AI1103: Assignment 2

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Download all python codes from

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https://github.com/tanmaygar/AI−Course/blob/main
    /Assignment2/codes/problem_47.py
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and latex-tikz codes from

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https://github.com/tanmaygar/AI−Course/blob/main
    /Assignment2/Assignment2.tex
```

Problem GATE-EC-2019-Q47:

A random variable X takes values -1 and +1 with probabilities 0.2 and 0.8, respectively. It is transmitted across a channel which adds noise N, so that the random variable at the channel output is $Y = X + N$. The noise N is independent of X, and is uniformly distributed over the interval [-2 , 2]. The receiver makes a decision

$$\hat{X} = \begin{cases} -1, & \text{if} \quad Y \leq \theta \\ +1, & \text{if} \quad Y \geq \theta \end{cases}$$

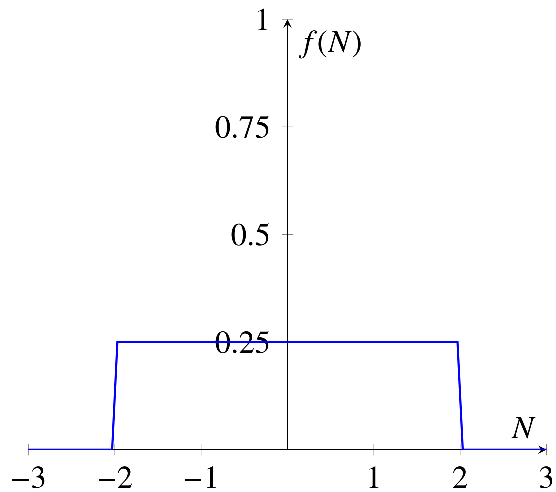
where the threshold $\theta \in [-1, 1]$ is chosen so as to minimize the probability of error $\Pr\left(\hat{X} \neq X\right)$. The minimum probability of error, rounded off to 1 decimal place, is?

Solution:

We know that

$$X \in \{-1, +1\} \tag{0.0.1}$$
$$N \in [-2, 2] \tag{0.0.2}$$
$$Y = X + N \tag{0.0.3}$$
$$\Pr(X = -1) = 0.2 \tag{0.0.4}$$
$$\Pr(X = +1) = 0.8 \tag{0.0.5}$$

Since N is uniformly distributed
$\therefore$ the probability distribution function of N is:



The cdf of this uniform probability distribution function is

$$F_X(x) = \int_{-\infty}^{x} f(N)dN \tag{0.0.6}$$
$$= \int_{-2}^{x} \frac{1}{4}dN \tag{0.0.7}$$
$$= \frac{x+2}{4} \tag{0.0.8}$$

For $X \neq \hat{X}$ we need to check for each case. Using equation (0.0.8)

$$\therefore \Pr(N > \theta + 1) = 1 - \Pr(N < \theta + 1) \tag{0.0.9}$$
$$= 1 - F_X(\theta + 1) \tag{0.0.10}$$
$$= 1 - \frac{\theta + 3}{4} \tag{0.0.11}$$
$$= \frac{1}{4}(1 - \theta) \tag{0.0.12}$$

$$\therefore \Pr(N < \theta - 1) = F_X(\theta - 1) \tag{0.0.13}$$
$$= \frac{1}{4}(1 + \theta) \tag{0.0.14}$$

The probability of error:

$$\Pr\left(\hat{X} \neq X\right) = P(-1) \cdot P(\theta < -1 + N) + P(1) \cdot P(\theta > N + 1) \tag{0.0.15}$$

Substituting (0.0.12) and (0.0.14) in (0.0.15). We get:

$$\Pr\left(\hat{X} \neq X\right) = 0.2 \cdot \frac{1}{4}(1-\theta) + 0.8 \cdot \frac{1}{4}(1+\theta) \tag{0.0.16}$$

On simplifying the equation we get

$$\Pr\left(\hat{X} \neq X\right) = \frac{1}{4} + \frac{3}{20}\theta \tag{0.0.17}$$

Since this is a linear equation in θ, the minimum will occur at boundary points. Putting $\theta = +1$, we get

$$\Pr\left(\hat{X} \neq X\right) = 0.4 \tag{0.0.18}$$

but on putting $\theta = -1$, we get

$$\Pr\left(\hat{X} \neq X\right) = 0.1 \tag{0.0.19}$$

Hence the value of probability of error is:

$$\therefore \Pr\left(\hat{X} \neq X\right) = 0.1 \tag{0.0.20}$$