Collected Problems: Computational

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§ 1 Introduction

I use the following scheme: 1 point is roughly AMC 10 8-14 level. 2 points is roughly AMC 10 # 15-17 level, 3 points are AMC 10 # 18-21 level, 4 points are AMC 10 # 22-23 level and 5 points are # 24-25 level. Furthermore, 6 points are #6-8 AIME level, 7 points are # 9-11 AIME level, 8 points are #12-13 AIME level, 9 points are #14-15 AIME level, and 10 points are a hypothetical #16-18 AIME level. 10 pointers are usually olympiad-style problems that require several major lemmas and have very long solutions.

Most of these problems are from more obscure contests that will serve as good AIME and AMC practice.

§ 2 Combinatorics

§ 2.1 Casework

[1] **Problem 1** (ARML Local 2014) Let A, B, and C be randomly chosen (not necessarily distinct) integers between 0 and 4 inclusive. Pat and Chris compute the value of $A + B \cdot C$ by two different methods. Pat follows the proper order of operations, computing $A + (B \cdot C)$. Chris ignores order of operations, choosing instead to compute $(A + B) \cdot C$. Compute the probability that Pat and Chris get the same answer.

Solution: $\boxed{\dfrac{9}{25}}$

[2] **Problem 2** (BmMT 2014) Call a positive integer top-heavy if at least half of its digits are in the set $\{7, 8, 9\}$. How many three digit top-heavy numbers exist? (No number can have a leading zero.)

Solution: Consider $7, 8, 9$ to be the same and assign actual values at end, do casework on number of $7, 8, 9$.

[3✐] **Problem 3** (PuMAC 2019) Suppose Alan, Michael, Kevin, Igor, and Big Rahul are in a running race. It is given that exactly one pair of people tie (for example, two people both get second place), so that no other pair of people end in the same position. Each competitor has equal skill; this means that each outcome of the race, given that exactly two people tie, is equally likely. The probability that Big Rahul gets first place (either by himself or he ties for first) can be expressed in the form m/n, where m, n are relatively prime, positive integers. Compute $m+n$.

Solution: First, find total number of outcomes. Then, casework on if Big Rahul wins by himself or if he ties for first

[3✐] **Problem 4** (PuMAC 2019) Prinstan Trollner and Dukejukem are competing at the game show WASS. Both players spin a wheel which chooses an integer from 1 to 50 uniformly at random, and this number becomes their score. Dukejukem then flips a weighted coin that lands heads with probability 3/5. If he flips heads, he adds 1 to his score. A player wins the game if their score is higher than the other player's score. The probability Dukejukem defeats the Trollner to win WASS equals m/n where m, n are co-prime positive integers. Compute $m+n$.

Solution: Casework on if Dukejukem flips heads or tails on coin.

[4✐] **Problem 5** (Purple Comet 2015 HS) Seven people of seven different ages are attending a meeting. The seven people leave the meeting one at a time in random order. Given that the youngest person leaves the meeting sometime before the oldest person leaves the meeting, the probability that the third, fourth, and fifth people to leave the meeting do so in order of their ages (youngest to oldest) is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution: Casework on if youngest or oldest in third, fourth, fifth block.

[5✐] **Problem 6** (AIME I 2020/5) Six cards numbered 1 through 6 are to be lined up in a row. Find the number of arrangements of these six cards where one of the cards can be removed leaving the remaining five cards in either ascending or descending order.

[5✐] **Problem 7** (AMC 12B 2017/22) Abby, Bernardo, Carl, and Debra play a game in which each of them starts with four coins. The game consists of four rounds. In each round, four balls are placed in an urn—one green, one red, and two white. The players each draw a ball at random without replacement. Whoever gets the green ball gives one coin to whoever gets the red ball. What is the probability that, at the end of the fourth round, each of the players has four coins?

[5✐] **Problem 8** (HMMT November 2013) Find the number of positive integer divisors of 12! that leave a remainder of 1 when divided by 3.

[6✐] **Problem 9** (HMMT November 2014) Consider the set of 5-tuples of positive integers at most 5. We say the tuple $(a_1, a_2, a_3, a_4, a_5)$ is perfect if for any distinct indices i, j, k, the three numbers a_i, a_j, a_k do not form an arithmetic progression (in any order). Find the number of perfect 5-tuples.

Solution: Casework on parity of exponent of $2, 5, 11$. Note that there can't be any 3 and 7 can be included or excluded without consequence.

§ 2.1.1 PIE

[1✎] **Problem 10** How many integers from 1 to 100 (inclusive) are multiples of 2 or 3?

Solution: There are 50 multiples of 2 and 33 multiples of 3. If and only if a number is both a multiple of 2 and 3, then it is a multiple of 6. There are 16 multiples of 6. Our answer is $50+33-16=\boxed{67}$.

[1✎] **Problem 11** (AMC 10B 2017/13) There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

[8✎] **Problem 12** (SLKK AIME 2020) Andy the Banana Thief is trying to hide from Sheriff Buffkin in a row of 6 distinct houses labeled 1 through 6. Andy and Sheriff Buffkin each pick a permutation of the 6 houses, chosen uniformly at random. On the n^{th} day, with $1 \leq n \leq 6$, Andy and Sheriff Buffkin visit the n^{th} house in their respective permutations, and Andy is caught by the Sheriff on the first day they visit the same house. For example, if Andy's permutation is $1,3,4,5,6,2$ and Sheriff Buffkin's permutation is $3,4,1,5,6,2$, Andy is caught on day 4. Given that Sheriff Buffkin catches Andy within 6 days and the expected number of days it takes to catch Andy can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b, find the remainder when $a+b$ is divided by 1000.

Solution: Notice that we can fix Sheriff Buffkin's permutation to just be $1,2,3,4,5,6$ because for each of Buffkin's permutation, the probability of being caught on the nth day clearly doesn't change. The number of Andy's permutations such that he does get caught is $6!-D_6=455$. Then, we do casework on day caught and then use complementary PIE. If he is caught on the nth day, the nth number inf Andy's permutation is clearly n. Then, we do PIE on the number of permutations such that at least one of the previous numbers are the same. This turns out to be $\binom{n-1}{1}\cdot 4! - \binom{n-1}{2}\cdot 3! + \binom{n-1}{3}\cdot 2! \cdots$. Then, the probability that he is caught on the nth day is $\frac{5!-\binom{n-1}{1}\cdot 4!+\binom{n-1}{2}\cdot 3!-\binom{n-1}{3}\cdot 2!\cdots}{6!}$.

Doing the computation for all days, we get $\frac{1331}{455} \implies \boxed{786}$.

§ 2.2 Perspectives

[1✎] **Problem 13** (AIME I 2002/1) Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left) is $\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution: Calculate the complementary probability. The probability that the three numbers aren't a palindrome is $1\cdot 1\cdot \frac{9}{10}=\frac{9}{10}$. The probability the three lettters aren't a palindrome is $1\cdot 1\cdot \frac{25}{26}=\frac{25}{26}$. Multiplying together, the probability both aren't a palindrome is $\frac{9\cdot 25}{10\cdot 26}=\frac{9\cdot 5}{2\cdot 26}=\frac{45}{52}$. So, our wanted probability is $1-\frac{45}{52}=\frac{7}{52} \implies \boxed{59}$.

[1✎] **Problem 14** (PuMAC 2019) How many ways can you arrange 3 Alice's, 1 Bob, 3 Chad's, and 1 David in a line if the Alice's are all indistinguishable, the Chad's are all indistinguishable, and Bob and David want to be adjacent to each other? (In other words, how many ways can you arrange 3 A's, 1 B, 3 C's, and 1 D in a row where the B and D are adjacent?)

Solution: Consider BD as one block of E and then multiply by $2! = 2$ at the end to permute the B and D's. We want to arrange 3 A's, 1 E, and 3 C's. There are $\frac{7!}{3!3!} = \frac{5040}{6 \cdot 6} = 140$ such permutations. So, the total number of ways is $\boxed{280}$

[2✎] **Problem 15** (MAθ 2016) The product of any two of the elements of the set $\{30, 54, N\}$ is divisible by the third. Find the number of possible values of N.

Solution: Consider the primes $2, 3, 5$ separately and get independent inequalities.

[2✎] **Problem 16** (2017 AMC 10B/17) Call a positive integer *monotonous* if it is a one-digit number or its digits, when read from left to right, form either a strictly increasing or a strictly decreasing sequence. For example, 3, 23578, and 987620 are monotonous, but 88, 7434, and 23557 are not. How many monotonous positive integers are there?

[2✎] **Problem 17** (AIME II 2002/9) Let $\mathcal{S}$ be the set $\{1, 2, 3, \dots, 10\}$ Let n be the number of sets of two non-empty disjoint subsets of $\mathcal{S}$. (Disjoint sets are defined as sets that have no common elements.) Find the remainder obtained when n is divided by 1000.

Solution: We count the number of ordered pairs for now and divide by two at the end. For each element, we have 3 choices: in either of the sets or in neither. But, we also need to subtract off when at least one of the subsets is empty. If at least one is empty, there's 2 choices for each element. If both are empty, there is 1 choice for each element. We also have to divide by two in the end to make it unordered (and note that it's impossible for the two sets to be the same).

$$\frac{3^{10} - 2 \cdot 2^{10} + 1}{2} = \frac{59049 - 2048 + 1}{2} = \frac{57002}{2} = 28\boxed{501}$$

[2✎] **Problem 18** (AMC 10B 2018/22) Real numbers x and y are chosen independently and uniformly at random from the interval $[0, 1]$. What is the probability that x, y, and 1 are the side lengths of an obtuse triangle?

Solution: Note that $1 > x, y$ so 1 is opposite the obtuse angle. This gives the inequalty $1 > x^2 + y^2$. Also, by the Triangle Inequalty, $1 < x + y$. Graphing this, we want the area under the quarter unit circle centered at $(0, 0)$ but above the line from $(0, 1)$ to $(1, 0)$. This has area $\boxed{\frac{\pi - 2}{4}}$

[2✎] **Problem 19** (AMC 10B 2020/23) Square $ABCD$ in the coordinate plane has vertices at the points $A(1, 1), B(-1, 1), C(-1, -1)$, and $D(1, -1)$. Consider the following four transformations:

- ✦ L, a rotation of 90° counterclockwise around the origin;
- ✦ R, a rotation of 90° clockwise around the origin;

- ✦ H, a reflection across the x-axis; and
- ✦ V, a reflection across the y-axis.

Each of these transformations maps the squares onto itself, but the positions of the labeled vertices will change. For example, applying R and then V would send the vertex A at $(1,1)$ to $(-1,-1)$ and would send the vertex B at $(-1,1)$ to itself. How many sequences of 20 transformations chosen from $\{L, R, H, V\}$ will send all of the labeled vertices back to their original positions? (For example, R, R, V, H is one sequence of 4 transformations that will send the vertices back to their original positions.)

Solution: Notice that after any sequence of 19 moves, we have a unique move that takes the square back to its original square. So, the total number of sequences is $4^{19} = \boxed{2^{38}}$.

[2✎] **Problem 20** (CNCM Online Round 1) Pooki Sooki has 8 hoodies, and he may wear any of them throughout a 7 day week. He changes hishoodie exactly 2 times during the week, and will only do so at one of the 6 midnights. Once he changes out of a hoodie, he never wears it for the rest of the week. The number of ways he can wear his hoodies throughout the week can be expressed as $\frac{8!}{2^k}$. Find k.

Solution: There is $8 \cdot 7 \cdot 6$ ways to choose the hoodies Pooki wears and there are $\binom{6}{2} = 15 = 3 \cdot 5$ ways to choose the midnights he changes. The number of ways is then $8 \cdot 7 \cdot 6 \cdot 5 \cdot 3 = \frac{8!}{4 \cdot 2} = \frac{8!}{2^3}$. Our answer is $\boxed{3}$.

[3✎] **Problem 21** (BmMT 2014) If you roll three regular six-sided dice, what is the probability that the three numbers showing will form an arithmetic sequence? (The order of the dice does matter, but we count both (1, 3, 2) and (1, 2, 3) as arithmetic sequences.)

Solution: We count the number of ordered triplets (a, b, c) such that two of the numbers average to the third number. WLOG $a \le b \le c$. Note that if a, b, c are distinct, then this corresponds to 6 ordered triplets. Otherwise, $a = b = c$ and this clearly corresponds to only 1 ordered triplet.

For the first case, notice that this is identical to choosing two numbers a, c (unordered) and seeing if their average is an integer. This happens if a, c are the same parity. Then, this gives $\binom{3}{2} + \binom{3}{2} = 6$ (a, b, c) such that $a \le b \le c$. We multiply by 6 to get the full number of ordered triplets which is 36.

For the second case, this clearly has 6 ordered (a, b, c).

Then, our probability is $\frac{42}{216} = \boxed{\dfrac{7}{36}}$.

[4✎] **Problem 22** (PuMAC 2019) Keith has 10 coins labeled 1 through 10, where the ith coin has weight 2^i. The coins are all fair, so the probability of flipping heads on any of the coins is $\frac{1}{2}$. After flipping all of the coins, Keith takes all of the coins which land heads and measures their total weight, W. If the probability that $137 \le W \le 1061$ is m/n for coprime positive integers m, n, determine $m + n$

Solution: This bijects to even binary numbers between 0 and 2046. There $\frac{1060-138}{2} + 1 = 462$ even numbers in the range $[137, 1061]$. Then, $\frac{462}{1024} = \frac{231}{512} \implies \boxed{743}$.

[4✎] **Problem 23** (PHS HMMT TST 2016) Compute the number of ordered triples of sets (A_1, A_2, A_3) that satisfy the following:

1. $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6\}$
2. $A_1 \cap A_2 \cap A_3 = \emptyset$

Solution: Consider each element separately and see where it can go in the Venn Diagram. It can go in $8 - 2 = 6$ sections as it can't be in all three or not in any. So, $\boxed{6^6}$.

[4✎] **Problem 24** (Magic Math AMC 10 2020) The county of Tropolis has five towns, with no roads built between any two of them. How many ways are there for the mayor of Tropolis to build five roads between five different pairs of towns such that it is possible to get from any town to any other town using the roads?

Solution: We do complementary counting and find the number of ways so there are two components in the graph. Note that it is impossible for the largest component to be a 3-component (or smaller) because there is at most 3 roads in the 3-component and at most 1 road between the two other cities. This means there is at most 4 roads which is a contradiction as there are 5 roads.

So, the largest component must be a 4-component (since we cannot have a 5-component). This means the graph looks like a connected 4-component and an isolated city. Note that no matter how we place roads in the 4-component, it is connected (by similar reasoning by before as there cannot be a 3-component). There are $\binom{4}{2} = 6$ possible roads. Then, there are $\binom{6}{5} = 6$ ways to create the roads.

Now, the total number of ways to addin roads without the condition the whole graph is connected is $\binom{\binom{5}{2}}{2} = 45$. So, the number of ways to have a connected graph after adding in the roads is $45 - 6 = \boxed{39}$.

[5✎] **Problem 25** (HMMT Feburary 2014) We have a calculator with two buttons that displays an integer x. Pressing the first button replaces x by $\lfloor \frac{x}{2} \rfloor$, and pressing the second button replaces x by $4x + 1$. Initially, the calculator displays 0. How many integers less than or equal to 2014 can be achieved through a sequence of arbitrary button presses? (It is permitted for the number displayed to exceed 2014 during the sequence. Here, byc denotes the greatest integer less than or equal to the real number y.)

Solution: Any number with 1 digits separated by one or more 0's is valid. Notice that for 2015 through 2047, the first two digits are 11 so they are not valid. Casework on number of digits now.

§ 2.2.1 Stars and Bars

[1✎] **Problem 26** (AMC 8 2019/25) Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each person (include Alice) has at least 2 apples?

Solution: We have $a + b + c = 24$ and $a \geq 2$ while $b, c \geq 0$. Let $a' = a - 2$. Then, $a' \geq 0$. We have $a' + b + c = 22$. By Stars and Bars, there are $\binom{24}{2} = 23 \cdot 12 = \boxed{276}$ ways to distribute.

[1✎] **Problem 27** (AMC 10A 2003/21) Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

Solution: Let a, b, c be the number of choclate chip, oatmetal, and peanut butter cookies repsecitvely. Then, $a+b+c=6$ has $\binom{8}{2} = \boxed{28}$ distributions.

[1✎] **Problem 28** (AMC 10A 2018/11) When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as

$$\frac{n}{6^7},$$

where n is a positive integer. What is n?

Solution: Let the ith dice roll x_i. Notice that $x_i \geq 1$. Let $x_i' = x_1 - 1$. Then, $x_1 + x_2 \cdots + x_7 = 10 \implies x_1 + x_2 \cdots + x_7 = 3$ has $\binom{9}{6} = \boxed{84}$ distributions. Note that it is impossible for any of the x_i's to exceed 6.

[4✎] **Problem 29** (AMC 12A 2006/25) How many non- empty subsets S of $\{1, 2, 3, \ldots, 15\}$ have the following two properties?

1. No two consecutive integers belong to S.
2. If S contains k elements, then S contains no number less than k.

Solution: We can do casework on the number of elements k. Then, we can only have numbers in the range $\{k, k+1 \cdots 15\}$. Now, let the elements in the subset be $a_1, a_2 \cdots a_k$. Let $d_1 = a_1 - k$, $d_2 = a_2 - a_1$, $d_3 = a_3 - a_2$ and so on until $d_k = a_k - a_{k-1}$ and $d_{k+1} = 15 - a_k$. We have that $d_1 + d_2 \cdots + d_k + d_{k+1} = 15 - k$.

Notice that $d_2, d_3 \cdots d_k \geq 2$ to satisfy that no two consecutive integers are in S. So, let $d_k' = d_k - 2$ for $k = 2, 3 \cdots k$. So, $d_1 + d_2' + d_3' \cdots + d_k' + d_{k+1} = 15 - k - 2(k-1) = 17 - 3k$. By Stars and Bars, there are $\binom{17-2k}{k}$ ordered $(k+1)$-tuplets $(d_1, d_2 \cdots d_{k+1})$. Notice that $(d_1, d_2 \cdots d_{k+1})$ determines $a_1, a_2 \cdots a_k$.

§ 2.2.2 Expected Value

[3✎] **Problem 30** (Math Prizes For Girls 2018) Maryam has a fair tetrahedral die, with the four faces of the die labeled 1 through 4. At each step, she rolls the die and records which number is on the bottom face. She stops when the current number is greater than or equal to the previous number. (In particular, she takes at least two steps.) What is the expected number (average number) of steps that she takes? Express your answer as a fraction in simplest form.

[5✎] **Problem 31** (SLKK AIME 2020) Woulard forms a 8 letter word by picking each letter from the set $\{w, o, u\}$ with equal probability. The score of a word is the nonnegative difference between the number of distinct occurrences of the three-letter word "uwu" and the number of distinct occurrences of the three-letter word "owo". For example, the string "owowouwu" has a score of 2 - 1 = 1. If the expected score of Woulard's string can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b, find the remainder when $a + b$ is divided by 1000.

Solution: Let the *value* of the string be the total number of uwu and owo's. Note that the value only differs from the score when there are both uwu and owo's. We can easily compute the value using Linearity of Expectation and do casework on when it differs.

[6 ✎] **Problem 32** (PuMAC 2019) . Marko lives on the origin of the Cartesian plane. Every second, Marko moves 1 unit up with probability 2/9, 1 unit right with probability 2/9, 1 unit up and 1 unit right with probability 4/9, and he doesn't move with probability 1/9. After 2019 seconds, Marko ends up on the point (A, B). What is the expected value of A · B?

[6 ✎] **Problem 33** (PuMAC 2019) Kelvin and Quinn are collecting trading cards; there are 6 distinct cards that could appear in a pack. Each pack contains exactly one card, and each card is equally likely. Kelvin buys packs until he has at least one copy of every card, then he stops buying packs. If Quinn is missing exactly one card, the probability that Kelvin has at least two copies of the card Quinn is missing is expressible as m/n for coprime positive integers m, n. Determine $m + n$.

Solution: Complementary counting, find probability that Kelvin has exactly one copy of the card Quinn is missing.

[7 ✎] **Problem 34** (2017 AMC 12B/25) A set of n people participate in an online video basketball tournament. Each person may be a member of any number of 5-player teams, but no two teams may have exactly the same 5 members. The site statistics show a curious fact: The average, over all subsets of size 9 of the set of n participants, of the number of complete teams whose members are among those 9 people is equal to the reciprocal of the average, over all subsets of size 8 of the set of n participants, of the number of complete teams whose members are among those 8 people. How many values n, $9 \leq n \leq 2017$, can be the number of participants?

Solution: Let t be the number of teams. We get that

$$\frac{t\binom{n-5}{4}}{\binom{n}{9}} = \frac{1}{\frac{t\binom{n-5}{3}}{\binom{n}{8}}}$$

by counting how many times each individual team gets counted. For the LHS, we can choose 4 other people out of the $n - 5$ non team members. The RHS follows similarly.

Then, after some bashing, we get

$$t = \frac{\frac{n!}{(n-5)!}}{2^5 \cdot 3^2 \cdot 5 \cdot 7}$$

We find that $\frac{n!}{(n-5)!}$ is a multiple of $9, 5$ always. Also, $\frac{n}{(n-5)!}$ is a multiple of 7 for 5 resdiues and (by some inelegant bashing) is a multiple of 32 for 16 residues. So, in each period $[224n, 224n+223]$, we have 80 successful n. So, in $[2, 2017]$ there are $80 \cdot 7 = 560$ successful n (we subtract out 1 which works and add in 2017 which works). We also have to subtract out $[2, 8]$ out of which 3 work. We get $\boxed{557}$.

§ 2.2.3 Recursion

[4✐] **Problem 35** (Math Prizes For Girls 2019) A $1{\times}5$ rectangle is split into five unit squares (cells) numbered 1 through 5 from left to right. A frog starts at cell 1. Every second it jumps from its current cell to one of the adjacent cells. The frog makes exactly 14 jumps. How many paths can the frog take to finish at cell 5?

[7✐] **Problem 36** (CNCM Online Round 2) On a chessboard with 6 rows and 9 columns, the Slow Rook is placed in the bottom-left corner and the Blind King is placed on the top-left corner. Then, 8 Sleeping Pawns are placed such that no two Sleeping Pawns are in the same column, no Sleeping Pawn shares a row with the Slow Rook or the Blind King, and no Sleeping Pawn is in the rightmost column. The Slow Rook can move vertically or horizontally 1 tile at a time, the Slow Rook cannot move into any tile containing a Sleeping Pawn, and the Slow Rook takes the shortest path to reach the Blind King. How many ways are there to place the Sleeping Pawns such that the Slow Rook moves exactly 15 tiles to get to the space containing the Blind King?

§ 2.3 Miscellaneous

[1✐] **Problem 37** (Mandelbrot Nationals Sample Test) Michael Jordan's probability of hitting any basketball shot is three times greater than mine, which never exceeds a third. To beat him in a game, I need to hit a shot myself and have Jordan miss the same shot. If I pick my shot optimally, what is the maximum probability of winning which I can attain?

Solution: The probability is $p(1-3p) = p - 3p^2 = 3(-p^2 + \frac{p}{3}) = 3(-(p-\frac{1}{6})^2 + \frac{1}{36})$. So, the maximium probability is $\frac{1}{36} \cdot 3 = \boxed{\frac{1}{12}}$.

[2✐] **Problem 38** (CNCM Online Round 2) Adi the Baller is shooting hoops, and makes a shot with probability p. He keeps shooting hoops until he misses. The value of p that maximizes the chance that he makes between 35 and 69 (inclusive) buckets can be expressed as $\frac{1}{\sqrt[a]{b}}$ for a prime a and positive integer b. Find $a + b$.

Solution:

[2✐] **Problem 39** (PHS ARML TST 2017) Consider a group of eleven high school students. To create a middle school math contest, they must pick a four-person committee to write problems and a four-person committee to proofread. Every student can be on neither committee, one committee, or both committees, except for one student who does not want to be on both. How many combinations of committees are possible?

Solution: Complementary counting, find how many committees have that student on both and how many committees without that restriction

[2✐] **Problem 40** (Mandelbrot Regionals 2009) Mr. Strump has formed three person groups in his math class for working on projects. Every student is in exactly two groups, and any two groups have at most one person in common. In fact, if two groups are chosen at random then the probability that they have exactly one person in common is one-third. How many students are there in Mr. Strump's class?

[3✐] **Problem 41** (ARML Local 2014) Bobby, Peter, Greg, Cindy, Jan, and Marcia line up for ice cream. In an acceptable lineup, Greg is ahead of Peter, Peter is ahead of Bobby, Marcia is ahead of Jan, and Jan is ahead of Cindy. For example, the lineup with Greg in front, followed by Peter, Marcia, Jan, Cindy, and Bobby, in that order, is an acceptable lineup. Compute the number of acceptable lineups.

Solution: The probability Greg, Peter, and Bobby line up in their specific order is $\frac{1}{6}$. Similarly, the probability that Marica, Jan, and Cindy line up in their specific order is $\frac{1}{6}$. So, the number of acceptable lineups is $720 \cdot \frac{1}{36} = \boxed{20}$

[3✐] **Problem 42** (AMC 12B 2017/17) A coin is biased in such a way that on each toss the probability of heads is $\frac{2}{3}$ and the probability of tails is $\frac{1}{3}$. The outcomes of the tosses are independent. A player has the choice of playing Game A or Game B. In Game A she tosses the coin three times and wins if all three outcomes are the same. In Game B she tosses the coin four times and wins if both the outcomes of the first and second tosses are the same and the outcomes of the third and fourth tosses are the same. Find the probability of winning game A minus the probability of winning game B.

[3✐] **Problem 43** (ARML Local 2009) Some students in a gym class are wearing blue jerseys, and the rest are wearing red jerseys. There are exactly 25 ways to pick a team of three players that includes at least one player wearing each color. Compute the number of students in the class.

Solution: $\boxed{7}$

[6✐] **Problem 44** (CRMT Team 2019) A deck of the first 100 positive integers is randomly shuffled. Find the expected number of
draws it takes to get a prime number if there is no replacement.

[6✐] **Problem 45** (CNCM PoTD) Find the remainder when $\sum_{n=0}^{333} \sum_{k=3n}^{999} \binom{k}{3n}$ is divided by 70.

§ 3 Number Theory

§ 3.1 Divisors

[1✐] **Problem 1** (ARML Local 2013) Compute the smallest positive integer n such that $n^2 + n^0 + n^1 + n^3$ is a multiple of 13.

Solution: $\boxed{5}$

[1✐] **Problem 2** (LMT Fall 2019) a and b are positive integers and $8^a 9^b$ has 578 factors. Find ab.

Solution: $\boxed{88}$

[1✐] **Problem 3** (MAθ 2018) How many distinct prime numbers are in the first 50 rows of Pascal's Triangle?

Solution: If $k \neq 1, n-1$ for $\binom{n}{k}$, then $\binom{n}{k}$ is composite by its explicit formula. So, it is clear only $\binom{p}{1} = p$ works. $\boxed{15}$

[2✎] **Problem 4** (AHSME 1984) How many triples (a, b, c) of positive integers satisfy the simultaneous equations:

$$ab + bc = 44$$

$$ac + bc = 23$$

[2✎] **Problem 5** (PHS ARML TST 2017) Compute the greatest prime factor of

$$3^8 + 2 \cdot 3^4 \cdot 4^4 + 2^{16}$$

Solution: Let $3^4 = x$ and $4^4 = y$. Then, this is just $x^2 + 2xy + y^2 = (x + y)^2$. So, it is $(81 + 256)^2 = (337)^2$. Note that 337 is prime as it is not divisible by $2, 3, 5, 7, 11, 13, 17$. So, the greatest prime factor is $\boxed{337}$.

[2✎] **Problem 6** (HMMT November 2014) Compute the greatest common divisor of $4^8 - 1$ and $8^{12} - 1$

[2✎] **Problem 7** (BMT 2018) . Suppose for some positive integers, that $\frac{p+\frac{1}{q}}{q+\frac{1}{p}} = 17$. What is the greatest integer n such that $\frac{p+q}{n}$ is always an integer?

Solution: $\frac{p+\frac{1}{q}}{q+\frac{1}{p}} = \frac{p}{q} \implies p = 17q$. Then $\frac{p+q}{n} = \frac{18p}{n}$. So, $n = \boxed{18}$.

[2✎] **Problem 8** (LMT Spring 2020) Let LMT represent a 3-digit positive integer where L and M are nonzero digits. Suppose that the 2-digit number MT divides LMT. Compute the difference between the maximum and minimum possible values of LMT .

[2✎] **Problem 9** (BMT 2018) How many multiples of 20 are also divisors of 17!?

[2✎] **Problem 10** (LMT Spring 2020) Suppose there are n ordered pairs of positive integers (a_i, b_i) such that $a_i + b_i = 2020$ and a_i, b_i is a multiple of 2020, where $1 \leq i \leq n$. Compute the sum

$$\sum_{i=1}^{n} a_i + b_i.$$

[3✎] **Problem 11** (HMMT Feburary 2018) Distinct prime numbers p, q, r satisfy the equation

$$2pqr + 50pq = 7pqr + 55pr = 8pqr + 12qr = A$$

for some positive integer A. What is A?

[3✎] **Problem 12** (Math Prizes For Girls 2019) How many positive integers less than 4000 are not divisible by 2, not divisible by 3, not divisible by 5, and not divisible by 7?

Solution: The amount of numbers in $[210n+1, 210n+210]$ that are not divisible by any of $2, 3, 5, 7$ is $\phi(210) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot 210 = 48$. Then, the amount of such numbers in $[1, 3990]$ is $19 \cdot 48 = 912$. In, $[3991, 4000]$, only 3991 is relatively prime to both $2, 3, 5, 7$. So, our answer is $\boxed{913}$.

[3 ✎] **Problem 13** (MAθ 2018) The number $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + 100 \cdot 100!$ ends with a string of 9s. How many consecutive 9s are at the end of the number?

Solution: $n \cdot n! = (n+1)! - n!$, then telescope to get $101! - 1!$. Then, we are finding how many 0's does 101! have at the end which is equivalent to the number of factors of 10. The number of factors of 10 is equivalent to the number of factors of 5. By Legendre's, there are $20 + 4 = \boxed{24}$ 5's.

[3 ✎] **Problem 14** (AMC 12B 2017/16) The number $21! = 51,090,942,171,709,440,000$ has over $60,000$ positive integer divisors. One of them is chosen at random. What is the probability that it is odd?

[3 ✎] **Problem 15** (BMT 2015) There exists a unique pair of positive integers k, n such that k is divisible by 6, and $\sum_{i=1}^{k} i^2 = n^2$. Find (k, n).

Solution: You get $k \cdot (6k+1) \cdot (12k+1) = n^2$. All of these are pairwise coprime so each are squares. Trying out the first couple of squares for k, we get $k = 4$ is a solution. $k = 4$ gives $2^2 \cdot 5^2 \cdot 7^2 = n^2 \implies n = 70$. Having both $b^2 = 6a^2 + 1$ and $c^2 = 12a^2 + 1$ be squares is not possible for larger a. So, our only solution is $\boxed{(4, 70)}$.

[4 ✎] **Problem 16** (LMT Spring 2020) Let $\phi(k)$ denote the number of positive integers less than or equal to k that are relatively prime to k. For example, $\phi(2) = 1$ and $\phi(10) = 4$. Compute the number of positive integers $n \leq 2020$ such that $\phi(n^2) = 2\phi(n)^2$.

Solution: Note that $\phi(n^2) = n \cdot \phi(n)$. This is clear as each of the intervals $[kn, kn+n-1]$ has $\phi(n)$ numbers and also from the explicit interval. Then, we have $n \cdot phi(n) = 2\phi(n)^2 \implies \phi(n) = \frac{n}{2}$. Using the explicit formula, the only prime factor of n is 2 clearly. Otherwise, consider the largest prime factor of n which is $p > 2$. Then, the denominator has factor p as p cannot be canceled out (it would have to be canceled out by a multiple of p in the numerator, implying a larger prime factor of n). There are $\boxed{10}$ powers of 2.

[4 ✎] **Problem 17** (Magic Math AMC 10 2020) Find the number of ordered pairs of positive integers (a, b) with $a < b < 2017$ such that $10a$ is divisible by b and $10b$ is divisible by a.

Solution: Let $\gcd(a, b = k$. Then, let $x = \frac{a}{k}$ and $y = \frac{b}{k}$. It is clear that $\gcd(x, y) = 1$. Now, $b|10a \iff yk|10xk \iff y|10x$. Similarly, $b|10a \iff x|10y$. Also, note that $n|m$ if and only if $\nu_p(n) \leq \nu_p(m)$ for all primes p. Now, consider $p = 2$. We get $\nu_2(x) \leq \nu_2(y)+1$ and $\nu_2(y) \leq \nu_2(x)+1$. Combining, we get $\nu_2(x) - 1 \leq \nu_2(y) \leq \nu_2(x) + 1$. Also, note that as $\gcd(x, y) = 1$, both $\nu_2(x)$ and $\nu_2(y)$ cannot be positive. So, at least one is zero. WLOG $\nu_2(x) = 0$, then $\nu_2(y) = 0, 1$. This gives the solutions $(\nu_2(x), \nu_2(y)) = (0, 0), (0, 1), (1, 0)$. Similarly, we get the solutions $(\nu_5(x), \nu_5(y)) = (0, 0), (0, 1), (1, 0)$. Note that ν_5 and ν_2 are essentially independent. Then, we get (x, y) can possibly be $(1, 1), (1, 2), (1, 5), (1, 10), (2, 1), (2, 5), (5, 1), (5, 2), (10, 1)$. But, also note $a < b$ so $x < y$ so only $(1, 2), (1, 5), (1, 10), (2, 5)$ are valid. We have $b < 2017 \implies yk < 2017 \implies y < \frac{2017}{k}$. This gives $1008 + 403 + 201 + 403 = \boxed{2015}$ solutions.

[4✐] **Problem 18** (HMMT Feburary 2016) For which integers $n \in \{1, 2, \ldots, 15\}$ is $n^n + 1$ a prime number?

Solution: Clearly, n must be even otherwise $n^n + 1$ is even. Furthermore, n cannot have any odd factors. Otherwise, expressing $n = mk$ for odd k gives $n^n + 1 = n^{mk} + 1 = (m^m)^k + 1^k = (m^m + 1)(m^{mk-m} \ldots)$ which means it cannot be prime. Then, n must be 2^x for some x. Then, $n = 1, 2, 4, 8$ are only possible choices now. Now, note $1^1 + 1, 2^2 + 1, 4^4 + 1$ are clearly prime, Now, $8^8 + 1 = 2^{24} + 1 = (2^8)^3 + 1^3$ so it is divisible by $2^8 + 1$ and clearly not prime.

Our answers are $\boxed{1, 2, 4}$.

[5✐] **Problem 19** (CNCM Online Round 1) Consider all possible pairs of positive integers (a, b) such that $a \geq b$ and both $\frac{a^2+b}{a-1}$ and $\frac{b^2+a}{b-1}$ are integers. Find the sum of all possible values of the product ab.

[6✐] **Problem 20** (HMMT Feburary 2017) Find all pairs (a, b) of positive integers such that $a^{2017}+b$ is a multiple of ab.

[7✐] **Problem 21** (HMMT Feburary 2017) . Kelvin the Frog was bored in math class one day, so he wrote all ordered triples (a, b, c) of positive integers such that $abc = 2310$ on a sheet of paper. Find the sum of all the integers he wrote down. In other words, compute

$$\sum_{\substack{abc=2310 \\ a,b,c\in\mathbb{N}}} (a + b + c),$$

where $\mathbb{N}$ denotes the set of positive integers.

§ 3.2 Modulo

[1✐] **Problem 22** (MAθ 2018) The number $4^{14} - 1$ is divisible by 29 but $2^{14} - 1$ is not. What is the remainder when $2^{14} - 1$ is divided by 29?

Solution: $4^{14} - 1 = (2^{14} - 1)(2^{14} + 1) = 0 \pmod{29}$. Since $2^{14} - 1 \neq 0 \pmod{29}$, $2^{14} + 1 = 0 \pmod{29} \implies 2^{14} - 1 = \boxed{27} \pmod{29}$

[1✐] **Problem 23** (AMC 12A 2003/18) Let n be a 5-digit number, and let q and r be the quotient and the remainder, respectively, when n is divided by 100. For how many values of n is $q + r$ divisible by 11?

Solution: We have $n = 100q + r \implies n \pmod{11} = q + r$. So, $q + r = 0 \pmod{11} \iff n = 0 \pmod{11}$. The smallest 5-digit multiple of 11 is 10010 and the largest is 99990. So, there are $\frac{99990-10010}{11} + 1 = 9090 - 910 + 1 = \boxed{8181}$.

[1✐] **Problem 24** (BMT 2018) Find the minimal N such that any N-element subset of {1, 2, 3, 4, . . . 7} has a subset S such that the sum of elements of S is divisible by 7.

Solution: First, note that $N \leq 4$. Any 4-element subset has all of at least one of the following: $\{1, 6\}, \{2, 5\}, \{3, 4\}, \{7\}$ by the Pigeonhole Principle. So, any 4-element subset has a subset that has a sum divisible by 7. Also, note $N > 3$ because of the counterexample: $6, 5, 4$. So, $N = \boxed{4}$.

[2✎] **Problem 25** (AMC 10B 2019/14) The base-ten representation for 19! is

$$121, 6T5, 100, 40M, 832, H00,$$

where T, M, and H denote digits that are not given. What is $T + M + H$?

Solution: Also, $19! = 0 \pmod{125}$ by Legendre's so $H = 0$. Note that $19! = 0 \pmod 9$ so $T + M + H + 33 = 0 \pmod 9 \implies T + M + H = 3 \pmod 9 \implies T + M = 3 \pmod 9 \implies T + M = 3, 12$. We also have $19! = 0 \pmod{11} \implies H - 2 + 3 - 8 + M - 0 + 4 - 0 + 0 - 1 + 5 - T + 6 - 1 + 2 - 1 = 0 \pmod{11} \implies H + M - T + 7 = 0 \pmod{11} \implies H + M - T = 4 \pmod{11} \implies M - T = 4 \pmod{11} \implies M - T = 4$ since $0 \leq M, T \leq 9$. Notice that $T + M = 3$ is impossible then. So, $T + M = 12 \implies H + T + M = \boxed{12}$

[2✎] **Problem 26** (BMT 2018) What is the remainder when 201820182018 . . . [2018 times] is divided by 15?

Solution: Let $N = 201820182018\ldots$ [2018 times]. We will find $N \pmod 3$ and $N \pmod 5$ and then use CRT. Clearly, $N \pmod 5 = 3$. Also, $N \pmod 3 = 2018 \cdot 2018 = (-1)^2 = 1$. Then, $N \pmod{15} = \boxed{13}$.

[2✎] **Problem 27** (CNCM PoTD) Find the number of positive integer x less than 100 such that

$$3^x + 5^x + 7^x + 11^x + 13^x + 17^x + 19^x$$

is prime.

Solution: Considering $\pmod 3$, we get $3(1)^x + 3(-1)^x = 0 \pmod 3$ which is impossible as the expression is clearly > 3. So, $\boxed{0}$.

[3✎] **Problem 28** (LMT Fall 2019) Determine the remainder when $13^{2020} + 11^{2020}$ is divided by 144.

[3✎] **Problem 29** (AMC 10B 2018/16) Let $a_1, a_2, \ldots, a_{2018}$ be a strictly increasing sequence of positive integers such that

$$a_1 + a_2 + \cdots + a_{2018} = 2018^{2018}.$$

What is the remainder when $a_1^3 + a_2^3 + \cdots + a_{2018}^3$ is divided by 6?

Solution: Notice that $n^3 = n \pmod 2$ and that $n^3 = n \pmod 3$ by Euler's Totient Theorem. So, $n^3 = n \pmod 6$ by CRT. Then, $a_1^3 + a_2^3 \cdots + a_{2018}^3 = a_1 + a_2 \cdots + a_{2018} \pmod 6 = 2018^{2018} \pmod 6$. Now, $2018^{2018} \pmod 2 = 0$ and $2018^{2018} \pmod 3 = (-1)^{2018} \pmod 3 = 1$. So, by CRT, $2018^{2018} = \boxed{4} \pmod 6$.

[3✎] **Problem 30** (AMC 10A 2020/18) Let (a, b, c, d) be an ordered quadruple of not necessarily distinct integers, each one of them in the set $0, 1, 2, 3$. For how many such quadruples is it true that $a \cdot d - b \cdot c$ is odd? (For example, $(0, 3, 1, 1)$ is one such quadruple, because $0 \cdot 1 - 3 \cdot 1 = -3$ is odd.)

Solution: We have two cases: ad is even and bc is odd or ad is odd and bc is even. If ad is even, then we can do complementary counting and get $4^2 - 2^2 = 12$ ways for (a, d). This is because if ad was odd, this is equivalent to both a, d being odd. If bc is odd, we get $2^2 = 4$ ways for (b, c). The other case follows similarly. Altogether, are $2 \cdot 12 \cdot 4 = \boxed{96}$ such (a, b, c, d).

[3✎] **Problem 31** (CNCM Online Round 2) An ordered pair (n, p) is juicy if $n^2 \equiv 1 \pmod{p^2}$ and $n \equiv -1 \pmod{p}$ for positive integer n and odd prime p. How many juicy pairs exist such that $n, p \leq 200$?

[3✎] **Problem 32** (HMMT Feburary 2018) There are two prime numbers p so that $5p$ can be expressed in the form $\lfloor \frac{n^2}{5} \rfloor$ for some positive integer n. What is the sum of these two prime numbers?

Solution: The two prime numbers are 5 and 29, consider $\pmod{25}$. Our answer is $\boxed{34}$

[3✎] **Problem 33** (BMT 2019) Compute the remainder when the product of all positive integers less than and relatively prime to 2019 is divided by 2019.

[4✎] **Problem 34** (AMC 12B 2017/21) Last year Isabella took 7 math tests and received 7 different scores, each an integer between 91 and 100, inclusive. After each test she noticed that the average of her test scores was an integer. Her score on the seventh test was 95. What was her score on the sixth test?

[4✎] **Problem 35** (BMT 2018) If r_i are integers such that $0 \leq r_i < 31$ and r_i satisfies the polynomial $x^4 + x^3 + x^2 + x \equiv 30 \pmod{31}$, find

$$\sum_{i=1}^{4} (r_i^2 + 1)^{-1} \pmod{31}.$$

where x^{-1} is the modulo inverse of x, that is, it is the unique integer y such that $0 < y < 31$ and $xy - 1$ is divisible by 31.

Solution: Note that $x^4 + x^3 + x^2 + x \equiv 30 \pmod{31} \iff x^4 + x^3 + x^2 + x + 1 \equiv 0 \pmod{31} \iff \frac{x^5-1}{x-1} \equiv 0 \pmod{31}$. So, r_i are the roots of $x^5 - 1 \equiv 0 \pmod{31}$ except for 1. Note that $31 = 2^5 - 1 \implies 2^5 \equiv 1 \pmod{31}$. So, 2 is a root. Then, $2^2 = 4, 2^3 = 8, 2^4 = 16$ are also roots.

Now, we want to compute

$$\begin{aligned}
&\frac{1}{5} + \frac{1}{17} + \frac{1}{65} + \frac{1}{257} \\
&= \frac{1}{5} + \frac{1}{17} + \frac{1}{3} + \frac{1}{9} \\
&= \frac{1}{5} + \frac{1}{9} + \frac{1}{17} + \frac{1}{3} \\
&= \frac{14}{45} + \frac{20}{51} \\
&= \frac{14}{14} + \frac{20}{20} \\
&= 1 + 1 \\
&= \boxed{2}
\end{aligned}$$

[4✎] **Problem 36** (SLLKK AIME 2020) Smush is a huge Kobe Bryant fan. Smush randomly draws n jerseys from his infinite collection of Kobe jerseys, each being either the recent #24 jersey or the throwback #8 jersey with equal probability. Let $p(n)$ be the probability that Smush can divide the n jerseys into two piles such that the sum of all jersey numbers in each pile is the same. If

[5✎] **Problem 37** (BMT 2018) Ankit wants to create a pseudo-random number generator using modular arithmetic. To do so he starts with a seed x_0 and a function $f(x) = 2x + 25 \pmod{31}$. To compute the kth pseudo random number, he calls $g(k)$ defined as follows:

$$g(k) = \begin{cases} x_0 & \text{if } k = 0 \\ f(g(k-1)) & \text{if } k > 0 \end{cases}$$

If x_0 is 2017, compute $\sum_{j=0}^{2017} g(j) \pmod{31}$.

Solution: $\boxed{21}$

[5✎] **Problem 38** (LMT Spring 2020) Compute the maximum integer value of k such that 2^k divides $3^{2n+3} + 40n - 27$ for any positive integer n.

Solution: Factorize it into $8(27(3^{2n-2} + 3^{2n-4} \cdots + 1) + 5n)$ and analyze the inner term $\pmod 8$ and $\pmod{16}$. Then, get $\boxed{6}$.

[6✎] **Problem 39** (PHS HMMT TST 2020) Find the largest integer $0 < n < 100$ such that $n^2 + 2n$ divides $4(n-1)! + n + 4$.

Solution: For n is even, $n^2 + 2n = (n)(n+2) = 4(\frac{n}{2})(\frac{n+2}{2})$. $4(n-1)! = 0 \pmod{4(\frac{n}{2})(\frac{n+2}{)}2}$, then we get $n + 4 = 0 \pmod{n^2 + 2n}$ which is impossible. For n is odd, $n^2 + 2n = (n)(n+2)$ and $\gcd(n, n+2) = 1$. If either of $n, n+2$ are composite, WLOG $n = 0 \pmod p$ for some prime $p < n$, then $(n-1)! = 0 \pmod p \implies n + 4 = 0 \pmod p \implies 4 = 0 \pmod p$ contradiction. Similar for $n + 2$. Then, we have that $n, n+2$ must be both primes. We show that this works. We have $4(n-1)! + n + 4 \pmod n = -4 + 4 = 0 \pmod n$ by Wilsons'. Also, $4(n-1)! + n + 4 \pmod{n+2} = 2 + 4\frac{-1}{(n+1)(n)} \pmod{n+2} = 2 + 4\frac{-1}{2} \pmod{n+2} = 2 - 2 = 0 \pmod{n+2}$. Since $\gcd(n, n+2) = 1$, we're done.

The largest twin primes in the range are $\boxed{71}$, 73.

[7✎] **Problem 40** (HMMT November 2014) Suppose that m and n are integers with $1 \le m \le 49$ and $n \ge 0$ such that m divides $n^{n+1} + 1$. What is the number of possible values of m?

[7✎] **Problem 41** (BMT 2018) How many $1 < n \le 2018$ such that the set $\{0, 1, 1+2, \ldots, 1+2+3+\cdots+i, \ldots, 1+2+\cdots+n-1\}$ is a permutation of $\{0, 1, 2, 3, 4, \cdots, n-1\}$ when reduced modulo n?

Solution: Only 2^k works so $\boxed{10}$.

[8✎] **Problem 42** (BMT 2018) Determine the number of ordered triples (a, b, c), with $0 \le a, b, c \le 10$ for which there exists (x, y) such that $ax^2 + by^2 \equiv c \pmod{11}$

[8✎] **Problem 43** (BMT 2018) Compute the following:

$$\sum_{i=0}^{99}(x^2+1)^{-1} \pmod{199}$$

where x^{-1} is the value $0 \le y \le 199$ such that $xy-1$ is divisible by 199

[10✎] **Problem 44** (SLKK AIME 2020) : Let p = 991 be a prime. Let S be the set of all lattice points (x, y), with $1 \le x, y \le p-1$. On each point (x, y) in S, Olivia writes the number x^2+y^2. Let $f(x, y)$ denote the product of the numbers written on all points in S that share at least one coordinate with (x, y). Find the remainder when

$$\sum_{i=1}^{p-2}\sum_{j=1}^{p-2} f(i,j)$$

is divided by p.

§ 3.3 Bases

[4✎] **Problem 45** (HMMT November 2014) Mark and William are playing a game with a stored value. On his turn, a player may either multiply the stored value by 2 and add 1 or he may multiply the stored value by 4 and add 3. The first player to make the stored value exceed 2100 wins. The stored value starts at 1 and Mark goes first. Assuming both players play optimally, what is the maximum number of times that William can make a move? (By optimal play, we mean that on any turn the player selects the move which leads to the best possible outcome given that the opponent is also playing optimally. If both moves lead to the same outcome, the player selects one of them arbitrarily.)

[4✎] **Problem 46** (HMMT November 2013) How many of the first 1000 positive integers can be written as the sum of finitely many distinct numbers from the sequence $3^0, 3^1, 3^2 \cdots$?

[6✎] **Problem 47** (HMMT November 2014) For any positive integers a and b, define $a \oplus b$ to be the result when adding a to b in binary (base 2), neglecting any carry-overs. For example, $20 \oplus 14 = 101002 \oplus 11102 = 110102 = 26$. (The operation $\oplus$ is called the exclusive or.) Compute the sum

$$\sum_{k=0}^{2^{2014}-1}(k \oplus \lfloor \frac{k}{2} \rfloor)$$

[3✎] **Problem 48** (CNCM Online Round 1) Akshar is reading a 500 page book, with odd numbered pages on the left, and even numbered pages on the right. Multiple times in the book, the sum of the digits of the two opened pages are 18. Find the sum of the page numbers of the last time this occurs.

Solution: We have three cases: $\overline{abc}, \overline{ab(c+1)}$ and $\overline{ab9}, \overline{a(b+1)0}$ and $\overline{a99}, \overline{(a+1)00}$. For the first case, the sum of digits is $2a+2b+2c+1 = 1 \pmod 2$ so it is impossible. For the second case, $2a+2b+10 = 18 \implies a+b=4$. This gives $b=4$ and $a=0$ as the maximium solution for this case. For the third case, the sum of digits is $2a+19>18$. So, the maximium solutions are $409, 410$ giving a sum of $\boxed{819}$.

[3✎] **Problem 49** (CNCM Online Round 1) Define $S(N)$ to be the sum of the digits of N when it is written in base 10, and take $S^k(N) = S(S(...(N)...))$ with k applications of S. The stability of a number N is defined to be the smallest positive integer K where $S^K(N) = S^{K+1}(N) = S^{K+2}(N) = \ldots$. Let T_3 be the set of all natural numbers with stability 3. Compute the sum of the two least entries of T_3.

Solution: $S^K(N) = S^{K+1}(N) \implies S^K(N) = S(S^K(N))$. Note that for $n \geq 10$, $S(n) < 10$. So, $S^N(N)$ is a one digit number. So, if a number has stability 3, this implies that the first time $S^k(N)$ is a one-digit number is when $k=3$. Then, $S^2(N) \geq 10$ and $S(N) \geq 19$. So, the two smallest are $199, 289$. Summing, we get $\boxed{488}$.

§ 3.4 Binomial Theorem

[1✎] **Problem 50** (BMT 2015) Compute the sum of the digits of 1001^{10}.

[7✎] **Problem 51** (AIME I 2020/12) Let n be the least positive integer for which $149^n - 2^n$ is divisible by $3^3 \cdot 5^5 \cdot 7^7$. Find the number of positive integer divisors of n.

§ 3.5 Series and Sequences

[1✎] **Problem 52** (ARML Local 2009) Let p be a prime number. If p years ago, the ages of three children formed a geometric sequence with a sum of p and a common ratio of 2, compute the sum of the children's current ages.

[3✎] **Problem 53** (ARML Local 2014) For each positive integer k, let S_k denote the infinite arithmetic sequence of integers with first term k and common difference k^2. For example, S_3 is the sequence $3, 12, 21, \ldots$ Compute the sum of all k such that 306 is an element of S_k

[4✎] **Problem 54** (ARML Local 2014) The arithmetic sequences $a_1, a_2, a_3, ..., a_{20}$ and $b_1, b_2, b_3, ..., b_{20}$ consist of 40 distinct positive integers, and $a_{20} + b_{14} = 1000$. Compute the least possible value for $b_{20} + a_{14}$.

Solution: $\boxed{10}$

§ 3.6 Miscellenous

[1✎] **Problem 55** (LMT Spring 2020) Compute the smallest nonnegative integer that can be written as the sum of 2020 distinct integers.

Solution: Note that $0 = (-1010+1010) + (-1009+1009) \cdots + (-1+1)$. So, our answer is $\boxed{0}$.

[3✎] **Problem 56** (BMT 2015) Find all integer solutions to

$$x^2 + 2y^2 + 3z^2 = 36$$

$$3x^2 + 2y^2 + z^2 = 84$$

$$xy + xz + yz = -7$$

Solution: Adding the first two and dividng by 4 gives $x^2 + y^2 + z^2 = 30$. Then, $(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + xz + yz) = 16 \implies x + y + z = \pm 4$. Also, $(x + y)^2 + (x + z)^2 + (y + z)^2 = 2(x^2 + y^2 + z^2) + 2(xy + xz + yz) = 46$.

Let $x + y = a, x + z = b, y + z = c$. We get $a + b + c = \pm 8$ and $a^2 + b^2 + c^2 = 46$. The only decomposition of 46 into squares is $36, 9, 1$. So, $(6, 3, -1)$ and $(-6, -3, 1)$ are our only solutions for (a, b, c) (including permutations since it is symmetric). Notice that (a, b, c) being a permutation of $(6, 3, -1)$ means (x, y, z) is a permutation of $(5, 1, -2)$. Similarily, (a, b, c) being a permutation of $(-6, -3, 1)$ means (x, y, z) is a permutation of $(-5, -1, 2)$.

We look at which permutations of $(5, 1, -2)$ work. We can biject any valid permutation of $(5, 1, -2)$ to get a corresponding solution $(-x, -y, -z)$ for permutations of $(-5, -1, 2)$. It's clear only x can be 5 otherwise $2y^2, 3z^2 > 36$. Then, $2y^2 + z^2 = 9 \implies y = 2, z = -1$. So the only valid solution is $(5, 2, -1)$ and the corresponding solution is $(-5, -2, 1)$.

We get $\boxed{(5, 2, -1), (-5, -2, 1)}$.

[3] **Problem 57** (Magic Math AMC 10 2020) Let $s(n)$ denote the sum of the digits of a positive integer n. Find the number of three-digit positive integers $k \le 500$ satisfying $s(k) = s(1000 - k)$.

Solution: We do casework on $k = \overline{abc}$, $k = \overline{ab0}$ or $k = \overline{a00}$ for non-zero a, b, c. We find that $k = \overline{a00}$ works for $a = 5$, $k = \overline{ab0}$ never works as it implies $2(a + b) = 19$ and $k = \overline{abc}$ works when $a + b + c = 14$. Then, $a = 1, 2 \cdots 4$ gives $6 + 7 + 8 + 9 = 30$. Our answer is $\boxed{31}$.

[4] **Problem 58** (BMT 2019) For a positive integer n, define $\phi(n)$ as the number of positive integers less than or equal to n that are relatively prime to n. Find the sum of all positive integers n such that $\phi(n) = 20$

[5] **Problem 59** (CNCM Online Round 2) Let S be the set of all ordered pairs (x, y) of integer solutions to the equation

$$6x^2 + y^2 + 6x = 3xy + 6y + x^2y.$$

S contains a unique ordered pair (a, b) with a maximal value of b. Compute $a + b$.

[5] **Problem 60** (PHS HMMT TST 2020) Find the unique triplet of integers (a, b, c) with $a > b > c$ such that $a + b + c = 95$ and $a^2 + b^2 + c^3 = 3083$.

[5] **Problem 61** (BMT 2019) 0. Let S(n) be the sum of the squares of the positive integers less than and coprime to n. For example, $S(5) = 1^2 + 2^2 + 3^2 + 4^2$, but $S(4) = 1^2 + 3^2$. Let $p = 2^7 - 1 = 127$ and $q = 2^5 - 1 = 31$ be primes. The quantity $S(pq)$ can be written in the form

$$\frac{p^2q^2}{6}(a - \frac{b}{c})$$

where a, b, and c are positive integers, with b and c coprime and $b < c$. Find a.

[6] **Problem 62** (CNCM PoTD) How many positive integers k are there such that $101 \le k \le 10000$ and $\lfloor \sqrt{k - 100} \rfloor$ is a divisor of k?

§ 4 Algebra

§ 4.1 Polynomials

Generally uses the following techniques: Vieta's, Binomial Theorem, Multinomial Theorem, Remainder Theorem, Newton's Sums, Reciprocal Roots Trick, Quadratic Formula (including using Determinant), Finite Differences, $x + \frac{k}{x}$ subsitution, Polynomial Interpolation

[1] **Problem 1** (CRMT Math Bowl 2019) Find the sum of all real numbers such that

$$\sqrt[4]{16x^4 - 32x^3 + 24x^2 - 8x + 1} = 5$$

Solution: Note that this is $\sqrt[4]{(2x-1)^4} = |2x-1|$ using the Binomial Theorem. Then, $|2x-1| = 5 \implies x = 3, -2$. The sum is $\boxed{1}$.

[2] **Problem 2** (TAMU 2019) In the expansion of $(1 + ax - x^2)^8$ where a is a positive constant, the coefficient of x^2 is 244. Find the value of a

Solution: By the Multinomial Theorem, the coefficient of x^2 is $\binom{8}{2} \cdot a^2 - \binom{8}{1} = 28a^2 - 8$. So, $28a^2 - 8 = 244 \implies a^2 = 9 \implies a = \boxed{3}$.

[3] **Problem 3** (HMMT November 2014) Let $f(x) = x^2 + 6x + 7$. Determine the smallest possible value of $f(f(f(f(x))))$ over all real numbers x.

[3] **Problem 4** (HMMT February 2014) Find the sum of all real numbers x such that $5x^4 + 10x^3 + 10x^2 + 5x + 11 = 0$

Solution: Note that $f(x) = 5x^4 + 10x^3 + 10x^2 + 5x + 11$ is symmetric about $-\frac{1}{2}$. This is clearer after you rewrite as $5(x^4 + 2x^3 + 2x^2 + x) + 11 = 5x(x+1)(x^2 + x + 1)$ and you can see $f(-1-x) = f(x)$. But after a rough sketch and since it is monotonically increasing after 0, there are only two real roots that sum to $\boxed{1}$.

[3] **Problem 5** (BmMT 2014) Consider the graph of $f(x) = x^3 + x + 2014$. A line intersects this cubic at three points, two of which have x-coordinates 20 and 14. Find the x-coordinate of the third intersection point

Solution: Let the line have the equation $y = mx + b$. Then, $mx + b = x^3 + x + 2014 \implies x^2 + (1-m)x + 2014 - b = 0$. Then, the sum of the roots are 0. So, the third intersection point is $\boxed{-34}$.

[3] **Problem 6** (ARML 2017) Compute the number of ordered pairs of integers (a, b) such that the polynomials $x^2 - ax + 24$ and $x^2 - bx + 36$ have one root in common

Solution: Let r be the common root. Then, $r^2 - ar + 24 = 0 \implies a = r + \frac{24}{r}$ and $r^2 - br + 36 = 0 \implies b = r + \frac{36}{r}$. Since a, b are integers, this implies $r | \gcd(24, 36) = 12$ (r can be both positive and negative divisors). Then, there are $2 \cdot 2 \cdot 3 = 12$ values for r which then determine (a, b). Also, clearly they don't have two roots in common as both are monic and the constant term is different. Our answer is $\boxed{12}$.

[3] **Problem 7** (AMC 10A 2015/23) The zeroes of the function $f(x) = x^2 - ax + 2a$ are integers. What is the sum of the possible values of a?

Solution: If the roots are integers, then the discriminant must be a perfect square; otherwise, the roots are irrational. Then, $a^2 - 8a = n^2$ for some positive integer n. This gives $(a-4)^2 - n^2 = 16 \implies (a-n-4)(a+n-4) = 16$. Note that the sum of the roots is $2a - 8$. We have factor pairs $(-16, -1), (-8, -2), (-4, -4), (1, 16), (2, 8), (4, 4)$. Note that if the roots are integers, a must also be a integer because a is the sum of the roots by Vieta's. From this, we get $a = -1, 0, 9, 8$. Trying, we get that all of these work. So, the sum is $\boxed{16}$.

[4] **Problem 8** (HMMT Feburary 2017) Let $Q(x) = a_0 + a_1x + \ + a_nx^n$ be a polynomial with integer coefficients, and $0 \leq ai < 3$ for all $0 \leq i \leq n$. Given that $Q(\sqrt{3}) = 20 + 17\sqrt{3}$, compute $Q(2)$

[4] **Problem 9** (TAMU 2018) Suppose f is a cubic polynomial with roots a, b, c such that

$$a = \frac{1}{3 - bc}$$

$$b = \frac{1}{5 - ac}$$

$$c = \frac{1}{7 - ab}$$

If $f(0) = 1$, find $f(abc + 1)$.

Solution: Let the leading coefficient be k. We have $f(0) = 1 \implies abc = \frac{-1}{k}$. Multiply out the expressions to get $3a - abc = 1 \implies 3a + \frac{1}{k} = 1 \implies a = \frac{1-\frac{1}{k}}{3}$. Similarily, $b = \frac{1-\frac{1}{k}}{5}, c = \frac{1-\frac{1}{k}}{7}$. Also, $f(abc) + 1 = k(abc - a)(abc - b)(abc - c)$.

[4] **Problem 10** (PuMAC 2019) Let $f(x) = x^2 + 4x + 2$. Let r be the difference between the largest and smallest real solutions of the equation $f(f(f(f(x)))) = 0$. Then $r = a^{\frac{p}{q}}$ for some positive integers a, p, q so a is square-free and p, q are relatively prime positive integers. Compute $a + p + q$

Solution: Some pattern finding gives $f^n(x) = 0$ has solutions $x = -2 \pm 2^{\frac{1}{2^n}}$.

[4] **Problem 11** (HMMT February 2014) Find all real numbers k such that $r^4 + kr^3 + r^2 + 4kr + 16 = 0$ is true for exactly one real number r.

Solution: Divide by r^2 and subsitute $t = r + \frac{4}{r}$.

[4] **Problem 12** (PHS HMMT TST 2020) Let a, b, c be the distinct real roots of $x^3 + 2x + 5$. Find $(8 - a^3)(8 - b^3)(8 - c^3)$.

[4] **Problem 13** (PuMAC 2019) Let Q be a quadratic polynomial. If the sum of the roots of $Q^{100}(x)$ (where $Q^i(x)$ is defined by $Q^1(x) = Q(x), Q^i(x) = Q(Q^{i-1}(x))$ for integers $i \geq 2$) is 8 and the sum of the roots of Q is S, compute $|\log_2(S)|$.

[7✐] **Problem 14** (HMMT Feburary 2017) A polynomial P of degree 2015 satisfies the equation $P(n) = \frac{1}{n^2}$ for $n = 1, 2, \dots, 2016$. Find $\lfloor 2017P(2017) \rfloor$

[8✐] **Problem 15** (AIME I 2014/14) Let m be the largest real solution to the equation

$$\frac{3}{x-3} + \frac{5}{x-5} + \frac{17}{x-17} + \frac{19}{x-19} = x^2 - 11x - 4$$

There are positive integers a, b, and c such that $m = a + \sqrt{b + \sqrt{c}}$. Find $a + b + c$.

§ 4.1.1 Newton's Sums

[3✐] **Problem 16** (BMT 2015) Let r, s, and t be the three roots of the equation $8x^3 + 1001x + 2008 = 0$. Find $(r+s)^3 + (s+t)^3 + (t+r)^3$

[4✐] **Problem 17** (BMT 2019) Let r_1, r_2, r_3 be the (possibly complex) roots of the polynomial $x^3 + ax^2 + bx + \frac{4}{3}$. How many pairs of integers a, b exist such that $r_1^3 + r_2^3 + r_3^3 = 0$?

[6✐] **Problem 18** (SLKK AIME 2020) Let a, b, and c be the three distinct solutions to $x^3 - 4x^2 + 5x + 1 = 0$. Find

$$(a^3 + b^3)(a^3 + c^3)(b^3 + c^3).$$

§ 4.1.2 Roots of Unity

[2✐] **Problem 19** (AMC 12B 2017/12) What is the sum of the roots of $z^{12} = 64$ that have a positive real part?

[5✐] **Problem 20** (BMT 2019) Let a_n be the product of the complex roots of $x^{2n} = 1$ that are in the first quadrant of the complex plane. That is, roots of the form $a + bi$ where $a, b > 0$. Let $r = a_1 \cdot a_2 \cdot \ldots \cdot a_{10}$. Find the smallest integer k such that r is a root of $x^k = 1$

[6✐] **Problem 21** (BMT 2015) Evaluate $\sum_{k=0}^{37} (-1)^k \binom{75}{2k}$.

Solution: Roots of Unity Filter

§ 4.1.3 Polynomial Interpolation

[6✐] **Problem 22** (AMC 12B 2017/23) The graph of $y = f(x)$, where $f(x)$ is a polynomial of degree 3, contains points $A(2, 4)$, $B(3, 9)$, and $C(4, 16)$. Lines AB, AC, and BC intersect the graph again at points D, E, and F, respectively, and the sum of the x-coordinates of D, E, and F is 24. What is $f(0)$?

§ 4.2 Complex Numbers

[4✐] **Problem 23** (HMMT Feburary 2016) Let z be a complex number such that $|z| = 1$ and $|z - 1.45| = 1.05$. Compute the real part of z.

[4✎] **Problem 24** (2019 AMC 12B) How many nonzero complex numbers z have the property that $0, z$, and z^3, when represented by points in the complex plane, are the three distinct vertices of an equilateral triangle?

[8✎] **Problem 25** (2015 AIME I/13) With all angles measured in degrees, the product

$$\prod_{k=1}^{45} \csc^2(2k-1)^\circ = m^n,$$

where m and n are integers greater than 1. Find $m+n$.

§ 4.3 Manipulation

[1✎] **Problem 26** (LMT Fall 2019) If the numerator of a certain simplfied fraction is added to the numerator and the denominator of the fraction, the result is $\frac{20}{19}$. What is the fraction?

[1✎] **Problem 27** (AMC 10A 2020/7) The 25 integers from -10 to 14, inclusive, can be arranged to form a 5-by-5 square in which the sum of the numbers in each row, the sum of the numbers in each column, and the sum of the numbers along each of the main diagonals are all the same. What is the value of this common sum?

Solution: Let the common sum be s. Note that if we sum each of the columns, we sum each integer in -10 to 14 exactly once. So, $5s = -10-9-8\cdots+13+14 = 11+12+13+14 = 50 \implies s = \boxed{10}$.

[1✎] **Problem 28** (2018 AMC 10A/10) Suppose that real number x satisfies

$$\sqrt{49-x^2} - \sqrt{25-x^2} = 3.$$

What is the value of $\sqrt{49-x^2} + \sqrt{25-x^2}$?

Solution: Multiplying $(\sqrt{49-x^2} + \sqrt{25-x^2})(\sqrt{49-x^2} - \sqrt{25-x^2})$ gives

$$(49-x^2) - (25-x^2) = 24.$$

We know that $\sqrt{49-x^2} - \sqrt{25-x^2} = 3$, so $\sqrt{49-x^2} + \sqrt{25-x^2}$ must be $\frac{24}{3} = \boxed{8}$.

[2✎] **Problem 29** (2018 BmMT) Let x be a positive real number so that $x - \frac{1}{x} = 1$. Compute $x^8 - \frac{1}{x^8}$.

Solution: We square twice to get $x^2 + \frac{1}{x^2} = 3$ and $x^4 + \frac{1}{x^4} = 7$. Then, note that $x^8 - \frac{1}{x^8} = (x^4 - \frac{1}{x^4})(x^4 + \frac{1}{x^4}) = 7(x^2 - \frac{1}{x^2})(x^2 + \frac{1}{x^2}) = 21(x - \frac{1}{x})(x + \frac{1}{x}) = 21(x + \frac{1}{x}) = 21(x + \frac{1}{x})$. To find $x + \frac{1}{x}$, we set it to a value n. $n^2 = x^2 + \frac{1}{x^2} + 2 \implies n^2 = 5$, and we know $x + \frac{1}{x}$ is positive, so $x + \frac{1}{x} = \sqrt{5}$. So, our final answer is $\boxed{21\sqrt{5}}$.

[2✎] **Problem 30** (LMT Spring 2020) Let a,b be real numbers satisfying $a^2 + b^2 = 3ab = 75$ and $a > b$. Compute $a^3 - b^3$.

[2] **Problem 31** (HMMT November 2013) Evaluate

$$\frac{1}{2-\frac{1}{2-\frac{1}{2-\cdots\frac{1}{2-\frac{1}{2}}}}}$$

where the digit 2 appears 2013 times

Solution: We can do some pattern finding when 2 appears n times. When $n = 1$, this is $\frac{1}{2}$. When $n = 2$, this is $\frac{2}{3}$. We conjecture that when 2 appears n times, the expresion is $\frac{n}{n+1}$. We can prove this by induction. With 2 appearing $n+1$ times, we have $\frac{1}{2-\frac{1}{2-\frac{1}{2\cdots}}} = \frac{1}{2-\frac{n}{n+1}} = \frac{n+1}{n+2}$. Our answer is then $\boxed{\frac{2013}{2014}}$.

[2] **Problem 32** (Mandelbrot) If $\frac{x^2}{y^2} = \frac{8y}{x} = z$, find the sum of all possible z.

Solution: Let $\frac{x}{y} = k$. Then, $k^2 = \frac{8}{k} \implies k^3 = 8 \implies k = 2$. Then, $k^2 = 4$ is only possible z. The sum is $\boxed{4}$

[2] **Problem 33** (AMC 10A 2018/14) What is the greatest integer less than or equal to

$$\frac{3^{100}+2^{100}}{3^{96}+2^{96}}?$$

Solution: Conider $\frac{3^{100}+2^{100}}{3^{96}+2^{96}} = 3^4 - \frac{3^4 2^{96} - 2^{100}}{3^{96}+2^{96}}$. Clearly, $3^4 2^{96} > 2^{100}$ but $3^4 2^{96} - 2^{100} = (81-16)2^{96} = 65 \cdot 2^{96}$ is much less than 3^{96}. So, the expression is less than 81 but greater than 80. Our answer is $\boxed{80}$.

[3] **Problem 34** (BMT 2016) Simplify $\frac{1}{\sqrt[3]{81}+\sqrt[3]{72}+\sqrt[3]{64}}$.

Solution: Let $\sqrt[3]{8} = a$ and $\sqrt[3]{9} = b$. Then, it is equivalent to $\frac{1}{a^2+ab+b^2} = \frac{a-b}{a^3-b^3} = \boxed{\sqrt[3]{9} - \sqrt[3]{8}}$.

[3] **Problem 35** (MAθ 2018) The solutions to $3\sqrt{2x^2-5x-3} + 2x^2 - 5x = 7$ can be written in the form $x = \frac{a \pm \sqrt{b}}{c}$ where a, b, c are positive integers and x is in simplest form. Find $a + b + c$.

Solution: Subsitute $\sqrt{2x^2-5x-3} = y$. We get $3y+y^2+3 = 7 \implies y^2+3y-4 = \implies (y+4)(y-1)$. Since y is positive, $y = 1$. Then, $2x^2 - 5x - 3 = 1 \implies 2x^2 - 5x - 4 = 0$. By the Quadratic Formula, we get $x = \frac{5\pm\sqrt{57}}{4}$. We get $5 + 57 + 4 = \boxed{66}$ as our answer.

[3] **Problem 36** (Mandelbrot Nationals 2008) Find the positive real number x for which $5\sqrt{1-x} + 5\sqrt{1+x} = 7\sqrt{2}$.

Solution: Let $\sqrt{1-x} = a$ and $\sqrt{1+x} = b$. We get $5a + 5b = 7\sqrt{2} \implies a + b = \frac{7\sqrt{2}}{5} \implies a^2 + 2ab + b^2 = \frac{98}{25}$. We also have $a^2 + b^2 = 2$. So, $ab = \frac{24}{25} \implies \sqrt{1-x^2} = \frac{24}{25} \implies 1 - x^2 = \frac{576}{625}$. This gives $x = \boxed{\dfrac{7}{25}}$.

[4] **Problem 37** (NEMO 2019) Suppose x and y are positive real numbers satisfying

$$\sqrt{xy} = x - y = \frac{1}{x+y} = k$$

Determine k.

Solution:

$$x - y = \frac{1}{x+y} \implies x^2 - y^2 = 1$$

$$x - y = k \implies x^2 - 2xy + y^2 = k^2 \implies x^2 + y^2 = 3k^2$$

Combining, we get $x^2 = \frac{3k^2+1}{2}$ and $y^2 = \frac{3k^2-1}{2}$. Then, $\sqrt{xy} = k \implies x^2y^2 = k^4 \implies \frac{9k^4-1}{4} = k^4 \implies k = 5^{-\frac{1}{4}}$.

[4] **Problem 38** (2014 November HMMT) Let a, b, c, x be reals with $(a+b)(b+c)(c+a) \neq 0$ that satisfy

$$\frac{a^2}{a+b} = \frac{a^2}{a+c} + 20, \frac{b^2}{b+c} = \frac{b^2}{b+a} + 14, \text{ and } \frac{c^2}{c+a} = \frac{c^2}{c+b} + x$$

Compute x.

Solution: Notice that $\frac{a^2}{a+b} = \frac{a^2}{a+c} + 20 \implies \frac{a^2(a+c)-a^2(a+b)}{(a+b)(a+c)} = 20 \implies \frac{a^2(c-b)}{(a+b)(a+c)} = 20$. Similarly, $\frac{b^2(a-c)}{(b+c)(b+a)} = 14$ and $\frac{c^2(b-a)}{(c+a)(c+b)} = x$. Summing, we get $\frac{a^2(c^2-b^2)+b^2(a^2-c^2)+c^2(b^2-a^2)}{(a+b)(b+c)(c+a)} = 0 = 34 + x \implies x = \boxed{-34}$.

[5] **Problem 39** (Magic Math AMC 10 2020) Determine the remainder when $\sum (-1)^{m+n} mn$ is divided by 1009, where the sum is taken over all pairs of integers (m, n)satisfying $1 \le m < n \le 2020$

Solution: Notice that by the Distributive Property, $(\sum_{m=1}^{2020}(-1)^m m)(\sum_{n=1}^{2020}(-1)^n n) = 2\sum_{1\le m<n\le 2020}(-1)^{m+n}mn +$ $\sum_{i=1}^{2020} i^2$. Note that $(\sum_{m=1}^{2020}(-1)^m m)(\sum_{n=1}^{2020}(-1)^n n) = (\sum_{m=1}^{2020}(-1)^m m)^2 = (-1+2-3+4\cdots - 2019+2020) = 1010^2 = 1 \pmod{1009}$. Also, $\sum_{i=1}^{2020} i^2 = \frac{2020\cdot 2021\cdot 4041}{6} = \frac{2\cdot 3\cdot 5}{6} \pmod{1009} = 5$.

Then, letting $x = \sum_{1\le m<n\le 2020}(-1)^{m+n}mn$,we have

$$1 = 2x + 5 \pmod{1009}$$

$$-2 = x \pmod{1009}$$

$$x = \boxed{1007} \pmod{1009}$$

[5✎] **Problem 40** (Math Prizes For Girls 2015) Let S be the sum of all distinct real solutions of the equation

$$\sqrt{x+2015} = x^2 - 2015.$$

Compute $\lfloor 1/S \rfloor$. Recall that if r is a real number, then $\lfloor r \rfloor$ (the floor of r) is the greatest integer that is less than or equal to r

Solution: Let $2015 = y$. Then, we have $\sqrt{x+y} = x^2 - y \implies x + y = x^4 - 2x^2y + y^2 \implies y^2 + (-2x^2 - 1)y + x^4 - x = 0$. Then, $y = \frac{2x^2+1\pm(2x+1)}{2}$. Now, we have $2015 = x^2 + x + 1$ or $2015 = x^2 - x$. These give $x = \frac{-1\pm\sqrt{8057}}{2}$ and $x = \frac{1\pm\sqrt{8061}}{2}$. Now, note that we have $x + y \geq 0 \implies x \geq -2015$ and $x^2 - y \geq 0 \implies |x| \geq \sqrt{2015}$.

We can see that $\frac{-1+\sqrt{8057}}{2} > \sqrt{2015}$ and that $\frac{-1-\sqrt{8057}}{2} < -\sqrt{2015}$. Also, $\frac{1-\sqrt{8061}}{2} > -\sqrt{2015}$ and $\frac{1+\sqrt{8061}}{2} > \sqrt{2015}$. So, we have that our two solutions are $\frac{-1-\sqrt{8057}}{2}$ and $\frac{1+\sqrt{8061}}{2}$.

Then, $\frac{1}{S} = \frac{2}{\sqrt{8061}-\sqrt{8057}} = \frac{\sqrt{8061}+\sqrt{8057}}{2}$. So, $89 < \frac{1}{S} < 90$ so our answer is $\boxed{89}$

§ 4.4 Series and Sequences

[1✎] **Problem 41** (djmathman Mock AMC 2013/9) Let p and q be numbers with $|p| < 1$ and $|q| < 1$ such that

$$p + pq + pq^2 + pq^3 + \cdots = 2 \quad\text{and}\quad q + qp + qp^2 + \cdots = 3.$$

What is $100pq$?

Solution: Using the formula for an infinite geometric series, we get $\frac{p}{1-q} = 2 \implies p = 2 - 2q$ and $\frac{q}{1-p} = 3 \implies q = 3 - 3p \implies p = \frac{3-q}{3}$. So, $\frac{3-q}{3} = 2 - 2q \implies 3 - q = 6 - 6q \implies 5q = 3 \implies q = \frac{3}{5} \implies p = \frac{4}{5}$. Then $100pq = 100 \cdot \frac{12}{25} = \boxed{48}$.

[1✎] **Problem 42** (PHS HMMT TST 2020) What is the value of $\frac{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\cdots}{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\cdots}$? Remember that $\frac{1}{1^2} + \frac{1}{2^2}\cdots = \frac{\pi^2}{6}$

Solution: Notice that every positive integer can be uniquely represented as $2^k \cdot o$ for some non-negative k and odd integer o. Then, $\frac{1}{1^2} + \frac{1}{2^2}\cdots = (\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{4^2}\cdots)(\frac{1}{1^2} + \frac{1}{3^2}\cdots) = \frac{4}{3}(\frac{1}{1^2} + \frac{1}{3^2}\cdots)$. So, our answer is $\boxed{\frac{4}{3}}$.

[4✎] **Problem 43** (Math Prizes For Girls 2019) For each integer from 1 through 2019, Tala calculated the product of its digits. Compute the sum of all 2019 of Tala's products.

[4✎] **Problem 44** (HMMT Feburary 2017) Find the value of

$$\sum_{1\leq a<b<c} \frac{1}{2^a 3^b 5^c}$$

(i.e the sum of $\frac{1}{2^a3^b5^c}$ over all triples of positive integers (a, b, c) satisfying $a < b < c$)

[5✐] **Problem 45** (HMMT Feburary 2016) Let A denote the set of all integers n such that $1 \leq n \leq 10000$, and moreover the sum of the decimal digits of n is 2. Find the sum of the squares of the elements of A.

[5✐] **Problem 46** (HMMT Feburary 2016) Determine the remainder when

$$\sum_{i=0}^{2015} \lfloor \frac{2^i}{25} \rfloor$$

is divided by 100, where $\lfloor x \rfloor$ denotes the largest integer not greater than x.

§ 4.4.1 Telescoping

[3✐] **Problem 47** (Purple Comet 2015 HS)

$$\left(1+\frac{1}{1+2^1}\right)\left(1+\frac{1}{1+2^2}\right)\left(1+\frac{1}{1+2^3}\right)\cdots\left(1+\frac{1}{1+2^{10}}\right)=\frac{m}{n},$$

where m and n are relatively prime positive integers. Find $m + n$.

§ 4.5 Trigonometry

[4✐] **Problem 48** (MAθ 1992) If A and B are both in $[0, 2\pi)$ and A and B satisfy the equations

$$\sin A + \sin B = \frac{1}{3}$$

$$\cos A + \cos B = \frac{4}{3}$$

find $\cos(A - B)$

[4✐] **Problem 49** (TAMU 2019) Simplify $\arctan \frac{1}{1+1+1^2} + \arctan \frac{1}{1+2+2^2} + \arctan \frac{1}{1+3+3^2} \cdots + \arctan \frac{1}{1+n+n^2}$

Solution: Use induction to show this is just $\boxed{\arctan \frac{n}{n+2}}$.

[6✐] **Problem 50** (Purple Comet 2015 HS) Let x be a real number between 0 and $\frac{\pi}{2}$ for which the function $3\sin^2 x + 8\sin x \cos x + 9\cos^2 x$ obtains its maximum value, M. Find the value of $M + 100\cos^2 x$.

[6✐] **Problem 51** (ARML Local 2009) If $6\tan^{-1} x + 4\tan^{-1}(3x) = \pi$, compute x^2

Solution: $\boxed{\frac{15-8\sqrt{3}}{33}}$

§ 4.6 Logarithms

[2✎] **Problem 52** (MAθ 1991) Given that $\log_{10} 2 \approx 0.3010$, how many digits are in 5^{44}?

Solution: Note $\log_{10}(10) - \log_{10}(2) = \log_{10}(5) \implies \log_{10}(5) \approx 0.699 \approx \frac{7}{10}$. Then, $5^{44} \approx 10^{44 \cdot \frac{7}{10}} = 10^{30.8}$. So, it has $\boxed{31}$ digits.

[3✎] **Problem 53** (ARML Local 2009) Compute all real values of x such that $\log_2(\log_2 x) = \log_4(\log_4 x)$. [3✎] **Problem 54** (PuMAC 2019) If x is a real number so $3^x = 27x$, compute $\log_3(\frac{3^{3^x}}{x^{3^3}})$.

Solution: Note that $3^x = 27x \implies x = 3 + \log_3(x)$. Also, $\log_3(\frac{3^{3^x}}{x^{3^3}}) = 3^x - 27\log_3(x) = 27x - 27(x-3) = \boxed{81}$.

[3✎] **Problem 55** (AMC 12B 2017/20) Real numbers x and y are chosen independently and uniformly at random from the interval $(0, 1)$. What is the probability that $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor$?

[4✎] **Problem 56** (MAθ 1991) Suppose a and b are positive numbers for which $\log_{4n} 40\sqrt{3} = \log_{3n} 45$, find n^3

Solution: Note $\log_{4n} 40\sqrt{3} = \log_{3n} 45 \implies \frac{\log 40\sqrt{3}}{\log 4n} = \frac{\log 45}{\log 3n}$ for an arbitary base. Then, we also get $\frac{\log 40\sqrt{3}}{\log 4 + \log n} = \frac{\log 45}{\log 3 + \log n} \implies \log 120\sqrt{3} + \log 40n\sqrt{3} = \log 180 + \log 45n \implies \log \frac{8n\sqrt{3}}{9} = \log \frac{3}{2\sqrt{3}} = \log \frac{\sqrt{3}}{2} \implies n = \frac{9}{16}$. So, $n^3 = \boxed{\frac{729}{4096}}$.

[4✎] **Problem 57** (MAθ 1992) Suppose a and b are positive numbers for which

$$\log_9 a = \log_{15} b = \log_{25}(a + 2b)$$

What is the value of $\frac{b}{a}$?

[4✎] **Problem 58** (PHS ARML TST 2017) Positive real numbers x, y, and z satisfy the following system of equations:

$$x^{\log(yz)} = 100$$
$$y^{\log(xz)} = 10$$
$$z^{\log(xy)} = 10\sqrt{10}$$

Compute the value of the expression $(\log(xyz))^2$

Solution: Let $x = 10^a, y = 10^b, z = 10^c$. Let $\log(xyz) = a + b + c = k$. Then, we have $a(b + c) = 2$ and $b(a + c) = 1$ and $c(a + b) = \frac{3}{2}$. Summing, we get $2\sum_{cyc} ab = \frac{9}{2} \implies \sum_{cyc} ab = \frac{9}{4}$. Then, subtraction from each of the equations gives $ab = \frac{3}{4}, ac = \frac{5}{4}, bc = \frac{1}{4}$. Then, multiplying all together, we get $abc = \frac{\sqrt{15}}{8}$. Solving, we get $a = \frac{\sqrt{15}}{2}, b = \frac{\frac{\sqrt{15}}{5}}{2}, c = \frac{\frac{\sqrt{15}}{3}}{2}$. Then, $a + b + c = \frac{23\sqrt{15}}{30}$. So, $(a + b + c)^2 = \frac{529 \cdot 15}{30^2} = \boxed{\frac{529}{60}}$.

[4 ✎] **Problem 59** (If $60^a = 3$ and $60^b = 5$, then find $12^{\frac{1-a-b}{2-2b}}$.)

[5 ✎] **Problem 60** (SLKK AIME 2020) Let x be a real number in the interval $(0, \frac{\pi}{2})$ such that $\log_{\sin^2(x)} \cos(x) + \log_{\cos^2(x)} \sin(x) = \frac{5}{4}$. If $\sin^2(2x)$ can be expressed as $m\sqrt{n} - p$, where m, n, and p are positive integers such that n is not divisible by the square of a prime, find $m + n + p$

§ 4.7 Sequences

[3 ✎] **Problem 61** (BMT 2015) Let $\{a_n\}$ be a sequence of real numbers with $a_1 = -1, a_2 = 2$ and for all $n \geq 3$, $a_{n+1} - a_n - a_{n+2} = 0$. Find $a_1 + a_2 + a_3 + ... + a_{2015}$.

Solution: Note that the condition is $a_{n+2} = a_{n+1} - a_n$ and that $a_3 = 3, a_4 = 1, a_5 = -2, a_6 = -3, a_7 = -1, a_8 = 2$. So, it repeats with period 6. The sum $a_1 + a_2 \cdots + a_6 = 0$. So, $a_1 + a_2 \cdots + a_{2015} = 0 - a_{2016} = 0 - a_6 = \boxed{3}$.

§ 4.8 Arbitary Functions

[3 ✎] **Problem 62** (Mandelbrot Nationals 2009) Let $f(x)$ be a function defined for all positive real numbers satisfying the conditions $f(x) > 0$ for all $x > 0$ and $f(x - y) = \sqrt{f(xy) + 1}$ for all $x > y > 0$. Determine $f(2009)$.

[2 ✎] **Problem 63** (LMT Spring 2020) A function $f(x)$ is such that for any integer x, $f(x) + xf(2 - x) = 6$. Compute $-2019f(2020)$.

Solution: Plug in $x = 2020$ and $x = -2018$ and solve the system of linear equations.

[5 ✎] **Problem 64** (HMMT Feburary 2017) Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(x)f(y) = f(x - y)$. Find all possible values of $f(2017)$.

§ 4.9 Inequalities

[1 ✎] **Problem 65** (PuMAC 2019) Let a, b be positive integers such that $a + b = 10$. Let $\frac{p}{q}$ be the difference between the maximum and minimum possible values of $\frac{1}{a} + \frac{1}{b}$, where p and q are relatively prime positive integers. Compute $p + q$.

Solution: Note that $\frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{10}{ab}$. To minimize this, we have to maximize ab which happens when a, b are equal or very close together (you can rigorously show this using a discrete smoothing arugment, i.e if they are not as close together as possible, then you can make the expression bigger by moving them together). That means $a = b = 5 \implies ab = 25$ so we have a minimum of $\frac{2}{5}$. The maximium of the expression occurs when a, b are very far apart (again by a discrete smoothing argument) when $a = 1, b = 9$, giving a maximium of $\frac{10}{9}$. Then, the difference is $\frac{10}{9} - \frac{2}{5} = \frac{50-18}{45} = \frac{32}{45}$, giving $\boxed{77}$.

[2 ✎] **Problem 66** (Math Prizes For Girls 2019) The degree measures of the six interior angles of a convex hexagon form an arithmetic sequence (not necessarily in cyclic order). The common difference of this arithmetic sequence can be any real number in the open interval $(-D, D)$. Compute the greatest possible value of D.

Solution: Let the angles be $a, a+d \cdots a+5d$. Notice that any negative common difference can be turned into a positive common difference by simply starting at the maximium instead of the minimium. This same applies vice versa. So, let $d \geq 0$. Then, they sum to $180(4) = 720$ giving $a + (a+d) \cdots + (a+5d) = 720 \implies (2a+5d)(3) = 720 \implies 2a + 5d = 240 \implies a = \frac{240-5d}{2}$. So, any $d < \frac{240}{5} = 48$ would corresond to a valid a. Notice that since the hexagon is convex, $a + 5d < 180 \implies \frac{240+5d}{2} < 180 \implies \frac{48+d}{2} < 36 \implies d < 24$. Now, it is clear that any $D \leq 24$ would work. So, the greatest possible value of D is $\boxed{24}$.

[3✎] **Problem 67** (Math Prizes For Girls 2019) Find the least real number K such that for all real numbers x and y, we have $(1+20x^2)(1+19y^2) \geq Kxy$. Express your answer in simplified radical form.

Solution: By AM-GM, $(1+20x^2) \geq 2\sqrt{5} \cdot |x|$ and $(1+19y^2) \geq \sqrt{19} \cdot |y|$. Then, we get $(1+20x^2)(1+19y^2) \geq 2\sqrt{95} \cdot |x| \cdot |y|$ as $|x|, |y|$ are positive. Now, if both x, y have the same sign, then $|x||y| = xy$ so $K = 2\sqrt{95}$ for that case. But, if x, y have opposite signs, $|x||y| = -xy$ so $(1+20x^2)(1+19y^2) \geq -19\sqrt{95}xy$. If K was any larger in this case, it would not work. So, the least K is $\boxed{-19\sqrt{95}}$.

[4✎] **Problem 68** (HMMT November 2013) Find the largest real number λ such that $a^2 + b^2 + c^2 + d^2 \geq ab + \lambda bc + cd$ for all real numbers a, b, c, d.

[5✎] **Problem 69** (HMMT February 2014) Suppose that x and y are positive real numbers such that $x^2 - xy + 2y^2 = 8$. Find the maximum possible value of $x^2 + xy + 2y^2$.

[5✎] **Problem 70** (BMT 2019) Find the number of ordered integer triplets x, y, z with absolute value less than or equal to 100 such that $2x^2 + 3y^2 + 3z^2 + 2xy + 2xz - 4yz < 5$

Solution: $(x+y)^2 + (x+z)^2 + 2(y-z)^2 < 5$

§ 4.10 Fake Algebra

[3✎] **Problem 71** (BMT 2019) Find the maximum value of $\frac{x}{y}$ if x and y are real numbers such that $x^2 + y^2 - 8x - 6y + 20 = 0$.

Solution: Let $\frac{x}{y} = k$. Then, $y = kx$. Plugging this in, we get $(k^2+1)x^2 - (8+6k)x + 20 = 0$. Clearly, the maximum k occurs when the determinant is 0. This implies $(8+6k)^2 - 4(20)(k^2+1) = (36k^2 + 96k + 64) - (80k^2 + 80) = -44k^2 + 96k - 16 = 0 \implies 11k^2 - 24k + 4 = 0 \implies (11k-2)(k-2) = 0$. Then, we get $k = 2$ and $k = \frac{2}{11}$. Our answer is $\boxed{2}$.

[3✎] **Problem 72** (BMT 2015) Let x and y be real numbers satisfying the equation $x^2 - 4x + y^2 + 3 = 0$. If the maximum and minimum values of $x^2 + y^2$ are M and m respectively, compute the numerical value of $M - m$.

Solution: We rewrite to get $(x-2)^2 + y^2 = 1$ which is the equation of a circle. Let $P = (x, y)$ and $O = (0, 0)$. Then, $x^2 + y^2 = PO^2$. So, we want to find the minimum of PO and maximum of PO. Let $C = (2, 0)$ which is the center of the circle. Clearly, the maximium and minimium occur where PC intersects the circle. So, the minimium is $2 - 1 = 1$ and the maximium is $2 + 1 = 3$. So, $M - m = 9 - 1 = \boxed{8}$.

[3✎] **Problem 73** (Magic Math AMC 10 2020) Find the number of ordered pairs of real numbers (x, y) satisfying $x = \frac{2y}{y^2-1}$ and $y = \frac{2x}{x^2-1}$.

[4✎] **Problem 74** (ARML Local 2014) Compute the area of the region defined by $x^2+y^2 \leq |x|+|y|$.

Solution: $\boxed{2+\pi}$

[4✎] **Problem 75** (PuMAC 2019) Let x and y be positive real numbers that satisfy $(\log x)^2 + (\log y)^2 = \log x^2 + \log y^2$. Compute the maximum possible value of $(\log xy)^2$.

Solution: Subsitute $\log x = a$, $\log y = b$. You get the equation of a circle $(a-1)^2 + (b-1)^2 = 2$. You want to find the y-intersect of tangent line with slope -1 on "top" of the circle. Draw a perpendicular to the line from the center to find the tangency point. This has slope 1 and it is $\sqrt{2}$ long. So, the coordinates of this tangency is $(2, 2)$ and $a + b = 4$. We get $(a + b)^2 = \boxed{16}$.

[7✎] **Problem 76** (HMMT February 2014) Given that a, b, and c are complex numbers satisfying

$$a^2 + ab + b^2 = 1 + i$$

$$b^2 + bc + c^2 = 2$$

$$c^2 + ca + a^2 = 1,$$

compute $(ab + bc + ca)^2$

§ 5 Geometry

§ 5.1 Coordinate Geometry

[1✎] **Problem 1** (BmMT 2014) Find the area of the convex quadrilateral with vertices at the points $(-1, 5), (3, 8), (3, -1)$, and $(-1, -2)$.

Solution: Direct application of Shoelace.

[2✎] **Problem 2** (CRMT Individuals 2019) Let S be the set of all distinct points in the coordinate plane that form an acute isosceles triangle with the points $(32, 33)$ and $(63, 63)$. Given that a line L crosses S a finite number of times, find the maximum number of times L can cross S.

Solution: Replace $(32, 33)$ and $(63, 63)$ by A and B. Then, we do casework on $AC = BC$ or $CB = AB$ or $CA = BA$. We get a line and two semicircles. A line can intersect a semicircle two times and a line one time. So, our answer is $4 + 1 = \boxed{5}$.

[2✎] **Problem 3** (HMMT November 2013) Plot points A, B, C at coordinates (0, 0), (0, 1), and (1, 1) in the plane, respectively. Let S denote the union of the two line segments AB and BC. Let X1 be the area swept out when Bobby rotates S counterclockwise 45 degrees about point A. Let X2 be the area swept out when Calvin rotates S clockwise 45 degrees about point A. Find $\frac{X_1+X_2}{2}$

[2✎] **Problem 4** (AMC 12B 2017/9) A circle has center $(-10, -4)$ and radius 13. Another circle has center $(3, 9)$ and radius $\sqrt{65}$. The line passing through the two points of intersection of the two circles has equation $x + y = c$. What is c?

[3✎] **Problem 5** (Math Prizes For Girls 2019) Two ants sit at the vertex of the parabola $y = x^2$. One starts walking northeast (i.e., upward along the line $y = x$) and the other starts walking northwest (i.e., upward along the line $y = -x$). Each time they reach the parabola again, they swap directions and continue walking. Both ants walk at the same speed. When the ants meet for the eleventh time (including the time at the origin), their paths will enclose 10 squares. What is the total area of these squares?

[5✎] **Problem 6** (BMT 2019) A regular hexagon has positive integer side length. A laser is emitted from one of the hexagon's corners, and is reflected off the edges of the hexagon until it hits another corner. Let a be the distance that the laser travels. What is the smallest possible value of a^2 such that $a > 2019$? You need not simplify/compute exponents.

[5✎] **Problem 7** (SLKK AIME 2020) Mr. Duck draws points $A = (a, 0), B = (0, b), C = (3, 5)$ and $O = (0, 0)$ such that $a, b > 0$ and $\angle ACB = 45°$. If the maximum possible area of $\triangle AOB$ can be expressed as $m - n\sqrt{p}$ where m, n, and p are positive integers such that p is not divisible by the square of a prime, find $m + n + p$

§ 5.2 3D Geometry

[1✎] **Problem 8** (AMC 12A 2009/5) One dimension of a cube is increased by 1, another is decreased by 1, and the third is left unchanged. The volume of the new rectangular solid is 5 less than that of the cube. What was the volume of the cube?

Solution: Let x be the original side length of the cube. Then, we have $(x + 1)(x - 1)(x) + 5 = x^3 \implies x^3 - x + 5 = x^3 \implies x = 5$. So, the original volume is $x^3 = \boxed{125}$.

[1✎] **Problem 9** (AMC 12A 2008/8) What is the volume of a cube whose surface area is twice that of a cube with volume 1?

Solution: The surface area of a cube with volume 1 is $6 \cdot 1^2 = 6$. Then, the surface area of the larger cube is 12 and so each face has area 2. This implies the side length is $\sqrt{2}$. The volume is $2\sqrt{2}$.

[1✎] **Problem 10** (BMT 2018) A cube has side length 5. Let S be its surface area and V its volume. Find $\frac{S^3}{V^2}$.

Solution: Note $S = 6 \cdot 5^2 = 150$ and $V = 5^3$. Then, $\frac{S^3}{V^2} = \frac{6^3 \cdot 5^6}{5^6} = \boxed{216}$.

[2✎] **Problem 11** (AMC 12B 2008/18) On a sphere with a radius of 2 units, the points A and B are 2 units away from each other. Compute the distance from the center of the sphere to the line segment AB.

Solution: Consider the great circle that goes through both A and B. Also, let the center of the sphere be O. Then, drop the perpendicular from O to AB with foot C. Then, C bisects AB with $AC = CB = 1$. Also, $OB = 2$. Using the Pythagorean Theorem, $OC = \boxed{\sqrt{3}}$.

[2✐] **Problem 12** (AMC 12B 2005/16) Eight spheres of radius 1, one per octant, are each tangent to the coordinate planes. What is the radius of the smallest sphere, centered at the origin, that contains these eight spheres?

Solution: We are basically finding the distance from the origin to the furtherest point on these spheres and that will be our radius. It is clear that a sphere with this radius will work and that if this distance is any smaller, the sphere will not completely contain all the other spheres. It is also clear that the spheres are symmetric so we only need to consider one of the spheres to find this furthest distance.

We consider the sphere in the positive x,y,z direction. Then the center C of this sphere is $(1, 1, 1)$. So, $OC = \sqrt{3}$. Now, it is clear that the furthest point P is the other intersection of OC with the sphere.

So, the furthest distance is $\boxed{1 + \sqrt{3}}$.

[3✐] **Problem 13** (AIME 1984/9) In tetrahedron $ABCD$, edge AB has length 3 cm. The area of face ABC is 15cm^2 and the area of face ABD is 12 cm^2. These two faces meet each other at a 30° angle. Find the volume of the tetrahedron in cm^3.

[3✐] **Problem 14** (HMMT February 2014) Let C be a circle in the xy plane with radius 1 and center $(0, 0, 0)$, and let P be a point in space with coordinates $(3, 4, 8)$. Find the largest possible radius of a sphere that is contained entirely in the slanted cone with base C and vertex P.

[3✐] **Problem 15** (AMC 12B 2017/14) An ice-cream novelty item consists of a cup in the shape of a 4-inch-tall frustum of a right circular cone, with a 2-inch-diameter base at the bottom and a 4-inch-diameter base at the top, packed solid with ice cream, together with a solid cone of ice cream of height 4 inches, whose base, at the bottom, is the top base of the frustum. What is the total volume of the ice cream, in cubic inches?

[4✐] **Problem 16** (ARML Local 2009) A cylinder with radius r and height h has volume 1 and total surface area 12. Compute $\frac{1}{r} + \frac{1}{h}$

Solution: $\boxed{6}$

§ 5.3 Inequalties

[2✐] **Problem 17** (Math Prizes For Girls) A paper equilateral triangle with area 2019 is folded over a line parallel to one of its sides. What is the greatest possible area of the overlap of folded and unfolded parts of the triangle?

[3✐] **Problem 18** (BMT 2018) If A is the area of a triangle with perimeter 1, what is the largest possible value of A^2?

Solution: Using Heron's Formula and a smoothing argument gives that an equilaterial triangle gives the largest area. So, we have $s = \frac{1}{3}$ and $A = \frac{s^2\sqrt{3}}{4} = \frac{\sqrt{3}}{36}$. So, $A^2 = \frac{3}{36^2} = \frac{1}{12 \cdot 36} = \boxed{\frac{1}{432}}$.

§ 5.4 Transformations

[3✐] **Problem 19** (HMMT February 2014) In quadrilateral ABCD, $\angle DAC = 98, \angle DBC = 82, \angle BCD = 70$, and $BC = AD$. Find $\angle ACD$.

Solution: Reflect.

[6✐] **Problem 20** (LMT Spring 2020) Let ABC be a triangle such that such that $AB = 14, BC = 13$, and $AC = 15$. Let X be a point inside triangle ABC. Compute the minimum possible value of $(\sqrt{2}AX + BX + CX)^2$.

[8✐] **Problem 21** (SLKK AIME 2020) Squares $ABCD$ and $DEFG$ are drawn in the plane with both sets of vertices A, B, C, D and D, E, F, G labeled counterclockwise. Let P be the intersection of lines AE and CG. If $DA = 35, DG = 20$, and $BF = 25\sqrt{2}$, find DP^2.

Solution: Spiral Similarity from ABCD to DEFG.

§ 5.5 General

[1✐] **Problem 22** (MAθ 2018) A parallelograms has diagonals of length 10 and 20. Find the area inclosed by the circle inscribed in the parallelogram.

[1✐] **Problem 23** (TAMU 2019) An acute isosceles triangle ABC is inscribed in a circle. Through B and C, tangents to the circle are drawn, meeting at D. If $\angle ABC = 2\angle CDB$, then find the radian measure of $\angle BAC$.

[1✐] **Problem 24** (PHS PuMAC TST 2017) In triangle ABC, let D and E be the midpoints of BC and AC. Suppose AD and BE meet at F. If the area of $\triangle DEF$ is 50, then what is the area of $\triangle CDE$?

[1✐] **Problem 25** (AMC 10A 2019/13) Let $\triangle ABC$ be an isosceles triangle with $BC = AC$ and $\angle ACB = 40°$. Construct the circle with diameter $\overline{BC}$, and let D and E be the other intersection points of the circle with the sides $\overline{AC}$ and $\overline{AB}$, respectively. Let F be the intersection of the diagonals of the quadrilateral $BCDE$. What is the degree measure of $\angle BFC$?

[2✐] **Problem 26** (AMC 10B 2011/17) In the given circle, the diameter $\overline{EB}$ is parallel to $\overline{DC}$, and $\overline{AB}$ is parallel to $\overline{ED}$. The angles AEB and ABE are in the ratio $4 : 5$. What is the degree measure of angle BCD?

[2✐] **Problem 27** (Brazil 2007) Let ABC be a triangle with circumcenter O. Let P be the intersection of straight lines BO and AC and ω be the circumcircle of triangle AOP. Suppose that $BO = AP$ and that the measure of the arc OP in ω, that does not contain A, is 40°. Determine the measure of the angle $\angle OBC$.

[2✐] **Problem 28** (BMT 2018) A 1 by 1 square ABCD is inscribed in the circle m. Circle n has radius 1 and is centered around A. Let S be the set of points inside of m but outside of n. What is the area of S?

[2✐] **Problem 29** (AMC 10B 2011/18) Rectangle $ABCD$ has $AB = 6$ and $BC = 3$. Point M is chosen on side AB so that $\angle AMD = \angle CMD$. What is the degree measure of $\angle AMD$?

[2✐] **Problem 30** (TAMU 2019) Let AA_1 be an altitude of triangle $\triangle ABC$, and let A_2 be the midpoint of the side BC. Suppose that AA_1 and AA_2 divide angle $\angle BAC$ into three equal angles. Find the product of the angles of $\triangle ABC$ when the angles are expressed in degrees.

Solution: Let $\angle BAA_1 = \angle A_1AA_2 = \angle A_2AC = \alpha$. We have that $\triangle ABA_2$ is an isosceles triangle as $\angle ABA_1 = \angle AA_2A_1 = 90 - \alpha$. Then, as AA_1 is an altitude, $BA_1 = A_1A_2$. Let $BA_1 = A_1A_2 = x$, then $A_2C = BA_2 = 2x$. Consider triangles BAA_1 and $\triangle A_1AC$. We have that $\tan\alpha = \frac{x}{AA_1}$ and $\tan 2\alpha = \frac{3x}{AA_1}$. So, $\frac{\tan 2\alpha}{\tan\alpha} = 3 \implies \tan\alpha = \frac{1}{\sqrt{3}} \implies \alpha = 30$. So, $\angle BAC = 90$, $\angle ABC = 60$, and $\angle ACB = 30$. The product is $\boxed{162000}$.

[2✐] **Problem 31** (PHS PuMAC TST 2017) A trapezoid has area 32, and the sum of the lengths of its two bases and altitude is 16. If one of the diagonals is perpendicular to both bases, then what is the length of the other diagonal?

[2✐] **Problem 32** (Autumn Mock AMC 10) Equilateral triangle ABC has side length 6. Points D, E, F lie within the lines AB, BC and AC such that $BD = 2AD$, $BE = 2CE$, and $AF = 2CF$. Let N be the numerical value of the area of triangle DEF. Find N^2.

[2✐] **Problem 33** (LMT Spring 2020) Three mutually externally tangent circles are internally tangent to a circle with radius 1. If two of the inner circles have radius $\frac{1}{3}$, the largest possible radius of the third inner circle can be expressed in the form $\frac{a+b\sqrt{c}}{d}$ where c is squarefree and $\gcd(a, b, d) = 1$. Find $a + b + c + d$.

[3✐] **Problem 34** (AHSME 1984/28) Triangle ABC has area 10. Points D, E, and F, all distinct from A, B, and C, are on sides AB, BC, and CA, respectively, and AD = 2, DB = 3. Triangle ABE and quadrilateral DBEF have equal areas s. Find s.

[3✐] **Problem 35** (FARML 2012/6) In triangle ABC, $AB = 7$, $AC = 8$, and $BC = 10$. D is on AC and E is on BC such that $\angle AEC = \angle BED = \angle B + \angle C$. Compute the length AD.

[3✐] **Problem 36** (HMMT November 2013) Let ABC be an isosceles triangle with $AB = AC$. Let D and E be the midpoints of segments AB and AC, respectively. Suppose that there exists a point F on ray $\overrightarrow{DE}$ outside of ABC such that triangle BFA is similar to triangle ABC. Compute $\frac{AB}{BC}$.

[3✐] **Problem 37** (Magic Math AMC 10 2020) Two concentric circles have radii $\sqrt{2}$ and 2. Three points A, B, C are chosen on the larger circle such that $\triangle ABC$ is equilateral. Find the area of the region in the smaller circle that is also inside the triangle.

[3✐] **Problem 38** (AMC 12B 2017/15) Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3 \cdot AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3 \cdot BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3 \cdot CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$?

[3✐] **Problem 39** (CNCM Online Round 2) There is a rectangle $ABCD$ such that $AB = 12$ and $BC = 7$. E and F lie on sides AB and CD respectively such that $\frac{AE}{EB} = 1$ and $\frac{CF}{FD} = \frac{1}{2}$. Call X the intersection of AF and DE. What is the area of pentagon $BCFXE$?

[3✐] **Problem 40** (LMT Spring 2020) Let $\triangle ABC$ be a triangle such that $AB = 6, BC = 8$, and $AC = 10$. Let M be the midpoint of BC. Circle ω passes through A and is tangent to BC at M.

Suppose ω intersects segments AB and AC again at points X and Y, respectively. If the area of $\triangle AXY$ can be expressed as $\frac{p}{q}$ where p, q are relatively prime integers, compute $p+q$.

Solution: Power of a Point, then Sine Triangle Area formula.

[3✎] **Problem 41** (AMC 12B 2017/18) The diameter AB of a circle of radius 2 is extended to a point D outside the circle so that $BD = 3$. Point E is chosen so that $ED = 5$ and line ED is perpendicular to line AD. Segment AE intersects the circle at a point C between A and E. What is the area of $\triangle ABC$?

[3✎] **Problem 42** (PHS ARML TST 2017) An algorithm starts with an equilateral triangle $A_0B_0C_0$ of side length 1. At step k, points A_k, B_k,and C_k are chosen on line segments $B_{k-1}C_{k-1}, C_{k-1}A_{k-1}$ and $A_{k-1}B_{k-1}$ respectively, such that

$$B_{k-1}A_k : A_kC_{k-1} = 1 : 1$$
$$C_{k-1}B_k : B_kA_{k-1} = 1 : 2$$
$$A_{k-1}C_k : C_kB_{k-1} = 1 : 3$$

What is the value of the infinite series:

$$\sum_{i=0}^{\infty} \text{Area}[\triangle A_kB_kC_k]$$

[3✎] **Problem 43** (ARML Local 2014) In triangle $ABC, a = 12, b = 17,$ and $c = 13$. Compute $bcosC - ccosB$.

Solution: Find the expression in terms of segments and use Pythaogrean Theorem and Difference of Squares to easily compute. $\boxed{10}$

[4✎] **Problem 44** (Mandelbrot Nationals 2009) Triangle ABC has sides of length $AB = \sqrt{41}, AC =$ 5,and $BC = 8$. Let O be the center of the circumcircle of $\triangle ABC$, and let A' be the point diametrically opposite A, as shown. Determine the area of $\triangle A'BC$.

[4✎] **Problem 45** (HMMT February 2014) Triangle ABC has sides $AB = 14, BC = 13,$ and $CA = 15$. It is inscribed in circle, which has center O. Let M be the midpoint of AB, let B' be the point on diametrically opposite B, and let X be the intersection of AO and MB'. Find the length of AX.

Solution: AX is the centroid of ABB'.

[4✎] **Problem 46** (AIME 1989) Triangle ABC has an right angle at B and contains a point P such that $AP = 10$, $BP = 6$, and $\angle APC = \angle CPB = \angle BPA$. Find CP.

Solution: Law of Cosines and Pythagorean Theorem gives $CP = 33$.

[4✎] **Problem 47** (106 Geometry Problems) In triangle ABC, medians BB_1 and CC_1 are perpendicular. Given that $AC = 19$ and $AB = 22$, find BC.

Solution: Let $BG = 2x, GB_1 = x$ and $CG = 2y, GC_1 = y$. Set systems of equations and solve.

[4✐] **Problem 48** (ARML Local 2014) In triangle $\triangle ABC$, $BC = 2$. Point D is on AC such that $AD = 1$ and $CD = 2$. If $m\angle BDC = 2m\angle A$, compute $\sin A$.

Solution: Note that $\angle DBA = \angle A$ so $BD = DA = 1$. We then apply Law of Cosines on $\triangle BDC$ and bash to get $\boxed{\dfrac{\sqrt{6}}{4}}$.

[4✐] **Problem 49** (AIME 2005) In quadrilateral $ABCD$, let $BC = 8, CD = 12, AD = 10$ and $\angle A = \angle B = 60°$. Given that $AB = p + \sqrt{q}$, where p and q are positive integers, find $p + q$.

Solution: Extend AD and BC to make an equilateral triangle and then Law of Cosines.

[4✐] **Problem 50** (HMMT November 2013) Let ABC be a triangle and D a point on BC such that $AB = \sqrt{2}, BC = \sqrt{3}, \angle BAD = 30°$, and$\angle CAD = 45°$. Find AD.

[4✐] **Problem 51** (CMIMC 2020) Let ABC be a triangle with centroid G and $BC = 3$. If ABC is similar to GAB, compute the area of ABC.

[4✐] **Problem 52** (Magic Math AMC 10 2020) Isosceles triangle $\triangle ABC$ has $AB = AC$ and circumcenter O. The circle with diameter AO intersects segment BC at P and Q, with P closer to B, such that $BP = QC = 4$ and $PQ = 6$. Compute the area of $\triangle A$

[5✐] **Problem 53** (PHS HMMT TST 2020) $\triangle$ ABC has side lengths $AB = 11, BC = 13, CA = 20$. A circle is drawn with diameter AC. Line AB intersects the circle at $D \neq A$, and line BC intersects the circle at $E \neq B$. Find the length of DE.

[5✐] **Problem 54** (CMIMC 2020) Let ABC be a triangle with centroid G and $BC = 3$. If ABC is similar to GAB, compute the area of ABC.

[5✐] **Problem 55** (Magic Math AMC 10 2020) Six points A, B, C, D, E, F are selected on a circle with center O such that $ABCDEF$ is an equiangular hexagon with $AB = CD = EF < BC = DE = FA$. Diagonals AD, BE, FC are drawn, intersecting at points X, Y, Z. The three circles passing through O and two of X, Y, Z are each internally tangent to O. Find BC/AB.

[6✐] **Problem 56** (AMC 12B 2017/24) Quadrilateral $ABCD$ has right angles at B and C, $\triangle ABC \sim \triangle BCD$, and $AB > BC$. There is a point E in the interior of $ABCD$ such that $\triangle ABC \sim \triangle CEB$ and the area of $\triangle AED$ is 17 times the area of $\triangle CEB$. What is $\frac{AB}{BC}$?

Solution: Coordinate bashable by setting $BC = 1$ and letting $B = (0, 0), A = (0, x), C = (1, \frac{1}{x})$. Then, $E = (\frac{1}{x^2+1}, \frac{x}{x^2+1})$. We apply Shoelace and find $x^2 = 9 + 2\sqrt{10} \implies x = \boxed{2 + \sqrt{5}}$.

[6✐] **Problem 57** (SLKK AIME 2020) Cyclic quadrilateral $AXBY$ is inscribed in circle ω such that AB is a diameter of ω. M is the midpoint of XY and $AM = 13, BM = 5$, and $AB = 16$. If the area of AXBY can be expressed as $m\sqrt{p} + n$, where m, n, and p are positive integers such that m and n are relatively prime and p is not divisible by the square of a prime, find the remainder when $m + n + p$ is divided by 1000.

§ 6 Misc

These are problems that don't really fall into any other category at all.

§ 6.1 General

[2✐] **Problem 1** (Math Prizes For Girls 2019) Let $a_1, a_2, \ldots, a_{2019}$ be a sequence of real numbers. For every five indices $i, j, k,$ and m from 1 through 2019, at least two of the numbers a_i, a_j, a_k and a_m have the same absolute value. What is the greatest possible number of distinct real numbers in the given sequence?

Solution: There can't be five or more distinct absolute values because then we can just take those five. It's clear that only having four distinct absolute values work. By the Pigeonhole Principle, at least two are the same. Then, from those four, we can have at most $4 \cdot 2 = \boxed{8}$ distinct real numbers.

[3✐] **Problem 2** (ARML Local 2014) The sequence of words $\{a_n\}$ is defined as follows: $a_1 = X, a_2 = O$, and for $n \geq 3$, an is a_{n-1} followed by the reverse of a_{n-2}. For example, $a_3 = OX, a_4 = OXO, a_5 = OXOXO$, and $a_6 = OXOXOOXO$. Compute the number of palindromes in the first 1000 terms of this sequence.

Solution: Only a_{3n} are not palindromes. We can prove this by induction. Let $\overline{a}$ be the reverse of a. Let $a_{3n+1} = x, a_{3n+2} = y$ and assume those are palindromes. Then, $a_{3n+3} = y\overline{x}$ which cannot be a palindrome (the length is less than twice of y) and $a_{3n+4} = y\overline{x}y$ which is a palindrome as y is a palindrome. Also, $a_{3n+5} = y\overline{x}yx\overline{y}$. This also a palindrome. Then, our answer is $1000 - 333 = \boxed{667}$.

§ 6.2 Games

[2✐] **Problem 3** (BmMT 2016) Suppose you have a 20×16 bar of chocolate squares. You want to break the bar into smaller chunks, so that after some sequence of breaks, no piece has an area of more than 5. What is the minimum possible number of times that you must break the bar?

Solution: By the Pigeon Hole Principle, at the end, there should be at least $\frac{320}{5} = 64$ pieces. It takes at least 63 moves to break it into 64 pieces as each move creates one more piece. We can also construct a sequence of moves by noticing that this is equalty case of the Pigeon Hole Principle so each piece must have exactly 5 squares. Break it into 16 of 20×1 pieces. This uses 15 moves. Then, break each of those pieces into 4 of 5×1 pieces. This uses $3 \cdot 16 = 48$ moves. So, our answer is $\boxed{63}$

[3✐] **Problem 4** (BMT 2015) Two players play a game with a pile with N coins is on a table. On a player's turn, if there are n coins, the player can take at most $\frac{n}{2} + 1$ coins, and must take at least one coin. The player who grabs the last coin wins. For how many values of N between 1 and 100 (inclusive) does the first player have a winning strategy?

§ 6.3 Logic

[2 ✎] **Problem 5** (BmMT 2014) Alice, Bob, Carl, and Dave are either lying or telling the truth. If the four of them make the following statements, who has the coin?

Alice: I have the coin.
Bob: Carl has the coin.
Carl: Exactly one of us is telling the truth.
Dave: The person who has the coin is male.

Solution: Analyzing, Carl must be lying since if he was telling the truth, everyone else are lying which means Alice can't have the coin but the person who has the coin isn't male which is a contradiction. Also note that all of them can't be lying by similar reasoning. So, either two, three, or four are telling the truth. Four telling the truth is impossible as Carl's statement is false. Three telling the truth means everyone but Carl is telling the truth which is impossible as Alice and Dave's statements conflict. So, two must be telling the truth. Dave's and Alice's statements are true if and only if the other is false. If Dave is false and Alice is true, then Bob must also be false which is a contradiction to the fact two are telling the truth. If Dave is true and Alice is false, then Bob must be telling the truth and $\boxed{Carl}$ has the coin. This is the only possible case.

[2 ✎] **Problem 6** (Berkeley Math Circle 2013) Ten people sit side by side at a long table, all facing the same direction. Each of them is either a knight (and always tells the truth) or a knave (and always lies). Each of the people announces: "There are more knaves on my left than knights on my right." How many knaves are in the line?

Solution: 5 knaves on the left, 5 knights on the right gives a valid solution. So, $\boxed{5}$.