

# Collected Problems: Olympiad

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## § 1 Introduction

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This is a collection of some of the olympiad problems that I have done. Because of my inexperience, the difficulty ratings are particularly subjective compared to the list of computational problems. Because of this, there's

## § 2 Combinatorics


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[2]  **Problem 1** (Romania TST) How many polynomials  $P$  with coefficients  $0, 1, 2, \text{ or } 3$  satisfy  $P(2) = n$ , where  $n$  is a given positive integer?


**Solution:** Generating Functions: You get  $\prod_{i=0} \frac{x^{4 \cdot 2^i} - 1}{x^{2^i} - 1} = \prod_{i=0} \frac{x^{2^{i+2}} - 1}{x^{2^i} - 1} = \frac{1}{(x-1)(x^2-1)} = \frac{-\frac{1}{4}}{x-1} + \frac{\frac{1}{2}}{(x-1)^2} + \frac{\frac{1}{4}}{x+1}$ .

## § 3 Number Theory

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[1]  **Problem 1** (AoPS Forum)  $a, b \in \mathbb{N}$  and  $ab \mid (a^2 + b^2)$ , then prove that  $a = b$ .

**Solution:** First,  $a = b$  is equivalent to  $\nu_p(a) = \nu_p(b)$  for all primes  $p$ . Also,  $ab \mid (a^2 + b^2)$  is equivalent to  $\nu_p(a) + \nu_p(b) \leq \nu_p(a^2 + b^2)$  for all primes  $p$ . Then, suppose  $x = \nu_p(a) > \nu_p(b) = y$  for some prime  $p$ . Then,  $\nu_p(a^2) = 2x$  and  $\nu_p(b^2) = 2y$ . So,  $\nu_p(a^2 + b^2) = 2x$ . Then, we have  $x + y \leq 2x$  which implies  $y \leq x$ . This is a contradiction to our assumption  $x > y$ . Similarly,  $\nu_p(a) < \nu_p(b)$  for some prime  $p$  is impossible. So,  $\nu_p(a) = \nu_p(b)$  for all primes  $p$ .

[1]  **Problem 2** (IMO 2005/4) Determine all positive integers relatively prime to all terms of the infinite sequence  $a_n = 2^n + 3^n + 6^n - 1$  for  $n \geq 1$ .

**Solution:** Notice that for  $p > 3$ ,  $a_{p-2} = 2^{p-2} + 3^{p-2} + 6^{p-2} - 1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} - 1 = 0 \pmod{p}$  using Fermat Little Theorem. Then, note that  $a_2 = 4 + 9 + 36 - 1 = 48$  which has prime factors of  $2, 3$ . So, if a number contains any prime factors, it is not relatively prime to all the terms of  $a_n$ . So, the

only possible positive integer that can be relatively prime to all terms is  $\boxed{1}$ . Clearly,  $\gcd(1, a_n) = 1$  and so it works.

**[1] Problem 3** (1978 IMO) Let  $m$  and  $n$  be positive integers such that  $1 \leq m < n$ . In their decimal representations, the last three digits of  $1978^m$  are equal, respectively, to the last three digits of  $1978^n$ . Find  $m$  and  $n$  such that  $m + n$  has its least value.

**Solution:** Notice that  $1978^m \equiv 1978^n \pmod{1000} \iff (-22)^m \equiv (-22)^n \pmod{1000} \iff (-22)^m \equiv (-22)^n \pmod{125, 8}$  by Chinese Remainder Theorem. If  $m < 3$ , then  $(-22)^m \equiv (-22)^n \pmod{8} \implies m = n$  which is a contradiction. Similarly,  $n < 3$  is impossible. So,  $m, n \geq 3$ . Then, both are  $0 \pmod{8}$ . Now, consider  $(-22)^m \equiv (-22)^n \pmod{125} \iff (-22)^{n-m} \equiv 1 \pmod{125}$ . We seek to minimize  $n - m$  to minimize  $m + n = (n - m) + 2m$ . Now, this is equivalent to finding  $\text{ord}_{125}(-22)$ . Note that  $\text{ord}_{25}(-22) = \text{ord}_{25}3 = \text{ord}_{25}20$ . So,  $20 = \text{ord}_{25}(-22) | \text{ord}_{125}(-22)$  clearly. By Euler's,  $\text{ord}_{125}(-22) | 100$ . We can compute  $(-22)^{20} \pmod{125}$  and find that it doesn't work. So, the order is 100 and  $n - m = 0 \pmod{100} \implies n - m \geq 100$ .

Our minimum  $m + n$  is  $3 + 103 = \boxed{106}$ .

**[1] Problem 4** (Euler) Prove that all factors of  $2^{2^{n+1}}$  are of the form  $k \cdot 2^{n+1} + 1$ .

**Solution:** This is equivalent to showing every factor is  $1 \pmod{2^{n+1}}$ .

Let  $p$  be a prime factor of  $2^{2^{n+1}}$ . Then,  $2^{2^n} + 1 = 0 \pmod{p} \implies 2^{2^n} = -1 \pmod{p}$ . Then, let  $d$  be the order of 2 modulo  $p$ . We have  $d | 2^{n+1}$  but  $d \nmid 2^n$ . This implies  $d = 2^{n+1}$ . Then, we have  $d | p - 1 \implies p = 1 \pmod{2^{n+1}}$ . Note that if  $b = a = 1 \pmod{2^{n+1}} \implies ab = 1 \pmod{2^{n+1}}$ . Then, as every factor of  $2^{2^{n+1}}$  is composed of those factors, every factor is  $1 \pmod{2^{n+1}}$ .

**[2] Problem 5** (Folklore) Let  $p > 5$  be a prime. In terms of  $p$ , compute the remainder when

$$\prod_{i=1}^{p-1} (i^2 + 1)$$

is divided by  $p$

**Solution:** Consider  $P(x) = (x+1)(x+2)\cdots(x+p-1)$  in  $\mathbf{F}_p$ . Then,  $P(x) = x^{p-1} - 1 = 0 \pmod{p}$  as the two polynomials have exactly the same roots and are the same degree. By a well-known property of  $\mathbf{F}_p$ , this means the two polynomials are equivalent. Now, our wanted expression is  $P(i) \cdot P(-i) = \boxed{(i^{p-1} - 1)((-i)^{p-1} - 1)}$ .

**[2] Problem 6** (Folklore) Find all positive integers  $n$  such that  $n$  divides  $2^n - 1$ .

**Solution:** Assume  $n \geq 2$ . Note that  $n$  can't be even because  $2^n - 1$  is odd and it is impossible for an even number to be a divisor of an odd number. Then, consider the minimal odd prime  $p$  such that  $p | n$ . Also, let  $\text{ord}_p 2 = d$ . Then,  $2^n - 1 = 0 \pmod{n} \implies 2^n = 1 \pmod{n} \implies d | n$ . Also, by Fermat's Little Theorem,  $2^{p-1} = 1 \pmod{p}$  so  $d | p-1$ . Then,  $d | \gcd(n, p-1)$ . If  $\gcd(n, p-1) = k > 1$ , then  $n$  has a prime divisor less than  $p-1$  which contradicts the minimality of  $p$ . So,  $\gcd(n, p-1) = 1$ . Then,  $n = (p-1)q + 1$  for some  $q$ . So,  $2^n - 1 \pmod{p} = (2^{p-1})^q \cdot 2 - 1 = 2 - 1 = 1$ . Since  $n = 0 \pmod{p}$ , we have  $1 = 0 \pmod{p}$  which is clearly impossible.

So,  $n = 1$  is the only possible solution. Checking,  $n = 1$  does work. So, our solutions are  $\boxed{1}$ .

[2] **Problem 7** (Folklore) Let  $p$  be a prime that is relative prime to 10, and  $n$  be an integer,  $0 < n < p$ . Let  $d$  be the order of 10 modulo  $p$ .

1. Show that if the length of the period of the decimal expansion of  $\frac{n}{p}$  is  $d$
2. Prove that if  $d$  is even, then the period of the decimal expansion of  $\frac{n}{p}$  can be divided into two halves whose sum is  $10^{\frac{d}{2}} - 1$ .

**Solution 1:** Let  $m$  be the period. Note that  $\frac{n}{p} = 0.\overline{a_1 a_2 \dots a_m} = \frac{a_1 a_2 \dots a_m}{99 \dots 9} = \frac{M}{10^m - 1} \implies n \cdot (10^m - 1) = p \cdot M \implies p \mid 10^m - 1$  as  $\gcd(n, p) = 1$ . So,  $10^m \equiv 1 \pmod{p} \implies d \mid m \implies m \geq d$ .

We now want to show  $m \leq d$ . Let  $M'$  such that  $\frac{n}{p} = \frac{M'}{10^d - 1} = M'(\frac{1}{10^d} + \frac{10^{2d}}{\dots})$ . Since  $M' < 10^d - 1$ , it clearly has a period of less than equal to  $d$ . So,  $m \leq d$ .

Then,  $m = d$ .

**Solution 2:** Let  $d = 2k$ . Then, let  $\frac{n}{p} = 0.\overline{a_1 a_2 \dots a_k a_{k+1} \dots a_{2k}}$ . We want to show  $\overline{a_1 a_2 \dots a_k} + \overline{a_{k+1} a_{k+2} \dots a_{2k}} = 10^{\frac{d}{2}} - 1$ . Call  $\overline{a_1 \dots a_k} = M_1$  and  $\overline{a_{k+1} \dots a_{2k}} = M_2$ . We have  $\frac{(10^{2k} - 1)n}{p} = M_1 \cdot 10^k + M_2$ .

So,  $\frac{(10^k + 1)n}{p} = 10^k \cdot \frac{n}{p} + \frac{n}{p} = a_1 \dots a_k . a_{k+1} \dots a_{2k} \overline{a_1 \dots a_k} + 0.\overline{a_1 \dots a_{2k}}$ . In order for it to come out as a integer,  $\overline{a_{k+1} \dots a_{2k} a_1 \dots a_k} + 0.\overline{a_1 \dots a_{2k}} = 0.\overline{9} = 1$ . Then,  $\frac{(10^k + 1)n}{p} = M_1 + 1$ . Then,  $M_1 + M_2 = 10^k - 1$  which is what we wanted.

[3] **Problem 8** (China TST 2006) Find all positive integers  $a$  and  $n$  such that  $\frac{(a+1)^n - a^n}{n}$  is an integer.

**Solution:** Assume  $n \geq 2$ . Then, let  $p$  be the smallest prime factor of  $n$ . Then,  $p \mid (a+1)^n - a^n$ . So,  $(a+1)^n - a^n \equiv 0 \pmod{p}$ . If  $a \equiv 0 \pmod{p}$ , then  $(a+1)^n - a^n \equiv 1 \pmod{p} \neq 0$ . If  $a \equiv -1 \pmod{p}$ ,  $(a+1)^n - a^n \equiv (-1)^{n+1} \pmod{p} \neq 0$ .

If  $a \not\equiv 0, 1 \pmod{p}$ , then  $(a+1)^n - a^n \equiv 0 \pmod{p} \implies (a+1)^n \equiv a^n \pmod{p} \implies (a+1)^n \cdot (a^{-1})^n \equiv -1 \pmod{p}$  where  $a^{-1}$  denotes the inverse of  $a$ . Note  $a^{-1}$  exists since  $a$  is relatively prime to  $p$ . Then, we get  $((a+1) \cdot a^{-1})^n \equiv 1 \pmod{p}$ . Let  $m$  be the order of  $(a+1) \cdot a^{-1}$  modulo  $n$ . Then, we have  $m \mid n$ . But, we also have  $m \mid p-1$  as  $((a+1) \cdot a^{-1})^{p-1} \equiv 1 \pmod{p}$  by Fermat's Little Theorem and both  $a+1$  and  $a^{-1}$  are not equal to 0  $\pmod{p}$ . This means  $m \leq p-1$  and  $m \mid n$  which is a contradiction of our assumption that  $p$  is the smallest prime factor. So,  $n \geq 2$  is impossible.

This means our only solutions are when  $n = 1$ . Clearly, for all positive integer  $a$ ,  $\frac{(a+1) - a}{1}$  is a positive integer.

$(1, a)$  for all positive integers  $a$

[3] **Problem 9** (IMO 1990/3) Determine all integers  $n > 1$  such that

$$\frac{2^n + 1}{n^2}$$

is an integer.

**Solution:** Note that  $n$  has to be odd as an even can't divide an odd. Let  $p$  be the smallest odd prime divisor of  $n$ . Then,  $p|n^2|2^n + 1$ . So,  $p|(2^n + 1)(2^n - 1)|2^{2n} - 1$ . By Fermat's Little Theorem,  $p|2^{p-1} - 1$ . So, by GCD Lemma,  $p|2^{\gcd(2n, p-1)} - 1$ . Note  $\gcd(n, p-1) = 1$  as  $p$  is the minimal prime divisor so there can't be any prime divisors of  $n$  less than or equal to  $p-1$ . Also,  $p-1$  is even and  $n$  is even so  $\gcd(2n, p-1) = 2$ . Then,  $p|2^2 - 1|3 \implies p = 3$ . Let  $\nu_3(n) = k$ .

Then,  $2^n + 1 = 2^{3 \cdot \frac{n}{3}} + 1 = 8^{\frac{n}{3}} - (-1)^{\frac{n}{3}}$  and  $8 + 1 = 9$  and by LTE,  $\nu_3(2^n + 1) = \nu_3(9) + \nu_3(\frac{n}{3}) = 2 + k - 1 = k + 1$ . However,  $\nu_3(n^2) = 2k$ . So, we have  $2k \leq k + 1 \implies k \leq 1$ . So,  $k = 1$ .

Let  $n = 3m$  where  $m \not\equiv 0 \pmod{3}$  and  $m > 2$ . Let  $q|m$  such that  $q$  is the smallest odd prime. So,  $q > 3$ . By Fermat's Little Theorem,  $q|2^{q-1} - 1$ . Also,  $q|n|n^2|2^n + 1 = 2^{3m} + 1|2^{6m} - 1$ . By GCD Lemma,  $q|\gcd(2^{q-1} - 1, 2^{6m} - 1) = 2^{\gcd(q-1, 6m)} - 1$ . Then, we have  $\gcd(q-1, m) = 1$  as  $p$  is the minimal prime divisor. Also,  $2|q-1$  as  $q$  is odd. Then  $\gcd(6m, q-1) = 2, 6$ . The first case gives  $q|3$  which is impossible. The second case gives  $q|63 \implies q = 7$ .

Now, if  $q = 7$ ,  $n^2 \equiv 0 \pmod{49}$  and  $2^n + 1 = 2^{3m} + 1 = 8^m + 1 \equiv 2 \pmod{7}$ . So, it is impossible. So, we  $m = 1$  is the only possible  $m$  and  $n = 3$ .

**[3] Problem 10 (IMO 1999)** Find all the pairs of positive integers  $(x, p)$  such that  $p$  is a prime,  $x \leq 2p$  and  $x^{p-1}$  is a divisor of  $(p-1)^x + 1$ .

**Solution:** Assume  $p \geq 3$ . This then implies  $x$  is odd.

$x^{p-1} | (p-1)^x + 1$ . Then, let  $q$  be the smallest prime divisor of  $x$ . Then,  $q|x|x^{p-1} | (p-1)^x + 1 | (p-1)^{2x} - 1$ . Assume  $\gcd(p-1, q) = 1$ . By Fermat's Little Theorem,  $((p-1)^2)^{q-1} \equiv 1 \pmod{q} \implies q|(p-1)^{2q-2}$ . By the GCD Lemma,  $q|((p-1)^2)^{\gcd(x, q-1)}$ . Note that  $\gcd(x, q-1) = 1$  because  $q$  is the minimal prime. So,  $q|(p-1)^2 - 1 = (p-1)(p-3)$ . Since  $\gcd(p, p-2) = \gcd(p, 2) = 1$ ,  $q|p$  or  $q|p-2$ . If  $q|p$ , then  $p = q$ . Then,  $x \equiv 0 \pmod{p}$  and  $x = p, 2p$ . Then,  $x \neq 2p$  because  $x$  can't have a prime factor less than  $p$ . Then,  $x = p$  gives  $p^{p-1}$  is a divisor of  $(p-1)^p + 1$  which implies  $p^{p-1} \leq (p-1)^p + 1$ . Clearly, this fails for large enough  $p$ . The only possible solutions are  $p = 3$ . This then gives  $x^2 | 2^x + 1$  where  $x \leq 6$ . We can easily check and see  $x = 1, 3$  are the only solutions.

If  $p \equiv 2 \pmod{q}$ . Then,  $q|(p-1)^x + 1 \equiv 2 \pmod{q}$  which is a contradiction.

If  $\gcd((p-1)^2, q) = q$ , then  $(p-1)^2 \equiv 0 \pmod{q} \implies p-1 \equiv 0 \pmod{q} \implies p \equiv 1 \pmod{q}$ . Then,  $(p-1)^x + 1 \equiv 1 \pmod{q}$ . However, it is a multiple of  $x^{p-1} \equiv 0 \pmod{q}$  which is a contradiction. By LTE, for odd  $x$ ,  $\nu_p((p-1)^x + 1) = \nu_p(p) + \nu_p(x) = 1 + \nu_p(x)$ .

Now, the only remaining case is  $p = 2$ . This gives  $x|2$  which gives  $x = 1, 2$ .

Our solutions are  $\boxed{(1, 2), (2, 2), (1, 3), (3, 3)}$ .

**[4] Problem 11 (MOP 2011)** Let  $p$  be a prime and  $n$  a positive integer. Suppose that  $p^1$  fully divides  $2^n - 1$  (meaning it is divisible by  $p$  but not  $p^2$ ). Prove that  $p^1$  fully divides  $2^{p-1} - 1$ .

**Solution:** Let  $d = \text{ord}_p(n)$ . Then, since  $2^n \equiv 1 \pmod{p}$ , we know  $d|n$ . Let  $n = dk$ . Then, by Lifting The Exponent,  $1 = \nu_p(2^n - 1) = \nu_p((2^d)^k - 1^k) = \nu_p(2^d - 1) + \nu_k$  as  $2^d - 1 \equiv 0 \pmod{p}$  by the definition of order. This then implies  $\nu_p(2^d - 1) \leq 1$ . Since  $2^d - 1 \equiv 0 \pmod{p}$ , we also have  $\nu_p(2^d - 1) \geq 1$ . Altogether,  $\nu_p(2^d - 1) = 1$ .

Now, note that since  $2^{p-1} \equiv 1 \pmod{p}$ , we have  $d|p-1$ . Let  $p-1 = dt$ . Then by Lifting The Exponent,  $\nu_p(2^{p-1} - 1) = \nu_p((2^d)^t - 1^t) = \nu_p(2^d - 1) + \nu_p(t)$  as  $2^d - 1 \equiv 0 \pmod{p}$ . Then, this is equal to  $1 + \nu_p(t)$  by (1). Note that since  $t|p-1 \implies t < p$ ,  $\nu_p(t) = 0$ . So, it is exactly equal to 1. This what we wanted so we're done.

[4] **Problem 12** (IMO SL 2006) Find all integer solutions of the equation

$$\frac{x^7 - 1}{x - 1} = y^5 - 1$$

**Lemma 1 (Lemma 1)** Let  $p$  be a prime,  $n$  a positive integer and  $a$  any integer. Suppose  $\Phi_n(a) \equiv 0 \pmod{p}$ . Then, either:

1.  $n|p - 1$
2.  $p|n$

Consider  $\Phi_7(x) = 1 + x + \dots + x^6 = \frac{x^7 - 1}{x - 1}$ . By the Main Theorem/Lemma,  $p|\Phi_7(x)$  implies  $p|n$  or  $n|p - 1$ . As a corollary,  $p|\Phi_q(x)$  we have  $p|q$  (which implies  $p = q$ ) or  $p = 1 \pmod{q}$ . So, if  $p|\Phi_7(x)$ ,  $p = 7$  or  $p = 1 \pmod{7}$ . Then, the LHS is  $0 \pmod{7}$  or  $1 \pmod{7}$ .

This then implies  $y - 1 = 0 \pmod{7}$  or  $y - 1 = 1 \pmod{7}$  as  $y - 1$  is a divisor of the RHS. These give  $y = 1, 2 \pmod{7}$ . For the first case,  $1 + y + \dots + y^4 = 5 \pmod{7}$  which is a contradiction as it's not equal to 1 or 0. If we plug in  $y = 2 \pmod{7}$ , we get  $1 + 2 + 2^2 + 2^3 + 2^4 = 31 = 3 \pmod{7}$ .

Therefore, there are no solutions.

[4] **Problem 13** (IMO 2000) Does there exist a positive integer  $n$  such that  $n$  has exactly 2000 prime divisors and  $n$  divides  $2^n + 1$ ?

**Solution:** Clearly, 2000 is arbitrary and we can replace by  $m$ . We do induction on  $m$ . For  $m = 1$ ,  $p|2^p + 1$  and  $p = 3$  works. Now, we do the induction step. By the inductive hypothesis, we have a  $n = p_1^{a_1} \dots p_m^{a_m}$  and  $n|2^n + 1$  where  $p_1, \dots, p_m$  are distinct primes and  $a_1 \dots a_m \geq 1$ . Let  $p_i^{a_i}|2^n + 1$ , suppose  $p_i^{b_i}||2^n + 1$  where  $||$  means it perfectly divides. Then,  $p_1^{b_1} \dots p_m^{b_m}||2^n + 1$ . By LTE,  $p_1^{b_1 + \ell} p_2^{b_2} \dots p_m^{b_m}||2^{np_1^\ell} + 1$ . Consider the sequence of  $2^{np_1^\ell} + 1$  for  $\ell = 1, 2, \dots$ . By Kobayashi's, there is some  $\ell$  such that  $2^{np_1^\ell} + 1$  has a prime factor  $p_{m+1}$  distinct from  $p_1, p_2, \dots, p_m$  because the sequence's terms contain an infinite number of prime factors. So,  $p_1^{b_1 + \ell} p_2^{b_2} \dots p_m^{b_m} p_{m+1}|2^{np_1^\ell} + 1$ .

[5] **Problem 14** (IMO 2003) Let  $p$  be a prime number. Prove that there exists a prime number  $q$  such that for every integer  $n$ , the number  $n^p - p$  is not divisible by  $q$ .

**Lemma 2 (Lemma 1)** Let  $p$  be a prime,  $n$  a positive integer and  $a$  any integer. Suppose  $\Phi_n(a) \equiv 0 \pmod{p}$ . Then, either:

1.  $n|p - 1$
2.  $p|n$

Consider  $q|\Phi_p(p)$  with  $p^2 \nmid q - 1$ . It is clear such a prime  $q$  exists because  $\Phi_p(p)$  is positive integer greater than 1 and that not all prime factors can be  $\pm 1 \pmod{p^2}$ . This is because  $\Phi_p(p) = 1 + p \pmod{p^2} \neq \pm 1$ . By the Main Lemma,  $p|q - 1$  or  $q|p$ . Clearly,  $q|p$  is impossible as both are distinct primes. Otherwise,  $p = q$  which implies  $p|p^p - 1$ . By way of contradiction,  $n^p = p$

$(\text{mod } q) \implies n = p^{\frac{1}{p}} \pmod{q}$ . By Fermat's Little Theorem,  $n^{q-1} = 1 \pmod{q} \implies p^{\frac{q-1}{p}} = 1 \pmod{q}$ .

So,  $q | \Phi_p(p) | p^p - 1$  and  $q | p^{\frac{q-1}{p}} - 1$ . By GCD Lemma,  $q | \gcd(p^p - 1, p^{\frac{q-1}{p}}) = p^{\gcd(p, \frac{q-1}{p})} - 1$ . Note  $\nu_p(q-1) = 1$  because we established before,  $\nu_p(q-1) \geq 1$  and by our assumption,  $\nu_p(q-1) < 2$ . So,  $\nu_p(q-1) = 1$ . Then,  $q | p^{\gcd(p, \frac{q-1}{p})} - 1 \implies q | p - 1$ .

Then, as  $q | p - 1$  and  $p | q - 1$  which give  $q \leq p - 1$  and  $p \leq q - 1$  which are clearly impossible.

## § 4 Algebra

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### § 4.1 Inequalities

[1] **Problem 1** (Canada MO 2017) For pairwise distinct nonnegative reals  $a, b, c$ , prove that

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(b-a)^2} > 2$$

**Solution:** WLOG  $a < b < c$  and let  $b = a + x$  and  $c = a + y$ . Then

$$\begin{aligned} & \frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(b-a)^2} \\ &= \frac{a^2}{(y-x)^2} + \frac{(a+x)^2}{y^2} + \frac{(b+y)^2}{x^2} \\ &\geq \frac{x^2}{y^2} + \frac{y^2}{x^2} \geq 2 \end{aligned}$$

by AM-GM

[1] **Problem 2** (Canada MO 2002) Let  $a, b, c$  be positive reals. Prove

$$\frac{a^3}{bc} + \frac{b^4}{ac} + \frac{c^4}{bc} \geq a + b + c.$$

**Solution:**

$$\begin{aligned} & \frac{a^3}{bc} + \frac{b^4}{ac} + \frac{c^4}{bc} \geq a + b + c \\ \iff & a^4 + b^4 + c^4 \geq a^2bc + b^2ac + c^2bc \end{aligned}$$

after multiplying by  $abc$  on both sides. We will show this last inequality.

Then, note that by Weighted AM-GM,  $2a^4 + b^4 + c^4 \geq 4a^2bc$ . Similarly,  $2b^4 + a^4 + c^4 \geq 4b^2ac$  and  $2c^4 + a^4 + b^4 \geq 4c^2ab$ . Then,

$$\begin{aligned} & 4a^4 + 4b^4 + c^4 \geq 4a^2bc + 4b^2ac + 4c^2ab \\ \iff & a^4 + b^4 + c^4 \geq a^2bc + b^2ac + c^2ab \end{aligned}$$

so it is true.

[2] **Problem 3** (How should  $n$  balls be put into  $k$  boxes to minimize the number of pairs of balls which are in the same box?)

**Solution:** Let the number of balls in the  $i$ th box be  $n_i$ . Then,  $n_1 + n_2 + \dots + n_k = n$  and we want to minimize  $\binom{n_1}{2} + \binom{n_2}{2} + \dots + \binom{n_k}{2}$ . If  $n_i - n_j \geq 2$  for some  $i, j$ , then we can replace  $n_i$  by  $n_i - 1$  and  $n_j$  by  $n_j + 1$  (which preserves  $n$ ) and decrease the number of pairs as:

$$\begin{aligned}\binom{n_i}{2} + \binom{n_j}{2} &\geq \binom{n_i - 1}{2} + \binom{n_j + 1}{2} \\ \iff n_i - n_j - 1 &\geq 0 \\ \iff n_i - n_j &\geq 1\end{aligned}$$

which is true.

So, all  $n_i, n_j$  are  $a, a + 1$  for some  $n$ . Note that one of  $a, a + 1$  is  $\lfloor \frac{k}{n} \rfloor$ .

[3] **Problem 4** (1974 USAMO) For  $a, b, c > 0$ , prove  $a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$ .

**Solution:**

$$a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$$

Taking the natural log of both sides,

$$\begin{aligned}\iff \ln(a^a b^b c^c) &\geq \ln((abc)^{\frac{a+b+c}{3}}) \\ \iff a \ln a + b \ln b + c \ln c &\geq \frac{a+b+c}{3} \ln(abc) \\ \iff \frac{a \ln a + b \ln b + c \ln c}{3} &\geq \frac{a+b+c}{3} \ln((abc)^{\frac{1}{3}})\end{aligned}$$

We will show the last inequality.

Let  $f(x) = x \ln x$ . Note that  $f'(x) = x' \ln(x) + x \ln'(x) = \ln(x) + 1$  and  $f''(x) = \ln'(x) = \frac{1}{x} \geq 0$  for positive  $x$ . So,  $f(x)$  is convex on positive numbers. By Jensen's,

$$\begin{aligned}\frac{f(a) + f(b) + f(c)}{3} &\geq f\left(\frac{a+b+c}{3}\right) \\ \frac{a \ln a + b \ln b + c \ln c}{3} &\geq \frac{a+b+c}{3} \ln\left(\frac{a+b+c}{3}\right)\end{aligned}$$

. Then, by AM-GM,  $\frac{a+b+c}{3} \geq \sqrt[3]{abc} = (abc)^{\frac{1}{3}} \implies \ln\left(\frac{a+b+c}{3}\right) \geq \ln((abc)^{\frac{1}{3}})$ . So,  $\frac{a+b+c}{3} \ln\left(\frac{a+b+c}{3}\right) \geq \ln((abc)^{\frac{1}{3}})$ . Combining the chain of inequalities, we get our desired

$$\iff \frac{a \ln a + b \ln b + c \ln c}{3} \geq \frac{a+b+c}{3} \ln((abc)^{\frac{1}{3}})$$

[3] **Problem 5** (1995 India) Let  $x_1, \dots, x_n$  be positive numbers whose sum is 1. Prove

$$\frac{x_1}{\sqrt{1-x_1}} + \dots + \frac{x_n}{\sqrt{1-x_n}} \geq \sqrt{\frac{n}{n-1}}$$

**Solution:** Let  $f(x) = \frac{x}{\sqrt{1-x}}$ . Then, note that  $f'(x) = \frac{x(\sqrt{1-x})' - x'(\sqrt{1-x})}{(\sqrt{1-x})^2} = \frac{x \cdot \frac{1}{2} \cdot (1-x)^{-\frac{1}{2}} - \sqrt{1-x}}{1-x} = \frac{\frac{1}{2} \cdot x \cdot (1-x)^{-\frac{3}{2}} - (1-x)^{\frac{1}{2}}}{1-x}$ . and that  $f''(x) = \frac{3x}{4}(1-x)^{-\frac{5}{2}} + \frac{1}{4}(1-x)^{-\frac{3}{2}}$ . This is positive when  $0 < x < 1 \iff 0 < 1-x < 1$ . So,  $f(x)$  is convex on  $(0, 1)$ .

Since  $0 < x_i < 1$ , we can apply Jensen's to get

$$\frac{f(x_1) \cdots + f(x_n)}{n} \geq f\left(\frac{x_1 \cdots + x_n}{n}\right) = f\left(\frac{1}{n}\right) = \frac{\frac{1}{n}}{\sqrt{1 - \frac{1}{n}}} = \frac{\frac{1}{\sqrt{n}}}{\sqrt{n-1}}$$

$$\implies f(x_1) \cdots + f(x_n) \geq \frac{\sqrt{n}}{\sqrt{n-1}}$$

$$\frac{x_1}{\sqrt{1-x_1}} + \cdots + \frac{x_n}{\sqrt{1-x_n}} \geq \sqrt{\frac{n}{n-1}}$$

[3] **Problem 6** (1998 USAMO) Let  $a_0, a_1 \cdots a_n$  be numbers from  $(0, \frac{\pi}{2})$  such that

$$\tan(a_0 - \frac{\pi}{4}) + \tan(a_1 - \frac{\pi}{4}) \cdots + \tan(a_n - \frac{\pi}{4}) \geq n-1$$

Prove that

$$\tan a_0 \tan a_1 \cdots \tan a_n \geq n^{n+1}$$

**Solution:** Let  $x_i = \tan(a_i - \frac{\pi}{4})$ . Note that  $-1 \leq x_i \leq 1$ . The condition translates to  $x_0 \cdots + x_n \geq n-1$ .

Also, note that  $\tan(a_i) = \frac{1+x_i}{1-x_i}$  using Tangent Addition Formula. Then,

$$\tan(a_0 - \frac{\pi}{4}) + \tan(a_1 - \frac{\pi}{4}) \cdots + \tan(a_n - \frac{\pi}{4}) \geq n-1$$

$$\iff \frac{x_0+1}{1-x_0} \cdots \cdot \frac{x_n+1}{1-x_n} \geq n^{n+1}$$

$$\iff \ln\left(\frac{x_0+1}{1-x_0} \cdots + \frac{x_n+1}{1-x_n}\right) \geq (n+1) \ln n$$

$$\iff \frac{1}{n+1} (\ln\left(\frac{x_0+1}{1-x_0} \cdots + \frac{x_n+1}{1-x_n}\right)) \geq \ln n$$

We will prove this last inequality.

Let  $f(x) = \ln\left(\frac{x+1}{1-x}\right)$ . Note that  $f'(x) = \frac{-2(1-x)}{(1+x)^2}$  and

$$f''(x) = \frac{(1-x)(x+1)' - (1-x)'(x+1)}{(x+1)^2} = \frac{1-x - (x+1)}{(x+1)^2} = \frac{-2x}{(x+1)^2}$$

So,  $f(x)$  is convex on  $(0, 1]$ . So, if all  $x_i$  are positive, it follows by Jensen's that

$$\frac{1}{n+1} (f(x_0) \cdots + f(x_n)) \geq f\left(\frac{x_0 \cdots + x_n}{n+1}\right)$$



$$\frac{1}{n+1}(\ln(\frac{x_0+1}{1-x_0}) \cdots + \ln(\frac{x_n+1}{1-x_n})) \geq \ln(\frac{\frac{x_0 \cdots x_n}{n+1} + 1}{1 - \frac{x_0 + \cdots + x_n}{n+1}}) \geq \ln(\frac{\frac{n-1}{n+1} + 1}{1 - \frac{n-1}{n+1}}) = \ln n$$

Without loss of generality,  $x_0 \leq x_1 \leq \cdots \leq x_n$ . Note that at most one of  $x_i$ 's are negative. Otherwise, it is strictly less than  $0 + 1(n-1) = n-1$ . If  $x_0$  is negative, we can replace  $x_0, x_1$  by  $\frac{x_0+x_1}{2}$  which is positive. Otherwise, since sum is preserved and is greater than  $n-1$ , it is impossible for there to be 2 or more negatives. Then, we can find that  $\frac{1}{n+1}(\ln(\frac{x_0+1}{1-x_0}) \cdots + \ln(\frac{x_n+1}{1-x_n}))$  decreases which implies that the minimum of the LHS is when all  $x_i$ 's are negative. We can apply the same reasoning as before.

## § 4.2 Misc

[3] **Problem 7** (M&IQ 1992) Prove that there are no positive integers  $n, m$  such that

$$(3 + 5\sqrt{2})^n = (5 + 3\sqrt{2})^m$$

## § 5 Geometry

[5] **Problem 1** (IMO SL 2005) Let  $ABC$  be a triangle such that  $M$  is the midpoint of  $BC$  and  $\gamma$  is the incircle of  $ABC$ .  $AM$  intersects  $\gamma$  at points  $K$  and  $L$ . Let lines passing through  $K$  and  $L$  parallel to  $BC$  intersect  $\gamma$  again at the points  $X$  and  $Y$ . Let lines  $AX$  and  $AY$  intersect  $BC$  again at points  $P$  and  $Q$ . Prove  $BP = CQ$ .

**Solution:** Note that  $BQ = CP \iff QM = PM \iff YL = LZ$  We will show this.

First, note that  $\triangle AKX \sim \triangle ALZ \implies \frac{KX}{LZ} = \frac{AK}{AL}$  (1). Also,  $AGKL$  is a harmonic bundle by a well-known property as  $AF, AE$  are tangents to  $\gamma$  and  $G, L$  pass through  $A$  and  $K$  is  $GL \cap \gamma$ . Then,  $\frac{KA}{KG} = \frac{LA}{LG}$  (2). Using (1)+(2), we get  $\frac{AL \cdot KX}{LZ \cdot KG} = \frac{LA}{LG} \implies \frac{KX}{LZ} = \frac{KG}{LG}$  (3). Also, let  $H = GX \cap YZ$ . Then,  $\triangle KXG \sim \triangle LHG$  as  $LH \parallel KX$  and  $\angle HGL = \angle KGX$ . Then,  $\frac{KG}{LG} = \frac{KH}{HL} \implies \frac{KX}{LZ} = \frac{KX}{HZ}$  by (3). Then, this implies  $LZ = HL$ . We will show  $Y = H$ . This is equivalent to showing  $X, G, Y$  are collinear.

We now prove a lemma: let  $ABC$  be a triangle with incircle  $\omega$  and incenter  $I$ . Then, let the tangency points of  $\omega$  with  $BC, AC, AB$  be  $D, E, F$  respectively. Also, let  $M$  be the midpoint of  $BC$ . Then,  $AM, EF, ID$  concur.

*Proof.* Let  $X = EF \cap ID$ . Then, we want to show  $A, X, M$  are collinear. Let  $U, V$  be the intersections of the line through  $X$  parallel to  $BC$  with  $AB$  and  $AC$  respectively. <sup>1</sup> Then,  $\angle IFX = \angle IUX = \angle IUV$  as  $XUFI$  is cyclic since  $IX \perp BC \implies IX \perp UV \implies \angle IXU = 90$  and  $IF \perp AB \implies \angle IFU = 90$ . This then implies  $\angle IFE = \angle IUV$ . Also,  $IXEV$  is cyclic as  $\angle IXV = 90$  and  $\angle IEV = 90$  so  $\angle IXV = \angle IEV$ , then  $\angle IEX = \angle IVX$ . Then, this implies  $\angle IEF = \angle IVU$ . By AA similarity,  $\triangle UIV \sim \triangle FIE$ . As  $\triangle FIE$  is isosceles, then  $\triangle UIV$  is isosceles

<sup>1</sup>Note that by the converse of the Simpson Line Theorem, since  $E, X, F$  are collinear and  $IF \perp AB = AU$ ,  $IE \perp AC = AV$ , and  $IX \perp BC \implies IX \perp UV$  since  $BC \parallel UV$ , that  $I \in (AUV)$ .

with  $UI = UV$ . As  $IV \perp UV$ , then  $UX = XV$ . So  $X$  is midpoint of  $UV$  and  $X$  is on the median  $AM$ . We are done.  $\square$

By the lemma,  $ID$  goes through  $G$ . Now, we want to show  $\angle GYL = \angle KXG$  which is equivalent to  $X, G, Y$  collinear by the converse of Alternate Interior Angles Theorem. Note that  $ID$  bisects  $YL$  as  $ID \perp BC \implies ID \perp YL$  and  $YL$  is a chord of  $\gamma$ . So,  $YD = DL$  and  $\triangle GYD \cong \triangle GLD \implies GY = GL$ . Then,  $\angle GYL = \angle GLY$ . By Alternative Interior Angles Theorem and as  $K, G, L$  are collinear,  $\angle GLX = \angle GKX$ . Similarly, letting  $ID$  intersect  $KX$  at  $P$ , we have  $KP = PX \implies \triangle KPG \cong \triangle XPG \implies GK = GX \implies \angle GKX = \angle KXG$ . So, using the chain of equalities, we get  $\angle GYL = \angle KXG$  which is what we want.