Applied Number Theory in Cryptography: RSA

I Am Curious

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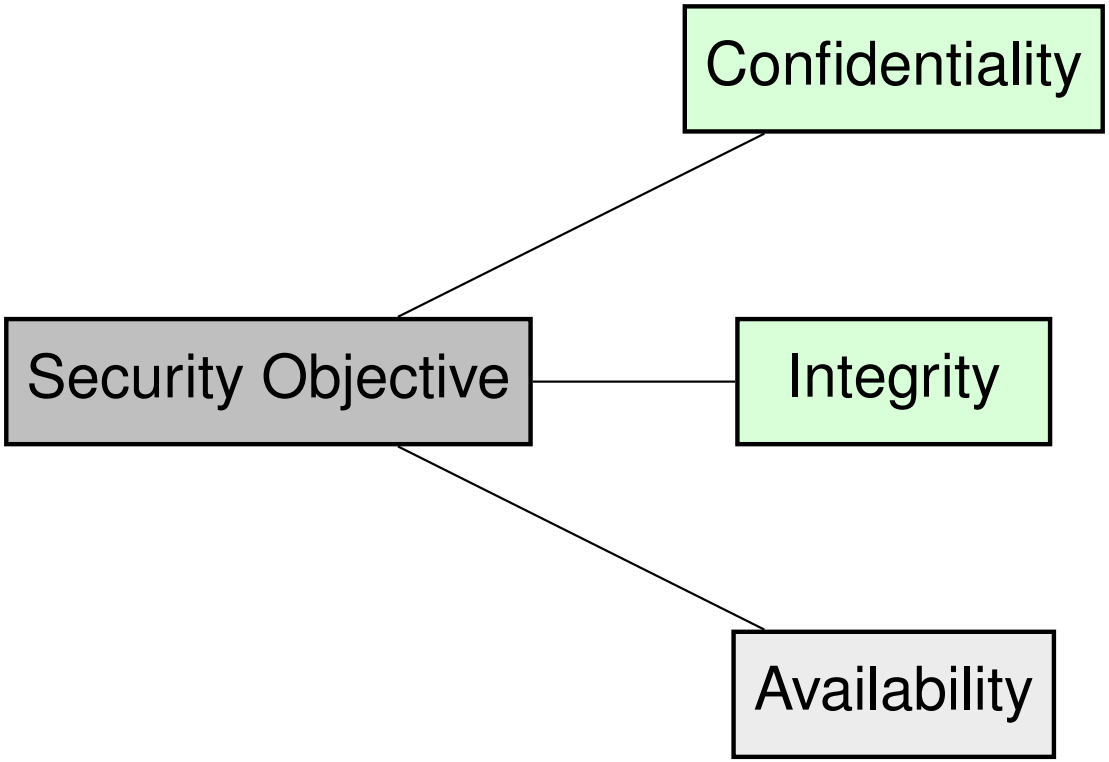
Learning objectives of this presentation

- In-depth understanding of RSA with sound mathematical basis
- Ability to assess constraints w.r.t. selection of various parameters in RSA (e.g. key length, set up of encryption key)
- Provide formal foundations for further reading in mathematical cryptography

Outline

Cryptography: Security Objectives



Cryptography Target

No Cryptography Target

The principle of cryptography

Cryptographic System

An encryption is a function $enc : K \times P \to C$ from the cartesian product of the set of keys K and the set of plaintexts P to the set cyphertexts C. A decryption is a function $dec : K \times C \to P$. A suitable decryption has a key k' such that

$$\forall k \in K \forall x \in P : dec(k', enc(k, x)) = x$$

Alternative notation for $enc(k, m)$ is $enc_k(m)$.

Desired Properties for Cryptographic Systems

1. $enc_k(m)$ must be efficiently computable
2. $dec_{k'}(m)$ must be efficiently computable
3. *Resilience against known ciphertext attack:* Given $c_1, ..., c_n \in C$, $dec_{k'}(c_i)$ must not be efficiently computable (without knowledge of k')
4. *Resilience against chosen ciphertext attack:* Given $(m_1, c_1), ..., (m_n, c_n)$, $dec_{k'}(c)$ with $c \neq c_i$ should not be efficiently computable

Symmetric and Asymmetric Cryptography

Symmetric and Asymmetric Ciphers

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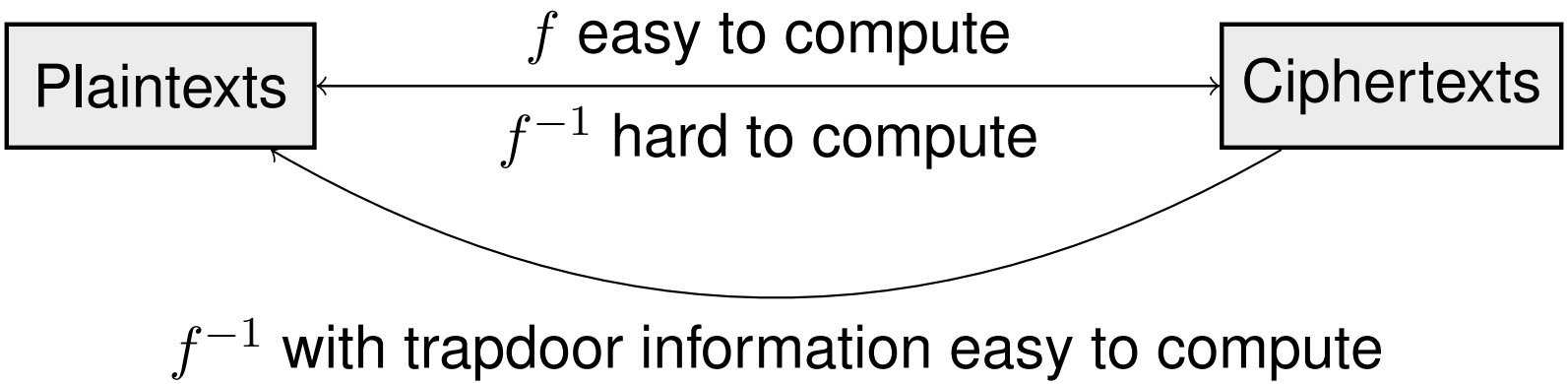

Outline

Now let's start with the math

Groups, Rings, and Fields

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- *Inverse Elements:* $\forall x \in G \exists y \in G : x + y = 0$. It is called *inverse* of x, also denoted as $-x$ or x^{-1} if multiplicatively denoted.

Groups, Rings, and Fields

Rings with 1

Let R be a set with at least two neutral elements $0, 1$ w.r.t. the operations $+, \cdot : R \times R \to R$. $(R, +, \cdot)$ is said to be a ring (with 1) if and only if the following properties hold.

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- $(R, +)$ is an Abelian group.
- For $(R, \cdot)$ Associativity and Commutativity hold and 1 is the neutral element.
- *Distributivity:* $\forall x, y, z \in R : (x + y) \cdot z = x \cdot z + y \cdot z$.

Example

An example is the ring of integers $\mathbb{Z}$.

Groups, Rings, and Fields

Fields

Let F be a set with at least two neutral elements $0, 1$ w.r.t. the operations $+, \cdot : F \times F \to F$. $(F, +, \cdot)$ is said to be a field if and only if the following properties hold.

- $(F, +)$ is an Abelian group.
- $(F, \cdot)$ is an Abelian group.
- Distributivity holds.

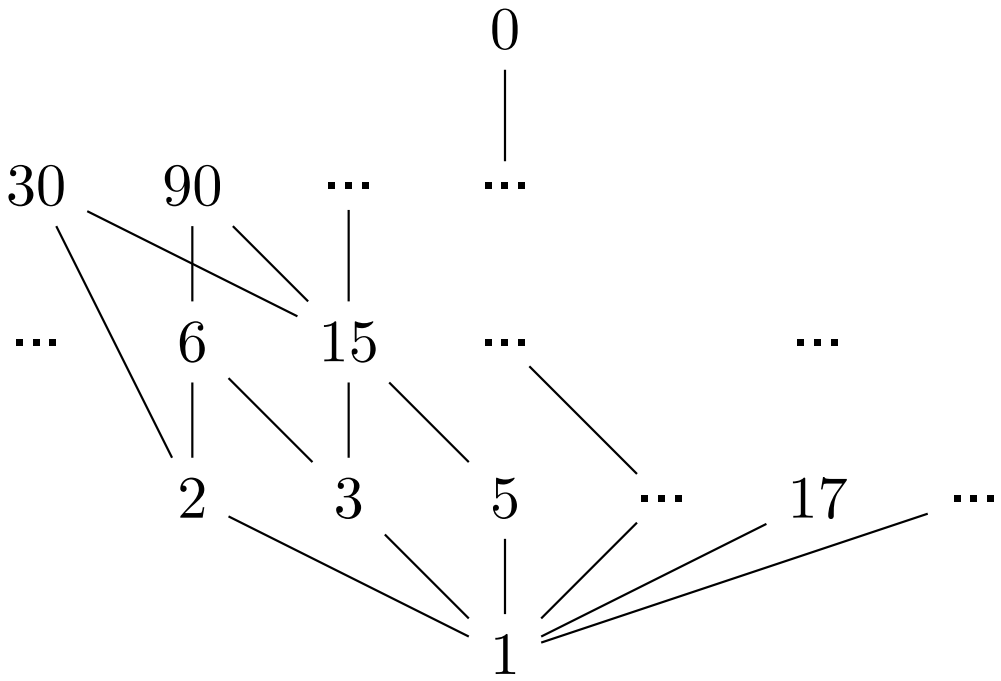
Example

Examples are the rational numbers $\mathbb{Q}$ and the real numbers $\mathbb{R}$.

Important relations on integers

Divisibility Relation

For $a, b \in \mathbb{Z}$ we define the divisibility relation $| \subset \mathbb{Z} \times \mathbb{Z}$ as $a|b :\Leftrightarrow \exists m : a \cdot m = b$. It is an order (reflexive, antisymmetric, transitive).



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For $a, b, n \in \mathbb{Z}$, we define the relation $\equiv \subset \mathbb{Z} \times \mathbb{Z}$ as $a \equiv b \mod n :\Leftrightarrow n|(a-b)$. It is an equivalence relation (reflexive, symmetric, transitive).

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An equivalence class for an element $a \in \mathbb{Z}$ is defined as

$$[a] := \{b \in \mathbb{Z} | a \equiv b \mod n\}$$

Elements in $[a]$ yield all the same residue by dividing through n. Note that

$$\forall x \in \mathbb{Z} \exists \xi, \eta \in \mathbb{Z} : x = \xi \cdot n + \eta \wedge 0 \leq \eta < n$$

and ξ, η are unique.

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We define operations $+, \cdot$ on it as

$$[a] + [b] := [a + b], [a] \cdot [b] := [a \cdot b]$$

This yields a ring $(\mathbb{Z}/n\mathbb{Z}, +, \cdot)$.

Subgroups and Lagrange's Theorem

Lagrange's theorem

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Corollary to Lagrange's Theorem

For every element $g \in G$ it holds $g^{ord(G)} = 1$.

Ring's multiplikative subgroups and Euler's phi

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Examples

$[1], [3] \in (\mathbb{Z}/4\mathbb{Z})^*$:
$[1]^{-1} = [1], [3]^{-1} = 3, [2] \notin (\mathbb{Z}/4\mathbb{Z})^*$ ($[2] \cdot [2] = [0], [2] \cdot [3] = [2]$)
$1, 2, 3, 4 \in (\mathbb{Z}/5\mathbb{Z})^*$

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Particularly if $n = p \cdot q$, we have

$$\varphi(n) = (p-1)\cdot(q-1)$$

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Now you no the basics, let's move on to the algorithm...

Encryption

Preparation - the public key

Select two primes p and q and calculate $n = p \cdot q$.

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Say, we encrypt $m = 8$. This yields $m^e = 8^7 \equiv 2 \mod 55$, which means 8 is encrypted by 2.

Decryption

Private Key

The private key d fulfills

$$d \cdot e \equiv 1 \mod \varphi(n)$$

d can be obtained via Enhanced Euclidean Algorithm.

Enhanced Euclidean Algorithm

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$$\begin{aligned}\gcd(40, 7) = 1 = 5 - 2 \cdot 2 = 5 - 2 \cdot (7 - 5) = -2 \cdot 7 + 3 \cdot 5 \\ = -2 \cdot 7 + 3 \cdot (40 - 5 \cdot 7) = 3 \cdot 40 - 17 \cdot 7\end{aligned}$$

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We were looking for d such that $d \cdot e \equiv 1 \mod \varphi(n)$. This translates in the example to $d = -17 \equiv 23 \mod 40$

Decryption Example

In our example, we have obtained $d = 23$. We had $m = 8$ and obtained $c = 2$. To decrypt, we calculate

$$c^d = 2^{23} \equiv 8 \mod 55$$

Soundness proof: From [complex looking math] it straightforwardly follows...

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Hence, $c^d = (m^e)^d = m^{e \cdot d} = m^{k \cdot \varphi(n)+1} = m^{k \cdot \varphi(n)} \cdot m = 1 \cdot m = m$

□

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Hence, $m^{k\cdot\varphi(n)+1} - m \equiv 0 \mod q$ and $m^{k\cdot\varphi(n)+1} - m \equiv 0 \mod p$.

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Hence, $m^{k\cdot\varphi(n)+1} - m \equiv 0 \mod q$ and $m^{k\cdot\varphi(n)+1} - m \equiv 0 \mod p$.
Since $\gcd(p, q) = 1$, this yields $m^{k\cdot\varphi(n)+1} - m \equiv 0 \mod p \cdot q$.

□

Are you still with me?

Outline

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$k^2 - n = 2589^2 - 6699557 = 58^2$. Hence,
$6699557 = 2589^2 - 58^2 = (2589 + 58)(2589 - 58) = 2647 \cdot 2531$

Small encryption exponent e

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Trivial: If $m < n^{1/e}$, then just take $e-$th root of c.
If the same message m is sent encrypted with small exponent e, e.g. 3 to different partners (using different moduli n_1, n_2, n_3), then the simultaneous congruence

$$x \equiv c_1 \mod n_1$$

$$x \equiv c_2 \mod n_2$$

$$x \equiv c_3 \mod n_3$$

can be solved by exploiting Chinese remainder theorem.
If $m^3 < n_1 n_2 n_3$, this also implies $x = m^3$, which in turn yields the plaintext m.
Solution: Salting (Add random bit string to message)

Other attacks

- Small decryption exponent d ($d < n^{1/4}$): Attackable if $\gcd(p-1, q-1)$ small. Then d can be calculated.
- Adaptive chosen cypher text attack due to homomorphic property: $(m_1 m_2)^e \equiv m_1^e m_2^e \equiv c_1 c_2 \mod n$.
- Coppersmith attacks: Exploit if messages depend linearly on each other, i.e., $m_1 = a \cdot m_2 + b$
- Cycling attacks: For $c = m^e \mod n$, there is a k such that $c^{e^k} \equiv c \mod n$. Hence, $c^{e^{k-1}} \equiv m \mod n$ (like taking $e-$th root of c). However, proven to be as hard (and therefore as low-likely as factorization)
- Message concealing: If $m^e \equiv m \mod n$ (very small probability).

Thank you for your attention

Any Questions?