mathbf{a}^\star_{i,t} = \begin{cases} +1, & \text{if } \mathbf{S}_{i,t} \geq 0, \\ -1, & \text{otherwise.} \end{cases} \tag{2}$$

B. Two Communicating Agents

The authors in [25] have considered more sophisticated scenarios involving two agents, one of whom (called j) observes the other's (called i) actions (unidirectionally) in addition to her own sequence of private signals. They distinguish two cases: in case one the more informed agent j only observes her neighbor's penultimate action $\mathbf{a}_{i,t-1}$; in case two she observes the whole sequence of actions taken by her neighbor, $(\mathbf{a}_{i,0}, \ldots, \mathbf{a}_{i,t-1})$, and use them in her decisions. The optimal (Bayesian) decision for agent j in both cases can be derived as threshold rules on the sum $\mathbf{S}_{j,t}$; however, the respective thresholds are time-varying and more complex. In particular, if agent j in addition to her private signals also observes the last action of agent i, who only observes private signals, then the optimal action of agent j involves a time-varying threshold expressed below, while the optimal action of agent i is the same as (3):

$$\mathbf{a}^{\star}_{j,t} = \begin{cases} \operatorname{sign}(\mathbf{S}_{j,t}), & \text{if } |\mathbf{S}_{j,t}| \geq \eta^{\star}_{t}, \\ \mathbf{a}_{i,t-1}, & \text{otherwise}, \end{cases} \tag{3}$$

where

$$\eta^{\star}_{t} = \left(\log \frac{\mathcal{P}_{\theta}\{\mathbf{a}_{i,t-1} = \theta\}}{\mathcal{P}_{\theta}\{\mathbf{a}_{i,t-1} \neq \theta\}}\right) \Big/ \left(\frac{p}{1-p}\right),$$

is set to optimally balance the probability of an by agent i with the strength of private signals of agent j; thus enabling the latter to decide whether to imitate her neighbor or else disregard her neighbors' action and act based on her own signals. If j observes the entire sequence of actions taken by i the optimal threshold depend also on the length of last run that is the number of time periods in which agent i has taken the same action as her last choice, cf. [25, Proposition 14]; the authors further highlight the difficulties in the case where both agents i and j observe each other's actions. In particular, while the optimal decisions in the bidirectional case are not known, the increased interaction and the more information that is available as a result of bidirectional communication do not lead to a faster rate of learning. The slower rate of learning is caused by a process of "Bayesian groupthink", when the agents' mistakes reinforce each other, thus preventing them from taking a correct action despite the ample evidence presented by their private signals.

In general, when a rational agent observes her neighbors in a network, she should compensate for repetitions in the sources of her information: the same neighbors' actions are repeatedly observed and neighboring actions may be affected by the past actions of the agent herself; thence major challenges of Bayesian inference for social learning are due to the private signals and third party interactions that are hidden from the agent. Moreover, existence of loops in the network cause dependencies and correlations in the information received from different neighbors, which further complicates the inference task. In the next section, we present a systematic search algorithm for calculating the Bayesian decisions in a social network.

C. Bayesian Calculations for Group Decision Making

Here we consider a group decision scenario where each agent $i \in [n]$ receive a private signal $\mathbf{s}_i$ at the beginning and engages in repeated interactions afterwards: choosing actions and observing their neighbor's choices at every decision epoch.

Associated with every agent i is an action space $\mathcal{A}_i$ that represents all the choices available to her at every point of time $t \in \mathbb{N}_0$, and a utility $u_i(\cdot,\cdot) : \mathcal{A}_i \times \Theta \to \mathbb{R}$ which in expectation represents her preferences regarding lotteries with independent draws from $\mathcal{A}_i$ and/or Θ. Subsequently, at every time $t \in \mathbb{N}_0$ each agent $i \in [n]$ chooses an action $\mathbf{a}_{i,t} \in \mathcal{A}_i$ and is rewarded $u_i(\mathbf{a}_{i,t}, \theta)$. We assume that the preferences of agents across time are myopic. At every time $t \in \mathbb{N}$, agents i takes action $\mathbf{a}_{i,t}$ to maximize her expected unitility, $\mathbb{E}_{i,t}\{u_i(\mathbf{a}_{i,t}, \theta)\}$.This myopia is rooted in the underlying group decision scenario that we are modeling: the agents goal for interacting with other group members is to come up with a decision that is more informed if they were to act solely based on their own private data; hence, by observing the recommendations of their neighboring agents $\mathbf{a}_{j,t}$ they hope to augment their information with what their neighbors as well as other agents in the network know that they do not. In particular, the agent do not have the freedom to learn form consequences of their recommendations, not before committing to a choice. Expressly in the group decision scenario the agents do not learn from the realized values of the utilities of their previous recommendations (unless they commit to their choice), rather the purpose of the group discussion is augment their information by learning from recommendations of other as much as possible before committing to a choice.

Accordingly, at every epoch t, agent i observes actions of her neighbors $\mathbf{a}_{j,t-1}$ for all $j \in \mathcal{N}_i$ and chooses an action $\mathbf{a}^{\star}_{i,t} \in \mathcal{A}_i$, maximizing her expected utility from point t onward (until time N) given her Bayesian posterior $\mathcal{M}_{i,t}(\cdot)$. For example in the case of two communicating agents the action of agent one at time three $\mathbf{a}_{i,3}$ is influenced by her own private signal $\mathbf{s}_1$ as well as the neighboring action at time two; part of the difficulty of the analysis is due to the fact that the action of agent two at time two is shaped not only by the private information of agent two but also by the action of agent one at time one, cf. Fig. 2. In more general, there also unobserved third party interactions that influence the decisions of agent two but are not available to agent one (and therefore should be inferred indirectly).

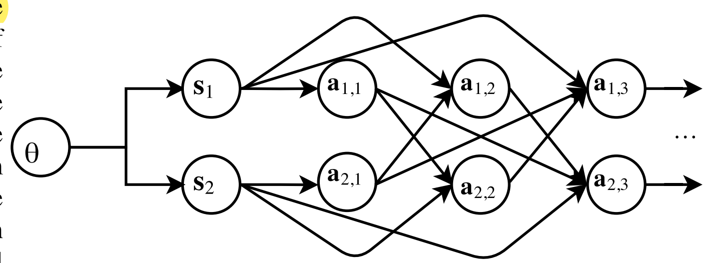

Fig. 2: The Decision Flow Diagram for Two Bayesian Agents

For each agent i, her history of observations $h_{i,t}$ is an element of the set:

$$\mathcal{H}_{i,t} = \mathcal{S}_i \times \left(\prod_{j \in \mathcal{N}_i} \mathcal{A}_j\right)^{t-1}.$$

At every time t, the expected reward to agent i given her choice of action a_i and observed history $\mathbf{h}_{i,t}$ is given by the expected reward function $r_{i,t} : \mathcal{A}_i \times \mathcal{H}_{i,t} \to \mathbb{R}$, as follows:

$$r_{i,t}(a_i, h_{i,t}) = \mathbb{E}_{\theta}\left\{u_i(a_i, \theta) \mid h_{i,t}\right\} = \int_{\Theta} u_i(a_i, \theta)\mathcal{M}_{i,t}(d\theta),$$

for all $h_{i,t} \in \mathcal{H}_{i,t}$, where $\mathcal{M}_{i,t}(d\theta)$ is the Bayesian posterior of agent i about the truth θ given the observed history $h_{i,t}$. The (myopic) optimal action of agent i at time t is then given by $\mathbf{a}^\star_{i,t} \hookleftarrow \arg\max_{a_i \in \mathcal{A}_i} r_{i,t}(a_i, \mathbf{h}_{i,t})$. Here for a set $\mathcal{A}$, we use the notation $\mathbf{a} \hookleftarrow \mathcal{A}$ to denote an arbitrary choice from the elements of $\mathcal{A}$ that is assigned to $\mathbf{a}$. Thus the agent's problem is to calculate her Bayesian posterior belief $\mathcal{M}_{i,t}(\cdot)$, given her history of past observations: $\mathbf{h}_{i,t} := \{\mathbf{s}_i, \mathbf{a}^\star_{j,\tau}, j \in \mathcal{N}_i, \tau \in [t-1]\}$.

Building on the prior works [28], we now describe in detail the Bayesian calculations that take place among the group member. To calculate her Bayesian posterior, each agent keeps track of a list of the possible combinations of private signals of all the other agents, and at each iteration after simulating the network at every possible combination of the private signals, she crosses out the entries in her list that are inconsistent with the actions that she has observed from her neighbors up to that point. In this scheme, the agent not only needs to keep track of the list of private signals that are consistent with her observations, but also to evaluate the consistency of each of these signal profiles she needs to consider what other agents regard as consistent with their own observations under the particular set of initial signals. The latter consideration enables the decision maker to calculate actions of other agents under any circumstances that arise at a fixed profile of initial signals, as she tries to evaluate the feasibility of that particular signal profile. Hence, the list of private signals can be regarded as the information set representing the current understanding of the agent about her environment and the way additional observations are informative is by trimming the current information set and reducing the ambiguity in the set of initial signals that have caused the agent's history of past observations.

To proceed, let $\overline{s} = (s_1, \ldots, s_n) \in \mathcal{S}_1 \times \ldots \times \mathcal{S}_n$ be a typical profile of initial signals observed by each agent across the network, and denote the set of all private signal profiles that agent i regards as feasible at time t, i.e. her information set at time t, by $\mathcal{I}_{i,t} \subset \mathcal{S}_1 \times \ldots \times \mathcal{S}_n$. Starting form $\mathcal{I}_{i,0} = \mathcal{S}^n$, at every decision epoch agent i removes those signal profiles in $\mathcal{I}_{i,t-1}$ that are not consistent with her history of observations $\mathbf{h}_{i,t}$ and comes up with a trimmed set of signal profiles $\mathcal{I}_{i,t} \subset \mathcal{I}_{i,t-1}$ to form her Bayesian posterior belief and make her decision at time t. As mentioned above we also need to consider the set of feasible signals for other agents $j \in [n]$ and subject to a particular profile of private signals $\overline{s}$. Given the profile of private signals $\overline{s}$, let $\mathcal{I}_{k,t}(\overline{s})$ be the set of signal profiles that agent k regards as feasible at time t under the assumption that the initial private signals are as prescribed by $\overline{s}$.

Each set of feasible signals $\mathcal{I}_{i,t}$ or $\mathcal{I}_{i,t}(\overline{s})$ can be mapped to a Bayesian posterior for agent i at time t as follows:

$$\mathcal{M}_{i,t}(d\theta) = \frac{\sum_{\overline{s} \in \mathcal{I}_{i,t}} \mathcal{L}(\overline{s}|\theta)\mathcal{V}(d\theta)}{\int_\Theta \sum_{\overline{s} \in \mathcal{I}_{i,t}} \mathcal{L}(\overline{s}|\theta)\mathcal{V}(d\theta)}, \tag{4}$$

which in turn enables the agent to choose an optimal (myopic) action given her observations:

$$a^\star_{i,t}(\overline{s}) \hookleftarrow \arg\max_{a_i \in \mathcal{A}_i} \int_\Theta u_i(a_i, \theta)\mathcal{M}_{i,t}(d\theta). \tag{5}$$

In this sense, the Bayesian posterior is a sufficient statistic for the history of observations and unlike the observation history, it does not grow in dimension.

We can thus simulate the actions of every other agent given what they regards as being a feasible set of initial signal profiles. Let $\overline{\mathcal{A}}_{j,t}(\overline{s}) = \mathcal{A}^\star_{j,1} \times \ldots \times \mathcal{A}^\star_{j,t}$ be the entire set of feasible trajectories calculated for agent j under the assumption that the initial private signals are prescribed by $\overline{s} = (s_1, \ldots, s_n)$, i.e.

$$\mathcal{A}^\star_{j,\tau} = \arg\max_{a_j \in \mathcal{A}_j} \frac{\sum_{\overline{s}' \in \mathcal{I}_{i,\tau}(\overline{s})} \mathcal{L}(\overline{s}'|\theta)\mathcal{V}(d\theta)}{\int_\Theta \sum_{\overline{s}' \in \mathcal{I}_{i,t}(\overline{s})} \mathcal{L}(\overline{s}'|\theta)\mathcal{V}(d\theta)}, \forall \tau \in [t].$$

Given $\mathcal{A}^\star_{j,t}$ for all $\overline{s} \in \mathcal{I}_{i,t-1}$ and every $j \in \mathcal{N}_i$, the agent can reject any $\overline{s}$ for which the observed neighboring action $\mathbf{a}_{j,t}$ for some $j \in \mathcal{N}(i)$ does not belong to the set of simulated feasible actions $\mathcal{A}^\star_{j,t}$.

For example at time 1 agent i learns her private signal $\mathbf{s}_i$, this reduces her list of feasible signals from $\mathcal{I}_{i,0} = \prod_{k \in [n]} \mathcal{S}_k$ to $\mathcal{I}_{i,1} = \{\mathbf{s}_i\} \times \prod_{k \in [n] \setminus \{i\}} \mathcal{S}_k$. Subsequently, her Bayesian posterior at time 1 is given by:

$$\mathcal{M}_{i,1}(d\theta) = \frac{\sum_{\overline{s} \in \mathcal{I}_{i,1}} \mathcal{L}(\overline{s}|\theta)\mathcal{V}(d\theta)}{\int_\Theta \sum_{\overline{s} \in \mathcal{I}_{i,1}} \mathcal{L}(\overline{s}|\theta)\mathcal{V}(d\theta)} = \frac{\ell_i(\mathbf{s}_i|\theta)\mathcal{V}(d\theta)}{\int_\Theta \ell_i(\mathbf{s}_i|\theta)\mathcal{V}(d\theta)}$$

and her optimal action (recommendation) at time 1 is as follows:

$$\mathbf{a}_{i,1} \hookleftarrow \arg\max_{a_i \in \mathcal{A}_i} \frac{\int_\Theta u_i(a_i,\theta)\ell_i(\mathbf{s}_i|\theta)\mathcal{V}(d\theta)}{\int_\Theta \ell_i(\overline{s}|\theta)\mathcal{V}(d\theta)}.$$

At time 2 having observed her neighbor's actions at time 1 agent i learns something about the private signals of each of her neighbors $\{\mathbf{s}_j, j \in \mathcal{N}_i\}$. In particular, from her observation of $\{\mathbf{a}_{j,0}, j \in \mathcal{N}_i\}$ she infers that $\mathbf{s}_j$ for each $j \in \mathcal{N}_i$ should necessarily satisfy:

$$\mathbf{s}_j \in \left\{ s_j \in \mathcal{S}_j : \mathbf{a}_{j,0} \in \arg\max_{a_j \in \mathcal{A}_j} \frac{\int_\Theta u_j(a_j,\theta)\ell_j(\mathbf{s}_j|\theta)\mathcal{V}(d\theta)}{\int_\Theta \ell_i(\overline{s}|\theta)\mathcal{V}(d\theta)} \right\}.$$

Subsequently, she crosses out any signal profile $\overline{s} \in \mathcal{I}_{i,1}$ for which s_j does not satisfy

$$\mathbf{a}_{j,1} \in \arg\max_{a_j \in \mathcal{A}_j} \frac{\int_\Theta u_j(a_j,\theta)\ell_j(s_j|\theta)\mathcal{V}(d\theta)}{\int_\Theta \ell_i(\overline{s}|\theta)\mathcal{V}(d\theta)},$$

thus pruning $\mathcal{I}_{i,1}$ into the smaller set $\mathcal{I}_{i,2}$. She then updates her Bayesian posterior $\mathcal{M}_{i,2}$ and changes her recommendation $\mathbf{a}_{i,2}$ according to (4) and (5), respectively. At time 3 the agent observes her neighbors recommendations $\{\mathbf{a}_{j,2}, j \in \mathcal{N}_i\}$ for a second time. The second interaction informs her about what actions her neighbor's neighbors have may have taken at time 1 and in turn what private signals they have observed at time 0; in addition, she also refines what she has already learned about private observations of her neighbors in $\mathcal{N}(i)$ based on their actions at time one, $\{\mathbf{a}_{j,1}, j \in \mathcal{N}(i)\}$, cf. Fig. 3.

To this end, she calculates the time one actions of all of the agents in $\mathcal{N}(\mathcal{N}(i))$ for each of the signal profiles belonging to $\mathcal{I}_{i,2}$ and use the result to calculate the time two actions of all her neighbors for each $\overline{s} \in \mathcal{I}_{i,2}$. Any $\overline{s}$ for which the

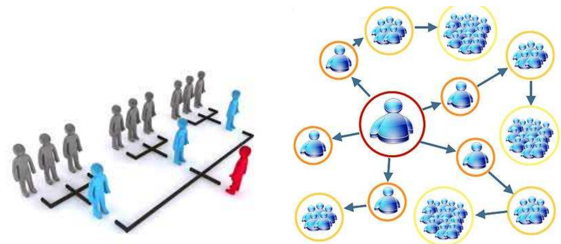

Fig. 3: Learning about the private signals of neighbors, neighbors of neighbors, and so on.

calculated time 2 action of some neighbor $j \in \mathcal{N}(i)$ does not agree with with the observed action $\mathbf{a}_{j,2}$ is subsequently removed from $\mathcal{I}_{i,2}$ and the updated list $\mathcal{I}_{i,3}$ is thus obtained. A similar set of calculations is repeated at time four: for every signal profile in $\mathcal{I}_{i,3}$ that have survived the pruning process up until $t = 4$, the agent starts by calculating the actions of agents in $\mathcal{N}(\mathcal{N}(\mathcal{N}(i)))$ at time one (as determined by the their private signals fixed in $\overline{s}$). These actions along with the rest of the private signal in turn determine the choices of the agents in $\mathcal{N}(\mathcal{N}(i))$ at time two as well those of $\mathcal{N}(i)$ at time three; subsequently, the agent can compare the latter calculated actions with her observation of the actions of her neighbors at time three and eliminate the signal profiles for which there is a mismatch.

These calculations of the Bayesian agent can also be characterized in the framework of a partially observed Markov decision processes, where the state space is $\mathcal{S}^n$: the space of all possible private signals. The decision maker only has access to partial observations, a deterministic function of the state, which is this case is the actions of her neighbors. The partially observed problem and its relations to the decentralized and team decision problems have been the subject of major classical contributions [29], [30]; in particular, the partially observed problem is known to be PSPACE-hard in the worst case [31, Theorem 6].

IV. Heuristic Decision Making

On the one hand, the properties of rational learning models are difficult to analyze beyond some simple asymptotic facts such as convergence. On the other hand, these models make unrealistic assumptions about the complexity and amount of computations that agents perform before committing to a decision. To avoid these shortcomings, an alternative "*non-Bayesian*" approach relies on simple and intuitive heuristics that are descriptive of how agents aggregate the reports of their neighbors before coming up with a decision.

The archetype of such results is the seminal work of [32] which suggests agents update their opinions to a convex combination of their neighbors' beliefs and reach a consensus. The authors [13] and [12] study many properties of this model, including the nature of limiting beliefs and inefficiencies that arise as a result of bounded rational behavior. Ref. [33] considers other variations of this model, where agents in addition in addition to observing the neighboring beliefs also receive private signals about the unknown state. More recent results aim at developing the behavioral and axiomatic foundations of social inferences using non-Bayesian update rules [34], [35], [36], [37].

A dual process theory for the psychology of mind and its operation identifies two systems of thinking [38]: one that is fast, intuitive, non-deliberative, habitual and automatic (system one); and a second one that is slow, attentive, effortful, deliberative, and conscious (system two).[1] Major advances in behavioral economics are due to incorporation of this dual process theory and the subsequent models of bounded rationality [42]. Reliance on heuristics for decision making is a distinctive feature of system one that avoids the computational burdens of a rational evaluation; system two on the other hand, is bound to deliberate on the options based on the available information before making recommendations. The interplay between these two systems and how they shape the individual decisions is of paramount importance [43].

As the agent experiences with her environment her initial response would engage her system two: she rationally evaluates the reports of her neighbors and use them to make a decision. However, after her initial experience and by engaging in repeated interactions with other group members her system one takes over her decision processes, implementing a heuristic that imitates her (rational/Bayesian) inferences from her initial experience; hence avoiding the burden of additional cognitive processing in the ensuing interactions with her neighbors.

Given her initial signal $\mathbf{s}_i$, agent i forms an initial Bayesian opinion $\boldsymbol{\mathcal{M}}_{i,0}(\cdot)$ about the value of unknown θ and chooses her action $\mathbf{a}_{i,0} \hookleftarrow \arg\max_{a_i \in \mathcal{A}_i} \int_{\Theta} u_i(a_i, \hat{\theta}) \boldsymbol{\mathcal{M}}_{i,0}(d\theta)$, maximizing her expected reward. Not being notified of the actual realized value for $u_i(\mathbf{a}_{i,0}, \theta)$, she then observes the actions that her neighbors have taken. Given her extended set of observations $\{\mathbf{a}_{j,0}, j \in \mathcal{N}_i\}$ at time $t = 1$, she refines her opinion into $\boldsymbol{\mathcal{M}}_{i,1}(\cdot)$ and makes a second, and possibly different, move $\mathbf{a}_{i,1}$ according to:

$$\mathbf{a}_{i,1} \hookleftarrow \arg\max_{a_i \in \mathcal{A}_i} \int_{\Theta} u_i(a_i, \theta) \boldsymbol{\mathcal{M}}_{i,1}(d\theta),$$

maximizing her expected pay off conditional on everything that she has observed thus far; i.e.

$$f_i(\mathbf{a}_{j,0} : j \in \mathcal{N}_i) := \mathbf{a}_{i,1} \hookleftarrow \arg\max_{a_i \in \mathcal{A}_i} \mathbb{E}_{i,1}\{u_i(a_i, \theta)\}.$$

Once the format of the mapping $f_i(\cdot)$ is obtained, it is then used as a heuristic for decision making in every future epoch. In any subsequent time instance $t > 1$ each agent $i \in [n]$ observes the preceding actions of her neighbors $\mathbf{a}_{j,t-1} : j \in \mathcal{N}_i$ and takes an option $\mathbf{a}_{i,t}$ out of the set $\mathcal{A}_i$. The agents update

[1]While many decision science applications focus on developing dual process theories of cognition and decision making (cf. [39], [40] and the references therein); other researchers identify multiple neural systems that derive decision making and action selection: ranging from reflexive and fast (Pavlovian) responses to deliberative and procedural (learned) ones; and these systems are in turn supported by several motoric, perceptual, situation-categorization and motivational routines which together comprise the decision making systems [41, Chapyter 6].

their action by choosing: $\mathbf{a}_{i,t} = f_i\left(\mathbf{a}_{j,t-1} : j \in \mathcal{N}_i\right), \forall t > 1$. We refer to the mappings $f_i : \prod_{j \in \mathcal{N}_i} \mathcal{A}_j \to \mathcal{A}_i$ thus obtained, as *Bayesian heuristics*. In the following sections, we describe the asymptotic outcome of various group decision processes, when the agents use such Bayesian heuristics to aggregate their information.

A. Binary Heuristics

Similar to Subsection III-A, we consider a binary state space $\Theta = \{\theta_1, \theta_2\}$ and label the states as $\theta_1 := +1$ and $\theta_2 := -1$. Suppose further that the agents have a common binary action space $\mathcal{A}_i = \{-1, 1\}$, $\forall i$ and let their utilities be given by $u_i(a, \boldsymbol{\theta}) = 2\mathbb{1}_{\boldsymbol{\theta}=a} - 1$, for any agent i and all $a \in \{-1, 1\}$. Subsequently, the agent is rewarded by $+1$ every time she correctly determines the value of $\boldsymbol{\theta}$ and is penalized by -1 otherwise.

The agents observe a sequence of private signals $\mathbf{s}_{i,t}$ and also observe their neighboring actions from the previous decision epoch $\mathbf{a}_{j,t-1}, j \in \mathcal{N}(i)$. The heuristic update of agent i given her observations of her neighboring choices and her private signal is as follows: $\mathbf{a}_{i,t} = \operatorname{sign}\left(\sum_{j \in \mathcal{N}(i)} w_j \mathbf{a}_{j,t-1} + \eta_i + \lambda(\mathbf{s}_{i,t})\right)$, initialized by $\mathbf{a}_{i,0} = \operatorname{sign}\left(\log(\nu_i(\theta_1)/\nu_i(\theta_2)) + \lambda(\mathbf{s}_{i,0})\right)$.[1,2] The evolution of action profiles $\overline{\mathbf{a}}_t = (\mathbf{a}_{1,t}, \ldots, \mathbf{a}_{n,t})^T$ under the preceding update specifies a finite Markov chain with jumps between the vertices of the Boolean hyper cube, $\{\pm 1\}^n$; the recurrent states of this Markov chain are reachable with positive probability and they can be used to characterize the evolution of the action profiles as well as the asymptotic behavior. In particular, the requirement of learning is for the agents to reach a consensus on truth, i.e. for consensus on truth to be an equilibrium; however, whenever consensus on truth is an equilibrium consensus on the untruth is also an equilibrium (cf. [34], [55]). Therefore, there would always be a non-trivial (away from zero or one) probability for agents to (mis-)learn.

[1]The exact expressions of the constants w_i, η_i and their derivations can be found in [34]. These constants depend only on the signal structure and initial prior of the agent and her neighbors. In particular, we can write $\eta_i := \log\left(\nu_i(\theta_1)/\nu_i(\theta_2)\right) + \log V_i$, where V_i is determined by the knowledge of agent i about her neighbors' signal structures and it is increasing in $\ell_j(s_j \mid \theta_1)$ and decreasing in $\ell_j(s_j \mid \theta_2)$ for any fixed signal $s_j \in \mathcal{S}_j$, $j \in \mathcal{N}(i)$. The constant w_j is a measure of observational ability of agent j as relates to our model and in [34] we establish that constants w_j are non-negative for every agent $j \in [n]$; signifying a case of positive externalities: an agent is more likely to choose an action if her neighbors make the same decision.

[2]Majority and threshold functions, such as the heuristic update rule obtained here, have a rich history. They are studied in the analysis of Boolean functions [44, Chapter 5] and several properties of them including their noise stability are of particular interest [45], [46], [47]. They also appear as the McCulloch-Pitts model of an artificial neuron [48], with important applications in neural networks and computing [49]. This update rule is also important in the study of the Glauber Dynamics in the Ising model, where the ± 1 states represent atomic spins. The spins are arranged in a graph and each spin configuration has a probability associated with it depending on the temperature and the interaction structure [50, Chapter 15], [51]. The Ising model provides a natural setting for the study of cooperative behavior in social networks. Recent studies have explored the applications of Ising model for analysis of social and economic phenomena such as rumor spreading [52], study of market equilibria [53], and opinion dynamics [54].

B. Linear Action Updates

Linear averaging rules have a long history as a tool to model opinion dynamics in mathematical sociology and social psychology [56]; their origins can be traced to French's seminal work on"A Formal Theory of Social Power" [57]. This was followed up by Harary's investigation of the mathematical properties of the averaging model, including the consensus criteria, and its relations to Markov chain theory [58]. This model was later generalized to belief exchange dynamics and popularized by DeGroot's seminal work [32] on linear opinion pools. In engineering literature, the possibility to achieve consensus in a distributed fashion (through local interactions and information exchanges between neighbors) is very desirable in a variety of applications such as load balancing [59], distributed detection and estimation [60], [61], [62], tracking [63], sensor networks and data fusion [64], [65], as well as distributed control and robotics networks [66], [67]. Early works on development of consensus algorithms originated in 1980s with the works of Tsitsiklis et.al [68] who propose a weighted average protocol based on a linear iterative approach for achieving consensus: each node repeatedly updates its value as a weighted linear combination of its own value and those received by its neighbors.

To obtain the DeGroot update as a Bayesian heuristic, we consider a measurable sample space $\mathcal{S}$ with a σ-finite measure $\mathcal{G}_s(\cdot)$, and a class of sampling functions $\{\mathcal{L}(\cdot|\theta;\sigma_i) \in \Delta\mathcal{S}$, parametrized by $\sigma_i > 0$ as a member of the k-dimensional exponential family:

$$\begin{aligned} \ell(s|\theta;\sigma_i) &:= \frac{d\mathcal{L}(\cdot|\theta;\sigma_i)}{d\mathcal{G}_s} \\ &= \sigma_i \left|\frac{\Lambda_k(\xi(ds))}{\mathcal{G}_s(ds)}\right| \tau\left(\sigma_i \xi(s)\right) e^{\sigma_i \theta^T \xi(s) - \gamma(\theta)}, \end{aligned} \tag{6}$$

where $\xi(s) : \mathcal{S} \to \mathbb{R}^k$ is a measurable function acting as a sufficient statistic for the random samples, $\tau : \mathbb{R}^k \times (0, +\infty) \to (0, +\infty)$ is a positive weighting function, and

$$\gamma(\theta) := \ln \int\limits_{s \in \mathcal{S}} \sigma_i \left|\frac{\Lambda_k(\xi(ds))}{\mathcal{G}_s(ds)}\right| \tau(\sigma_i \xi(s)) e^{\sigma_i \theta^T \xi(s)} \mathcal{G}_s(ds),$$

is a normalization factor which is defined over the parameter space:

$$\Theta := \{\theta \in \mathbb{R}^k : \int\limits_{s \in \mathcal{S}} \left|\frac{\Lambda_k(\xi(ds))}{\mathcal{G}_s(ds)}\right| \tau\left(\xi(s)\right) e^{\theta^T \xi(s)} \mathcal{G}_s(ds) < \infty\}.$$

We assume that every agent $i \in [n]$ observes n_i i.i.d. private samples $\mathbf{s}_{i,p}, p \in [n_i]$ from the common sample space $\mathcal{S}$ and that the random samples are distributed according to the law $\mathcal{L}(\cdot|\theta;\sigma_i,\delta_i)$ given by (6). We further assume that every agent starts from a conjugate prior $\mathcal{V}_i(\cdot) = \mathcal{V}(\cdot;\alpha_i,\beta_i)$ belonging to the following conjugate family of priors:

$$\mathcal{F}_\gamma := \Bigg\{ \mathcal{V}(\theta;\alpha,\beta) \in \Delta\Theta, \alpha \in \mathbb{R}^k, \beta_i > 0 : \\ \nu(\theta;\alpha,\beta) := \frac{d\mathcal{V}(\cdot;\alpha,\beta)}{d\mathcal{G}_\theta} = \left|\frac{\Lambda_k(d\theta)}{\mathcal{G}_\theta(d\theta)}\right| \frac{e^{\theta^T\alpha - \beta\gamma(\theta)}}{\kappa(\alpha,\beta)}, \\ \kappa(\alpha,\beta) := \int\limits_{\theta\in\Theta} \left|\frac{\Lambda_k(d\theta)}{\mathcal{G}_\theta(d\theta)}\right| e^{\theta^T\alpha-\beta\gamma(\theta)}\mathcal{G}_\theta(d\theta) < \infty \Bigg\}.$$

Starting from any prior distribution $\mathcal{V}(\cdot) \in \mathcal{F}_\gamma$ and for any signal $s \in \mathcal{S}$, the posterior distribution given the observation $\mathbf{s} = s$ belongs to $\mathcal{F}$, i.e. the conjugate family is closed under the Bayesian updating with their respective exponential family likelihoods. Moreover, the members of the conjugate family $\mathcal{F}_\gamma$ satisfy a linearity property for the Bayes estimates of their sufficient statistics in terms of the distributional parameters α and β. In particular, we assume that agents take actions in $\mathbb{R}^k$, and that they aim for a minimum variance estimation of the regression function or conditional expectation (given θ) of the sufficient statistic $\xi(\mathbf{s}_i)$; i.e. $u_i(a,\theta) = -(a - m_{i,\theta})^T(a - m_{i,\theta})$, $\forall a \in \mathcal{A}_i = \mathbb{R}^k$, where $m_{i,\theta} := \mathbb{E}_{i,\theta}\{\xi(\mathbf{s_i})\}$ $:= \int_{s\in\mathcal{S}} \xi(s)\mathcal{L}(ds|\theta;\sigma_i) \in \mathbb{R}^k$.

The above lays ground for an affine action update with constant additive terms that depend on priors and neighboring likelihood parameters, as well as linearity coefficients that do not necessarily add to one, when summed over all the neighboring agents. If the agents have non-informative priors, then the constant terms vanish and the affine action updates become linear. To have the neighboring coefficients to sum to one at every row (at every neighborhood), we need each agent $i \in [n]$ to observe n_i i.i.d. samples belonging to the same exponential family signal-utility structure; requiring $\sigma_i = \sigma$ for all i, thus allowing heterogeneity only in the sample sizes, but not in the distribution of each sample. The Bayesian heuristics are then given by: $\mathbf{a}_{i,t} = \sum_{j\in\mathcal{N}_i} T_{ij}\mathbf{a}_{j,t-1}$, where $T_{ij} = n_j/\sum_{p\in\mathcal{N}_i} n_p$. By following these update rules, the agents reach a consensus. The efficient aggregate of the initial signals weights them in accordance with their sample sizes and it is given by $\mathbf{a}^\star = \sum_{j=1}^n \left(\mathbf{a}_{j,0}n_j/\sum_{p=1}^n n_p\right)$. A question of interest is whether or not the consensus action is efficient, following Bayesian heuristics.

From the analysis of convergence for DeGroot model in [13, Proporition 1], we know that for a strongly connected network $\mathcal{G}$ if it is aperiodic (meaning that one is the greatest common divisor of the lengths of all its circles; and it is the case for us, since the diagonal entries of T are all non-zero), then $\lim_{\tau\to\infty} T^\tau = \mathbb{1}\overline{s}^T$, where $\overline{s} := (s_1,\ldots,s_n)^T$ is the unique left eigenvector associated with the unit eigenvalue of T and satisfying $\sum_{i=1}^n s_i = 1$, $s_i > 0, \forall i$. Hence, starting from non-informative priors agents follow the DeGroot update and if $\mathcal{G}$ is also strongly connected, then they reach a consensus at $\overline{s}^T\overline{\mathbf{a}}_0 = \sum_{i=1}^n s_i(\sum_{p=1}^{n_i}\xi(\mathbf{s}_{i,p})/n_i)$. Now by expanding the eigenvector condition $\overline{s}^T T = \overline{s}^T$ we obtain that in order for the consensus action $\overline{s}^T\overline{\mathbf{a}}_0$ to agree with the efficient consensus $\mathbf{a}^\star$, it is necessary and sufficient to have that for all j:

$$\sum_{i=1}^n s_i T_{ij} = \sum_{i=1}^n \left(\frac{n_i}{\sum_{j=1}^n n_j}\right)\frac{n_j[I+A]_{ij}}{\sum_{p\in\mathcal{N}_i} n_p} = s_j = \frac{n_j}{\sum_{j=1}^n n_j}, \tag{7}$$

or equivalently,

$$\sum_{i:j\in\mathcal{N}_i} \frac{n_i}{\sum_{p\in\mathcal{N}_i} n_p} = \sum_{i\in\mathcal{N}_j^{out}} \frac{n_i}{\sum_{p\in\mathcal{N}_i} n_p} = 1, \tag{8}$$

for all j, where by $\mathcal{N}_j^{out}$ we mean the set of all agents who listen to j; this is in contrast to her neighborhood $\mathcal{N}_j$, which is the set of all agents whom she listens to. Both sets $\mathcal{N}_j$ and $\mathcal{N}_j^{out}$ include agent j as a member.

It is instructive to view the weights $T_{ij} = n_j/(\sum_{p\in\mathcal{N}_i} n_p)$ as transition probabilities of a node-weighted random walk on the social network graph, cf. [69, Section 5]; where each node $i \in [n]$ is weighted by $w_i = n_i$. Such a random walk is a special case of the more common type of random walks on weighted graphs where the edge weights determine the jump probabilities; indeed, if for any edge $(i,j) \in \mathcal{E}$ we set its weight equal to $w_{i,j} = w_i w_j$ then the random walk on the edge-weighted graph reduces to a random walk on the node-weighted graph with node weights $w_i, i \in [n]$. If the social network graph is undirected and connected (so that $w_{i,j} = w_{j,i}$ for all i,j), then the edge-weighted (whence also the node-weighted) random walks are time-reversible and their stationary distributions $(s_1,\ldots,s_n)^T$ can be calculated in closed form as follows [70, Section 3.2]:

$$s_i = \frac{\sum_{j\in\mathcal{N}_i} w_{i,j}}{\sum_{i=1}^n \sum_{j\in\mathcal{N}_i} w_{i,j}}. \tag{9}$$

In a node-weighted random walk we can replace $w_{i,j} = w_i w_j$ for all $j \in \mathcal{N}_i$ and (9) simplifies into

$$s_i = \frac{w_i \sum_{j\in\mathcal{N}_i} w_j}{\sum_{i=1}^n w_i \sum_{j\in\mathcal{N}_i} w_j}.$$

Similarly to (7), the consensus action will be efficient if and only if

$$s_i = \frac{w_i \sum_{j\in\mathcal{N}_i} w_j}{\sum_{i=1}^n w_i \sum_{j\in\mathcal{N}_i} w_j} = \frac{w_i}{\sum_{k=1}^n w_k}, \forall i,$$

or equivalently:

$$(\sum_{k=1}^n w_k)\sum_{j\in\mathcal{N}_i} w_j = \sum_{i=1}^n (w_i \sum_{j\in\mathcal{N}_i} w_j), \forall i,$$

which holds true only if $\sum_{j\in\mathcal{N}_i} w_j$ is a common constant that is the same for all agents, i.e. $\sum_{j\in\mathcal{N}_i} w_j = \sum_{j\in\mathcal{N}_i} n_j = C' > 0$ for all $i \in [n]$. Next replacing in (8) yields that, in fact, $C' = \sum_{i\in\mathcal{N}_j^{out}} n_i$ for all j. Hence, the consensus action is efficient if, and only if, $\sum_{p\in\mathcal{N}_j^{out}} n_p = \sum_{p\in\mathcal{N}_i} n_p$ for all i and j.

C. Log-Linear Belief Updates

Consider an environment where the state space is a finite set of cardinality m and agents take actions over the $(m-1)$-simplex of probability measures, truthfully announcing their beliefs to each other at every time step. In particular, if we denote the belief probability mass functions $\nu_i(\cdot) := d\mathcal{V}_i/d\mathcal{K}$ and $\mu_{i,t}(\cdot) := d\boldsymbol{\mathcal{M}}_{i,t}/d\mathcal{K}$ for all t, then the heuristic belief update is as follows:

$$\boldsymbol{\mu}_{i,t}(\hat{\theta}) = \frac{\boldsymbol{\mu}_{i,t-1}(\hat{\theta})\left(\prod_{j\in\mathcal{N}_i\setminus\{i\}}\frac{\boldsymbol{\mu}_{j,t-1}(\hat{\theta})}{\nu_j(\hat{\theta})}\right)}{\sum_{\tilde{\theta}\in\Theta}\boldsymbol{\mu}_{i,t-1}(\tilde{\theta})\left(\prod_{j\in\mathcal{N}_i\setminus\{i\}}\frac{\boldsymbol{\mu}_{j,t-1}(\tilde{\theta})}{\nu_j(\tilde{\theta})}\right)}, \quad (10)$$

for all $\hat{\theta}\in\Theta$ and at any $t>1$.

It is notable that the Bayesian heuristic in (11) has a log-linear structure. Geometric averaging and logarithmic opinion pools have a long history in Bayesian analysis and behavioral decision models [71], [72] and they can be also justified under specific behavioral assumptions [36]. The are also quite popular as a non-Bayesian update rule in engineering literature for addressing problems such as distributed detection and estimation [73], [74], [75], [76], [77]. In [77] the authors use a logarithmic opinion pool to combine the estimated posterior probability distributions in a Bayesian consensus filter; and show that as a result: the sum of KullbackLeibler divergences between the consensual probability distribution and the local posterior probability distributions is minimized. Minimizing the sum of KullbackLeibler divergences as a way to globally aggregate locally measured probability distributions is proposed in [78], [79] where the corresponding minimizer is dubbed the KullbackLeibler average. Similar interpretations of the log-linear update are offered in [80] as a gradient step for minimizing either the KullbackLeibler distance to the true distribution, or in [81] as a posterior incorporation of the most recent observations, such that the sum of KullbackLeibler distance to the local priors is minimized; indeed, the Bayes' rule itself has a product form and the Bayesian posterior can be characterized as the solution of an optimization problem involving the KullbackLeibler divergence to the prior distribution and subjected to the observed data [82].

Following the Bayesian heuristic belief updates in (11), $\lim_{t\to\infty}\boldsymbol{\mu}_{i,t}(\tilde{\theta}) = 1/|\Theta^{\diamondsuit}|$ for all $i\in[n]$ and any $\tilde{\theta}\in\Theta^{\star}$, where $\Theta^{\diamondsuit} := \arg\max_{\tilde{\theta}\in\Theta}\sum_{i=1}^{n}\alpha_i\log(\ell_i(\mathbf{s}_i|\tilde{\theta}))$. In particular, if the sum of signal log-likelihoods weighted by node centralities is uniquely maximized by $\theta^{\diamondsuit}$, i.e. $\{\theta^{\diamondsuit}\}=\Theta^{\diamondsuit}$, then $\lim_{t\to\infty}\boldsymbol{\mu}_{i,t}(\theta^{\diamondsuit})=1$ almost surely for all $i\in[n]$. Hence, the agents effectively become certain about the truth state of $\theta^{\diamondsuit}$, in spite of their essentially bounded aggregate information and in contrast with the rational (optimal) belief $\boldsymbol{\mu}^{\star}$ that is given by the Bayes rule:

$$\boldsymbol{\mu}^{\star}(\hat{\theta}) = \frac{\prod_{j\in\mathcal{N}_i}\ell_j(\mathbf{s}_j|\hat{\theta})}{\sum_{\tilde{\theta}\in\Theta}\prod_{j\in\mathcal{N}_i}\ell_j(\mathbf{s}_j|\tilde{\theta})},$$

and do not discredit or reject any of the less probable states.[1]

The heuristic group decision outcome demonstrates also a second form of departure from optimality in that the agents reach consensus on a belief that is supported over $\Theta^{\diamondsuit} := \arg\max_{\tilde{\theta}\in\Theta}\sum_{i=1}^{n}\alpha_i\log(\ell_i(\mathbf{s}_i|\tilde{\theta}))$, as opposed to the global (network-wide) likelihood maximizer $\Theta^{\star} := \arg\max_{\tilde{\theta}\in\Theta}\mu^{\star}(\tilde{\theta}) = \arg\max_{\tilde{\theta}\in\Theta}\sum_{i=1}^{n}\log(\ell_i(\mathbf{s}_i|\tilde{\theta}))$; note that the signal log-likelihoods in the case of $\Theta^{\diamondsuit}$ are weighted by the centralities, α_i, of their respective nodes. The fact that log-likelihoods in $\Theta^{\diamondsuit}$ are weighted by the node centralities is a source of inefficiency for the asymptotic outcome of the group decision process. This inefficiency is warded off in especially symmetric typologies, where in and out degrees of all nodes in the network are the same. In these so-called balanced regular digraphs, there is a fixed integer d such that all agents receive reports from exactly d agents, and also send their reports to some other d agents; d-regular graphs are a special case, since all links are bidirectional and each agent sends her reports to and receive reports from the same d agents. In such structures $\overline{\alpha} = (1/n)\mathbb{1}$ so that $\Theta^{\star} = \Theta^{\diamondsuit}$ and the support of the consensus belief identifies the global maximum likelihood estimator (MLE); i.e. the maximum likelihood estimator of the unknown θ, given the entire set of observations from all agents in the network.

When the agents observe a sequence of private signals $\mathbf{s}_{i,t}$ in addition to hearing their neighboring beliefs at every decision epoch, then update rules are given as follows:

$$\boldsymbol{\mu}_{i,t}(\hat{\theta}) = \frac{\nu_i(\hat{\theta})l_i(\mathbf{s}_{i,t}\mid\hat{\theta})\left(\prod_{j\in\mathcal{N}(i)}\frac{\boldsymbol{\mu}_{j,t-1}(\hat{\theta})}{\nu_j(\hat{\theta})}\right)}{\sum_{\tilde{\theta}\in\Theta}\nu_i(\tilde{\theta})l_i(\mathbf{s}_{i,t}\mid\tilde{\theta})\left(\prod_{j\in\mathcal{N}(i)}\frac{\boldsymbol{\mu}_{j,t-1}(\tilde{\theta})}{\nu_j(\tilde{\theta})}\right)}, \quad (11)$$

for all $\hat{\theta}\in\Theta$ and at any $t>1$.

A chief question of interest to [34] is whether the agents, after being exposed to sequence of private observations and while communicating with each other, can learn the truth using the Bayesian without recall update rules (11). Naivety of agents in these cases impedes their ability to learn; except in simple social structures such as cycles or rooted trees (cf. [83]). In [84] we show that learning in social network with complex neighborhood structures can be achieved if agents choose a neighbor randomly at every round and restrict their belief update to the selected neighbor each time. In [85], [86] we further investigate the resilience of the log-linear structure and its versatility in incorporating intermittent observations.

V. Discussion and Conclusions

After studying the decision problems of Bayesian agents in social learning and group decision scenarios, we shifted focus to a model of inference and heuristic decision making in groups that is rooted in the Bayes rule but avoids

[1]Indeed, such a response can be achieved in practice if one follows Kahneman's advice and collect each individual's information privately before combining them or allowing the individuals to engage in public discussions [9, Chapter 23].

the complexities of rational inference in partially observed environments with incomplete information. According to our model, the group members behave rationally at the initiation of their interactions with each other; however, in the ensuing decision epochs they rely on a heuristic that replicates their experiences from the first stage. Subsequently, the agents use their time-one Bayesian update and repeat it for all future time-steps; hence, updating their actions using a so-called *Bayesian heuristic*. This model avoids the complexities of fully rational inference and also provides a behavioral and normative foundation for non-Bayesian updating. It is also consistent with a dual-process psychological theory of decision making, where a controlled (conscious/slow) system develops the Bayesian heuristic at the beginning, and an automatic (unconscious/fast) system takes over the task of heuristic decision making in the sequel.

We specialized this model to a group decision scenario where private observations are received at the beginning, and agents aim to take the best action given the aggregate observations of all group members. We present the implications of the choices of signal structure and action space for such agents. We show that for a wide class of distributions from the exponential family the Bayesian heuristics take the form of an affine update in the self and neighboring actions. Furthermore, if the priors are non-informative (and possibly improper), then these action updates become a linear combination. We investigate the requirements on the modeling parameters for the action updates to constitute a convex combination as in the DeGroot model. The results reveal the nature of assumptions that are implicit in the DeGroot updating and highlights the fragility and restrictions of such assumptions; in particular, we show that for a linear action update to constitute a convex combination the precision or accuracy of private observations should be balanced among all neighboring agents, requiring a notion of social harmony or homogeneity in their observational abilities. Following the DeGroot model, agents reach a consensus asymptotically. We derive the requirements on the signal structure and network topology such that the consensus action aggregates information efficiently. This involves additional restrictions on the signal likelihoods and network structure. In the particular case that all agents observe the same number of i.i.d. samples from the same distribution, then efficiency arise in degree-regular balanced structures, where all nodes listen to and hear from the same number of neighbors.

We next shifted attention to a finite state model, in which agents take actions over the probability simplex; thus revealing their beliefs to each other. We show that the Bayesian heuristics in this case prescribe a log-linear update rule, where each agent's belief is set proportionally to the product of her own and neighboring beliefs. We analyze the evolution of beliefs under this rule and show that agents reach a consensus. The consensus belief is supported over the maximizers of a weighted sum of the log-likelihoods of the initial observations. Since the weights of the signal likelihoods coincide with the network centralities of their respective agents, these weights can be equalized in degree-regular and balanced topologies, where all nodes have the same in and out degrees. Therefore, such highly symmetric structures the support of the consensus belief coincides with the maximum likelihood estimators (MLE) of the truth state; and here again, balanced regular structures demonstrate a measure of efficiency. Nevertheless, the asymptotic beliefs systematically reject the less probable alternatives in spite of the limited initial data, and in contrast with the optimal (Bayesian) belief of an observer with complete information of the environment and private signals. The latter would assign probabilities proportionally to the likelihood of every state, without rejecting any of the possible alternatives. The asymptotic rejection of less probable alternatives indicates a case of group polarization, i.e. overconfidence in the group aggregate. Unlike the linear action updates and the DeGroot model which entail a host of knife-edge conditions on the signal structure and model parameters, we observe that the belief updates are unweighted; not only they effectively internalize the heterogeneity of the private observations, but also they compensate for the individual priors. Thence, we are led to the conclusion that multiplicative belief updates, when applicable provide a relatively robust description of the decision making behavior.

Our results indicate certain deviations from the globally efficient outcomes, when consensus is being achieved through the Bayesian heuristics. This inefficiency of Bayesian heuristics in globally aggregating the observations is attributed to the agents' naivety in inferring the sources of their information, which makes them vulnerable to structural network influences: the share of centrally located agents in shaping the asymptotic outcome is more than what is warranted by the quality of their data. Another source of inefficiency is in the group polarization that arise as a result of repeated group interactions; in case of belief updates, this is manifested in the structure of the (asymptotic) consensus beliefs. The latter assigns zero probability to any alternative that scores lower than the maximum in the weighted likelihoods scale: the agents reject the possibility of less probable alternatives with certainty, in spite of their limited initial data. This overconfidence in the group aggregate and shift toward more extreme beliefs is a key indicator of group polarization and is demonstrated very well by the asymptotic outcome of the group decision process.

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