Assignment 6

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https://github.com/gaureeshk/assignment6_2/blob/main/assignment6.tex

1 Problem

gov/stats/2018/STATISTICS-PAPER-2 , Q.58

Let X_1 and X_2 be i.i.d random variables with poisson. Then $(X_1 + 2X_2)$ is not sufficient because

1) $\Pr(X_1 = 1, X_2 = 1 | T = 3)$ depends on λ
2) $X_1 + 2X_2$ is poisson
3) $X_1 + 2X_2$ is not poisson
4) $\Pr(X_1 = 1, X_2 = 1 | T = 3)$ is Poisson with parameter one

Where T=$(X_1 + 2X_2)$

2 Solution

Definition 1. *Statistic : A statistic is a function $T = r(X_1, X_2, \dots, X_n)$ of the random sample $X_1, X_2, \dots, X_n$.*

Definition 2. *Sufficient Statistics : A statistic $t = T(X)$ is sufficient for θ if the conditional probability distribution of data X, given the statistic $t = T(X)$, doesn't depend on the parameter θ.*

$$\Pr(\theta | T(X)) = \Pr(\theta | X) \tag{2.0.1}$$

Theorem 2.1 (Factorization theorem)**.** *: Let X_1, X_2, X_n form a random sample from either a continuous distribution or a discrete distribution for which the pdf or the point mass function is $f(x|\theta)$, where the value of θ is unknown and belongs to a given parameter space Θ. A statistic $T(X_1, X_2, X_n)$ is a sufficient statistic for θ if and only if the joint pdf or the joint point mass function $f_n(x|\theta)$ X_1, X_2, X_n can be factorized as follows for all values of $x = (X_1, X_2, X_n) \to R^n$ and all values of $\theta \in \Theta$: $f_n(x|\theta) = u(x)v(T(x), \theta)$.*

Here the function u may depend on x but does not depend on θ, and the function v depends on θ but will depend on the observed value x only through the value of the statistic T(x).

Lemma 2.1. *Suppose that X_1 and X_2 are i.i.d poisson random variables with parameter λ, $T = X_1 + 2X_2$ does not follow poisson distribution.*

Proof. Let $\Phi_{X_1}(\omega)$, $\Phi_{2X_2}(\omega)$ and $\Phi_T(\omega)$be the characteristic functions of probability density function of random variables X_1, $2X_2$ and T respectively.

$$\Phi_{X_1}(\omega)=\mathrm{E}(e^{i\omega X_1}) = \sum_{x=0}^{\infty} \Pr(X_1 = x)\, e^{i\omega x} \tag{2.0.2}$$
$$= \sum_{x=0}^{\infty} \frac{e^{i\omega x-\lambda}\lambda^x}{x!} \tag{2.0.3}$$
$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^{i\omega}\lambda)^x}{x!} \tag{2.0.4}$$
$$= e^{\lambda\left(e^{i\omega}-1\right)} \tag{2.0.5}$$

Similarly,

$$\begin{aligned}\Phi_{2X_2}(\omega)&=\mathrm{E}\left(e^{i\omega 2X_2}\right)\\ &= \sum_{x=0}^{\infty} \Pr\left(X_2 = \frac{x}{2}\right) e^{i\omega x}\\ &= \sum_{x=0,2,4....}^{\infty} \frac{\left(e^{i\omega x-\lambda}\right)\lambda^{\frac{x}{2}}}{(\frac{x}{2})!}\\ &= e^{-\lambda} \sum_{\frac{x}{2}=0,1,2..}^{\infty} \frac{\left(e^{2i\omega}\lambda\right)^{\frac{x}{2}}}{(\frac{x}{2})!}\\ &= e^{\lambda\left(e^{2i\omega}-1\right)}\end{aligned} \tag{2.0.6}$$

$$\Phi_T(\omega) = \Phi_{X_1}(\omega) \times \Phi_{2X_2}(\omega) \tag{2.0.7}$$
$$= e^{\lambda\left(e^{i\omega}+e^{2i\omega}-2\right)} \tag{2.0.8}$$
$$\neq e^{\mu\left(e^{i\omega}-1\right)} \tag{2.0.9}$$

Hence the characteristic function of T ($\Phi_T(\omega)$) is not in the form of the characteristic function of a poisson random variable (for any value of the parameter μ).

□

Proof that T is not sufficient statistic:-
Joint p.m.f of X_1 and X_2 is,

$$f_{X_1X_2}(x_1, x_2) = \frac{e^{-2\lambda}\lambda^{(x_1+x_2)}}{x_1!x_2!} \tag{2.0.10}$$
$$= \frac{1}{x_1!x_2!} \times \lambda^{T(X_1,X_2)} e^{-2\lambda}\lambda^{-x_2} \tag{2.0.11}$$

As we can see (2.0.10) cannot be expressed in the form of $u(x)v\left(T(x), \theta\right)$
Hence using theorem 2.1, T is not a sufficient statistic

1) We know T=3 when (X_1, X_2) have values (1,1) and (3,0)

$$\Pr\left(X_1 = 1, X_2 = 1 | T = 3\right)$$

$$= \frac{\Pr\left(X_1 = 1, X_2 = 1 \cap T = 3\right)}{\Pr\left(T = 3\right)}$$

$$= \frac{\Pr\left(X_1 = 1, X_2 = 1\right)}{\Pr\left(X_1 = 1, X_2 = 1\right) + \Pr\left(X_1 = 3, X_2 = 0\right)}$$

$$= \frac{e^{-2\lambda}\lambda^2}{e^{-2\lambda}\lambda^2 + \frac{e^{-3\lambda}\lambda^3}{6}}$$

$$= \frac{6}{6+\lambda} \neq \frac{e^{-1}1^{\lambda}}{\lambda!} \tag{2.0.12}$$

Hence $\Pr\left(X_1 = 1, X_2 = 1 | T = 3\right)$ depends on λ but is not poisson with parameter 1. $\implies$ option 4 is incorrect and option 1 is correct.

2) Using lemma 2.0.10, $T = X_1 + 2X_2$ is not poisson.
$\implies$ option 2 is incorrect, option 3 is correct
Hence options 1 and 3 are correct.