

AI1103: Assignment 3

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Download all python codes from

[https://github.com/tanmaygar/AI-Course/blob/main/Assignment3/codes/GATE-2007-\(MA\)-Q14.py](https://github.com/tanmaygar/AI-Course/blob/main/Assignment3/codes/GATE-2007-(MA)-Q14.py)

and latex-tikz codes from

<https://github.com/tanmaygar/AI-Course/blob/main/Assignment3/Assignment3.tex>

PROBLEM GATE 2007 (MA), Q. 14:

Let X and Y be jointly distributed random variables having the joint probability density function

$$f(x, y) = \begin{cases} \frac{1}{\pi}, & \text{if } x^2 + y^2 \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Then $\Pr(Y > \max(X, -X))$ is

SOLUTION:

The pdf of X and Y are:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy \quad (0.0.1)$$

$$= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy \quad (0.0.2)$$

$$= \frac{2\sqrt{1-x^2}}{\pi} \quad (0.0.3)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx \quad (0.0.4)$$

$$= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi} dx \quad (0.0.5)$$

$$= \frac{2\sqrt{1-y^2}}{\pi} \quad (0.0.6)$$

The cdf of Y is:

$$F_Y(y) = \int_{-\infty}^y f_Y(y) dy \quad (0.0.7)$$

$$= \int_{-1}^y \frac{2\sqrt{1-y^2}}{\pi} dy \quad (0.0.8)$$

$$= \frac{2}{\pi} \left(\frac{\sin^{-1} y + y\sqrt{1-y^2}}{2} + \frac{\pi}{4} \right) \quad (0.0.9)$$

The value of $\Pr(-X < Y < X)$ is:

$$\Pr(-X < Y < X) = F_Y(X) - F_Y(-X) \quad (0.0.10)$$

$$= \frac{2}{\pi} \left(\sin^{-1} X + X\sqrt{1-X^2} \right) \quad (0.0.11)$$

Integrating our probability over all of X we get the value of $\Pr(Y > \max(X, -X))$:

$$= \int_{-\infty}^{\infty} f_X(x) \Pr(-x < Y < x) dx \quad (0.0.12)$$

$$= \left(\frac{2}{\pi} \right)^2 \int_0^1 \sqrt{1-x^2} \left(\sin^{-1} x + x\sqrt{1-x^2} \right) dx \quad (0.0.13)$$

Substituting

$$u = \sin^{-1} x + x\sqrt{1-x^2} \quad (0.0.14)$$

$$\frac{du}{dx} = 2\sqrt{1-x^2} \quad (0.0.15)$$

$$= \left(\frac{2}{\pi} \right)^2 \int_0^{\frac{\pi}{2}} \frac{u}{2} du \quad (0.0.16)$$

$$= \left(\frac{2}{\pi} \right)^2 \left(\frac{u^2}{4} \Big|_0^{\frac{\pi}{2}} \right) \quad (0.0.17)$$

$$= \left(\frac{2}{\pi} \right)^2 \left(\frac{\pi^2}{16} - 0 \right) \quad (0.0.18)$$

$$= \frac{4 \cdot \pi^2}{\pi^2 \cdot 16} \quad (0.0.19)$$

$$= \frac{1}{4} \quad (0.0.20)$$

The probability for:

$$\Pr(Y > \max(X, -X)) = \frac{1}{4} \quad (0.0.21)$$

