First OIMC 2021

Online International Mathematical Cup

Day II Problems

Problem 4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ with the property that any real numbers x, y satisfy

$$f(x+y)f(x+f(y)) = x^2 + f(y^2).$$

Problem 5. Let H be the orthocenter of $\triangle ABC$ and let AH intersect the circumcircle of $\triangle ABC$ at H_A. Let A' be the reflection of A across BC, let H_B, H_C be the feet of perpendiculars from H onto $A'B$ and $A'C$, respectively. Let T be the intersection of lines BH_C and CH_B. Prove that TH_A is tangent to the circumcircle of $\triangle ABC$.

Problem 6. Let $n \geq 2$ be a positive integer. On a table of area 1, there are $2n$ pieces of paper, which can have any shape, whose areas add up to n. The pieces of paper are completely contained within the table's surface. Prove that one can find two pieces of paper whose intersection has an area of at least

$$\frac{1}{4}\left(1 - \frac{1}{n}\right).$$

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Note. Each problem is worth 7 points. You have 4 hours and 30 minutes to solve the problems and write down your solutions. The use of calculators, engines and any other form of external help is forbidden. For more information, see the official guidelines. In order to receive full marks for problem 5, you must provide a diagram.