Moment Generating Function (MGF) in Statistics

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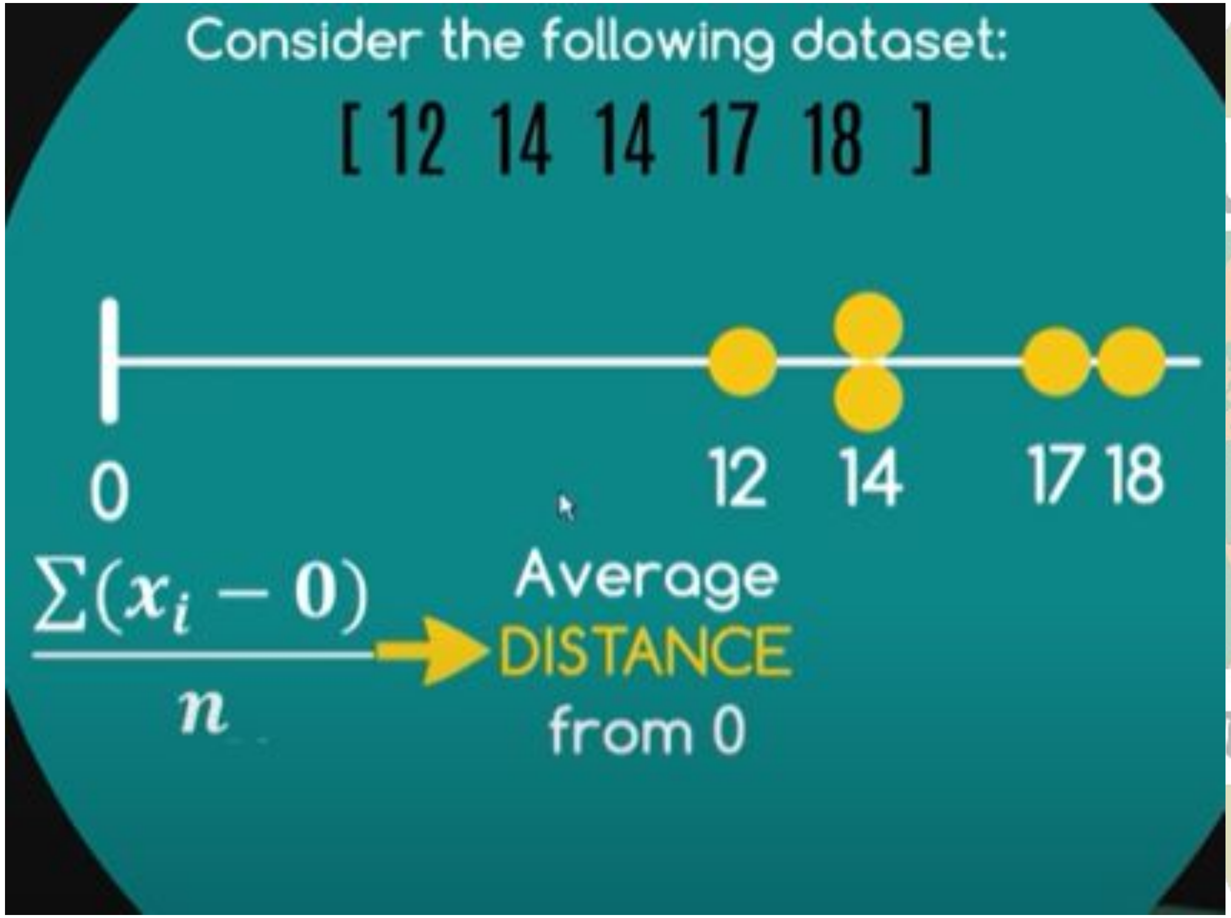
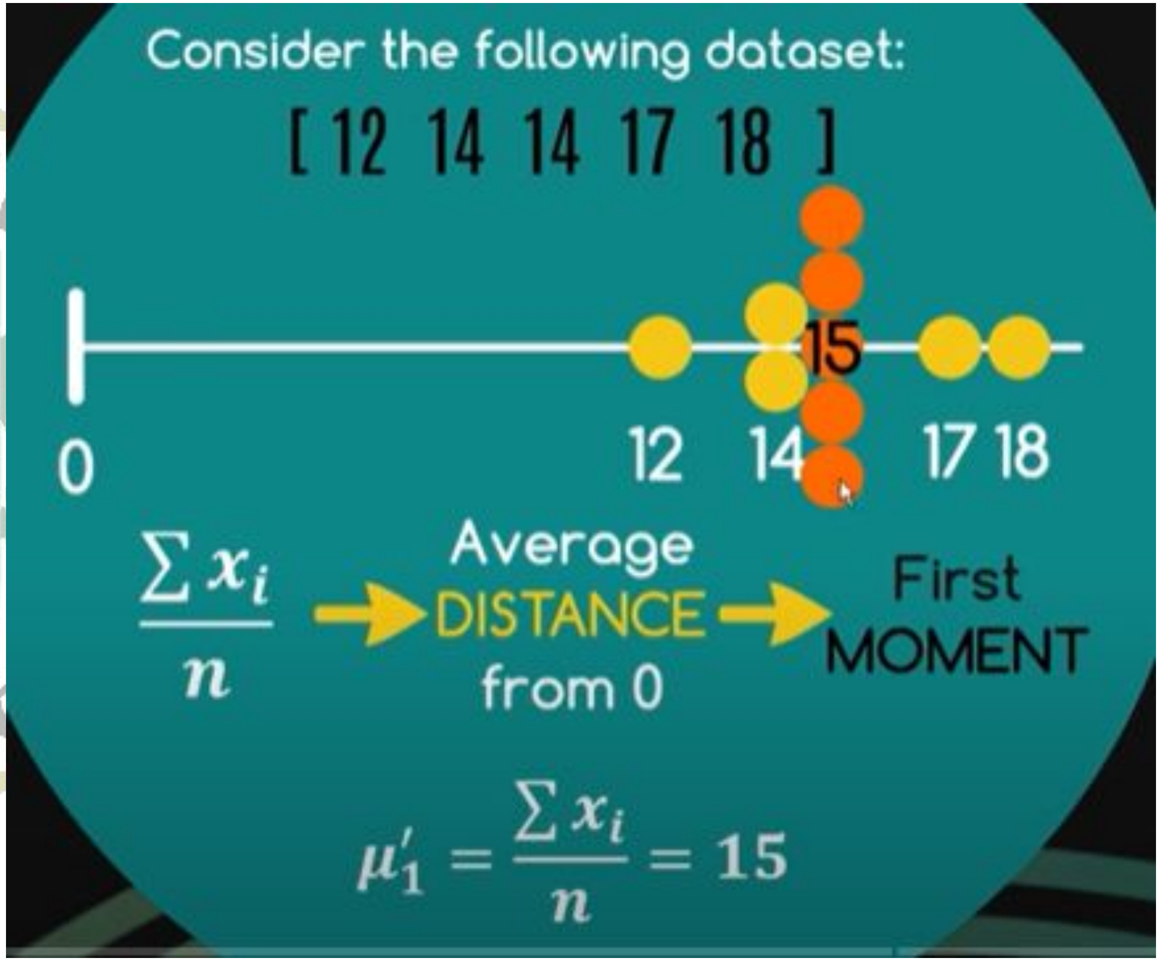
What is a “Moment” in Statistics?

What does it mean?

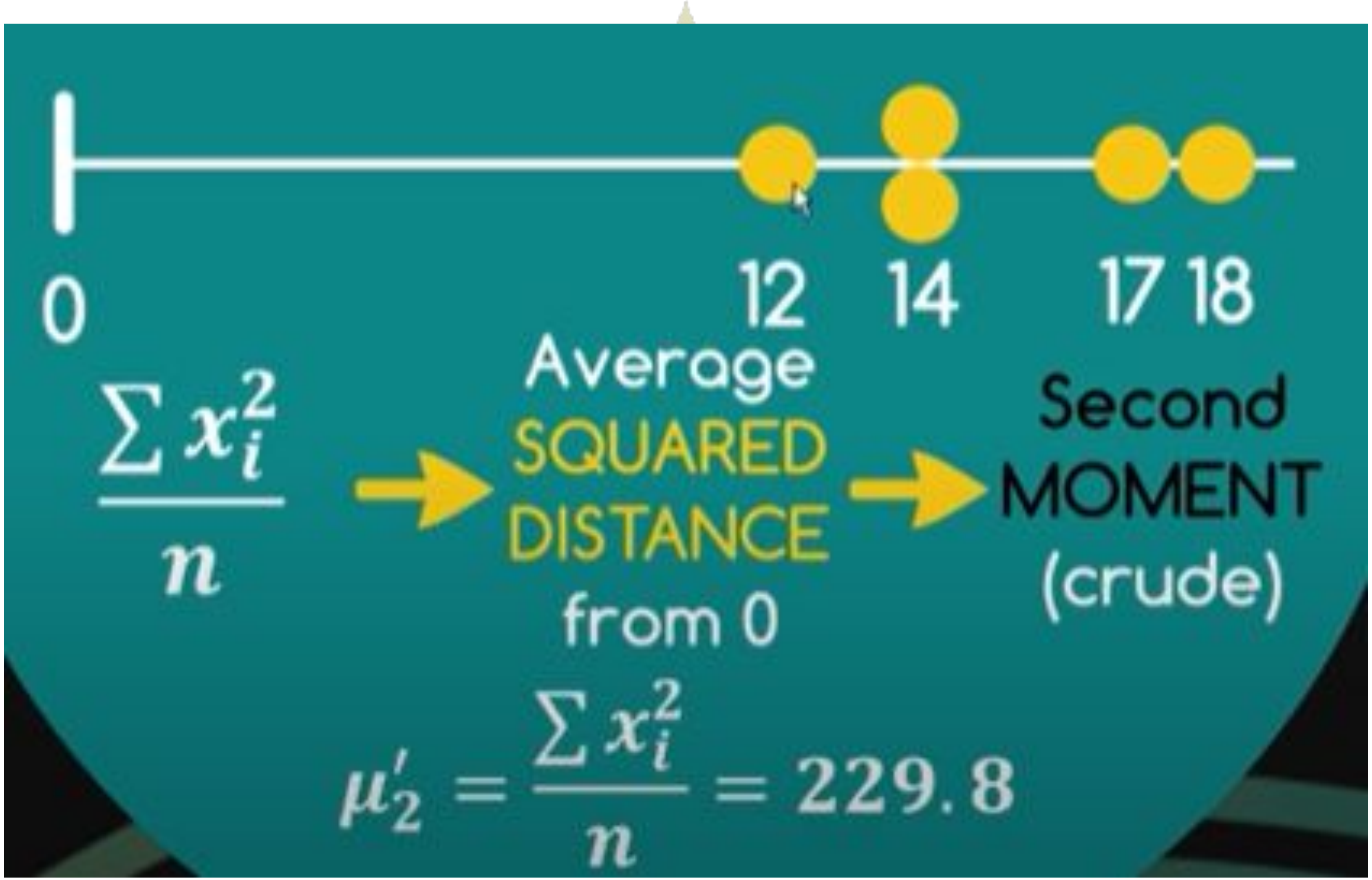
First Moment = Mean of the data





"moment" refers to how probability mass is distributed.

Second Moment of data – Now we square the values and take their sum



Moment is denoted by E(X)

General formula for calculating moment

"s" th moment-

The *s*th moment of the data set with values $x_1, x_2, x_3, \ldots, x_n$ is given by the formula:

$$(x_1^s + x_2^s + x_3^s + \ldots + x_n^s)/n$$

First Moment

For the first moment, we set $s = 1$. The formula for the first moment is thus:

$$E(x) = (x_1 + x_2 + x_3 + \ldots + x_n)/n$$

This is identical to the formula for the sample mean.

Other Moments of Data

For the second moment we set s = 2. The formula for the second moment is:

$$E(x^2) = (x_1^{\ 2} + x_2^{\ 2} + x_3^{\ 2} + \ldots + x_n^{\ 2})/n$$

For the third moment we set s = 3. The formula for the third moment is:

$$E(x^3) = (x_1^{\ 3} + x_2^{\ 3} + x_3^{\ 3} + \ldots + x_n^{\ 3})/n$$

For the fourth moment we set s = 4. The formula for the fourth moment is:

$$E(x^4) = (x_1^{\ 4} + x_2^{\ 4} + x_3^{\ 4} + \ldots + x_n^{\ 4})/n$$

And so on

Moments can also be calculated around a measurement

For example, let's take moments around mean:

1. First, calculate the mean of the values.
2. Next, subtract this mean from each value.
3. Then raise each of these differences to the *s*th power.
4. Now add the numbers from step #3 together.
5. Finally, divide this sum by the number of values we started with.

The formula for the *s*th moment about the mean m of the values values $x_1, x_2, x_3, \ldots, x_n$ is given by:

$$m_s = ((x_1 - m)^s + (x_2 - m)^s + (x_3 - m)^s + \ldots + (x_n - m)^s)/n$$

- First Moment About the Mean

$$m_1 = ((x_1 - m) + (x_2 - m) + (x_3 - m) + ... + (x_n - m))/n = ((x_1 + x_2 + x_3 + ... + x_n) - nm)/n = m - m = 0 \text{ (zero)}.$$

- Second Moment About the Mean

$$m_2 = ((x_1 - m)^2 + (x_2 - m)^2 + (x_3 - m)^2 + ... + (x_n - m)^2)/n$$

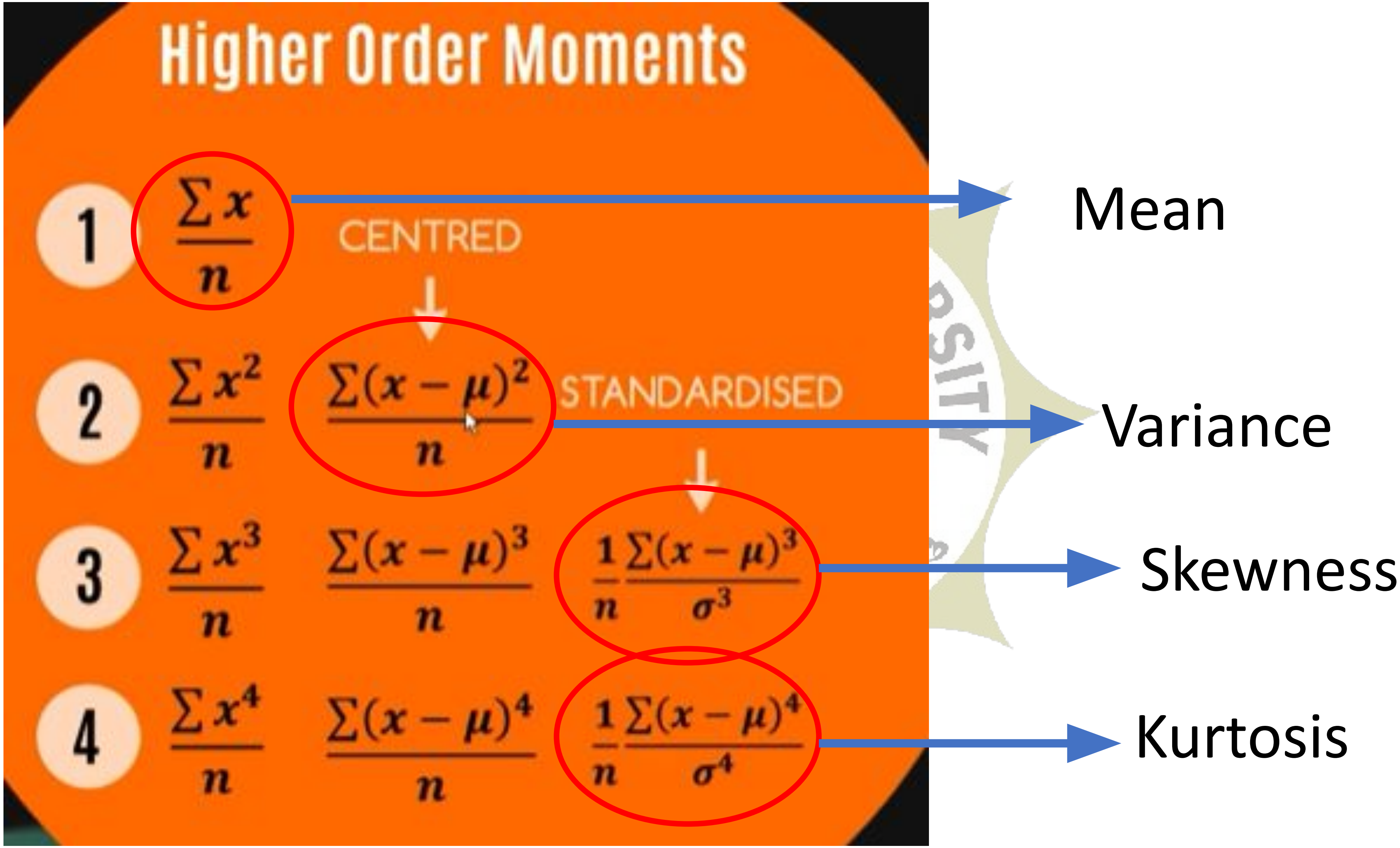
This formula is equivalent to that for the sample variance.

If we consider mean E(x) as:

$$\mu = E(X)$$

And if we consider standard deviation to be σ

Then We can find higher order moments as:


Higher Order Moments
1 $\frac{\sum x}{n}$
CENTRED
2 $\frac{\sum x^2}{n}$ $\frac{\sum (x-\mu)^2}{n}$
STANDARDISED
3 $\frac{\sum x^3}{n}$ $\frac{\sum (x-\mu)^3}{n}$ $\frac{1}{n}\frac{\sum (x-\mu)^3}{\sigma^3}$
4 $\frac{\sum x^4}{n}$ $\frac{\sum (x-\mu)^4}{n}$ $\frac{1}{n}\frac{\sum (x-\mu)^4}{\sigma^4}$
Mean
Variance
Skewness
Kurtosis

Then what is Moment Generating Function (MGF)?

As its name hints, MGF is the function that generates the moments — **E(X), E(X²), E(X³), … , E(X^n).**

$$MGF_X(t) := E[e^{tX}] = \begin{cases} \sum_x e^{tx} \cdot \overset{\text{PMF}}{P(x)} & X: \text{discrete} \\ \int_x e^{tx} \cdot \underset{\text{PDF}}{f(x)}\, dx & X: \text{continuous} \end{cases}$$

If you look at the definition of MGF, you might say…

"I'm not interested in knowing E(e^tx). I want E(X^n)."

Take a derivative of MGF n times and plug t = 0 in. **Then, you will get E(X^n)**.

This is how you get the moments from the MGF.

$$E(X^n) = \frac{d^n}{dt^n} MGF_X(t) \Big|_{t=0}$$

n-th moment

e.g.

$$E(X) = \frac{d}{dt} MGF_X(t) \Big|_{t=0} = MGF'_X(0)$$

$$E(X^2) = \frac{d^2}{dt^2} MGF_X(t) \Big|_{t=0} = MGF''_X(0)$$

$$\vdots$$

t is a helper variable. We introduced **t** in order to be able to use calculus (derivatives) and make the terms (that we are not interested in) zero.

Q) but we can calculate moments using the definition of expected values. Why do we need MGF exactly?

A)For convenience.
We want the MGF in order to calculate moments easily, for **Binomial(n, p)**, **Poisson(λ)**, **Exponential(λ)**, **Normal(0, 1)**, etc.

MGF of various probability distributions

Distribution	Moment-generating function $M_X(t)$
Degenerate δ_a	e^{ta}
Bernoulli $P(X=1)=p$	$1-p+pe^t$
Geometric $(1-p)^{k-1}\,p$	$\dfrac{pe^t}{1-(1-p)e^t}$ $\forall t < -\ln(1-p)$
Binomial $B(n,p)$	$\left(1-p+pe^t\right)^n$
Negative binomial $\mathrm{NB}(r,p)$	$\left(\dfrac{pe^t}{1-e^t+pe^t}\right)^r$
Poisson $\mathrm{Pois}(\lambda)$	$e^{\lambda(e^t-1)}$
Uniform (continuous) $\mathrm{U}(a,b)$	$\dfrac{e^{tb}-e^{ta}}{t(b-a)}$
Uniform (discrete) $\mathrm{DU}(a,b)$	$\dfrac{e^{at}-e^{(b+1)t}}{(b-a+1)(1-e^t)}$
Laplace $L(\mu,b)$	$\dfrac{e^{t\mu}}{1-b^2t^2},\ \lvert t\rvert < 1/b$
Normal $N(\mu,\sigma^2)$	$e^{t\mu+\frac{1}{2}\sigma^2t^2}$
Chi-squared χ^2_k	$(1-2t)^{-\frac{k}{2}}$
Noncentral chi-squared $\chi^2_k(\lambda)$	$e^{\lambda t/(1-2t)}(1-2t)^{-\frac{k}{2}}$
Gamma $\Gamma(k,\theta)$	$(1-t\theta)^{-k},\ \forall t < \frac{1}{\theta}$
Exponential $\mathrm{Exp}(\lambda)$	$\left(1-t\lambda^{-1}\right)^{-1},\ t < \lambda$
Multivariate normal $N(\mu,\Sigma)$	$e^{\mathbf{t}^{\mathrm{T}}\left(\boldsymbol{\mu}+\frac{1}{2}\boldsymbol{\Sigma}\mathbf{t}\right)}$

References

> **What are "moments" in statistics? An intuitive video! - zedstatistics, YouTube**

https://www.youtube.com/watch?v=ISaVvSO_3Sg

> **Moment Generating Function Explained -** Ms Aerin**, Towards Data Science**

https://towardsdatascience.com/moment-generating-function-explained-27821a739035

> **Moment Generating Function - Wikipedia**

https://en.wikipedia.org/wiki/Moment-generating_function

Thank you.

Hope you liked this presentation.

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