

AI1103: Assignment 2

Tanmay Garg

CS20BTECH11063 EE20BTECH11048

Download all python codes from

https://github.com/tanmaygar/AI-Course/blob/main/Assignment2/codes/problem_47.py

and latex-tikz codes from

<https://github.com/tanmaygar/AI-Course/blob/main/Assignment2/Assignment2.tex>

PROBLEM GATE-EC-2019-Q47:

A random variable X takes values -1 and $+1$ with probabilities 0.2 and 0.8 , respectively. It is transmitted across a channel which adds noise N , so that the random variable at the channel output is $Y = X + N$. The noise N is independent of X , and is uniformly distributed over the interval $[-2, 2]$. The receiver makes a decision

$$\hat{X} = \begin{cases} -1, & \text{if } Y \leq \theta \\ +1, & \text{if } Y \geq \theta \end{cases}$$

where the threshold $\theta \in [-1, 1]$ is chosen so as to minimize the probability of error $\Pr(\hat{X} \neq X)$. The minimum probability of error, rounded off to 1 decimal place, is?

SOLUTION:

We know that

$$X \in \{-1, +1\} \quad (0.0.1)$$

$$N \in [-2, 2] \quad (0.0.2)$$

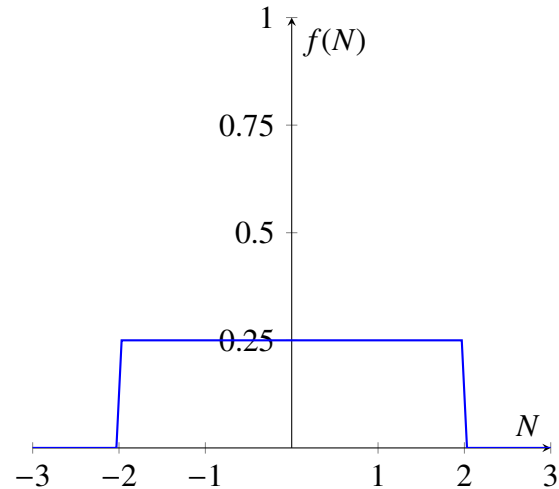
$$Y = X + N \quad (0.0.3)$$

$$\Pr(X = -1) = 0.2 \quad (0.0.4)$$

$$\Pr(X = +1) = 0.8 \quad (0.0.5)$$

Since N is uniformly distributed

\therefore the probability distribution function of N is:



The cdf of this uniform probability distribution function is

$$F_X(x) = \int_{-\infty}^x f(N) dN \quad (0.0.6)$$

$$= \int_{-2}^x \frac{1}{4} dN \quad (0.0.7)$$

$$= \frac{x+2}{4} \quad (0.0.8)$$

For $X \neq \hat{X}$ we need to check for each case. Integrating the p.d.f to find the probability for each case

$$\therefore \Pr(\theta < -1 + N) = \int_{\theta+1}^2 \frac{1}{4} dN \quad (0.0.9)$$

$$= \frac{1}{4}(1 - \theta) \quad (0.0.10)$$

$$\therefore \Pr(\theta > N + 1) = \int_{-2}^{\theta-1} \frac{1}{4} dN \quad (0.0.11)$$

$$= \frac{1}{4}(1 + \theta) \quad (0.0.12)$$

The probability of error:

$$\begin{aligned} \Pr(\hat{X} \neq X) &= P(-1) \cdot P(\theta < -1 + N) \\ &\quad + P(1) \cdot P(\theta > N + 1) \end{aligned} \quad (0.0.13)$$

Substituting (0.0.10) and (0.0.12) in (0.0.13). We get:

$$\Pr(\hat{X} \neq X) = 0.2 \cdot \frac{1}{4}(1 - \theta) + 0.8 \cdot \frac{1}{4}(1 + \theta) \quad (0.0.14)$$

On simplifying the equation we get

$$\Pr(\hat{X} \neq X) = \frac{1}{4} + \frac{3}{20}\theta \quad (0.0.15)$$

Since this is a linear equation in θ , the minimum will occur at boundary points. Putting $\theta = +1$, we get

$$\Pr(\hat{X} \neq X) = 0.4 \quad (0.0.16)$$

but on putting $\theta = -1$, we get

$$\Pr(\hat{X} \neq X) = 0.1 \quad (0.0.17)$$

$$(0.0.18)$$

Hence the value of probability of error is:

$$\therefore \Pr(\hat{X} \neq X) = 0.1 \quad (0.0.19)$$