Collected Problems: Olympiad

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§ 1 Introduction

This is a collection of some of the olympiad problems that I have done. Because of my inexperience, the difficulty ratings are particularly subjective compared to the list of computational problems. Because of this, there's

§ 2 Combinatorics

[2] **Problem 1** (Romania TST) How many polynomials P with coefficients $0, 1, 2, or 3$ satisfy $P(2) = n$, where n is a given positive integer?

Solution: Generating Functions: You get $\prod_{i=0} \frac{x^{4\cdot 2^n}-1}{x^{2^n}-1} = \prod_{i=0} \frac{x^{2^{n+2}}-1}{x^{2^n}-1} = \frac{1}{(x-1)(x^2-1)} = \frac{-\frac{1}{4}}{x-1} + \frac{\frac{1}{2}}{(x-1)^2} + \frac{\frac{1}{4}}{x+1}$.

§ 3 Number Theory

[1] **Problem 1** (IMO 2005/4) Determine all positive integers relatively prime to all terms of the infinite sequence $a_n = 2^n + 3^n + 6^n - 1$ for $n \geq 1$.

Solution: Notice that for $p > 3$, $a_{p-2} = 2^{p-2} + 3^{p-2} + 6^{p-2} - 1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} - 1 = 0 \pmod{p}$ using Fermat Little Theorem. Then, note that $a_2 = 4 + 9 + 36 - 1 = 48$ which has prime factors of $2, 3$. So, if a number contains any prime factors, it is not relatively prime to all the terms of a_n. So, the only possible positive integer that can be relatively prime to all terms is $\boxed{1}$. Clearly, $\gcd(1, a_n) = 1$ and so it works.

[1] **Problem 2** (1978 IMO) Let m and n be positive integers such that $1 \leq m < n$. In their decimal representations, the last three digits of 1978^m are equal, respectively, to the last three digits of 1978^n. Find m and n such that $m + n$ has its least value.

Solution: Notice that $1978^m \equiv 1978^n \pmod{1000} \iff (-22)^m \equiv (-22)^n \pmod{1000} \iff (-22)^m \equiv (-22)^n \pmod{125, 8}$ by Chinese Remainder Theorem. If $m < 3$, then $(-22)^m \equiv (-22)^n \pmod 8 \implies m = n$ which is a contradiction. Similarily, $n < 3$ is impossible. So, $m, n \geq 3$. Then, both are $0 \pmod 8$. Now, consider $(-22)^m \equiv (-22)^n \pmod{125} \iff (-22)^{n-m} \equiv 1 \pmod{125}$. We seek to minimize $n - m$ to minimize $m + n = (n - m) + 2m$. Now, this is equivalent to finding $\text{ord}_{125} - 22$. Note that $\text{ord}_{25} - 22 = \text{ord}_{25} 3 = \text{ord}_{25} 20$. So, $20 = \text{ord}_{25}(-22) | \text{ord}_{125}(-22)$ clearly. By Euler's, $\text{ord}_{125}(-22)|100$. We can compute $(-22)^{20} \pmod{125}$ and find that it doesn't works. So, the order is 100 and $n - m = 0 \pmod{100} \implies n - m \geq 100$.

Our minimium $m + n$ is $3 + 103 = \boxed{106}$.

[2] **Problem 3** (Folklore) Find all positive integers n such that n divides $2^n - 1$.

Solution: Assume $n \geq 2$. Note that n can't be even because $2^n - 1$ is odd and it is impossible for an even number to be a divisor of an odd number. Then, consider the minimal odd prime p such that $p|n$. Also, let $\text{ord}_p 2 = d$. Then, $2^n - 1 = 0 \pmod n \implies 2^n = 1 \pmod n \implies d|n$. Also, by Fermat's Little Theorem, $2^{p-1} = 1 \pmod p$ so $d|p-1$. Then, $d|\gcd(n, p-1)$. If $\gcd(n, p-1) = k > 1$, then n has a prime divisor less than $p-1$ which contradicts the minimality of p. So, $\gcd(n, p-1) = 1$. Then, $n = (p-1)q + 1$ for some q. So, $2^n - 1 \pmod p = (2^{p-1})^q \cdot 2 - 1 = 2 - 1 = 1$. Since $n = 0 \pmod p$, we have $1 = 0 \pmod p$ which is clearly impossible.

So, $n = 1$ is the only possible solution. Checking, $n = 1$ does work. So, our solutions are $\boxed{1}$.

[2] **Problem 4** (Folklore) Let p be a prime that is relative prime to 10, and n be an integer, $0 < n < p$. Let d be the order of 10 modulo p.

1. Show that if the length of the period of the decimal expansion of $\frac{n}{p}$ is d
2. Prove that if d is even, then the period of the decimal expansion of $\frac{n}{p}$ can be divided into two halves whose sum is $10^{\frac{d}{2}} - 1$.

Solution 1: Let m be the period. Note that $\frac{n}{p} = 0.\overline{a_1 a_2 \cdots a_m} = \frac{a_1 a_2 \cdots a_m}{99 \cdots 9} = \frac{M}{10^m - 1} \implies n \cdot (10^m - 1) = p \cdot M \implies p | 10^m - 1$ as $\gcd(n, p) = 1$. So, $10^m = 1 \pmod p \implies d|m \implies m \geq d$.

We now want to show $m \leq d$. Let M' such that $\frac{n}{p} = \frac{M'}{10^d - 1} = M'(\frac{1}{10^d} + \frac{10^{2d}}{\cdots})$. Since $M' < 10^d - 1$, it clearly has a period of less than equal to d. So, $m \leq d$.

Then, $m = d$.

Solution 2: Let $d = 2k$. Then, let $\frac{n}{p} = 0.\overline{a_1 a_2 \cdots a_k a_{k+1} \cdots a_{2k}}$. We want to show $\overline{a_1 a_2 \cdots a_k} + \overline{a_{k+1} a_{k+2} \cdots a_{2K}} = 10^{\frac{d}{2}} - 1$ Call $\overline{a_1 \cdots a_k} = M_1$ and $\overline{a_{k+1} \cdots a_{2k}} = M_2$. We have $\frac{(10^{2k} - 1)n}{p} = M_1 \cdot 10^k + M_2$.

So, $\frac{(10^k + 1)n}{p} = 10^k \cdot \frac{n}{p} + \frac{n}{p} = a_1 \cdots_k .a_{k+1} \cdots a_{2k} \overline{a_1 \cdots a_k} + 0.\overline{a_1 \cdots a_{2k}}$. In order for it to come out as a integer, $.\overline{a_{k+1} \cdots a_{2k} a_1 \cdots a_k} + 0.\overline{a_1 \cdots a_{2k}} = 0.\overline{9} = 1$. Then, $\frac{(10^k + 1)n}{p} = M_1 + 1$. Then, $M_1 + M_2 = 10^k - 1$ which is what we wanted.

[3] **Problem 5** (China TST 2006) Find all positive integers a and n such that $\frac{(a+1)^n - a^n}{n}$ is an integer.

Solution: Assume $n \geq 2$. Then, let p be the smallest prime factor of n. Then, $p|b|(a+1)^n - a^n$. So, $(a+1)^n - a^n = 0 \pmod p$. if $a = 0 \pmod p$, then $(a+1)^n - a^n = 1 \pmod p \neq 0$. If $a = -1 \pmod p$, $(a+1)^n - a^n = (-1)^{n+1} \pmod p \neq 0$.

If $a \neq 0, 1 \pmod p$, then $(a+1)^n - a^n = 0 \pmod p \implies (a+1)^n = a^n \pmod p \implies (a+1)^n \cdot (a^{-1})^n = -1 \pmod p$ where a^{-1} denotes the inverse of a. Note a^{-1} exists since a is relatively prime to p. Then, we get $((a+1) \cdot a^{-1})^n = 1 \pmod p$. Let m be the order of $(a+1) \cdot a^{-1}$ modulo n. Then, we have $m|n$. But, we also have $m|p-1$ as $((a+1) \cdot a^{-1})^{p-1} = 1 \pmod p$ by Fermat's Little Theorem and both $a+1$ and a^{-1} are not equal to $0 \pmod p$. This means $m \leq p-1$ and $m|n$ which is a contradiction of our assumption that p is the smallest prime factor. So, $n \geq 2$ is impossible.

This means our only solutions are when $n = 1$. Clearly, for all positive integer a, $\frac{(a+1)-a}{1}$ is a positive integer.

$\boxed{(1, a) \text{ for all positive integers a}}$

[3] **Problem 6** (IMO 1990/3) Determine all integers $n > 1$ such that

$$\frac{2^n + 1}{n^2}$$

is an integer.

Solution: Note that n has to be odd as an even can't divide an odd. Let p be the smallest odd prime divisor of n. Then, $p|n^2|2^n + 1$. So, $p|(2^n + 1)(2^n - 1)|2^{2n} - 1$. By Fermat's Little Theorem, $p|2^{p-1} - 1$. So, by GCD Lemma, $p|2^{\gcd(2n,p-1)} - 1$. Note $\gcd(n, p-1) = 1$ as p is the minimal prime divisor so there can't be any prime divisors of n less than or equal to $p-1$. Also, $p-1$ is even and n is even so $\gcd(2n, p-1) = 2$. Then, $p|2^2 - 1|3 \implies p = 3$. Let $\nu_3(n) = k$.

Then, $2^n + 1 = 2^{3 \cdot \frac{n}{3}} + 1 = 8^{\frac{n}{3}} - (-1)^{\frac{n}{3}}$ and $8 + 1 = 9$ and by LTE, $\nu_3(2^n + 1) = \nu_3(9) + \nu_3(\frac{n}{3}) = 2 + k - 1 = k + 1$. However, $\nu_3(n^2) = 2k$. So, we have $2k \leq k + 1 \implies k \leq 1$. So, $k = 1$.

Let $n = 3m$ where $m \neq 0 \pmod 3$ and $m > 2$. Let $q|m$ such that q is the smallest odd prime. So, $q > 3$. By Fermat's Little Theorem, $q|2^{q-1} - 1$. Also, $q|n|n^2|2^n + 1 = 2^{3m} + 1|2^{6m} - 1$. By GCD Lemma, $q|\gcd(2^{q-1} - 1, 2^{6m} - 1) = 2^{\gcd(q-1,6m)} - 1$. Then, we have $\gcd(q-1, m) = 1$ as p is the minimal prime divisor. Also, $2|q-1$ as q is odd. Then $\gcd(6m, q-1) = 2, 6$. The first case gives $q|3$ which is impossible. The second case gives $q|63 \implies q = 7$.'

Now, if $q = 7$, $n^2 = 0 \pmod{49}$ and $2^n + 1 = 2^{3m} + 1 = 8^m + 1 = 2 \pmod 7$. So, it is impossible. So, we $m = 1$ is the only possible m and $n = 3$.

[3] **Problem 7** (IMO 1999) Find all the pairs of positive integers (x, p) such that p is a prime, $x \leq 2p$ and x^{p-1} is a divisor of $(p-1)^x + 1$.

Solution: Assume $p \geq 3$. This then implies x is odd.

$x^{p-1}|(p-1)^x + 1$. Then, let q be the smallest prime divisor of x. Then, $q|x|x^{p-1}|(p-1)^x + 1|(p-1)^{2x} - 1$. Assume $\gcd(p-1, q) = 1$. By Fermat's Little Theorem, $((p-1)^2)^{q-1} \equiv 1 \pmod q \implies q|(p-1)^{2q-2}$. By the GCD Lemma, $q|((p-1)^2)^{\gcd(x,q-1)}$. Note that $\gcd(x, q-1) = 1$ because q is the minimal prime. So, $q|(p-1)^2 - 1 = (p)(p-2)$. Since $\gcd(p, p-2) = \gcd(p, 2) = 1$, $q|p$ or $q|p-2$. If $q|p$, then $p = q$. Then, $x = 0 \pmod p$ and $x = p, 2p$. Then, $x \neq 2p$ because x can't have a prime factor less than p. Then, $x = p$ gives p^{p-1} is a divisor of $(p-1)^p + 1$ which implies

$p^{p-1} \leq (p-1)^p + 1$. Clearly, this fails for large enough p. The only possible solutions are $p = 3$. This then gives $x^2 | 2^x + 1$ where $x \leq 6$. We can easily check and see $x = 1, 3$ are the only solutios.

If $p = 2 \pmod q$. Then, $q | (p-1)^x + 1 = 2 \pmod q$ which is a contradiction.

If $\gcd((p-1)^2, q) = q$, then$(p-1)^2 = 0 \pmod q \implies p - 1 = 0 \pmod q \implies p = 1 \pmod q$. Then, $(p-1)^x + 1 = 1 \pmod q$. However, it is a multiple of $x^{p-1} = 0 \pmod q$ which is a contradiction. By LTE, for odd x, $\nu_p((p-1)^x + 1) = \nu_p(p) + \nu_p(x) = 1 + \nu_p(x)$.

Now, the only remaining case is $p = 2$. This gives $x | 2$ which gives $x = 1, 2$.

Our solutions are $\boxed{(1,2), (2,2), (1,3), (3,3)}$.

[4✎] **Problem 8** (MOP 2011) Let p be a prime and n a positive integer. Suppose that p^1 fully divides $2^n - 1$ (meaning it is divisible by p but not p^2). Prove that p^1 fully divides $2^{p-1} - 1$.

Solution: Let $d = \text{ord}_p(n)$. Then, since $2^n = 1 \pmod p$, we know $d | n$. Let $n = dk$. Then, by Lifting The Exponent, $1 = \nu_p(2^n - 1) = \nu_p((2^d)^k - 1^k) = \nu_p(2^d - 1) + \nu_k$ as $2^d - 1 = 0 \pmod p$ by the definition of order. This then implies $\nu_p(2^d - 1) \leq 1$. Since $2^d - 1 = 0 \pmod p$, we also have $\nu_p(2^d - 1) \geq 1$. Altogether, $\nu_p(2^d - 1) = 1(1)$.

Now, note that since $2^{p-1} = 1 \pmod p$, we have $d | p - 1$. Let $p - 1 = dt$. Then by Lifting The Exponent, $\nu_p(2^{p-1} - 1) = \nu_p((2^d)^t - 1^t) = \nu_p(2^d - 1) + \nu_p(t)$ as $2^d - 1 = 0 \pmod p$. Then, this is equal to $1 + \nu_p(t)$ by (1). Note that since $t | p - 1 \implies t < p$, $\nu_p(t) = 0$. So, it is exactly equal to 1. This what we wanted so we're done.

[4✎] **Problem 9** (IMO SL 2006) Find all integer solutions of the equation

$$\frac{x^7 - 1}{x - 1} = y^5 - 1$$

Lemma 1 (Lemma 1) Let p be a prime, n a positive intger and a any integer. Suppose $\Phi_n(a) \equiv 0 \pmod p$. Then, either:

1. $n | p - 1$
2. $p | n$

Consider $\Phi_7(x) = 1 + x \cdots + x^6 = \frac{x^7-1}{x-1}$. By the Main Theorem/Lemma, $p | \Phi_n(x)$ implies $p | n$ or $n | p - 1$. As a corollary, $p | \Phi_q(x)$ we have $p | q$ (which implies $p = q$) or $p = 1 \pmod q$. So, if $p | \Phi_7(x)$, $p = 7$ or $p = 1 \pmod 7$. Then, the LHS is $0 \pmod 7$ or $1 \pmod 7$.

This then implies $y - 1 = 0 \pmod 7$ or $y - 1 = 1 \pmod 7$ as $y - 1$ is a divisor of the RHS. These give $y = 1, 2 \pmod 7$. For the first case, $1 + y \cdots + y^4 = 5 \pmod 7$ which is a contradiction as it's not equal to 1 or 0. If we plug in $y = 2 \pmod 7$, we get $1 + 2 + 2^2 + 2^3 + 2^4 = 31 = 3 \pmod 7$.

Therefore, there are no solutions.

[4✎] **Problem 10** (IMO 2000) Does there exist a positive integer n such that n has exactly 2000 prime divisors and n divides $2^n + 1$?

Solution: Clearly, 2000 is arbitary and we can replace by m. We do induction on m. For $m = 1$, $p|2^p + 1$ and $p = 3$ works. Now, we do the induction step. By the inductive hypothesis, we have a $n = p_1^{a_1} \cdots p_m^{a_m}$ and $n|2^n + 1$ where $p_1, \cdots p_m$ are distinct primes and $a_1 \cdots a_m \geq 1$. Let $p_i^{a_i}|2^n + 1$, suppose $p_i^{b_i}||2^n + 1$ where $||$ means it perfectly divides. Then, $p_1^{b_1} \cdots p_m^{b_m}||2^n + 1$. By LTE, $p_1^{b_1+\ell}p_2^{b_2} \cdots p_m^{b_m}||2^{np_1^\ell} + 1$. Consider the sequence of $2^{np_1^\ell} + 1$ for $\ell = 1, 2 \cdots$. By Kobayashi's, there is some ℓ such that $2^{np_1^\ell} + 1$ has a prime factor p_{m+1} distinct from $p_1, p_2 \cdots p_m$ because the sequence's terms contain an infinite number of prime factors. So, $p_1^{b_1+\ell}p_2^{b_2} \cdots p_m^{b_m}p_{m+1}|2^{np_1^\ell} + 1$.

[5✎] **Problem 11** (IMO 2003) Let p be a prime number. Prove that there exists a prime number q such that for every intger n, the number $n^p - p$ is not divisible by q.

Lemma 2 (Lemma 1) Let p be a prime, n a positive intger and a any integer. Suppose $\Phi_n(a) \equiv 0 \pmod p$. Then, either:

1. $n|p-1$
2. $p|n$

Consider $q|\Phi_p(p)$ with $p^2 \nmid q-1$. It is clear such a prime q exists because $\Phi_p(p)$ is positive integer greater than 1 and that not all prime factors can be $\pm 1 \pmod{p^2}$. This is because $\Phi_p(p) = 1 + p \pmod{p^2} \neq \pm 1$. By the Main Lemma, $p|q-1$ or $q|p$. Clearly, $q|p$ is impossible as both are distinct primes. Otherwise, $p = q$ which implies $p|p^p - 1$. By way of contradiction, $n^p = p \pmod q \implies n = p^{\frac{1}{p}} \pmod q$. By Fermat's Little Theorem, $n^{q-1} = 1 \pmod q \implies p^{\frac{q-1}{p}} = 1 \pmod q$.

So, $q|\Phi_p(p)|p^p - 1$ and $q|p^{\frac{q-1}{p}} - 1$. By GCD Lemma, $q|\gcd(p^p - 1, p^{\frac{q-1}{p}}) = p^{\gcd(p, \frac{q-1}{p})} - 1$. Note $\nu_p(q-1) = 1$ because we established before, $\nu_p(q-1) \geq 1$ and by our assumption, $\nu_p(q-1) < 2$. So, $\nu_p(q-1) = 1$. Then, $q|p^{\gcd(p, \frac{q-1}{p})} - 1 \implies q|p-1$.

Then, as $q|p-1$ and $p|q-1$ which give $q \leq p-1$ and $p \leq q-1$ which are clearly impossible.

§ 4 Algebra

§ 4.1 Inequalities

[1✎] **Problem 1** (Canada MO 2017) For pairwise distinct nonnegative reals a, b, c, prove that

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(b-a)^2} > 2$$

Solution: WLOG $a < b < c$ and let $b = a + x$ and $c = a + y$. Then

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(b-a)^2}$$

$$= \frac{a^2}{(y-x)^2} + \frac{(a+x)^2}{y^2} + \frac{(b+y)^2}{x^2}$$

$$\geq \frac{x^2}{y^2} + \frac{y^2}{x^2} \geq 2$$

by AM-GM

[1] **Problem 2** (Canada MO 2002) Let a, b, c be positive reals. Prove

$$\frac{a^3}{bc} + \frac{b^4}{ac} + \frac{c^4}{bc} \geq a+b+c.$$

Solution:

$$\frac{a^3}{bc} + \frac{b^4}{ac} + \frac{c^4}{bc} \geq a+b+c$$

$$\iff a^4 + b^4 + c^4 \geq a^2bc + b^2ac + c^2bc$$

after multiplying by abc on both sides. We will show this last inequality.

Then, note that by Weighted AM-GM, $2a^4 + b^4 + c^4 \geq 4a^2bc$. Similarly, $2b^4 + a^2 + c^2 \geq 4b^2ac$ and $2c^2 + a^2 + b^2 \geq 4c^2ab$. Then,

$$4a^4 + 4b^4 + c^4 \geq 4a^2bc + 4b^2ac + 4c^2ab$$

$$\iff a^4 + b^4 + c^4 \geq a^2bc + b^2ac + c^2ab$$

so it is true.

[2] **Problem 3** (How should n balls be put into k boxes to mimize the number of pairs of balls which are in the same box?)

Solution: Let the number of balls in the ith box be n_i. Then, $n_1 + n_2 \cdots + n_k = n$ and we want to minimize $\binom{n_1}{2} + \binom{n_2}{2} \cdots + \binom{n_k}{2}$. If $n_i - n_j \geq 2$ for some i, j, then we can replace n_i by $n_i - 1$ and $n_j + 1$ (which preserves n) and decrease the number of pairs as:

$$\binom{n_i}{2} + \binom{n_j}{2} \geq \binom{n_i - 1}{2} + \binom{n_j + 1}{2}$$

$$\iff n_i -_j -1 \geq 0$$

$$\iff n_i - n_j \geq 1$$

which is true.

So, all n_i, n_j are $a, a+1$ for some n. Note that one of $a, a+1$ is $\lfloor \frac{k}{n} \rfloor$.

[3] **Problem 4** (1974 USAMO) For $a, b, c > 0$, prove $a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$.

Solution:

$$a^a b^b c^c \geq (abc)^{\frac{a+b+c}{3}}$$

Taking the natural log of both sides,

$$\iff \ln(a^a b^b c^c) \geq \ln((abc)^{\frac{a+b+c}{3}})$$

$$\iff a \ln a + b \ln b + c \ln c \geq \frac{a+b+c}{3} \ln(abc)$$

$$\iff \frac{a \ln a + b \ln b + c \ln c}{3} \geq \frac{a+b+c}{3} \ln((abc)^{\frac{1}{3}})$$

We will show the last inequalty.

Let $f(x) = x \ln x$. Note that $f'(x) = x' \ln(x) + x \ln'(x) = \ln(x) + 1$ and $f''(x) = \ln'(x) = \frac{1}{x} \geq 0$ for positive x. So, $f(x)$ is convex on positive numbers. By Jensens',

$$\frac{f(a) + f(b) + f(c)}{3} \geq f(\frac{a+b+c}{3})$$

$$\frac{a \ln a + b \ln b + c \ln c}{3} \geq \frac{a+b+c}{3} \ln(\frac{a+b+c}{3})$$

. Then, by AM-GM, $\frac{a+b+c}{3} \geq \sqrt[3]{abc} = (abc)^{\frac{1}{3}} \implies \ln(\frac{a+b+c}{3}) \geq \ln(abc)^{\frac{1}{3}}$. So, $\frac{a+b+c}{3} \ln(\frac{a+b+c}{3}) \geq \ln((abc)^{\frac{1}{3}})$. Combining the chain of inequalties, we get our desired

$$\iff \frac{a \ln a + b \ln b + c \ln c}{3} \geq \frac{a+b+c}{3} \ln((abc)^{\frac{1}{3}})$$

[3] **Problem 5** (1995 India) Let $x_1, \cdots x_n$ be positive numbers whose sum is 1. Prove

$$\frac{x_1}{\sqrt{1-x_1}} + \cdots + \frac{x_n}{\sqrt{1-x_n}} \geq \sqrt{\frac{n}{n-1}}$$

Solution: Let $f(x) = \frac{x}{\sqrt{1-x}}$. Then, note that $f'(x) = \frac{x(\sqrt{1-x})' - x'(\sqrt{1-x})}{(\sqrt{1-x})^2} = \frac{x \cdot \frac{1}{2} \cdot (1-x)^{-\frac{1}{2}} - \sqrt{1-x}}{1-x} = \frac{1}{2} \cdot x \cdot (1-x)^{-\frac{3}{2}} - (1-x)^{\frac{1}{2}}$. and that $f''(x) = \frac{3x}{4}(1-x)^{-\frac{5}{2}} + \frac{1}{4}(1-x)^{-\frac{3}{2}}$. This is positive when $0 < x < 1 \iff 0 < 1-x < 1$. So, $f(x)$ is convex on $(0, 1)$.

Since $0 < x_i < 1$, we can apply Jensen's to get

$$\frac{f(x_1) \cdots + f(x_n)}{n} \geq f(\frac{x_1 \cdots + x_n}{n}) = f(\frac{1}{n}) = \frac{\frac{1}{n}}{\sqrt{1 - \frac{1}{n}}} = \frac{\frac{1}{\sqrt{n}}}{\sqrt{n-1}}$$

$$\implies f(x_1) \cdots + f(x_n) \geq \frac{\sqrt{n}}{\sqrt{n-1}}$$

$$\frac{x_1}{\sqrt{1-x_1}} + \cdots + \frac{x_n}{\sqrt{1-x_n}} \geq \sqrt{\frac{n}{n-1}}$$

[3✎] **Problem 6** (1998 USAMO) Let $a_0, a_1 \cdots a_n$ be numbers from $(0, \frac{\pi}{2}$ such that

$$\tan(a_0 - \frac{\pi}{4}) + \tan(a_1 - \frac{\pi}{4}) \cdots + \tan(a_n - \frac{\pi}{4}) \geq n - 1$$

Prove that

$$\tan a_0 \tan a_1 \cdots \tan a_n \geq n^{n+1}$$

Solution: Let $x_i = \tan(a_i - \frac{\pi}{4})$. Note that $-1 \leq x_i \leq 1$. The condition translates to $x_0 \cdots + x_n \geq n - 1$.

Also, note that $\tan(a_i) = \frac{1+x_i}{1-x_i}$ using Tangent Addition Formula. Then,

$$\tan(a_0 - \frac{\pi}{4}) + \tan(a_1 - \frac{\pi}{4}) \cdots + \tan(a_n - \frac{\pi}{4}) \geq n - 1$$

$$\iff \frac{x_0 + 1}{1 - x_0} \cdots \cdot (\frac{x_n + 1}{1 - x_n} \geq n^{n+1}$$

$$\iff \ln(\frac{x_0 + 1}{1 - x_0} \cdots + \frac{x_n + 1}{1 - x_n} \geq (n + 1) \ln n$$

$$\iff \frac{1}{n+1}(\ln(\frac{x_0 + 1}{1 - x_0} \cdots + \ln \frac{x_n + 1}{1 - x_n}) \geq \ln n$$

We will prove this last inequalty.

Let $f(x) = \ln(\frac{x+1}{1-x})$. Note that $f'(x) = \frac{-2(1-x)}{(1+x)}$ and

$$f''(x) = \frac{(1 - x)(x + 1)' - (1 - x)'(x + 1)}{(x + 1)^2} = \frac{1 - x - (x + 1)}{(x + 1)^2} = \frac{-2x}{(x + 1)^2}$$

.

So, $f(x)$ is convex on $(0, 1]$. So, if all x_i are positive, it follows by Jensen's that

$$\frac{1}{n+1}(f(x_0) \cdots + f(x_n) \geq f(\frac{x_0 \cdots + x_n}{n + 1})$$

$$\frac{1}{n+1}(\ln(\frac{x_0 + 1}{1 - x_0} \cdots + \ln \frac{x_n + 1}{1 - x_n}) \geq \ln(\frac{\frac{x_0 \cdots + x_n}{n+1} + 1}{1 - \frac{x_0 + \cdots + x_n}{n+1}}) \geq \ln(\frac{\frac{n-1}{n+1} + 1}{1 - \frac{n-1}{n+1}}) = \ln n$$

Without loss of generality, $x_0 \leq x_1 \ldots \leq x_n$. Note that at most one of x_i's are negative. Otherwise, it is strictly less than $0 + 1(n - 1) = n - 1$. If x_0 is negative, we can replace x_0, x_1 by $\frac{x_0+x_1}{2}$ which is positive. Otherwise, since sum is preserved and is greater than $n-1$, it is impossible for there to be 2 or more negatives. Then, we can find that $\frac{1}{n+1}(\ln(\frac{x_0+1}{1-x_0} \cdots + \ln \frac{x_n+1}{1-x_n})$ decreases which implies that the minimium of the LHS is when all x_i's are negative. We can apply the same reasoning as before.

§ 4.2 Misc

[3✎] **Problem 7** (M&IQ 1992) Prove that there are no positive integers n, m such that

$$(3 + 5\sqrt{2})^n = (5 + 3\sqrt{2})^m$$

§ 5 Geometry

[5] **Problem 1** (IMO SL 2005) Let ABC be a triangle such that M is the midpoint of BC and γ is the incircle of ABC. AM intersects γ at points K and L. Let lines passing through K and L parallel to BC intersect γ again at the points X and Y. Let lines AX and AY intersect BC again at points P and Q. Prove $BP = CQ$.

Solution: Note that $BQ = CP \iff QM = PM \iff YL = LZ$ We will show this.

First, note that $\triangle AKX \sim \triangle ALZ \implies \frac{KX}{LZ} = \frac{AK}{AL}(1)$. Also, $AGKL$ is a harmonic bundle by a well-known property as AF, AE are tangents to γ and G, L pass through A and K is $GL \cap \gamma$. Then, $\frac{KA}{KG} = \frac{LA}{LG}(2)$. Using (1)+(2), we get $\frac{\frac{AL \cdot KX}{LZ}}{KG} = \frac{LA}{LG} \implies \frac{KX}{LZ} = \frac{KG}{LG}(3)$. Also, let $H = GX \cap YZ$. Then, $\triangle KXG \sim LHG$ as $LH||KX$ and $\angle HGL = \angle KGX$. Then, $\frac{KG}{LG} = \frac{KX}{HL} \implies \frac{KX}{LZ} = \frac{KX}{HZ}$ by (3). Then, this implies $LZ = HL$. We will show $Y = H$. This is equivalent to showing X, G, Y are collinear.

We now prove a lemma: let ABC be a triangle with incircle ω and incenter I. Then, let the tangency points of ω with BC, AC, AB be D, E, F respectively. Also, let M be the midpoint of BC. Then, AM, EF, ID concur.

Proof. Let $X = EF \cap ID$. Then, we want to show A, X, M are collinear. Let U, V be the intersections of the line through X parallel to BC with AB and AC respectively. [1] Then, $\angle IFX = \angle IUX = IUV$ as $XUFI$ is cyclic since $IX \perp BC \implies IX \perp UV \implies IXU = 90$ and $IF \perp AB \implies \angle IFU = 90$ This then implies $\angle IFE = \angle IUV$. Also, $IXEV$ is cyclic as $\angle IXV = 90$ and $\angle IEV = 90$ so $\angle IXV = \angle IEV$, then $\angle IEX = \angle IVX$. Then, this implies $\angle IEF = \angle IVU$. By AA similarity, $\triangle UIV \sim \triangle FIE$. As $\triangle FIE$ is isosceles, then $\triangle UIV$ is isoscles with $UI = UV$. As $IV \perp UV$, then $UX = XV$. So X is midpoint of UV and X is on the median AM. We are done. $\square$

By the lemma, ID goes through G. Now, we want to show $\angle GYL = \angle KXG$ which is equivalent to X, G, Y collinear by the converse of Alternate Interior Angles Theorem. Note that ID bisects YL as $ID \perp BC \implies ID \perp YL$ and YL is a chord of γ. So, $YD = DL$ and $\triangle GYD \cong \triangle GLD \implies GY = GL$. Then, $\angle GYL = \angle GLY$. By Alternative Interior Angles Theorem and as K, G, L are collinear, $\angle GLX = \angle GKX$. Similarly, letting ID intersect KX at P, we have $KP = PX \implies \triangle KPG \cong XPG \implies GK = GX \implies \angle GKX = \angle KXG$. So, using the chain of equalities, we get $\angle GYL = \angle KXG$ which is what we want.

[1] Note that by the converse of the Simpson Line Theorem, since E, X, F are collinear and $IF \perp AB = AU$, $IE \perp AC = AV$, and $IX \perp BC \implies IX \perp UV$ since $BC||UV$, that $I \in (AUV)$.