

First OIMC 2021

Online International Mathematical Cup

Day I Problems

Problem 1. Let $\ell_1, \ell_2, \dots, \ell_k$ be $k \geq 4$ parallel lines in the plane. Let \mathcal{P} be a set of points situated on those lines. On each ℓ_i there are at least two points. Moreover, for any $A, B, C \in \mathcal{P}$, which are not collinear and no two of which lie on the same ℓ_i , there exists $P \in \mathcal{P}$ such that P is situated on the boundary of $\triangle ABC$. Prove that \mathcal{P} is infinite.

Problem 2. Let $p > 3$ be a prime number and m a positive integer such that p divides $m^2 - m + 1$. Prove that p divides the numerator of $P(m)$ written in lowest terms, where P is a polynomial defined as follows

$$P(x) := \sum_{i=1}^{p-1} \frac{x^i}{i}.$$

Problem 3. Let $ABCDEFGH$ be a regular heptagon, and let \mathcal{P} be a parabola tangent to lines AB, CD, DE and EF . Prove that the tangency point of \mathcal{P} with the line EF lies on the line AD .

Note: A parabola is the locus of points X such that the distance from X to a fixed point Y is the same as the distance from X to a fixed line ℓ . Moreover, $Y \notin \ell$.

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Note. Each problem is worth 7 points. You have 4 hours and 30 minutes to solve the problems and write down your solutions. The use of calculators, engines and any other form of external help is forbidden. For more information, see the official guidelines.