Collected Problems: Olympiad

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§ 1 Introduction

This is a collection of some of the olympiad problems that I have done. Because of my inexperience, the difficulty ratings are particularly subjective compared to the list of computational problems. Because of this, there's

§ 2 Combinatorics

[2 ✎] **Problem 1** (Romania TST) How many polynomials P with coefficients $0, 1, 2, or 3$ satisfy $P(2) = n$, where n is a given positive integer?

Solution: Generating Functions: You get $\prod_{i=0} \frac{x^{4 \cdot 2^n} - 1}{x^{2^n} - 1} = \prod_{i=0} \frac{x^{2^{n+2}} - 1}{x^{2^n} - 1} = \frac{1}{(x-1)(x^2-1)} = \frac{-\frac{1}{4}}{x-1} + \frac{\frac{1}{2}}{(x-1)^2} + \frac{\frac{1}{4}}{x+1}$. ■

§ 3 Number Theory

§ 4 Algebra

§ 4.1 Inequalities

[1 ✎] **Problem 1** (Canada MO 2017) For pairwise distinct nonnegative reals a, b, c, prove that

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(b-a)^2} > 2$$

Solution: WLOG $a < b < c$ and let $b = a + x$ and $c = a + y$. Then

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(b-a)^2}$$

$$= \frac{a^2}{(y-x)^2} + \frac{(a+x)^2}{y^2} + \frac{(b+y)^2}{x^2}$$

$$\geq \frac{x^2}{y^2} + \frac{y^2}{x^2} \geq 2$$

by AM-GM ■

[1] **Problem 2** (Canada MO 2002) Let a, b, c be positive reals. Prove

$$\frac{a^3}{bc} + \frac{b^4}{ac} + \frac{c^4}{bc} \geq a + b + c.$$

Solution:

$$\frac{a^3}{bc} + \frac{b^4}{ac} + \frac{c^4}{bc} \geq a + b + c$$
$$\iff a^4 + b^4 + c^4 \geq a^2bc + b^2ac + c^2bc$$

after multiplying by abc on both sides. We will show this last inequality.

Then, note that by Weighted AM-GM, $2a^4 + b^4 + c^4 \geq 4a^2bc$. Similarily, $2b^4 + a^2 + c^2 \geq 4b^2ac$ and $2c^2 + a^2 + b^2 \geq 4c^2ab$. Then,

$$4a^4 + 4b^4 + c^4 \geq 4a^2bc + 4b^2ac + 4c^2ab$$

$$\iff a^4 + b^4 + c^4 \geq a^2bc + b^2ac + c^2ab$$

so it is true. ■

[2] **Problem 3** (How should n balls be put into k boxes to mimize the number of pairs of balls which are in the same box?)

Solution: Let the number of balls in the ith box be n_i. Then, $n_1 + n_2 \cdots + n_k = n$ and we want to minimize $\binom{n_1}{2} + \binom{n_2}{2} \cdots + \binom{n_k}{2}$. If $n_i - n_j \geq 2$ for some i, j, then we can replace n_i by $n_i - 1$ and $n_j + 1$ (which preserves n) and decrease the number of pairs as:

$$\binom{n_i}{2} + \binom{n_j}{2} \geq \binom{n_i - 1}{2} + \binom{n_j + 1}{2}$$

$$\iff n_i -_j -1 \geq 0$$

$$\iff n_i - n_j \geq 1$$

which is true.

So, all n_i, n_j are $a, a+1$ for some n. Note that one of $a, a+1$ is $\lfloor \frac{k}{n} \rfloor$. ■

[3] **Problem 4** (1974 USAMO) For $a, b, c > 0$, prove $a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$.

Solution:

$$a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$$

Taking the natural log of both sides,

$$\iff \ln(a^a b^b c^c) \geq \ln((abc)^{\frac{a+b+c}{3}})$$

$$\iff a\ln a + b\ln b + c\ln c \geq \frac{a+b+c}{3}\ln(abc)$$

$$\iff \frac{a\ln a + b\ln b + c\ln c}{3} \geq \frac{a+b+c}{3}\ln((abc)^{\frac{1}{3}})$$

We will show the last inequalty.

Let $f(x) = x\ln x$. Note that $f'(x) = x'\ln(x) + x\ln'(x) = \ln(x) + 1$ and $f''(x) = \ln'(x) = \frac{1}{x} \geq 0$ for positive x. So, $f(x)$ is convex on positive numbers. By Jensens',

$$\frac{f(a)+f(b)+f(c)}{3} \geq f(\frac{a+b+c}{3})$$

$$\frac{a\ln a + b\ln b + c\ln c}{3} \geq \frac{a+b+c}{3}\ln(\frac{a+b+c}{3})$$

. Then, by AM-GM, $\frac{a+b+c}{3} \geq \sqrt[3]{abc} = (abc)^{\frac{1}{3}} \implies \ln(\frac{a+b+c}{3}) \geq \ln(abc)^{\frac{1}{3}}$. So, $\frac{a+b+c}{3}\ln(\frac{a+b+c}{3}) \geq \ln((abc)^{\frac{1}{3}})$. Combining the chain of inequalties, we get our desired

$$\iff \frac{a\ln a + b\ln b + c\ln c}{3} \geq \frac{a+b+c}{3}\ln((abc)^{\frac{1}{3}})$$

■

[3] **Problem 5** (1995 India) Let $x_1, \cdots x_n$ be positive numbers whose sum is 1. Prove

$$\frac{x_1}{\sqrt{1-x_1}} + \cdots + \frac{x_n}{\sqrt{1-x_n}} \geq \sqrt{\frac{n}{n-1}}$$

Solution: Let $f(x) = \frac{x}{\sqrt{1-x}}$. Then, note that $f'(x) = \frac{x(\sqrt{1-x})' - x'(\sqrt{1-x})}{(\sqrt{1-x})^2} = \frac{x\cdot\frac{1}{2}\cdot(1-x)^{-\frac{1}{2}} - \sqrt{1-x}}{1-x} = \frac{1}{2}\cdot x\cdot(1-x)^{-\frac{3}{2}} - (1-x)^{\frac{1}{2}}$. and that $f''(x) = \frac{3x}{4}(1-x)^{-\frac{5}{2}} + \frac{1}{4}(1-x)^{-\frac{3}{2}}$. This is positive when $0 < x < 1 \iff 0 < 1-x < 1$. So, $f(x)$ is convex on $(0,1)$.

Since $0 < x_i < 1$, we can apply Jensen's to get

$$\frac{f(x_1)\cdots + f(x_n)}{n} \geq f(\frac{x_1\cdots + x_n}{n}) = f(\frac{1}{n}) = \frac{\frac{1}{n}}{\sqrt{1-\frac{1}{n}}} = \frac{\frac{1}{\sqrt{n}}}{\sqrt{n-1}}$$

$$\implies f(x_1)\cdots + f(x_n) \geq \frac{\sqrt{n}}{\sqrt{n-1}}$$

$$\frac{x_1}{\sqrt{1-x_1}} + \cdots + \frac{x_n}{\sqrt{1-x_n}} \geq \sqrt{\frac{n}{n-1}}$$

■

[3 ✎] **Problem 6** (1998 USAMO) Let $a_0, a_1 \cdots a_n$ be numbers from $(0, \frac{\pi}{2}$ such that

$$\tan(a_0 - \frac{\pi}{4}) + \tan(a_1 - \frac{\pi}{4}) \cdots + \tan(a_n - \frac{\pi}{4}) \geq n - 1$$

Prove that

$$\tan a_0 \tan a_1 \cdots \tan a_n \geq n^{n+1}$$

Solution: Let $x_i = \tan(a_i - \frac{\pi}{4})$. Note that $-1 \leq x_i \leq 1$. The condition translates to $x_0 \cdots + x_n \geq n - 1$.

Also, note that $\tan(a_i) = \frac{1+x_i}{1-x_i}$ using Tangent Addition Formula. Then,

$$\tan(a_0 - \frac{\pi}{4}) + \tan(a_1 - \frac{\pi}{4}) \cdots + \tan(a_n - \frac{\pi}{4}) \geq n - 1$$

$$\iff \frac{x_0+1}{1-x_0} \dots \cdot (\frac{x_n+1}{1-x_n} \geq n^{n+1}$$

$$\iff \ln(\frac{x_0+1}{1-x_0} \cdots + \frac{x_n+1}{1-x_n} \geq (n+1)\ln n$$

$$\iff \frac{1}{n+1}(\ln(\frac{x_0+1}{1-x_0} \cdots + \ln\frac{x_n+1}{1-x_n}) \geq \ln n$$

We will prove this last inequalty.

Let $f(x) = \ln(\frac{x+1}{1-x})$. Note that $f'(x) = \frac{-2(1-x)}{(1+x)}$ and

$$f''(x) = \frac{(1-x)(x+1)' - (1-x)'(x+1)}{(x+1)^2} = \frac{1-x-(x+1)}{(x+1)^2} = \frac{-2x}{(x+1)^2}$$

.

So, $f(x)$ is convex on $(0, 1]$. So, if all x_i are positive, it follows by Jensen's that

$$\frac{1}{n+1}(f(x_0) \cdots + f(x_n) \geq f(\frac{x_0 \cdots + x_n}{n+1})$$

$$\frac{1}{n+1}(\ln(\frac{x_0+1}{1-x_0} \cdots + \ln\frac{x_n+1}{1-x_n}) \geq \ln(\frac{\frac{x_0\cdots+x_n}{n+1}+1}{1-\frac{x_0+\cdots+x_n}{n+1}}) \geq \ln(\frac{\frac{n-1}{n+1}+1}{1-\frac{n-1}{n+1}}) = \ln n$$

Without loss of generality, $x_0 \leq x_1 \ldots \leq x_n$. Note that at most one of x_i's are negative. Otherwise, it is strictly less than $0 + 1(n-1) = n-1$. If x_0 is negative, we can replace x_0, x_1 by $\frac{x_0+x_1}{2}$ which is positive. Otherwise, since sum is preserved and is greater than $n-1$, it is impossible for there to be 2 or more negatives. Then, we can find that $\frac{1}{n+1}(\ln(\frac{x_0+1}{1-x_0} \cdots + \ln\frac{x_n+1}{1-x_n})$ decreases which implies that the minimium of the LHS is when all x_i's are negative. We can apply the same reasoning as before. ■

§ 4.2 Misc

[3✎] **Problem 7** (M&IQ 1992) Prove that there are no positive integers n, m such that

$$(3+5\sqrt{2})^n = (5+3\sqrt{2})^m$$

§ 5 Geometry

[5✎] **Problem 1** (IMO SL 2005) Let ABC be a triangle such that M is the midpoint of BC and γ is the incircle of ABC. AM intersects γ at points K and L. Let lines passing through K and L parallel to BC intersect γ again at the points X and Y. Let lines AX and AY intersect BC again at points P and Q. Prove $BP = CQ$.

Solution: Note that $BQ = CP \iff QM = PM \iff YL = LZ$ We will show this.

First, note that $\triangle AKX \sim \triangle ALZ \implies \frac{KX}{LZ} = \frac{AK}{AL}(1)$. Also, $AGKL$ is a harmonic bundle by a well-known property as AF, AE are tangents to γ and G, L pass through A and K is $GL \cap \gamma$. Then, $\frac{KA}{KG} = \frac{LA}{LG}(2)$. Using (1)+(2), we get $\frac{\frac{AL \cdot KX}{LZ}}{KG} = \frac{LA}{LG} \implies \frac{KX}{LZ} = \frac{KG}{LG}(3)$. Also, let $H = GX \cap YZ$. Then, $\triangle KXG \sim LHG$ as $LH || KX$ and $\angle HGL = \angle KGX$. Then, $\frac{KG}{LG} = \frac{KX}{HL} \implies \frac{KX}{LZ} = \frac{KX}{HZ}$ by (3). Then, this implies $LZ = HL$. We will show $Y = H$. This is equivalent to showing X, G, Y are collinear.

We now prove a lemma: let ABC be a triangle with incircle ω and incenter I. Then, let the tangency points of ω with BC, AC, AB be D, E, F respectively. Also, let M be the midpoint of BC. Then, AM, EF, ID concur.

Proof. Let $X = EF \cap ID$. Then, we want to showing A, X, M are collinear. Let U, V be the intersections of the line through X parallel to BC with AB and AC respectively. [1] Then, $\angle IFX = \angle IUX = IUV$ as $XUFI$ is cyclic since $IX \perp BC \implies IX \perp UV \implies IXU = 90$ and $IF \perp AB \implies \angle IFU = 90$ This then implies $\angle IFE = \angle IUV$. Also, $IXEV$ is cyclic as $\angle IXV = 90$ and $\angle IEV = 90$ so $\angle IXV = \angle IEV$, then $\angle IEX = \angle IVX$. Then, this implies $\angle IEF = \angle IVU$. By AA similarity, $\triangle UIV \sim \triangle FIE$. As $\triangle FIE$ is isosceles, then $\triangle UIV$ is isoscles with $UI = UV$. As $IV \perp UV$, then $UX = XV$. So X is midpoint of UV and X is on the median AM. We are done. □

By the lemma, ID goes through G. Now, we want to show $\angle GYL = \angle KXG$ which is equivalent to X, G, Y collinear by the converse of Alternate Interior Angles Theorem. Note that ID bisects YL as $ID \perp BC \implies ID \perp YL$ and YL is a chord of γ. So, $YD = DL$ and $\triangle GYD \cong \triangle GLD \implies GY = GL$. Then, $\angle GYL = \angle GLY$. By Alternative Interior Angles Theorem and as K, G, L are collinear, $\angle GLX = \angle GKX$. Similarly, letting ID intersect KX at P, we have $KP = PX \implies \triangle KPG \cong XPG \implies GK = GX \implies \angle GKX = \angle KXG$. So, using the chain of equalities, we get $\angle GYL = \angle KXG$ which is what we want. ■

[1] Note that by the converse of the Simpson Line Theorem, since E, X, F are collinear and $IF \perp AB = AU$, $IE \perp AC = AV$, and $IX \perp BC \implies IX \perp UV$ since $BC || UV$, that $I \in (AUV)$.

§ 6 Associated Solutions

§ 6.1 Combinatorics

§ 6.2 Number Theory

§ 6.3 Algebra

§ 6.4 Geometry