Assignment 5

Gaureesha Kajampady - EP20BTECH11005

Download latex-tikz codes from

https://github.com/gaureeshk/assignment5/blob/main/assignment5.tex

1 Problem

(CSIR UGC NET EXAM (Dec 2014), Q.105)
Let $X_1, X_2, X_3, ...$ be independent random variables with $E(X_k) = 0$ and Var(X_k)=k. Let $S_n \sum_{k=1}^{n} X_k$.Then as n $\to \infty$,

1) $\frac{S_n}{n^{\frac{3}{2}}}$ $\to$0 in probability
2) $\frac{S_n}{n^{\frac{3}{2}}}$ $\to$0 in distribution
3) $\frac{S_n X_n}{n^{\frac{5}{2}}}$ $\to$0 in distribution
4) $\frac{S_n X_n}{n^{\frac{5}{2}}}$ $\to$0 in probability

2 Solution

Definition 1. *(convergence in probability)*
Let $X_1, X_2, \dots$ be an infinite sequence of random variables, and let Y be another random variable. Then the sequence $\{X_n\}$ converges in probability to Y, if for all $\epsilon > 0$,

$$\lim_{n\to\infty} \Pr\left(|X_n - Y| \geq \epsilon\right) = 0 \tag{2.0.1}$$

And we write as $n \to \infty$, $X_n \to Y$ in probability.

Definition 2. *(Convergence in Distribution)*
Let $X, X_1, X_2, \dots$ be random variables. Then we say that the sequence $\{X_n\}$ converges to X, if $\forall x \in R^1$ such that $\Pr(X = x) = 0$, we have

$$\lim_{n\to\infty} \Pr\left(X_n \leq x\right) = \Pr\left(X \leq x\right). \tag{2.0.2}$$

Theorem 2.1. *If $X_n \to X$ in probability, $X_n \to X$ in distribution.*

Theorem 2.2. *(Chebyshev's inequality)*
Let X be a random variable with finite expected value μ and finite non-zero variance σ^2. Then for any real number k ¿ 0,

$$\Pr\left(|X - \mu| \geq \epsilon\right) \leq \frac{\sigma^2}{\epsilon^2} \tag{2.0.3}$$

We know that,
Since S_n is a sum of random variables, it itself is a random variable. We also know that,

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) \tag{2.0.4}$$

Hence,

$$E\left(S_n\right) = \sum_{i=0}^{n} 0 = 0 \tag{2.0.5}$$

$$\implies E\left(\frac{S_n}{n^{\frac{3}{2}}}\right) = 0 \tag{2.0.6}$$

Since the random variables X_1, X_2,.. are independent

$$Var(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} Var(X_i) \tag{2.0.7}$$

$$Var(S_n) = \sum_{i=1}^{n} i = \frac{n \times (n+1)}{2} \tag{2.0.8}$$

$$\implies Var\left(\frac{S_n}{n^{\frac{3}{2}}}\right) = \frac{Var(S_n)}{n^3} \tag{2.0.9}$$

$$= \frac{(n+1)}{2n^2} \tag{2.0.10}$$

Hence as n $\to \infty$,

$$E\left(\frac{S_n}{n^{\frac{3}{2}}}\right) \to 0 \tag{2.0.11}$$

$$Var\left(\frac{S_n}{n^{\frac{3}{2}}}\right) \to 0 \tag{2.0.12}$$

$$\tag{2.0.13}$$

Using theorem 2.2 (Chebyshev's inequality),

$$\Pr\left(\left|\frac{S_n}{n^{\frac{3}{2}}} - \mu\right| \geq \epsilon\right) \leq \frac{\sigma^2}{\epsilon^2} \tag{2.0.14}$$

As n$\rightarrow\infty$, $\mu \rightarrow 0$ and $\sigma \rightarrow 0$

$$\implies \frac{\sigma^2}{\epsilon^2} \rightarrow 0 \qquad (2.0.15)$$

$$\implies \Pr\left(\left|\frac{S_n}{n^{\frac{3}{2}}} - 0\right| \geq \epsilon\right) \rightarrow 0 \qquad (2.0.16)$$

Hence $\frac{S_n}{n^{\frac{3}{2}}}$ converges to 0 in probability.
By using theorem 2.1,
$\implies \frac{S_n}{n^{\frac{3}{2}}}$ converges to 0 in distribution.