Collected Problems

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§ 1 Introduction

I use the following scheme: 1 point is roughly AMC 10 8-14 level. 2 points is roughly AMC 10 # 15-17 level. 3 points is AMC 10 # 18-21 level. 4 and 5 points are AMC 10 # 22-25 level. Points above 5 scale similarly.

Most of these problems are from more obscure contests that will serve as good AIME and AMC practice.

§ 2 Combinatorics

§ 2.1 Casework

[2✎] **Problem 1** (BmMT 2014) Call a positive integer top-heavy if at least half of its digits are in the set 7, 8, 9. How many three digit top-heavy numbers exist? (No number can have a leading zero.)

Solution: Consider $7, 8, 9$ to be the same and assign actual values at end, do casework on number of $7, 8, 9$.

[3✎] **Problem 2** (PuMAC 2019) Suppose Alan, Michael, Kevin, Igor, and Big Rahul are in a running race. It is given that exactly one pair of people tie (for example, two people both get second place), so that no other pair of people end in the same position. Each competitor has equal skill; this means that each outcome of the race, given that exactly two people tie, is equally likely. The probability that Big Rahul gets first place (either by himself or he ties for first) can be expressed in the form m/n, where m, n are relatively prime, positive integers. Compute $m + n$.

Solution: First, find total number of outcomes. Then, casework on if Big Rahul wins by himself or if he ties for first

[3✎] **Problem 3** (PuMAC 2019) Prinstan Trollner and Dukejukem are competing at the game show WASS. Both players spin a wheel which chooses an integer from 1 to 50 uniformly at random, and this number becomes their score. Dukejukem then flips a weighted coin that lands heads with probability 3/5. If he flips heads, he adds 1 to his score. A player wins the game if their score is higher than the other player's score. The probability Dukejukem defeats the Trollner to win WASS equals m/n where m, n are coprime positive integers. Compute $m+n$.

Solution: Casework on if Dukejukem flips heads or tails on coin.

[4 ✎] **Problem 4** (Purple Comet 2015 HS) Seven people of seven different ages are attending a meeting. The seven people leave the meeting one at a time in random order. Given that the youngest person leaves the meeting sometime before the oldest person leaves the meeting, the probability that the third, fourth, and fifth people to leave the meeting do so in order of their ages (youngest to oldest) is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution: Casework on if youngest or oldest in third, fourth, fifth block.

[5 ✎] **Problem 5** (HMMT November 2014) Consider the set of 5-tuples of positive integers at most 5. We say the tuple $(a_1, a_2, a_3, a_4, a_5)$ is perfect if for any distinct indices i, j, k, the three numbers a_i, a_j, a_k do not form an arithmetic progression (in any order). Find the number of perfect 5-tuples.

[5 ✎] **Problem 6** (HMMT November 2013) Find the number of positive integer divisors of 12! that leave a remainder of 1 when divided by 3.

Solution: Casework on parity of exponent of $2, 5, 11$. Note that there can't be any 3 and 7 can be included or excluded without consequence.

[7 ✎] **Problem 7** (SLKK AIME 2020) Andy the Banana Thief is trying to hide from Sheriff Buffkin in a row of 6 distinct houses labeled 1 through 6. Andy and Sheriff Buffkin each pick a permutation of the 6 houses, chosen uniformly at random. On the n^{th} day, with $1 \leq n \leq 6$, Andy and Sheriff Buffkin visit the n^{th} house in their respective permutations, and Andy is caught by the Sheriff on the first day they visit the same house. For example, if Andy's permutation is $1, 3, 4, 5, 6, 2$ and Sheriff Buffkin's permutation is $3, 4, 1, 5, 6, 2$, Andy is caught on day 4. Given that Sheriff Buffkin catches Andy within 6 days and the expected number of days it takes to catch Andy can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b, find the remainder when $a+b$ is divided by 1000.

Solution: Casework on day caught and then use complementary PIE.

§ 2.2 Perspectives

[1 ✎] **Problem 8** (PuMAC 2019) How many ways can you arrange 3 Alice's, 1 Bob, 3 Chad's, and 1 David in a line if the Alice's are all indistinguishable, the Chad's are all indistinguishable, and Bob and David want to be adjacent to each other? (In other words, how many ways can you arrange 3 A's, 1 B, 3 C's, and 1 D in a row where the B and D are adjacent?)

[2 ✎] **Problem 9** (MAθ 2016) The product of any two of the elements of the set $\{30, 54, N\}$ is divisible by the third. Find the number of possible values of N.

Solution: Consider the primes $2, 3, 5$ separately and get independent inequalities.

[3 ✎] **Problem 10** (BmMT 2014) If you roll three regular six-sided dice, what is the probability that the three numbers showing will form an arithmetic sequence? (The order of the dice does matter, but we count both (1, 3, 2) and (1, 2, 3) as arithmetic sequences.)

[4✎] **Problem 11** (PuMAC 2019) Keith has 10 coins labeled 1 through 10, where the ith coin has weight 2^i. The coins are all fair, so the probability of flipping heads on any of the coins is $\frac{1}{2}$. After flipping all of the coins, Keith takes all of the coins which land heads and measures their total weight, W. If the probability that $137 \leq W \leq 1061$ is m/n for coprime positive integers m, n, determine $m + n$

Solution: This bijects to even binary numbers between 0 and 2046. There $\frac{1060-138}{2} + 1 = 462$ even numbers in the range $[137, 1061]$. Then, $\frac{462}{1024} = \frac{231}{512} \implies \boxed{743}$.

[4✎] **Problem 12** (PHS HMMT TST 2016) Compute the number of ordered triples of sets (A_1, A_2, A_3) that satisfy the following:

1. $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6\}$
2. $A_1 \cap A_2 \cap A_3 = \emptyset$

Solution: Consider each element separately and see where it can go in the Venn Diagram. It can go in $8 - 2 = 6$ sections as it can't be in all three or not in any. So, $\boxed{6^6}$.

[5✎] **Problem 13** (HMMT Feburary 2014) We have a calculator with two buttons that displays an integer x. Pressing the first button replaces x by $\lfloor \frac{x}{2} \rfloor$, and pressing the second button replaces x by $4x + 1$. Initially, the calculator displays 0. How many integers less than or equal to 2014 can be achieved through a sequence of arbitrary button presses? (It is permitted for the number displayed to exceed 2014 during the sequence. Here, byc denotes the greatest integer less than or equal to the real number y.)

Solution: Any number with 1 digits separated by one or more 0's is valid. Notice that for 2015 through 2047, the first two digits are 11 so they are not valid. Casework on number of digits now.

§ 2.2.1 Expected Value

[5✎] **Problem 14** (SLKK AIME 2020) Woulard forms a 8 letter word by picking each letter from the set $\{w, o, u\}$ with equal probability. The score of a word is the nonnegative difference between the number of distinct occurrences of the three-letter word "uwu" and the number of distinct occurrences of the three-letter word "owo". For example, the string "owowouwu" has a score of 2 - $1 = 1$. If the expected score of Woulard's string can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b, find the remainder when $a + b$ is divided by 1000.

Solution: Let the *value* of the string be the total number of uwu and owo's. Note that the value only differs from the score when there are both uwu and owo's. We can easily compute the value using Linearity of Expectation and do casework on when it differs.

[6✎] **Problem 15** (PuMAC 2019) . Marko lives on the origin of the Cartesian plane. Every second, Marko moves 1 unit up with probability 2/9, 1 unit right with probability 2/9, 1 unit up and 1 unit right with probability 4/9, and he doesn't move with probability 1/9. After 2019 seconds, Marko ends up on the point (A, B). What is the expected value of A · B?

Solution: Need to do this

[6] **Problem 16** (PuMAC 2019) Kelvin and Quinn are collecting trading cards; there are 6 distinct cards that could appear in a pack. Each pack contains exactly one card, and each card is equally likely. Kelvin buys packs until he has at least one copy of every card, then he stops buying packs. If Quinn is missing exactly one card, the probability that Kelvin has at least two copies of the card Quinn is missing is expressible as m/n for coprime positive integers m, n. Determine $m + n$.

Solution: Complementary counting, find probability that Kelvin has exactly one copy of the card Quinn is missing.

§ 2.3 Miscellaneous

[1] **Problem 17** (Mandelbrot Nationals Sample Test) Michael Jordan's probability of hitting any basketball shot is three times greater than mine, which never exceeds a third. To beat him in a game, I need to hit a shot myself and have Jordan miss the same shot. If I pick my shot optimally, what is the maximum probability of winning which I can attain?

Solution: What's the max of $p(1-3p)$?

[3] **Problem 18** (PHS ARML TST 2017) Consider a group of eleven high school students. To create a middle school math contest, they must pick a four-person committee to write problems and a four-person committee to proofread. Every student can be on neither committee, one committee, or both committees, except for one student who does not want to be on both. How many combinations of committees are possible?

Solution: Complementary counting, find how many committees have that student on both and how many committees without that restriction

[3] **Problem 19** (Mandelbrot Regionals 2009) Mr. Strump has formed three person groups in his math class for working on projects. Every student is in exactly two groups, and any two groups have at most one person in common. In fact, if two groups are chosen at random then the probability that they have exactly one person in common is one-third. How many students are there in Mr. Strump's class?

[6] **Problem 20** (CRMT Team 2019) A deck of the first 100 positive integers is randomly shuffled. Find the expected number of draws it takes to get a prime number if there is no replacement.

[6] **Problem 21** (CNCM PoTD) Find the remainder when $\sum_{n=0}^{333} \sum_{k=3n}^{999} \binom{k}{3n}$ is divided by 70.

§ 3 Number Theory

§ 3.1 Divisors

[1] **Problem 22** (MAθ 2018) How many distinct prime numbers are in the first 50 rows of Pascal's Triangle?

Solution: If $k \neq 1, n-1$ for $\binom{n}{k}$, then $\binom{n}{k}$ is composite by its explicit formula. So, $\binom{n}{1} = n$, how many of those are prime for $1 \leq n \leq 50$?

[2 ✎] **Problem 23** (AHSME 1984) How many triples (a, b, c) of positive integers satisify the simultaneous equations:

$$ab + bc = 44$$

$$ac + bc = 23$$

[2 ✎] **Problem 24** (PHS ARML TST 2017) Compute the greatest prime factor of

$$3^8 + 2 \cdot 2^4 \cdot 4^4 + 2^{16}$$

Solution: Let $3^4 = x$ and $4^4 = y$. Then, this is just $x^2 + 2xy + y^2 = (x+y)^2$.

[2 ✎] **Problem 25** (HMMT November 2014) Compute the greatest common divisor of $4^8 - 1$ and $8^{12} - 1$

[3 ✎] **Problem 26** (MAθ 2018) The number $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + 100 \cdot 100!$ ends with a string of 9s. How many consecutive 9s are at the end of the number?

Solution: $n \cdot n! = (n+1)! - n!$, then telescope to get $101! - 1!$. How many 0's does 101! have?

[5 ✎] **Problem 27** (BMT 2015) There exists a unique pair of positive integers k, n such that k is divisible by 6, and $\sum_{i=1}^{k} i^2 = n^2$. Find (k, n).

Solution: You get $k \cdot (6k+1) \cdot (12k+1) = n^2$. All of these are pairwise coprime so each are squares. Trying out the first couple of squares for k, we get $k = 4$ is a solution. Having both $b^2 = 6a^2 + 1$ and $c^2 = 12a^2 + 1$ be squares is not possible for larger a.

§ 3.2 Modulo

[1 ✎] **Problem 28** (MAθ 2018) The number $4^{14} - 1$ is divisible by 29 but $2^{14} - 1$ is not. What is the remainder when $2^{14} - 1$ is divided by 29?

Solution: $4^{14} - 1 = (2^{14} - 1)(2^{14} + 1) = 0 \pmod{29}$. Since $2^{14} - 1 \neq 0 \pmod{29}$, $2^{14} + 1 = 0 \pmod{29} \implies 2^{14} - 1 = 27 \pmod{29}$

[3 ✎] **Problem 29** (CNCM PoTD) Find the number of positive integer x less than 100 such that

$$3^x + 5^x + 7^x + 11^x + 13^x + 17^x + 19^x$$

is prime.

Solution: Considering $\pmod 3$, we get $3(1)^x + 3(-1)^x = 0 \pmod 3$ which is impossible as the expression is clearly > 3. So, $\boxed{0}$.

[4 ✎] **Problem 30** (SLLKK AIME 2020) Smush is a huge Kobe Bryant fan. Smush randomly draws n jerseys from his infinite collection of Kobe jerseys, each being either the recent #24 jersey or the throwback #8 jersey with equal probability. Let $p(n)$ be the probability that Smush can divide the n jerseys into two piles such that the sum of all jersey numbers in each pile is the same. If

[5] **Problem 31** (BMT 2019) Compute the remainder when the product of all positive integers less than and relatively prime to 2019 is divided by 2019.

[5] **Problem 32** (PHS HMMT TST 2020) Find the largest integer $0 < n < 100$ such that $n^2 + 2n$ divides $4(n-1)! + n + 4$.

Solution: For n is even, $n^2 + 2n = (n)(n+2) = 4(\frac{n}{2})(\frac{n+2}{2})$. $4(n-1)! = 0 \pmod{4(\frac{n}{2})(\frac{n+2}{)}2)}$, then we get $n + 4 = 0 \pmod{n^2 + 2n}$ which is impossible. For n is odd, $n^2 + 2n = (n)(n+2)$ and $\gcd(n, n+2) = 1$. If either of $n, n+2$ are composite, WLOG $n = 0 \pmod p$ for some prime $p < n$, then $(n-1)! = 0 \pmod p \implies n + 4 = 0 \pmod p \implies 4 = 0 \pmod p$ contradiction. Similar for $n + 2$. Then, we have that $n, n+2$ must be both primes. We show that this works. We have $4(n-1)! + n + 4 \pmod n = -4 + 4 = 0 \pmod n$ by Wilsons'. Also, $4(n-1)! + n + 4 \pmod{n+2} = 2 + 4\frac{-1}{(n+1)(n)} \pmod{n+2} = 2 + 4\frac{-1}{2} \pmod{n+2} = 2 - 2 = 0 \pmod{n+2}$. Since $\gcd(n, n+2) = 1$, we're done.

The largest twin primes in the range are $\boxed{71}$, 73.

[7] **Problem 33** (HMMT November 2014) Suppose that m and n are integers with $1 \le m \le 49$ and $n \ge 0$ such that m divides $n^{n+1} + 1$. What is the number of possible values of m?

[9] **Problem 34** (SLKK AIME 2020) : Let p = 991 be a prime. Let S be the set of all lattice points (x, y), with $1 \le x, y \le p - 1$. On each point (x, y) in S, Olivia writes the number $x^2 + y^2$. Let $f(x, y)$ denote the product of the numbers written on all points in S that share at least one coordinate with (x, y). Find the remainder when

$$\sum_{i=1}^{p-2} \sum_{j=1}^{p-2} f(i, j)$$

is divided by p.

§ 3.3 Bases

[4] **Problem 35** (HMMT November 2014) Mark and William are playing a game with a stored value. On his turn, a player may either multiply the stored value by 2 and add 1 or he may multiply the stored value by 4 and add 3. The first player to make the stored value exceed 2100 wins. The stored value starts at 1 and Mark goes first. Assuming both players play optimally, what is the maximum number of times that William can make a move? (By optimal play, we mean that on any turn the player selects the move which leads to the best possible outcome given that the opponent is also playing optimally. If both moves lead to the same outcome, the player selects one of them arbitrarily.)

[4] **Problem 36** (HMMT November 2013) How many of the first 1000 positive integers can be written as the sum of finitely many distinct numbers from the sequence $3^0, 3^1, 3^2 \cdots$?

[6] **Problem 37** (HMMT November 2014) For any positive integers a and b, define $a \oplus b$ to be the result when adding a to b in binary (base 2), neglecting any carry-overs. For example,

$20 \oplus 14 = 10100_2 \oplus 1110_2 = 11010_2 = 26$. (The operation $\oplus$ is called the exclusive or.) Compute the sum

$$\sum_{k=0}^{2^{2014}-1} (k \oplus \lfloor \frac{k}{2} \rfloor)$$

§ 3.4 Binomial Theorem

[1✎] **Problem 38** (BMT 2015) Compute the sum of the digits of 1001^{10}.

§ 3.5 Miscellenous

[3✎] **Problem 39** (BMT 2015) . Find all integer solutions to

$$x^2 + 2y^2 + 3z^2 = 36$$

$$3x^2 + 2y^2 + z^2 = 84$$

$$xy + xz + yz = -7$$

[4✎] **Problem 40** (BMT 2019) For a positive integer n, define $\phi(n)$ as the number of positive integers less than or equal to n that are relatively prime to n. Find the sum of all positive integers n such that $\phi(n) = 20$

[5✎] **Problem 41** (PHS HMMT TST 2020) Find the unique triplet of integers (a, b, c) with $a > b > c$ such that $a + b + c = 95$ and $a^2 + b^2 + c^3 = 3083$.

[5✎] **Problem 42** (BMT 2019) 0. Let S(n) be the sum of the squares of the positive integers less than and coprime to n. For example, $S(5) = 1^2 + 2^2 + 3^2 + 4^2$, but $S(4) = 1^2 + 3^2$. Let $p = 2^7 - 1 = 127$ and $q = 2^5 - 1 = 31$ be primes. The quantity $S(pq)$ can be written in the form

$$\frac{p^2q^2}{6}(a - \frac{b}{c})$$

where a, b, and c are positive integers, with b and c coprime and $b < c$. Find a.

[6✎] **Problem 43** (CNCM PoTD) How many positive integers k are there such that $101 \leq k \leq 10000$ and $\lfloor \sqrt{k - 100} \rfloor$ is a divisor of k?

§ 4 Algebra

§ 4.1 Polynomials

Generally uses the following techniques: Vieta's, Binomial Theorem, Multinomial Theorem, Remainder Theorem, Newton's Sums, Reciprocal Roots Trick, Quadratic Formula (including using Determinant), [1 ✎] **Problem 44** (CRMT Math Bowl 2019) Find the sum of all real numbers such that

$$\sqrt[4]{16x^4 - 32x^3 + 24x^2 - 8x + 1} = 5$$

[2 ✎] **Problem 45** (TAMU 2019) In the expansion of $(1 + ax - x^2)^8$ where a is a positive constant, the coefficient of x^2 is 244. Find the value of a

[3 ✎] **Problem 46** (HMMT November 2014) Let $f(x) = x^2+6x+7$. Determine the smallest possible value of $f(f(f(f(x))))$ over all real numbers x.

[3 ✎] **Problem 47** (HMMT February 2014) Find the sum of all real numbers x such that $5x^4 + 10x^3 + 10x^2 + 5x + 11 = 0$

Solution: Symmetric about 1. Monotonic after 1 so only two real roots that sum to 1.

[3 ✎] **Problem 48** (BmMT 2014) Consider the graph of $f(x) = x^3 + x + 2014$. A line intersects this cubic at three points, two of which have x-coordinates 20 and 14. Find the x-coordinate of the third intersection point

Solution: Vieta's!

[4 ✎] **Problem 49** (TAMU 2018) Suppose f is a cubic polynomial with roots a, b, c such that

$$a = \frac{1}{3 - bc}$$
$$b = \frac{1}{5 - ac}$$
$$c = \frac{1}{7 - ab}$$

If $f(0) = 1$, find $f(abc + 1)$.

Solution: Let the leading coefficient be k. We have $f(0) = 1 \implies abc = \frac{-1}{k}$. Multiply out the expressions to get $3a - abc = 1 \implies 3a + \frac{1}{k} = 1 \implies a = \frac{1-\frac{1}{k}}{3}$. Similarily, $b = \frac{1-\frac{1}{k}}{5}, c = \frac{1-\frac{1}{k}}{7}$. Also, $f(abc) + 1 = k(abc - a)(abc - b)(abc - c)$.

[4 ✎] **Problem 50** (PuMAC 2019) Let $f(x) = x^2 + 4x + 2$. Let r be the difference between the largest and smallest real solutions of the equation $f(f(f(f(x)))) = 0$. Then $r = a^{\frac{p}{q}}$ for some positive integers a, p, q so a is square-free and p, q are relatively prime positive integers. Compute $a+p+q$

Solution: Some pattern finding gives $f^n(x) = 0$ has solutions $x = -2 \pm 2^{\frac{1}{2^n}}$.

[4 ✎] **Problem 51** (HMMT February 2014) Find all real numbers k such that $r^4 + kr^3 + r^2 + 4kr + 16 = 0$ is true for exactly one real number r.

Solution: Divide by r^2 and subsitute $t = r + \frac{4}{r}$.

[4 ✎] **Problem 52** (PHS HMMT TST 2020) Let a, b, c be the distinct real roots of $x^3 + 2x + 5$. Find $(8 - a^3)(8 - b^3)(8 - c^3)$.

[4 ✎] **Problem 53** (PuMAC 2019) Let Q be a quadratic polynomial. If the sum of the roots of $Q^{100}(x)$ (where $Q^i(x)$ is defined by $Q^1(x) = Q(x), Q^i(x) = Q(Q^{i-1}(x))$ for integers $i \geq 2$) is 8 and the sum of the roots of Q is S, compute $|\log_2(S)|$.

§ 4.1.1 Newton's Sums

[3 ✎] **Problem 54** (BMT 2015) Let r, s, and t be the three roots of the equation $8x^3+1001x+2008 = 0$. Find $(r+s)^3+(s+t)^3+(t+r)^3$

[4 ✎] **Problem 55** (BMT 2019) Let r_1, r_2, r_3 be the (possibly complex) roots of the polynomial $x^3+ax^2+bx+\frac{4}{3}$. How many pairs of integers a, b exist such that $r_1^3+r_2^3+r_3^3 = 0$?

[6 ✎] **Problem 56** (SLKK AIME 2020) Let a, b, and c be the three distinct solutions to $x^3-4x^2+5x+1 = 0$. Find

$$(a^3+b^3)(a^3+c^3)(b^3+c^3).$$

§ 4.1.2 Roots of Unity

[5 ✎] **Problem 57** (BMT 2019) Let a_n be the product of the complex roots of $x^{2n} = 1$ that are in the first quadrant of the complex plane. That is, roots of the form $a+bi$ where $a, b > 0$. Let $r = a_1 \cdot a_2 \cdot ... \cdot a_{10}$. Find the smallest integer k such that r is a root of $x^k = 1$

[6 ✎] **Problem 58** (BMT 2015) Evaluate $\sum_{k=0}^{37}(-1)^k\binom{75}{2k}$.

Solution: Roots of Unity Filter

§ 4.2 Manipulation

[2 ✎] **Problem 59** (PHS HMMT TST 2020) What is the value of $\frac{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\cdots}{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\cdots}$? Remember that $\frac{1}{1^2}+\frac{1}{2^2}\cdots = \frac{\pi^2}{6}$

[2 ✎] **Problem 60** (HMMT November 2013) Evaluate

$$\frac{1}{2-\frac{1}{2-\frac{1}{2-\cdots\frac{1}{2-\frac{1}{2}}}}}$$

where the digit 2 appears 2013 times

[2 ✎] **Problem 61** (Mandelbrot) If $\frac{x^2}{y^2} = \frac{8y}{x} = z$, find the sum of all possible z.

[3 ✎] **Problem 62** (MAθ 2018) The solutions to $3\sqrt{2x^2-5x-3}+2x^2-5x = 7$ can be written in the form $x = \frac{a\pm\sqrt{b}}{c}$ where a, b, c are positive integers and x is in simplest form. Find $a+b+c$.

[3 ✎] **Problem 63** (BMT 2016) Simplify $\frac{1}{\sqrt[3]{81}+\sqrt[3]{72}+\sqrt[3]{64}}$.

[3 ✎] **Problem 64** (Mandelbrot Nationals 2008) Find the positive real number x for which $5\sqrt{1-x}+5\sqrt{1+x} = 7\sqrt{2}$.

[4 ✎] **Problem 65** (2014 November HMMT) Let a, b, c, x be reals with $(a+b)(b+c)(c+a) \neq 0$ that satisfy

$$\frac{a^2}{a+b} = \frac{a^2}{a+c}+20, \frac{b^2}{b+c} = \frac{b^2}{b+a}+14, \text{ and } \frac{c^2}{c+a} = \frac{c^2}{c+b}+x$$

Compute x.

[5✎] **Problem 66** (Math Prizes For Girls 2015) Let S be the sum of all distinct real solutions of the equation

$$\sqrt{x+2015} = x^2 - 2015.$$

Compute $\lfloor 1/S \rfloor$. Recall that if r is a real number, then $\lfloor r \rfloor$ (the floor of r) is the greatest integer that is less than or equal to r

Solution: Let $2015 = y$. Then, we have $\sqrt{x+y} = x^2 - y \implies x + y = x^4 - 2x^2y + y^2 \implies y^2 + (-2x^2 - 1)y + x^4 - x = 0$. Then, $y = \frac{2x^2+1\pm(2x+1)}{2}$. Now, we have $2015 = x^2 + x + 1$ or $2015 = x^2 - x$. These give $x = \frac{-1\pm\sqrt{8057}}{2}$ and $x = \frac{1\pm\sqrt{8061}}{2}$. Now, note that we have $x + y \geq 0 \implies x \geq -2015$ and $x^2 - y \geq 0 \implies |x| \geq \sqrt{2015}$.

We can see that $\frac{-1+\sqrt{8057}}{2} > \sqrt{2015}$ and that $\frac{-1-\sqrt{8057}}{2} < -\sqrt{2015}$. Also, $\frac{1-\sqrt{8061}}{2} > -\sqrt{2015}$ and $\frac{1+\sqrt{8061}}{2} > \sqrt{2015}$. So, we have that our two solutions are $\frac{-1-\sqrt{8057}}{2}$ and $\frac{1+\sqrt{8061}}{2}$.

Then, $\frac{1}{S} = \frac{2}{\sqrt{8061}-\sqrt{8057}} = \frac{\sqrt{8061}+\sqrt{8057}}{2}$. So, $89 < \frac{1}{S} < 90$ so our answer is $\boxed{89}$

§ 4.3 Telescoping

[3✎] **Problem 67** (Purple Comet 2015 HS)

$$\left(1+\frac{1}{1+2^1}\right)\left(1+\frac{1}{1+2^2}\right)\left(1+\frac{1}{1+2^3}\right)\cdots\left(1+\frac{1}{1+2^{10}}\right) = \frac{m}{n},$$

where m and n are relatively prime positive integers. Find $m + n$.

§ 4.4 Trigonometry

[4✎] **Problem 68** (MAθ 1992) If A and B are both in $[0, 2\pi)$ and A and B satisfy the equations

$$\sin A + \sin B = \frac{1}{3}$$

$$\cos A + \cos B = \frac{4}{3}$$

find $\cos(A - B)$

[4✎] **Problem 69** (TAMU 2019) Simplify $\arctan\frac{1}{1+1+1^2} + \arctan\frac{1}{1+2+2^2} + \arctan\frac{1}{1+3+3^2}\cdots + \arctan\frac{1}{1+n+n^2}$

[6✎] **Problem 70** (Purple Comet 2015 HS) Let x be a real number between 0 and $\frac{\pi}{2}$ for which the function $3\sin^2 x + 8\sin x\cos x + 9\cos^2 x$ obtains its maximum value, M. Find the value of $M + 100\cos^2 x$.

§ 4.5 Logarithms

[3] **Problem 71** (PuMAC 2019) If x is a real number so $3^x = 27x$, compute $\log_3(\frac{3^{3^x}}{x^{3^3}})$.

[4] **Problem 72** (PHS ARML TST 2017) Positive real numbers x, y, and z satisfy the following system of equations:

$$x^{\log(yz)} = 100$$
$$y^{\log(xz)} = 10$$
$$z^{\log(xy)} = 10\sqrt{10}$$

Compute the value of the expression $(\log(xyz))^2$

[4] **Problem 73** (SLKK AIME 2020) Let x be a real number in the interval $(0, \frac{\pi}{2})$ such that $\log_{\sin^2(x)} \cos(x) + \log_{\cos^2(x)} \sin(x) = \frac{5}{4}$. If $\sin^2(2x)$ can be expressed as $m\sqrt{n} - p$, where m, n, and p are positive integers such that n is not divisible by the square of a prime, find $m + n + p$

§ 4.6 Sequences

[3] **Problem 74** (BMT 2015) Let $\{a_n\}$ be a sequence of real numbers with $a_1 = -1, a_2 = 2$ and for all $n \geq 3$, $a_{n+1} - a_n - a_{n+2} = 0$. Find $a_1 + a_2 + a_3 + ... + a_{2015}$.

Solution: Note that the condition is $a_{n+2} = a_{n+1} - a_n$ and that $a_3 = 3, a_4 = 1, a_5 = -2, a_6 = -3, a_7 = -1, a_8 = 2$. So, it repeats with period 6.

§ 4.7 Functions

[3] **Problem 75** (Mandelbrot Nationals 2009) Let $f(x)$ be a function defined for all positive real numbers satisfying the conditions $f(x) > 0$ for all $x > 0$ and $f(x - y) = \sqrt{f(xy) + 1}$ for all $x > y > 0$. Determine $f(2009)$.

§ 4.8 Inequalities

[1] **Problem 76** (PuMAC 2019) Let a, b be positive integers such that $a + b = 10$. Let $\frac{p}{q}$ be the difference between the maximum and minimum possible values of $\frac{1}{a} + \frac{1}{b}$, where p and q are relatively prime positive integers. Compute $p + q$.

[4] **Problem 77** (HMMT February 2014) Suppose that x and y are positive real numbers such that $x^2 - xy + 2y^2 = 8$. Find the maximum possible value of $x^2 + xy + 2y^2$.

[4] **Problem 78** (HMMT November 2013) Find the largest real number λ such that $a^2 + b^2 + c^2 + d^2 \geq ab + \lambda bc + cd$ for all real numbers a, b, c, d.

[5] **Problem 79** (BMT 2019) Find the number of ordered integer triplets x, y, z with absolute value less than or equal to 100 such that $2x^2 + 3y^2 + 3z^2 + 2xy + 2xz - 4yz < 5$

Solution: $(x + y)^2 + (x + z)^2 + 2(y - z)^2 < 5$

§ 4.9 Fake Algebra

[3✎] **Problem 80** (BMT 2019) Find the maximum value of $\frac{x}{y}$ if x and y are real numbers such that $x^2 + y^2 - 8x - 6y + 20 = 0$.

[3✎] **Problem 81** (BMT 2015) Let x and y be real numbers satisfying the equation $x^2-4x+y^2+3 = 0$. If the maximum and minimum values of x^2+y^2 are M and m respectively, compute the numerical value of $M - m$.

[4✎] **Problem 82** (PuMAC 2019) Let x and y be positive real numbers that satisfy $(\log x)^2 + (\log y)^2 = \log x^2 + \log y^2$. Compute the maximum possible value of $(\log xy)^2$.

Solution: Subsitute $\log x = a$, $\log y = b$. You get the equation of a circle $(a - 1)^2 + (b - 1)^2 = 2$. You want to find the y-intersect of tangent line with slope -1 on "top" of the circle. Draw a perpendicular to the line from the center to find the tangency point. This has slope 1 and it is $\sqrt{2}$ long. So, the coordinates of this tangency is $(2, 2)$ and $a + b = 4$.

[6✎] **Problem 83** (HMMT February 2014) Given that a, b, and c are complex numbers satisfying

$$a^2 + ab + b^2 = 1 + i$$

$$b^2 + bc + c^2 = 2$$

$$c^2 + ca + a^2 = 1,$$

compute $(ab + bc + ca)^2$

§ 5 Geometry

§ 5.1 Coordinate Geometry

[1✎] **Problem 84** (BmMT 2014) Find the area of the convex quadrilateral with vertices at the points $(-1, 5), (3, 8), (3, -1)$, and $(-1, -2)$.

Solution: Direct application of Shoelace.

[2✎] **Problem 85** (CRMT Individuals 2019) Let S be the set of all distinct points in the coordinate plane that form an acute isosceles triangle with the points $(32, 33)$ and $(63, 63)$. Given that a line L crosses S a finite number of times, find the maximum number of times L can cross S.

Solution: Replace $(32, 33)$ and $(63, 63)$ by A and B. Then, we do casework on $AC = BC$ or $CB = AB$ or $CA = BA$. We get a line and two semicircles. A line can intersect a semicircle two times and a line one time.

[3✎] **Problem 86** (HMMT November 2013) Plot points A, B, C at coordinates (0, 0), (0, 1), and (1, 1) in the plane, respectively. Let S denote the union of the two line segments AB and BC. Let X1 be the area swept out when Bobby rotates S counterclockwise 45 degrees about point A. Let X2 be the area swept out when Calvin rotates S clockwise 45 degrees about point A. Find $\frac{X_1+X_2}{2}$

[5] **Problem 87** (BMT 2019) A regular hexagon has positive integer side length. A laser is emitted from one of the hexagon's corners, and is reflected off the edges of the hexagon until it hits another corner. Let a be the distance that the laser travels. What is the smallest possible value of a^2 such that a ¿ 2019? You need not simplify/compute exponents.

[5] **Problem 88** (SLKK AIME 2020) Mr. Duck draws points $A = (a, 0), B = (0, b), C = (3, 5)$ and $O = (0, 0)$ such that $a, b > 0$ and $\angle ACB = 45°$. If the maximum possible area of $\triangle AOB$ can be expressed as $m - n\sqrt{p}$ where m, n, and p are positive integers such that p is not divisible by the square of a prime, find $m + n + p$

§ 5.2 3D Geometry

[4] **Problem 89** (HMMT February 2014) Let C be a circle in the xy plane with radius 1 and center $(0, 0, 0)$, and let P be a point in space with coordinates $(3, 4, 8)$. Find the largest possible radius of a sphere that is contained entirely in the slanted cone with base C and vertex P.

§ 5.3 General

[1] **Problem 90** (MAθ 2018) A parallelograms has diagonals of length 10 and 20. Find the area inclosed by the circle inscribed in the parallelogram.

[1] **Problem 91** (TAMU 2019) An acute isosceles triangle ABC is inscribed in a circle. Through B and C, tangents to the circle are drawn, meeting at D. If $\angle ABC = 2\angle CDB$, then find the radian measure of $\angle BAC$.

[1] **Problem 92** (PHS PuMAC TST 2017) In triangle ABC, let D and E be the midpoints of BC and AC. Suppose AD and BE meet at F. If the area of $\triangle DEF$ is 50, then what is the area of $\triangle CDE$?

[2] **Problem 93** (TAMU 2019) Let AA_1 be an altitude of triangle $\triangle ABC$, and let A_2 be the midpoint of the side BC. Suppose that AA_1 and AA_2 divide angle $\angle BAC$ into three equal angles. Find the product of the angles of $\triangle ABC$ when the angles are expressed in degrees.

Solution: Let $\angle BAA_1 = \angle A_1AA_2 = \angle A_2AC = \alpha$. We have that $\triangle ABA_2$ is an isosceles triangle as $\angle ABA_1 = \angle AA_2A_1 = 90 - \alpha$. Then, as AA_1 is an altitude, $BA_1 = A_1A_2$. Let $BA_1 = A_1A_2 = x$, then $A_2C = BA_2 = 2x$. Consider triangles BAA_1 and $\triangle A_1AC$. We have that $\tan\alpha = \frac{x}{AA_1}$ and $\tan 2\alpha = \frac{3x}{AA_1}$. So, $\frac{\tan 2\alpha}{\tan\alpha} = 3 \implies \tan\alpha = \frac{1}{\sqrt{3}} \implies \alpha = 30$. So, $\angle BAC = 90$, $\angle ABC = 60$, and $\angle ACB = 30$. The product is $\boxed{162000}$.

[2] **Problem 94** (PHS PuMAC TST 2017) A trapezoid has area 32, and the sum of the lengths of its two bases and altitude is 16. If one of the diagonals is perpendicular to both bases, then what is the length of the other diagonal?

[3] **Problem 95** (AHSME 1984/28) Triangle ABC has area 10. Points D, E, and F, all distinct from A, B, and C, are on sides AB, BC, and CA, respectively, and AD = 2, DB = 3. Triangle ABE and quadrilateral DBEF have equal areas s. Find s.

[3] **Problem 96** (HMMT February 2014) In quadrilateral ABCD, $\angle DAC = 98, \angle DBC = 82, \angle BCD = 70$, and $BC = AD$. Find $\angle ACD$.

Solution: Reflect.

[3✎] **Problem 97** (HMMT November 2013) Let ABC be an isosceles triangle with $AB = AC$. Let D and E be the midpoints of segments AB and AC, respectively. Suppose that there exists a point F on ray $\overrightarrow{DE}$ outside of ABC such that triangle BFA is similar to triangle ABC. Compute $\frac{AB}{BC}$.

[4✎] **Problem 98** (Mandelbrot Nationals 2009) Triangle ABC has sides of length $AB = \sqrt{41}, AC =$ 5,and $BC = 8$. Let O be the center of the circumcircle of $\triangle ABC$, and let A' be the point diametrically opposite A, as shown. Determine the area of $\triangle A'BC$.

[4✎] **Problem 99** (HMMT February 2014) Triangle ABC has sides $AB = 14, BC = 13$, and $CA = 15$. It is inscribed in circle, which has center O. Let M be the midpoint of AB, let B' be the point on diametrically opposite B, and let X be the intersection of AO and MB'. Find the length of AX.

Solution: AX is the centroid of ABB'.

[4✎] **Problem 100** (AIME 1989) Triangle ABC has an right angle at B and contains a point P such that $AP = 10$, $BP = 6$, and $\angle APC = \angle CPB = \angle BPA$. Find CP.

Solution: Law of Cosines and Pythagorean Theorem gives $CP = 33$.

[4✎] **Problem 101** (106 Geometry Problems) In triangle ABC, medians BB_1 and CC_1 are perepndicular. Given that $AC = 19$ and $AB = 22$, find BC.

Solution: Let $BG = 2x, GB_1 = x$ and $CG = 2y, GC_1 = y$. Set systems of equations and solve.

[4✎] **Problem 102** (PHS ARML TST 2017) An algorithm starts with an equilateral triangle A0B0C0 of side length 1. At step k, points A_k, B_k,and C_k are cosehn on line segments $B_{k-1}C_{k-1}, C_{k-1}A_{k-1}$ and $A_{k-1}B_{k-1}$ respectively, such that

$$B_{k-1}A_k : A_kC_{k-1} = 1 : 1$$

$$C_{k-1}B_k : B_kA_{k-1} = 1 : 2$$

$$A_{k-1}C_k : C_kB_{k-1} = 1 : 3$$

What is the value of the infinite series:

$$\sum_{i=0}^{\infty} \text{Area}[\triangle A_kB_kC_k]$$

[4✎] **Problem 103** (AIME 2005) In quadrilateral $ABCD$, let $BC = 8, CD = 12, AD = 10$ and $\angle A = \angle B = 60°$.

Solution: Extend AD and BC to make an equilateral triangle and then Law of Cosines.

[4✎] **Problem 104** (HMMT November 2013) Let ABC be a triangle and D a point on BC such that $AB = \sqrt{2}, BC = \sqrt{3}, \angle BAD = 30°$, and$\angle CAD = 45°$. Find AD.

[4✎] **Problem 105** (PHS HMMT TST 2020) $\triangle$ ABC has side lengths $AB = 11, BC = 13, CA =$ 20. A circle is drawn with diameter AC. Line AB intersects the circle at $D \neq A$, and line BC intersects the circle at $E \neq B$. Find the length of DE.

[6 ✎] **Problem 106** (SLKK AIME 2020) Cyclic quadrilateral $AXBY$ is inscribed in circle ω such that AB is a diameter of ω. M is the midpoint of XY and $AM = 13, BM = 5$, and $AB = 16$. If the area of AXBY can be expressed as $m\sqrt{p} + n$, where m, n, and p are positive integers such that m and n are relatively prime and p is not divisible by the square of a prime, find the remainder when $m + n + p$ is divided by 1000.

[8 ✎] **Problem 107** (SLKK AIME 2020) Squares $ABCD$ and $DEFG$ are drawn in the plane with both sets of vertices A, B, C, D and D, E, F, G labeled counterclockwise. Let P be the intersection of lines AE and CG. If $DA = 35, DG = 20$, and $BF = 25\sqrt{2}$, find DP^2.

Solution: Spiral Similarity from ABCD to DEFG.

§ 6 Misc

These are problems that don't really fall into any other category at all.

§ 6.1 Games

[2 ✎] **Problem 108** (BmMT 2016) Suppose you have a 20×16 bar of chocolate squares. You want to break the bar into smaller chunks, so that after some sequence of breaks, no piece has an area of more than 5. What is the minimum possible number of times that you must break the bar?

[3 ✎] **Problem 109** (BMT 2015) Two players play a game with a pile with N coins is on a table. On a player's turn, if there are n coins, the player can take at most $\frac{n}{2} + 1$ coins, and must take at least one coin. The player who grabs the last coin wins. For how many values of N between 1 and 100 (inclusive) does the first player have a winning strategy?

§ 6.2 Logic

[2 ✎] **Problem 110** (BmMT 2014) Alice, Bob, Carl, and Dave are either lying or telling the truth. If the four of them make the following statements, who has the coin?

Alice: I have the coin.
Bob: Carl has the coin.
Carl: Exactly one of us is telling the truth.
Dave: The person who has the coin is male.

Solution: Analyzing, Carl must be lying since if he was telling the truth, everyone else are lying which means Alice can't have the coin but the person who has the coin isn't male which is a contradiction. Also note that all of them can't be lying by similar reasoning. So, either two, three, or four are telling the truth. Four telling the truth is impossible as Carl's statement is false. Three telling the truth means everyone but Carl is telling the truth which is impossible as Alice and Dave's statements conflict. So, two must be telling the truth. Dave's and Alice's statements are true if and only if the other is false. If Dave is false and Alice is true, then Bob must also be false which is a contradiction to the fact two are telling the truth. If Dave is true and Alice is false, then Bob must be telling the truth and Carl has the coin. This is the only possible case.

[4✎] **Problem 111** (Berkeley Math Circle 2013) Ten people sit side by side at a long table, all facing the same direction. Each of them is either a knight (and always tells the truth) or a knave (and always lies). Each of the people announces: "There are more knaves on my left than knights on my right." How many knaves are in the line?

§ 7 Associated Solutions

§ 7.1 Combinatorics

§ 7.2 Number Theory

§ 7.3 Algebra

§ 7.4 Geometry