Circuits and Transforms

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Contents

***Abstract*—This manual provides a simple introduction to Transforms**

1 Definitions

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\,dt \tag{1.2}$$

2 Laplace Transform

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.

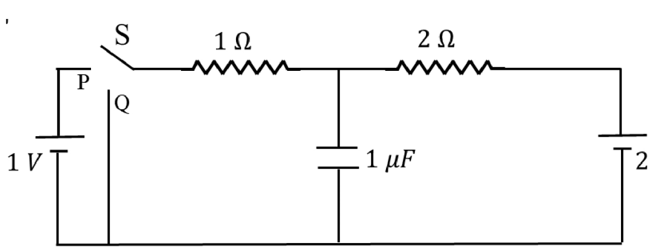

Fig. 2.1

2. Draw the circuit using latex-tikz.
Solution:

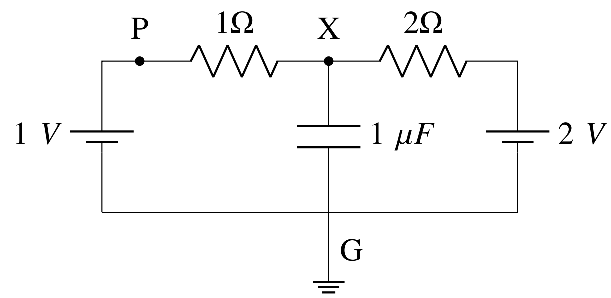

Fig. 2.2

3. Find q_1.
Solution: After a long time, the capacitor starts to behave like an open switch, which means that no current will flow through the capacitor. Assume that the circuit is grounded at the negative terminals of the battery and current in the circuit is i. Applying KVL in the loop:

$$1 + i + 2i - 2 = 0 \tag{2.1}$$

$$\implies i = \frac{1}{3}A \tag{2.2}$$

$$\frac{q_1\mu}{C} = 1 + \frac{1}{3} \tag{2.3}$$

$$\implies q_1 = \frac{4}{3} \tag{2.4}$$

4. Find q_1.
Solution:
After infinite time with switch at P
The capacitor is charged
Applying KCL at X:

$$\frac{V_x - 1}{1} = \frac{2 - V_x}{2} \tag{2.5}$$

$$\implies V_x = \frac{4}{3}\text{ V} \tag{2.6}$$

$$q_1 = CV = 1\ \mu\text{C} \tag{2.7}$$

5. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.

Solution:

$$\mathcal{U}(s) = \int_0^\infty u(t)e^{-st}dt \tag{2.8}$$

$$= \int_0^0 \frac{1}{2}e^{-st}dt + \int_0^\infty e^{-st}dt = \frac{1}{s} \tag{2.9}$$

$$\text{R.O.C: } Re(s) > 0 \tag{2.10}$$

6. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad a > 0 \tag{2.11}$$

and find the ROC.
Solution:

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \int_0^\infty u(t)e^{-(s+a)t}dt \tag{2.12}$$

$$= \frac{1}{s+a} \tag{2.13}$$

$$\text{R.O.C: } Re(s) > -a \tag{2.14}$$

7. Now consider the following resistive circuit transformed from Fig. 2.1 where

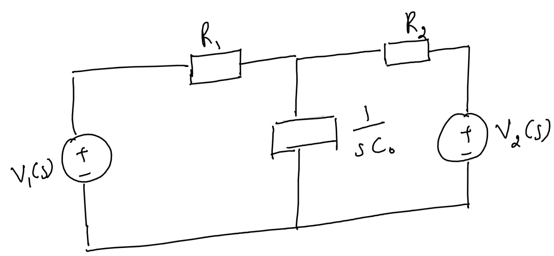

Fig. 2.3

$$u(t) \xleftrightarrow{\mathcal{L}} V_1(s) \tag{2.15}$$

$$2u(t) \xleftrightarrow{\mathcal{L}} V_2(s) \tag{2.16}$$

Find the voltage across the capacitor $V_{C_0}(s)$.

Solution: Applying KCL at X:

$$\frac{V_x - \frac{1}{s}}{R_1} + s(VC_0) = \frac{\frac{2}{s} - V_x}{R_2} \tag{2.17}$$

$$V(s) = \frac{\frac{1}{R_1} + \frac{2}{R_2}}{s\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right)} \tag{2.18}$$

$$= \frac{2R_1 + R_2}{R_1 + R_2}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right) \tag{2.19}$$

$$= \frac{4}{3}\left(\frac{1}{s} - \frac{1}{\frac{3}{2C_0} + s}\right) \tag{2.20}$$

8. Find $v_{C_0}(t)$. Plot using python.

$$v_{C_0}(t) = \frac{2R_1 + R_2}{R_1 + R_2}u(t)\left(1 - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}\right) \tag{2.21}$$

$$v_{C_0}(t) = \frac{4}{3}\left(1 - e^{-\left(1.5\times10^6\right)t}\right)u(t) \tag{2.22}$$

9. Verify your result using ngspice.
Solution:

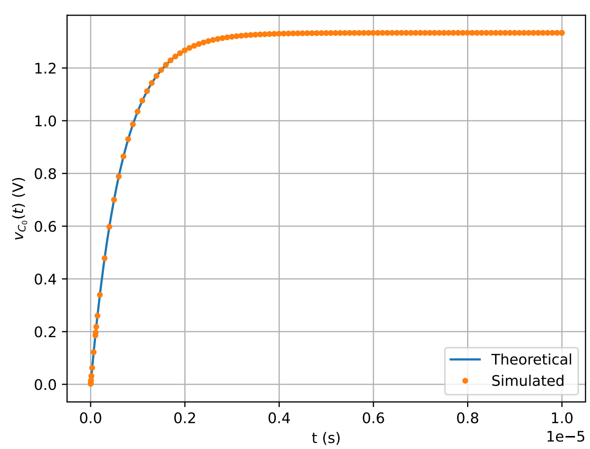

Fig. 2.4: $v_{C_0}(t)$ before the switch is flipped

10. Obtain Fig. 2.3 using the equivalent differential equation.

3 Initial Conditions

1. Find q_2 in Fig. 2.1.
Solution: The circuit at steady state when the

switch is at Q:
At steady state: Capacitor is charged

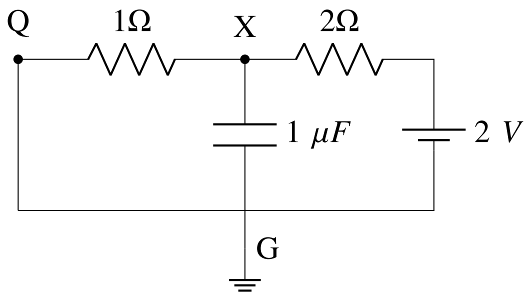

Fig. 3.1

Applying KCL at X.

$$\frac{V-0}{1}+\frac{V-2}{2}=0$$
$$\implies V=\frac{2}{3}V$$
$$q_2=\frac{2}{3}\mu C$$

1) Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz.
Solution:

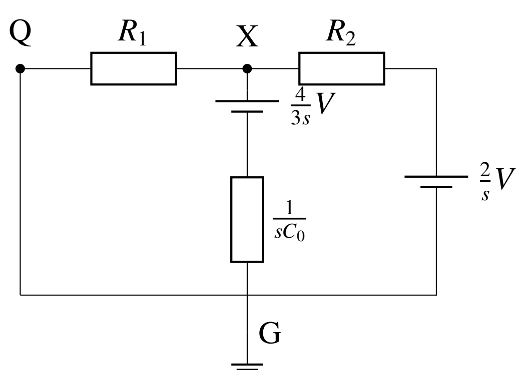

Fig. 3.2

2) $V_{C_0}(s)=$? **Solution:**

Applying KCL at node X in Fig. 3.2

$$\frac{V-0}{R_1}+\frac{V-\frac{2}{s}}{R_2}+sC_0\left(V-\frac{4}{3s}\right)=0 \quad (3.1)$$

$$\implies V_{C_0}(s)=\frac{\frac{2}{sR_2}+\frac{4C_0}{3}}{\frac{1}{R_1}+\frac{2}{R_2}+sC_0} \quad (3.2)$$

$$V_{C_0}(s)=\frac{4}{3}\left(\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right)$$
$$+\frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(\frac{1}{s}-\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) \quad (3.3)$$

3) $v_{C_0}(t)=$? Plot using python. Taking an inverse Laplace Transform,

$$v_{C_0}(t)=\frac{4}{3}e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t)$$
$$+\frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(1-e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}\right)u(t) \quad (3.4)$$

Substituting values gives

$$v_{C_0}(t)=\frac{2}{3}\left(1+e^{-\left(1.5\times10^6\right)t}\right)u(t) \quad (3.5)$$

The python code used to plot Fig. 3.3

https://github.com/gaureeshk/sig/cktsig/codes/e3.4.py

4) Verify your result using ngspice.
Solution: The following ngspice script simu-

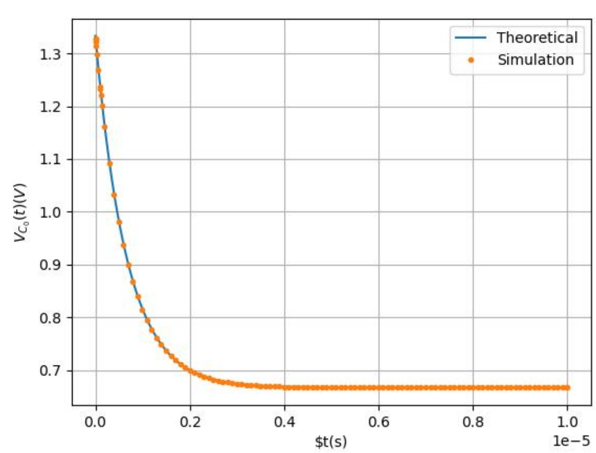

Fig. 3.3: $v_{C_0}(t)$ after the switch is flipped

lates the given circuit

https://github.com/gaureeshk/sig/cktsig/codes/e3.cir

5) Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.
Solution:

$$v_{C_0}(0-) = \lim_{t\to 0-} v_{C_0}(t) = \frac{q_1}{C} = \frac{4}{3}\text{ V} \tag{3.6}$$

$$v_{C_0}(0+) = \lim_{t\to 0+} v_{C_0}(t) = \frac{4}{3}V \tag{3.7}$$

$$v_{C_0}(\infty) = \lim_{t\to \infty} v_{C_0}(t) = \frac{2}{3}V \tag{3.8}$$

6) Obtain the Fig. in problem 1 using the equivalent differential equation.
Solution: The equivalent circuit in the t-domain is shown below.

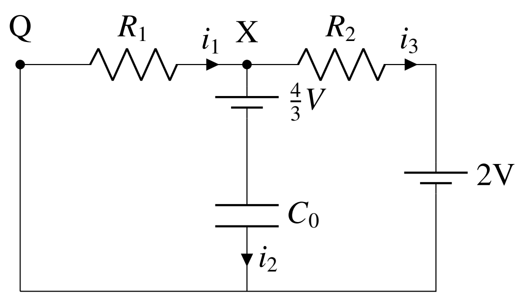

Fig. 3.4

From KCL and KVL,

$$i_1 = i_2 + i_3 \tag{3.9}$$

$$i_1R_1 + \frac{4}{3} + \frac{1}{C_0}\int_0^t i_2dt = 0 \tag{3.10}$$

$$\frac{4}{3} + \frac{1}{C_0}\int_0^t i_2dt - i_3R_2 - 2 = 0 \tag{3.11}$$

Taking Laplace Transforms on both sides and using the properties of Laplace Transforms,

$$\text{Let } i(t) \xleftrightarrow{\mathcal{L}} I(s) \tag{3.12}$$

$$\implies I_1 = I_2 + I_3 \tag{3.13}$$

$$I_1R_1 + \frac{4}{3} + \frac{1}{sC_0}I_2 = 0 \tag{3.14}$$

$$\frac{4}{3} + \frac{1}{sC_0}I_2 - I_3R_2 - 2 = 0 \tag{3.15}$$

4 Bilinear Transform

1. In Fig. 2.1, consider the case when S is switched to Q right in the beginning. Formulate the differential equation.

Solution: The equivalent circuit in the t-domain is shown below.

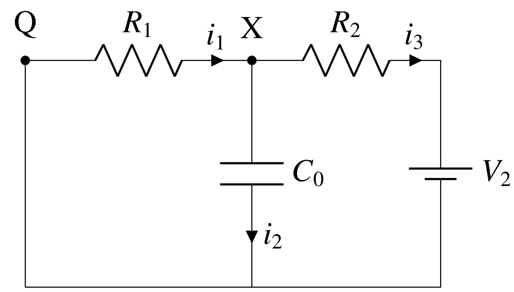

Fig. 4.1

Applying KCL and KVL,

$$i_1 = i_2 + i_3 \tag{4.1}$$

$$i_1R_1 + \frac{1}{C_0}\int_0^t i_2\,dt = 0 \tag{4.2}$$

$$i_3R_2 + 2 - \frac{1}{C_0}\int_0^t i_2\,dt = 0 \tag{4.3}$$

Differentiating the above equations,

$$\frac{di_1}{dt} = \frac{di_2}{dt} + \frac{di_3}{dt} \tag{4.4}$$

$$R_1\frac{di_1}{dt} + \frac{i_2}{C_0} = 0 \tag{4.5}$$

$$R_2\frac{di_3}{dt} - \frac{i_2}{C_0} = 0 \tag{4.6}$$

Using (4.4) and (4.6) in (4.5),

$$R_1\left(\frac{di_2}{dt} + \frac{di_3}{dt}\right) + \frac{i_2}{C_0} = 0 \tag{4.7}$$

$$R_1\frac{di_2}{dt} + \left(1 + \frac{R_1}{R_2}\right)\frac{i_2}{C_0} = 0 \tag{4.8}$$

$$\frac{di_2}{dt} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{i_2}{C_0} = 0 \tag{4.9}$$

$$\frac{di_2}{dt} + \frac{i_2}{\tau} = 0 \tag{4.10}$$

where $\tau = \frac{C_0R_1R_2}{R_1+R_2}$ is the RC time constant $i_2(0) = \dfrac{V_2}{R_2}$ and $i_2 = C_0\dfrac{dV}{dt}$, where $V = V_{C_0}$ is the voltage of the capacitor.

$$C_0\frac{dV}{dt} - \frac{V_2}{R_2} + \frac{C_0 V}{\tau} = 0 \tag{4.11}$$

$$\implies \frac{dV}{dt} + \frac{V}{\tau} = \frac{V_2}{C_0 R_2} \tag{4.12}$$

2. Find $H(s)$ considering the ouput voltage at the capacitor.
Solution:

$$H(s) = \frac{V_{C_0}(s)}{V_2(s)} \tag{4.13}$$

In the s domain:

$$\frac{V_{C_0}}{R_1} + \frac{V_{C_0}}{\frac{1}{sC_0}} + \frac{V_{C_0} - V_2}{R_2} = 0 \tag{4.14}$$

$$H(s)\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{1}{R_2} \tag{4.15}$$

$$H(s) = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} = \frac{1}{3 + sC_0} \tag{4.16}$$

3. Plot $H(s)$. What kind of filter is it?

https://github.com/gaureeshk/sig/cktsig/codes/e4.3.py

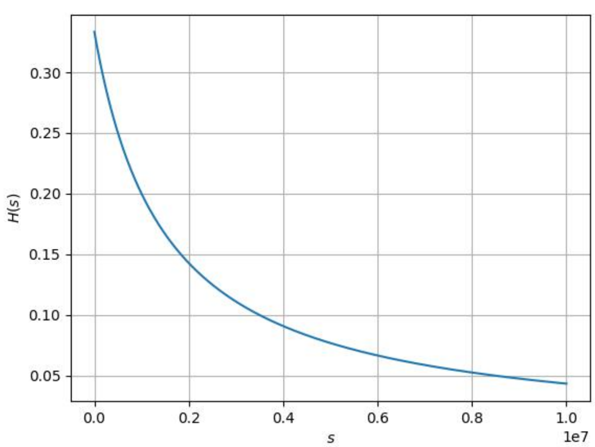

Fig. 4.2: Plot of $H(s)$.

It is a Low-Pass filter

4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \tag{4.17}$$

Solution:
$v_{in} = v_2(n) = 2u(n)$ and $v_{out} = v(n)$
Integrating (4.12) from n to n+1 and using (2.16):

$$v(n+1) - v(n) + \frac{v(n+1) + v(n)}{2\tau} = \frac{2}{C_0R_2}\frac{(u(n+1) + u(n))}{2} \tag{4.18}$$

$$v(n+1)(2\tau + 1) + v(n)(1 - 2\tau) = \frac{2}{C_0R_2}\tau(u(n+1) + u(n)) \tag{4.19}$$

$$v(n)(2\tau + 1) + v(n-1)(1 - 2\tau) = \frac{2}{C_0R_2}\tau(u(n) + u(n-1)) \tag{4.20}$$

5. Find $H(z)$.
Solution:

$$H(z) = \frac{V(z)}{V_2(z)} \tag{4.21}$$

We know that

$$v_2(n) = 2u(n) \implies V_2(z) = \frac{2}{1 - z^{-1}} \tag{4.22}$$

Applying $\mathcal{Z}$-transform on (4.20)

$$V(z)(2\tau + 1) + z^{-1}V(z)(1 - 2\tau) = \frac{2\tau(1 + z^{-1})}{C_0R_2(1 - z^{-1})} \tag{4.23}$$

$$V(z)\left(2\tau + 1 - z^{-1}(2\tau - 1)\right) = \frac{\tau V_2(z)(1 + z^{-1})}{C_0R_2} \tag{4.24}$$

$$H(z) = \frac{V(z)}{V_2(z)} = \frac{\tau(1 + z^{-1})}{C_0R_2\left(2\tau + 1 - z^{-1}(2\tau - 1)\right)} \tag{4.25}$$

$$H(z) = \frac{\tau(1 + z^{-1})}{R_2C_0\left(2\tau + 1 - z^{-1}(2\tau - 1)\right)} \tag{4.26}$$

$$= \frac{\tau(z + 1)}{R_2C_0(2\tau(z - 1) + z + 1)} \tag{4.27}$$

$$= \frac{1}{R_2C_0\left(2\frac{z - 1}{z + 1} + \frac{1}{\tau}\right)} \tag{4.28}$$

But $\tau = \frac{C_0 R_1 R_2}{R_1 + R_2} = \frac{2C_0}{3}$ and $R_2 = 2$

$$\implies H(z) = \frac{1}{2C_0\left(2\frac{z-1}{z+1} + \frac{3}{2C_0}\right)} \tag{4.29}$$

$$H(z) = \frac{1}{3 + 4C_0\frac{z-1}{z+1}} \tag{4.30}$$

R.O.C: $|z| < 1$

6. How can you obtain $H(z)$ from $H(s)$?
Solution:
Apply a Bilinear Transform

$$s \longrightarrow \frac{2}{T}\frac{z-1}{z+1} \tag{4.31}$$

$$H(z) = \frac{1}{3 + C_0\frac{2}{T}\frac{z-1}{z+1}} \tag{4.32}$$

Putting T = $\frac{1}{2}$ will give (4.30)

7. Find $v(n)$. Verify using ngspice and differential equation
Solution:

$$V(z) = H(z)V_2(z) \tag{4.33}$$

$$= \frac{1}{3 + \frac{2}{T}C_0\frac{1-z^{-1}}{1+z^{-1}}}\frac{2}{1-z^{-1}} \tag{4.34}$$

$$= \frac{2}{3}\left(\frac{1}{1 + \frac{\tau}{T}\frac{1-z^{-1}}{1+z^{-1}}}\frac{1}{1-z^{-1}}\right) \tag{4.35}$$

$$= \frac{2}{3}\left(\frac{1+z^{-1}}{1+z^{-1}+\frac{\tau}{T}\left(1-z^{-1}\right)}\frac{1}{1-z^{-1}}\right) \tag{4.36}$$

$$= \frac{2}{3}\left(\frac{1}{1-z^{-1}} - \frac{\frac{\tau}{T}}{1+z^{-1}+\frac{\tau}{T}\left(1-z^{-1}\right)}\right) \tag{4.37}$$

$$= \frac{2}{3}\left(\frac{1}{1-z^{-1}} - \frac{\frac{\tau}{T}}{1+\frac{\tau}{T}+z^{-1}\left(1-\frac{\tau}{T}\right)}\right) \tag{4.38}$$

$$= \frac{2}{3}\left(\frac{1}{1-z^{-1}} - \frac{\frac{\tau}{T}}{1+\frac{\tau}{T}}\frac{1}{1-\left(\frac{\frac{\tau}{T}-1}{\frac{\tau}{T}+1}\right)z^{-1}}\right) \tag{4.39}$$

Taking Inverse $\mathcal{Z}$ transform on both sides

$$v(n) = \frac{2}{3}u(n)\left[1 - \frac{\frac{\tau}{T}}{\frac{\tau}{T}+1}\left(\frac{\frac{\tau}{T}-1}{\frac{\tau}{T}+1}\right)^n\right] \tag{4.40}$$

Python code used to plot theoretical and simulated values of $v(n)$ can be found at:

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https://github.com/gaureeshk/sig/cktsig/codes/
    e4.7.py
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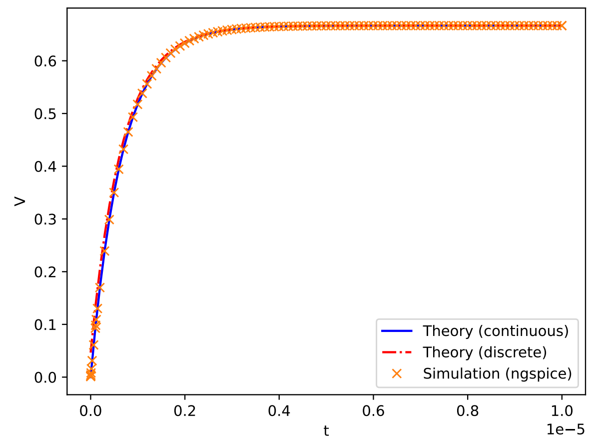

Fig. 4.3