AI1103: Assignment 3

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Download all latex-tikz codes from

https://github.com/tanmaygar/AI−Course/blob/main/Assignment3/Assignment3.tex

Problem GATE 2007 (MA), Q. 14:

Let X and Y be jointly distributed random variables having the joint probability density function

$$f(x,y) = \begin{cases} \frac{1}{\pi}, & \text{if} \quad x^2+y^2 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Then $P(Y > \max(X, -X))$ is

Solution:

The pdf of X and Y are:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy \tag{0.0.1}$$
$$= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}dy \tag{0.0.2}$$
$$= \frac{2\sqrt{1-x^2}}{\pi} \tag{0.0.3}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx \tag{0.0.4}$$
$$= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi}dx \tag{0.0.5}$$
$$= \frac{2\sqrt{1-y^2}}{\pi} \tag{0.0.6}$$

The cdf of Y is:

$$F_Y(y) = \int_{-\infty}^{y} f_Y(y)dy \tag{0.0.7}$$
$$= \int_{-1}^{y} \frac{2\sqrt{1-y^2}}{\pi}dy \tag{0.0.8}$$
$$= \frac{2}{\pi}\left(\frac{\sin^{-1} y + y\sqrt{1-y^2}}{2} + \frac{\pi}{4}\right) \tag{0.0.9}$$

The value of $\Pr(-X < Y < X)$ is:

$$\Pr(-X < Y < X) = F_Y(X) - F_Y(-X) \tag{0.0.10}$$
$$= \frac{2}{\pi}\left(\sin^{-1} X + X\sqrt{1-X^2}\right) \tag{0.0.11}$$

Integrating our probability over all of X we get the value of $\Pr(Y > \max(X, -X))$:

$$= \int_{-\infty}^{\infty} f_X(x) \Pr(-x < Y < x)\, dx \tag{0.0.12}$$
$$= \left(\frac{2}{\pi}\right)^2 \int_0^1 \sqrt{1-x^2}\left(\sin^{-1} x + x\sqrt{1-x^2}\right) dx \tag{0.0.13}$$

Substituting

$$u = \sin^{-1} x + x\sqrt{1-x^2} \tag{0.0.14}$$
$$\frac{du}{dx} = 2\sqrt{1-x^2} \tag{0.0.15}$$
$$= \left(\frac{2}{\pi}\right)^2 \int_0^{\frac{\pi}{2}} \frac{u}{2}du \tag{0.0.16}$$
$$= \left(\frac{2}{\pi}\right)^2 \left(\left.\frac{u^2}{4}\right|_0^{\frac{\pi}{2}}\right) \tag{0.0.17}$$
$$= \left(\frac{2}{\pi}\right)^2 \left(\frac{\pi^2}{16} - 0\right) \tag{0.0.18}$$
$$= \frac{4 \cdot \pi^2}{\pi^2 \cdot 16} \tag{0.0.19}$$
$$= \frac{1}{4} \tag{0.0.20}$$

The probability for:

$$\Pr(Y > \max(X, -X)) = \frac{1}{4} \tag{0.0.21}$$