First OJIMC 2021

Online International Mathematical Cup

Day I Problems

Problem 1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that any real numbers x, y satisfy

$$xf(y) + f(x + y) \geq (y + 1)f(x) + f(y).$$

Problem 2. Call a polynomial $P \in \mathbb{Z}[x]$ i-good if exactly one of i and $P(i)$ can be written as the sum of two squares. Prove that for any positive integer n there exists some P which is i-good for any $i \in \{1, 2, ..., n\}$. Does the exist a polynomial which is i-good for any $i \in \mathbb{N}$?

Problem 3. Alice has n (possibly empty) bags of coins labelled 1 to n. She performs moves as such: At the k^{th} move, she lists all the bags which have (strictly) less coins than the k^{th} bag (indices reduced modulo n), but differ by (strictly) less than n coins from it.

If there is at least one listed bag, Alice takes one coin from each such bag and puts them all in the k^{th} bag. Otherwise, she takes $n-1$ coins from the k^{th} bag and places one in each of the other bags.

If at some point Alice cannot perform a move, the process ends. Find the greatest total number of coins Alice can have, such that the process will end regardless of the initial distribution of the coins in the bags.

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Note. Each problem is worth 7 points. You have 4 hours and 30 minutes to solve the problems and write down your solutions. The use of calculators, engines and any other form of external help is forbidden. For more information, see the official guidelines.