AI1103: Assignment 2

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PROBLEM GATE 2007 (MA), Q. 14:

Let X and Y be jointly distributed random variables having the joint probability density function

$$f(x, y) = \begin{cases} \frac{1}{\pi}, & \text{if } \quad x^2 + y^2 \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Then $P(Y > \max(X, -X))$ is

SOLUTION:

The pdf of X and Y are:

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy \qquad (0.0.1)$$

$$= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy \qquad (0.0.2)$$

$$= \frac{2\sqrt{1-x^2}}{\pi} \qquad (0.0.3)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx \qquad (0.0.4)$$

$$= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi} dx \qquad (0.0.5)$$

$$= \frac{2\sqrt{1-y^2}}{\pi} \qquad (0.0.6)$$

The value of $\Pr(-X < Y < X)$ is:

$$\Pr(-X < Y < X) = \int_{-X}^{X} f_Y(y) dy$$

$$= \int_{-X}^{X} \frac{2\sqrt{1-y^2}}{\pi} dy \qquad (0.0.7)$$

$$= \frac{2}{\pi} \left(\sin^{-1} X + X\sqrt{1-X^2} \right) \qquad (0.0.8)$$

Integrating our probability over all of X we get the value of $\Pr(Y > \max(X, -X))$:

$$= \int_{-\infty}^{\infty} f_X(x) \Pr(-x < Y < x) dx \qquad (0.0.9)$$

$$= \left(\frac{2}{\pi}\right)^2 \int_0^1 \sqrt{1-x^2}\left(\sin^{-1} x + x\sqrt{1-x^2}\right) dx \qquad (0.0.10)$$

$$= \left(\frac{2}{\pi}\right)^2 \left(\frac{\left(\sin^{-1} x + x\sqrt{1-x^2}\right)^2}{4}\bigg|_0^1 \right) \qquad (0.0.11)$$

$$= \left(\frac{2}{\pi}\right)^2 \left(\frac{\pi^2}{16} - 0\right) \qquad (0.0.12)$$

$$= \frac{4 \cdot \pi^2}{\pi^2 \cdot 16} \qquad (0.0.13)$$

$$= \frac{1}{4} \qquad (0.0.14)$$

The probability for:

$$\Pr(Y > \max(X, -X)) = \frac{1}{4} \qquad (0.0.15)$$