Digital Signal Processing

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Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt-get update
sudo apt-get install libffi-dev libsndfile1 python3
    -scipy python3-numpy python3-matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://raw.githubusercontent.
    com/gadepall/
EE1310/master/filter/codes/
    Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2), \quad y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. 3.2.

```
wget https://github.com/gadepall/
    EE1310/raw/master/filter/codes/
    xnyn.py
```

3.3 Repeat the above exercise using a C code.

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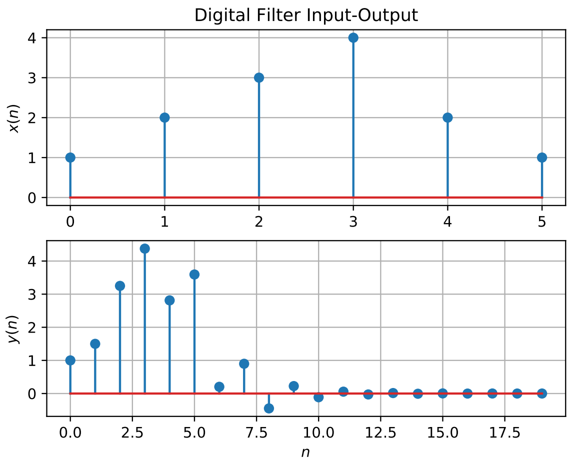

Fig. 3.2

4 Z-TRANSFORM

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.7),

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1} \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

4.2 Obtain X(z) for x(n) defined in the problem 3.1
Solution:

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.7}$$

$$= 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \frac{2}{z^5} + \frac{1}{z^6} \tag{4.8}$$

$$\tag{4.9}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.10}$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.11}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.12}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.13}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.14}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.15}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.16}$$

and from (4.14),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.17}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.18}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.19}$$

Solution: The Z-transform of $a^n u(n)$ given by,

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{n=\infty} a^n u(n) z^{-n} \tag{4.20}$$

$$= \sum_{n=0}^{n=\infty} a^n z^{-n} \tag{4.21}$$

$$= \frac{1}{1 - az^{-1}}, \text{ for } \left|\frac{a}{z}\right| < 1 \tag{4.22}$$

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.23}$$

Plot $|H(e^{j\omega})|$. Comment. $H(e^{j\omega})$ is known as the *Discrete Time Fourier Transform* (DTFT) of $x(n)$.

Solution: Using (4.12),

$$H\left(e^{j\omega}\right) = H(z = e^{j\omega}) \tag{4.24}$$

$$= \frac{1 + \left(e^{j\omega}\right)^{-2}}{1 + \frac{\left(e^{j\omega}\right)^{-1}}{2}} \tag{4.25}$$

$$= \frac{1 + e^{-2j\omega}}{1 + \frac{e^{-j\omega}}{2}} \tag{4.26}$$

$$\left|H\left(e^{j\omega}\right)\right| = \left|\frac{e^{j\omega} + e^{-j\omega}}{e^{j\omega} + \frac{1}{2}}\right| \tag{4.27}$$

$$= \left|\frac{2cos\left(\omega\right)}{cos\left(\omega\right) + \frac{1}{2} + jsin\left(\omega\right)}\right| \tag{4.28}$$

$$= \left|\frac{2cos\left(\omega\right)}{\sqrt{\left(cos\left(\omega\right) + \frac{1}{2}\right)^2 + sin^2\left(\omega\right)}}\right| \tag{4.29}$$

$$= \left|\frac{2cos\left(\omega\right)}{\sqrt{\frac{5}{4} + cos\left(\omega\right)}}\right| \tag{4.30}$$

The period of both numerator and denominator is 2π hence the period of $H\left(e^{j\omega}\right)$ is 2π.

The following code plots Fig. 4.6.

```
wget https://raw.githubusercontent.com/gadepall/EE1310/master/filter/codes/dtft.py
```

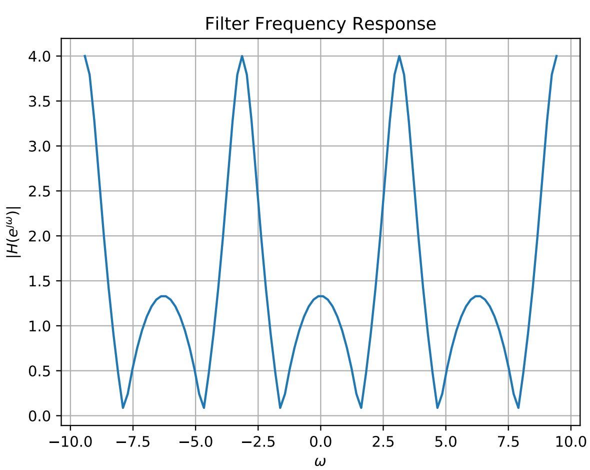

Fig. 4.6: $|H(e^{j\omega})|$

4.7 Express $h(n)$ in terms of $H(e^{j\omega})$. **Solution:**

$$H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{n=\infty} h\left(n\right) e^{-nj\omega} \tag{4.31}$$

Multiplying both sides with $e^{kj\omega}$ and integrating from $\omega = -\pi$ to $\omega = +\pi$ we get,

$$\int_{\omega=-\pi}^{\pi} H\left(e^{j\omega}\right) e^{kj\omega} d\omega = \sum_{n=-\infty}^{n=\infty} \int_{\omega=-\pi}^{\pi} h\left(n\right) e^{(k-n)j\omega} d\omega \tag{4.32}$$

Using property

$$\int_{-\pi}^{\pi} e^{(k-n)j\omega} = 2\pi\delta_{kn}$$

$$\int_{\omega=-\pi}^{\pi} H\left(e^{j\omega}\right) e^{kj\omega} d\omega = \sum_{n=-\infty}^{n=\infty} \int_{\omega=-\pi}^{\pi} h\left(n\right) e^{(k-n)j\omega} d\omega$$

$$= 2\pi \sum_{n=-\infty}^{n=\infty} h\left(n\right) \delta_{kn}$$

$$= 2\pi h\left(k\right)$$

$$\implies h\left(n\right) = \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} H\left(e^{j\omega}\right) e^{nj\omega} d\omega \tag{4.33}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.12). **Solution:**

$$\begin{array}{r|l}
 & 1 + \frac{-z^{-1}}{2} + \frac{5z^{-2}}{4} + \frac{-5z^{-3}}{8} + \frac{5z^{-4}}{16} + \dots \\ \hline
1 + \frac{z^{-1}}{2} & 1 + 0z^{-1} + z^{-2} + 0z^{-3} + 0z^{-4} + \quad \dots \\
 & 1 + \frac{z^{-1}}{2} \\ \hline
 & -\frac{z^{-1}}{2} + z^{-2} \\
 & \frac{-z^{-1}}{2} - \frac{z^{-2}}{4} \\ \hline
 & \frac{5z^{-2}}{4} \\
 & \frac{5z^{-2}}{4} + \frac{5z^{-3}}{8} \\ \hline
 & \frac{-5z^{-3}}{8} \\
 & \frac{-5z^{-3}}{8} - \frac{5z^{-4}}{16} \\ \hline
 & \frac{5z^{-4}}{16} + \frac{5z^{-5}}{32} \\
 & \vdots
\end{array}$$

$$\implies \sum_{n=-\infty}^{n=\infty} h(n)z^{-n} = 1 - \frac{z^{-1}}{2} + \frac{5z^{-2}}{4} - \frac{5z^{-3}}{8} + \frac{5z^{-4}}{16} + \frac{\frac{5z^{-3}}{16}}{1+\frac{z^{-1}}{2}} \tag{5.2}$$

Comparing coefficients,

$$h(n) = \begin{cases} 0 & if \quad n<0 \\ 1, & if \quad n=0 \\ -\frac{1}{2}, & if \quad n=1 \\ 5\left(-\frac{1}{2}\right)^n, & if \quad n\geq 2 \end{cases} \tag{5.3}$$

$$\implies h(n) = 0 \quad \forall \quad n<0 \tag{5.4}$$

$$h(0) = 1, h(1) = -\frac{1}{2}, h(2) = \frac{5}{4}, h(3) = -\frac{5}{8}, \tag{5.5}$$

$$h(4) = \frac{5}{16} \tag{5.6}$$

$$H(z) = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.7}$$

$$\tag{5.8}$$

ROC:

$$\left|\frac{z^{-1}}{2}\right| < 1 \tag{5.9}$$

$$\implies |z| > \frac{1}{2} \tag{5.10}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.11}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.12),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.12}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.13}$$

using (4.19) and (4.6).

5.3 Sketch $h(n)$. Is it bounded?Justify theoretically.
Solution: The following code plots Fig. 5.3.

```
wget https://raw.githubusercontent.
    com/gadepall/EE1310/master/
    filter/codes/hn.py
```

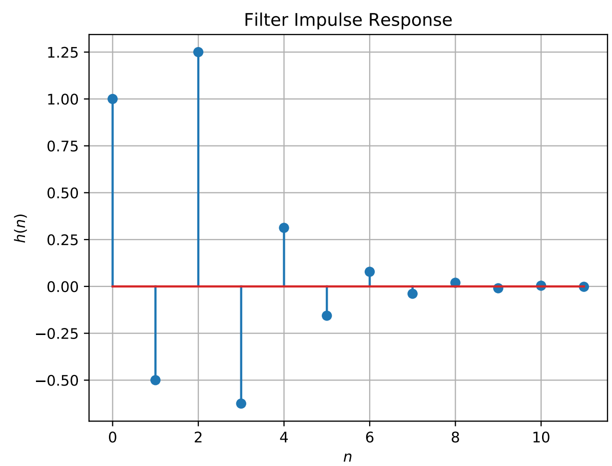

Fig. 5.3: $h(n)$ as the inverse of $H(z)$

As we can see from the graph, h(n) is bounded in [-0.625,1.25]

5.4 Convergent? Justify using ratio test **Solution:**

$$Lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \left|\frac{\left(\frac{-1}{2}\right)^{n+1} + \left(\frac{-1}{2}\right)^{n-1}}{\left(\frac{-1}{2}\right)^{n} + \left(\frac{-1}{2}\right)^{n-2}}\right| \tag{5.14}$$

$$= \frac{1}{2} \tag{5.15}$$

Using ratio test,the system is convergent

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.16}$$

Is the system defined by (3.2) stable for the impulse response in (5.11)?
Solution:

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n + \sum_{n=2}^{\infty}\left(\frac{-1}{2}\right)^{n-2} \tag{5.17}$$

$$= \frac{2}{3} + \frac{2}{3} \tag{5.18}$$

$$= \frac{4}{3} < \infty \tag{5.19}$$

Hence the system is stable.

5.6 Verify the above result with a python code
Solution: The Following code computes and proves the aboves result

```
wget https://github.com/gaureeshk/
    signal_processing/blob/main/hn.
    py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.20}$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

```
wget https://raw.githubusercontent.
    com/gadepall/EE1310/master/
    filter/codes/hndef.py
```

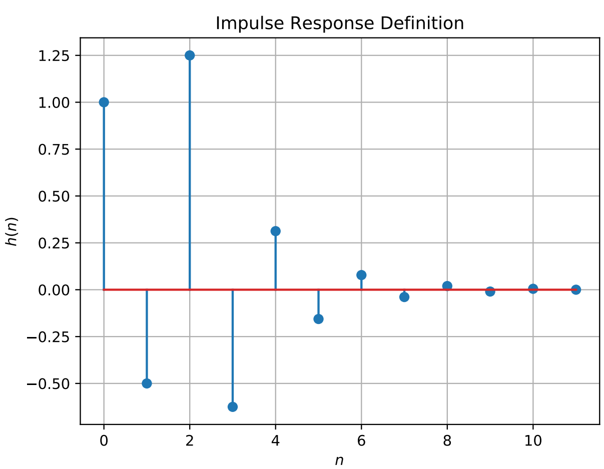

Fig. 5.7: $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \tag{5.21}$$

Comment. The operation in (5.21) is known as *convolution*.
Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://raw.githubusercontent.
    com/gadepall/EE1310/master/
    filter/codes/ynconv.py
```

5.9 Express the above convolution using a Teoplitz matrix. **Solution:** Since $x(n) = \{1, 2, 3, 4, 2, 1\}$

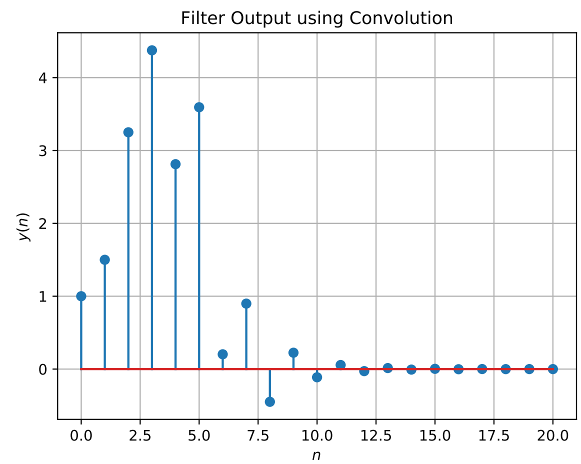

Fig. 5.8: $y(n)$ from the definition of convolution

From (5.21) $y(n) = x(n) * h(n)$

$$\begin{pmatrix} x(1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x(2) & x(1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x(3) & x(2) & x(1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x(4) & x(3) & x(2) & x(1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x(5) & x(4) & x(3) & x(2) & x(1) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x(6) & x(5) & x(4) & x(3) & x(2) & x(1) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x(6) & x(5) & x(4) & x(3) & x(2) & x(1) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x(6) & x(5) & x(4) & x(3) & x(2) & x(1) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x(6) & x(5) & x(4) & x(3) & x(2) & x(1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x(6) & x(5) & x(4) & x(3) & x(2) & x(1) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x(6) & x(5) & x(4) & x(3) & x(2) & x(1) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x(6) & x(5) & x(4) & x(3) & x(2) & x(1) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x(6) & x(5) & x(4) & x(3) & x(2) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x(6) & x(5) & x(4) & x(3) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x(6) & x(5) & x(4) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x(6) & x(5) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x(6) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} h(1) \\ h(2) \\ h(3) \\ h(4) \\ h(5) \\ h(6) \\ h(7) \\ h(8) \\ h(9) \\ h(10) \\ h(11) \\ h(12) \end{pmatrix} \tag{5.22}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 4 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 4 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1. \\ -0.5 \\ 1.25 \\ -0.625 \\ 0.3125 \\ -0.15625 \\ 0.078125 \\ -0.0390625 \\ 0.01953125 \\ -0.00976562 \\ 0.00390625 \\ -0.00195312 \end{pmatrix} \tag{5.23}$$

$$= \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ 4.375 \\ 2.8125 \\ 3.59375 \\ 0.203125 \\ 0.8984375 \\ -0.44921875 \\ 0.224609375 \\ -0.112304688 \\ 0.0561523438 \\ -0.0280761719 \\ 0.0140380859 \\ -7.01904297\times10^{-3} \\ 3.50952148\times10^{-3} \\ -1.75476074\times10^{-3} \\ 8.77380371\times10^{-4} \\ -4.38690186\times10^{-4} \\ 0 \end{pmatrix} \tag{5.24}$$

5.10 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.25}$$

Solution: Put $k' = n - k$ in (5.25)

$$\sum_{k'=\infty}^{-\infty} x(k')h(n-k') = \sum_{k'=-\infty}^{\infty} x(k')h(n-k') \tag{5.26}$$

$$= y(n) \tag{5.27}$$

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.

Solution:

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution:

Fig. 6.2: $Y(k)$ using DFT

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://raw.
      githubusercontent.com/
      gadepall/EE1310/master/
      filter/codes/yndft.py
```

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution: Download the code from

```
wget https://raw.githubusercontent.com/
    RaghavJuyal/EE3900/blob/main/Sound/
    codes/e6_4.py
```

Observe that Fig. (6.4) is the same as $y(n)$ in Fig. (3.2).

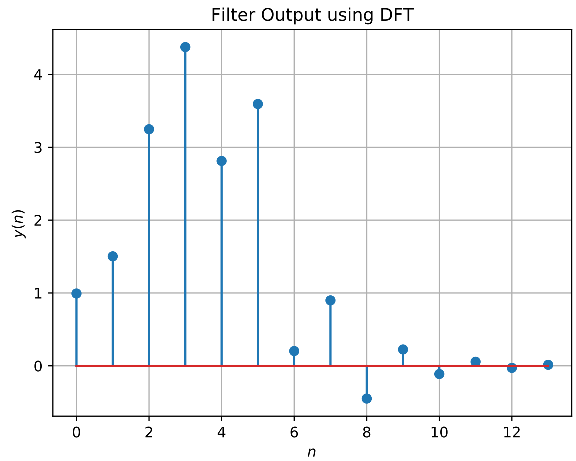

Fig. 6.3: $y(n)$ from the DFT

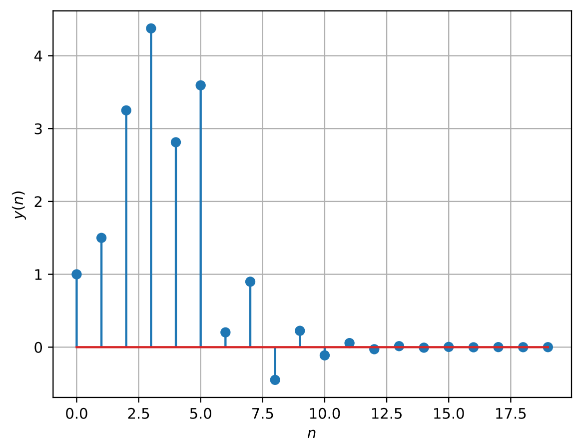

Fig. 6.4: $y(n)$ using FFT and IFFT

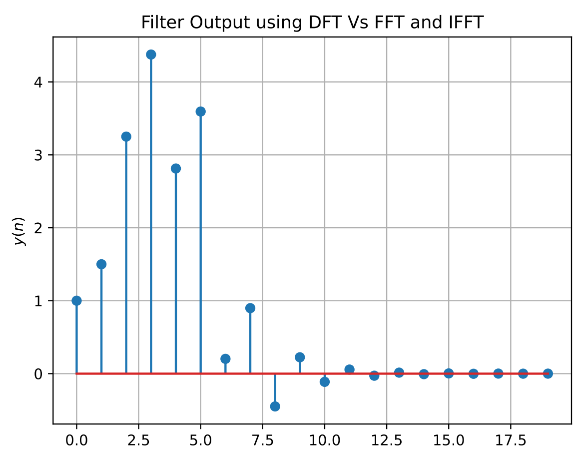

Fig. 6.4: $y(n)$ by DFT vs FTT and IFFT

6.5 Wherever possible, express all the above equations as matrix equations. **Solution:** We use the DFT Matrix, where $\omega = e^{-\frac{j2k\pi}{N}}$, which is given by

$$\mathbf{W} = \begin{pmatrix} \omega^0 & \omega^0 & \dots & \omega^0 \\ \omega^0 & \omega^1 & \dots & \omega^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^0 & \omega^{N-1} & \dots & \omega^{(N-1)(N-1)} \end{pmatrix} \tag{6.4}$$

i.e. $W_{jk} = \omega^{jk}$, $0 \le j, k < N$. Hence, we can write any DFT equation as

$$\mathbf{X} = \mathbf{W}\mathbf{x} = \mathbf{x}\mathbf{W} \tag{6.5}$$

where

$$\mathbf{x} = \begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(n-1) \end{pmatrix} \tag{6.6}$$

Using (**??**), the inverse Fourier Transform is given by

$$\mathbf{x} = \mathcal{F}^{-1}(\mathbf{X}) = \mathbf{W}^{-1}\mathbf{X} = \frac{1}{N}\mathbf{W}^{\mathbf{H}}\mathbf{X} = \frac{1}{N}\mathbf{X}\mathbf{W}^{\mathbf{H}} \tag{6.7}$$

$$\implies \mathbf{W}^{-1} = \frac{1}{N}\mathbf{W}^{\mathbf{H}} \tag{6.8}$$

where H denotes hermitian operator. We can rewrite (**??**) using the element-wise multiplication operator as

$$\mathbf{Y} = \mathbf{H} \cdot \mathbf{X} = (\mathbf{W}\mathbf{h}) \cdot (\mathbf{W}\mathbf{x}) \tag{6.9}$$

The plot of $y(n)$ using the DFT matrix in Fig. (5) is the same as $y(n)$ in Fig. (3.2). Download the code using

```
wget https://raw.
    githubusercontent.com/
    RaghavJuyal/EE3900/
    blob/main/Sound/codes/
    e6_5.py
```

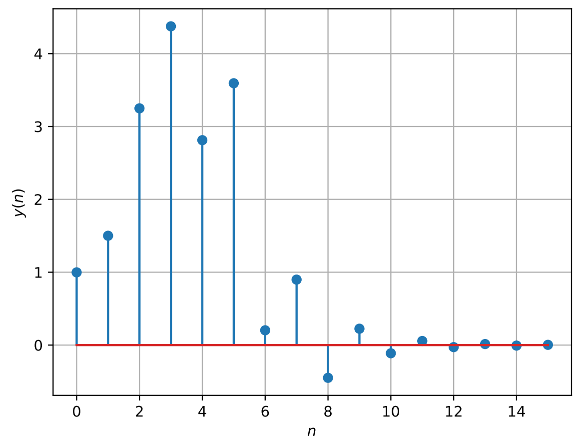

Fig. 5: $y(n)$ using the DFT matrix

7 FFT

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{7.1}$$

2. Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

Then the N-point *DFT matrix* is defined as

$$\vec{F}_N = [W_N^{mn}], \quad 0 \le m, n \le N-1 \tag{7.3}$$

where W_N^{mn} are the elements of $\vec{F}_N$.

3. Let

$$\vec{I}_4 = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^2 & \vec{e}_4^3 & \vec{e}_4^4 \end{pmatrix} \tag{7.4}$$

be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as

$$\vec{P}_4 = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^3 & \vec{e}_4^2 & \vec{e}_4^4 \end{pmatrix} \tag{7.5}$$

4. The 4 point *DFT diagonal matrix* is defined as

$$\vec{D}_4 = diag\begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{7.6}$$

5. Show that

$$W_N^2 = W_{N/2} \tag{7.7}$$

Solution:

$$W_N = e^{-j2\pi/N} \tag{7.8}$$

$$\implies W_{N/2} = e^{-j2\pi/(N/2)} \tag{7.9}$$

$$W_N^2 = e^{2(-j2\pi/N)} = e^{-j2\pi/(N/2)} = W_{N/2} \tag{7.10}$$

(7.11)

6. Show that

$$\vec{F}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \vec{P}_4 \tag{7.12}$$

7. Show that
$$\vec{F}_N = \begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix} \begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix} \vec{P}_N \tag{7.13}$$
8. Find
$$\vec{P}_4 \vec{x} \tag{7.14}$$
9. Show that
$$\vec{X} = \vec{F}_N \vec{x} \tag{7.15}$$
where $\vec{x}, \vec{X}$ are the vector representations of $x(n), X(k)$ respectively.
10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on
$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.16}$$
$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.17}$$
4-point FFTs into 2-point FFTs
$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.18}$$
$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.19}$$
$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.20}$$
$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.21}$$
$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.22}$$
$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.23}$$
$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.24}$$
Therefore,
$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.25}$$
$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.26}$$
$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.27}$$
$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.28}$$
11. For
$$\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.29}$$
compte the DFT using (7.15)
12. Repeat the above exercise using the FFT after zero padding $\vec{x}$.
13. Write a C program to compute the 8-point FFT.

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

8.1 The command

```
output_signal = signal.
    lfilter(b, a, input_signal
    )
```

in Problem 2.3 is executed through the following difference equation
$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{8.1}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

8.2 Repeat all the exercises in the previous sections for the above a and b.

8.3 What is the sampling frequency of the input signal?

Solution: Sampling frequency(fs)=44.1kHZ.

8.4 What is type, order and cutoff-frequency of the above butterworth filter

Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

8.5 Modifying the code with different input parameters and to get the best possible output.