Collected Problems

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§ 1 Introduction

I use the following scheme: 1 point is roughly AMC 10 8-14 level. 2 points is roughly AMC 10 # 15-17 level. 3 points is AMC 10 # 18-21 level. 4 and 5 points are AMC 10 # 22-25 level. Points above 5 scale similarily.

Most of these problems are from more obscure contests that will serve as good AIME and AMC practice.

§ 2 Combinatorics

§ 2.1 Casework

[4✎] **Problem 1** (Purple Comet 2015 HS) Seven people of seven different ages are attending a meeting. The seven people leave the meeting one at a time in random order. Given that the youngest person leaves the meeting sometime before the oldest person leaves the meeting, the probability that the third, fourth, and fifth people to leave the meeting do so in order of their ages (youngest to oldest) is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

[5✎] **Problem 2** (HMMT November 2014) Consider the set of 5-tuples of positive integers at most 5. We say the tuple $(a_1, a_2, a_3, a_4, a_5)$ is perfect if for any distinct indices i, j, k, the three numbers a_i, a_j, a_k do not form an arithmetic progression (in any order). Find the number of perfect 5-tuples.

[5✎] **Problem 3** (HMMT November 2013) Find the number of positive integer divisors of 12! that leave a remainder of 1 when divided by 3.

[7✎] **Problem 4** (SLKK AIME 2020) Andy the Banana Thief is trying to hide from Sheriff Buffkin in a row of 6 distinct houses labeled 1 through 6. Andy and Sheriff Buffkin each pick a permutation of the 6 houses, chosen uniformly at random. On the n^{th} day, with $1 \le n \le 6$, Andy and Sheriff Buffkin visit the n^{th} house in their respective permutations, and Andy is caught by the Sheriff on the first day they visit the same house. For example, if Andy's permutation is $1, 3, 4, 5, 6, 2$ and Sheriff Buffkin's permutation is $3, 4, 1, 5, 6, 2$, Andy is caught on day 4. Given that Sheriff Buffkin catches Andy within 6 days and the expected number of days it takes to catch Andy can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b, find the remainder when $a + b$ is divided by 1000.

§ 2.2 Perspectives

[2✐] **Problem 5** (MAθ 2016) The product of any two of the elements of the set $\{30, 54, N\}$ is divisible by the third. Find the number of possible values of N.

Solution: Consider the primes $2, 3, 5$ separately and get independent inequalities.
[4✐] **Problem 6** (PHS HMMT TST 2016) Compute the number of ordered triples of sets (A_1, A_2, A_3) that satisfy the following:

1. $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6\}$
2. $A_1 \cap A_2 \cap A_3 = \emptyset$

[5✐] **Problem 7** (HMMT Feburary 2014) We have a calculator with two buttons that displays an integer x. Pressing the first button replaces x by $\lfloor \frac{x}{2} \rfloor$, and pressing the second button replaces x by $4x + 1$. Initially, the calculator displays 0. How many integers less than or equal to 2014 can be achieved through a sequence of arbitrary button presses? (It is permitted for the number displayed to exceed 2014 during the sequence. Here, byc denotes the greatest integer less than or equal to the real number y.)

Solution: Any number with 1 digits separated by one or more 0's is valid. Notice that for 2015 through 2047, the first two digits are 11 so they are not valid. Casework on number of digits now.

§ 2.2.1 Expected Value

[5✐] **Problem 8** (SLKK AIME 2020) Woulard forms a 8 letter word by picking each letter from the set $\{w, o, u\}$ with equal probability. The score of a word is the nonnegative difference between the number of distinct occurrences of the three-letter word "uwu" and the number of distinct occurrences of the three-letter word "owo". For example, the string "owowouwu" has a score of 2 - $1 = 1$. If the expected score of Woulard's string can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b, find the remainder when $a + b$ is divided by 1000.

§ 2.3 Miscellaneous

[1✐] **Problem 9** (Mandelbrot Nationals Sample Test) Michael Jordan's probability of hitting any basketball shot is three times greater than mine, which never exceeds a third. To beat him in a game, I need to hit a shot myself and have Jordan miss the same shot. If I pick my shot optimally, what is the maximum probability of winning which I can attain?

Solution: What's the max of $p(1 - 3p)$?
[3✐] **Problem 10** (PHS ARML TST 2017) Consider a group of eleven high school students. To create a middle school math contest, they must pick a four-person committee to write problems and a four-person committee to proofread. Every student can be on neither committee, one committee, or both committees, except for one student who does not want to be on both. How many combinations of committees are possible?

Solution: Complementary counting, find how many committees have that student on both and how many committees without that restriction

[3 ✎] **Problem 11** (Mandelbrot Regionals 2009) Mr. Strump has formed three person groups in his math class for working on projects. Every student is in exactly two groups, and any two groups have at most one person in common. In fact, if two groups are chosen at random then the probability that they have exactly one person in common is one-third. How many students are there in Mr. Strump's class?

[6 ✎] **Problem 12** (CRMT Team 2019) A deck of the first 100 positive integers is randomly shuffled. Find the expected number of draws it takes to get a prime number if there is no replacement.

[6 ✎] **Problem 13** (CNCM PoTD) Find the remainder when $\sum_{n=0}^{333} \sum_{k=3n}^{999} \binom{k}{3n}$ is divided by 70.

§ 3 Number Theory

§ 3.1 Divisors

[1 ✎] **Problem 1** (MAθ 2018) How many distinct prime numbers are in the first 50 rows of Pascal's Triangle?

Solution: If $k \neq 1, n-1$ for $\binom{n}{k}$, then $\binom{n}{k}$ is composite by its explicit formula. So, $\binom{n}{1} = n$, how many of those are prime for $1 \leq n \leq 50$?

[2 ✎] **Problem 2** (AHSME 1984) How many triples (a, b, c) of positive integers satisify the simultaneous equations:

$$ab + bc = 44$$

$$ac + bc = 23$$

[2 ✎] **Problem 3** (PHS ARML TST 2017) Compute the greatest prime factor of

$$3^8 + 2 \cdot 2^4 \cdot 4^4 + 2^{16}$$

Solution: Let $3^4 = x$ and $4^4 = y$. Then, this is just $x^2 + 2xy + y^2 = (x+y)^2$.

[2 ✎] **Problem 4** (HMMT November 2014) Compute the greatest common divisor of $4^8 - 1$ and $8^{12} - 1$

[3 ✎] **Problem 5** (MAθ 2018) The number $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + 100 \cdot 100!$ ends with a string of 9s. How many consecutive 9s are at the end of the number?

Solution: $n \cdot n! = (n+1)! - n!$, then telescope to get $101! - 1!$. How many 0's does $101!$ have?

§ 3.2 Modulo

[1] **Problem 6** (MAθ 2018) The number $4^{14} - 1$ is divisible by 29 but $2^{14} - 1$ is not. What is the remainder when $2^{14} - 1$ is divided by 29?

Solution: $4^{14} - 1 = (2^{14} - 1)(2^{14} + 1) = 0 \pmod{29}$. Since $2^{14} - 1 \neq 0 \pmod{29}$, $2^{14} + 1 = 0 \pmod{29} \implies 2^{14} - 1 = 27 \pmod{29}$

[3] **Problem 7** (CNCM PoTD) Find the number of positive integer x less than 100 such that

$$3^x + 5^x + 7^x + 11^x + 13^x + 17^x + 19^x$$

is prime.

Solution: Considering $\pmod 3$, we get $3(1)^x + 3(-1)^x = 0 \pmod 3$ which is impossible as the expression is clearly > 3. So, $\boxed{0}$.

[4] **Problem 8** (SLLKK AIME 2020) Smush is a huge Kobe Bryant fan. Smush randomly draws n jerseys from his infinite collection of Kobe jerseys, each being either the recent #24 jersey or the throwback #8 jersey with equal probability. Let $p(n)$ be the probability that Smush can divide the n jerseys into two piles such that the sum of all jersey numbers in each pile is the same. If

[5] **Problem 9** (BMT 2019) Compute the remainder when the product of all positive integers less than and relatively prime to 2019 is divided by 2019.

[5] **Problem 10** (PHS HMMT TST 2020) Find the largest integer $0 < n < 100$ such that $n^2 + 2n$ divides $4(n-1)! + n + 4$.

Solution: For n is even, $n^2 + 2n = (n)(n+2) = 4(\frac{n}{2})(\frac{n+2}{2})$. $4(n-1)! = 0 \pmod{4(\frac{n}{2})(\frac{n+2}{)}2}$, then we get $n + 4 = 0 \pmod{n^2 + 2n}$ which is impossible. For n is odd, $n^2 + 2n = (n)(n+2)$ and $\gcd(n, n+2) = 1$. If either of $n, n+2$ are composite, WLOG $n = 0 \pmod p$ for some prime $p < n$, then $(n-1)! = 0 \pmod p \implies n + 4 = 0 \pmod p \implies 4 = 0 \pmod p$ contradiction. Similar for $n+2$. Then, we have that $n, n+2$ must be both primes. We show that this works. We have $4(n-1)! + n + 4 \pmod n = -4 + 4 = 0 \pmod n$ by Wilsons'. Also, $4(n-1)! + n + 4 \pmod{n+2} = 2 + 4\frac{-1}{(n+1)(n)} \pmod{n+2} = 2 + 4\frac{-1}{2} \pmod{n+2} = 2 - 2 = 0 \pmod{n+2}$. Since $\gcd(n, n+2) = 1$, we're done.

The largest twin primes in the range are $\boxed{71}$, 73.

[7] **Problem 11** (HMMT November 2014) Suppose that m and n are integers with $1 \leq m \leq 49$ and $n \geq 0$ such that m divides $n^{n+1} + 1$. What is the number of possible values of m?

§ 3.3 Bases

[4] **Problem 12** (HMMT November 2014) Mark and William are playing a game with a stored value. On his turn, a player may either multiply the stored value by 2 and add 1 or he may multiply the stored value by 4 and add 3. The first player to make the stored value exceed 2100 wins. The stored value starts at 1 and Mark goes first. Assuming both players play optimally, what is the maximum number of times that William can make a move? (By optimal play, we mean that on any turn the player selects the move which leads to the best possible outcome given that the opponent

is also playing optimally. If both moves lead to the same outcome, the player selects one of them arbitrarily.)

[4✎] **Problem 13** (HMMT November 2013) How many of the first 1000 positive integers can be written as the sum of finitely many distinct numbers from the sequence $3^0, 3^1, 3^2 \cdots$?

[6✎] **Problem 14** (HMMT November 2014) For any positive integers a and b, define $a \oplus b$ to be the result when adding a to b in binary (base 2), neglecting any carry-overs. For example, $20 \oplus 14 = 101002 \oplus 11102 = 110102 = 26$. (The operation $\oplus$ is called the exclusive or.) Compute the sum

$$\sum_{k=0}^{2^{2014}-1} (k \oplus \lfloor \frac{k}{2} \rfloor)$$

§ 3.4 Miscellenous

[5✎] **Problem 15** (PHS HMMT TST 2020) Find the unique triplet of integers (a, b, c) with $a > b > c$ such that $a + b + c = 95$ and $a^2 + b^2 + c^3 = 3083$.

[4✎] **Problem 16** (BMT 2019) For a positive integer n, define $\phi(n)$ as the number of positive integers less than or equal to n that are relatively prime to n. Find the sum of all positive integers n such that $\phi(n) = 20$

[5✎] **Problem 17** (BMT 2019) 0. Let S(n) be the sum of the squares of the positive integers less than and coprime to n. For example, $S(5) = 1^2 + 2^2 + 3^2 + 4^2$, but $S(4) = 1^2 + 3^2$. Let $p = 2^7 - 1 = 127$ and $q = 2^5 - 1 = 31$ be primes. The quantity $S(pq)$ can be written in the form

$$\frac{p^2q^2}{6}(a - \frac{b}{c})$$

where $a, b,$ and c are positive integers, with b and c coprime and $b < c$. Find a.

[6✎] **Problem 18** (CNCM PoTD) How many positive integers k are there such that $101 \leq k \leq 10000$ and $\lfloor \sqrt{k - 100} \rfloor$ is a divisor of k?

§ 4 Algebra

§ 4.1 Polynomials

Generally uses the following techniques: Vieta's, Binomial Theorem, Multinomial Theorem, Remainder Theorem, Newton's Sums, Reciprocal Roots Trick, Quadratic Formula (including using Determinant),
[1✎] **Problem 1** (CRMT Math Bowl 2019) Find the sum of all real numbers such that

$$\sqrt[4]{16x^4 - 32x^3 + 24x^2 - 8x + 1} = 5$$

[2 ✎] **Problem 2** (TAMU 2019) In the expansion of $(1 + ax - x^2)^8$ where a is a positive constant, the coefficient of x^2 is 244. Find the value of a

[3 ✎] **Problem 3** (HMMT November 2014) Let $f(x) = x^2 + 6x + 7$. Determine the smallest possible value of $f(f(f(f(x))))$ over all real numbers x.

[4 ✎] **Problem 4** (TAMU 2018) Suppose f is a cubic polynomial with roots a, b, c such that

$$a = \frac{1}{3 - bc}$$

$$b = \frac{1}{5 - ac}$$

$$c = \frac{1}{7 - ab}$$

If $f(0) = 1$, find $f(abc + 1)$.

[3 ✎] **Problem 5** (HMMT February 2014) Find the sum of all real numbers x such that $5x^4 + 10x^3 + 10x^2 + 5x + 11 = 0$

[3 ✎] **Problem 6** (BMT 2014) Consider the graph of $f(x) = x^3 + x + 2014$. A line intersects this cubic at three points, two of which have x-coordinates 20 and 14. Find the x-coordinate of the third intersection point

[4 ✎] **Problem 7** (HMMT February 2014) Find all real numbers k such that $r^4 + kr^3 + r^2 + 4kr + 16 = 0$ is true for exactly one real number r.

Solution: Symmetric about 1. Monotonic after 1 so only two real roots that sum to 1.

[4 ✎] **Problem 8** (PHS HMMT TST 2020) Let a, b, c be the distinct real roots of $x^3 + 2x + 5$. Find $(8 - a^3)(8 - b^3)(8 - c^3)$.

§ 4.1.1 Newton's Sums

[4 ✎] **Problem 9** (BMT 2019) Let r_1, r_2, r_3 be the (possibly complex) roots of the polynomial $x^3 + ax^2 + bx + \frac{4}{3}$. How many pairs of integers a, b exist such that $r_1^3 + r_2^3 + r_3^3 = 0$?

[6 ✎] **Problem 10** (SLKK AIME 2020) Let a, b, and c be the three distinct solutions to $x^3 - 4x^2 + 5x + 1 = 0$. Find

$$(a^3 + b^3)(a^3 + c^3)(b^3 + c^3).$$

§ 4.2 Roots of Unity

[5 ✎] **Problem 11** (BMT 2019) Let a_n be the product of the complex roots of $x^{2n} = 1$ that are in the first quadrant of the complex plane. That is, roots of the form $a + bi$ where $a, b > 0$. Let $r = a_1 \cdot a_2 \cdot \ldots \cdot a_{10}$. Find the smallest integer k such that r is a root of $x^k = 1$

§ 4.3 Manipulation

[2 ✎] **Problem 12** (PHS HMMT TST 2020) What is the value of $\frac{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\cdots}{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\cdots}$? Remember that $\frac{1}{1^2}+\frac{1}{2^2}\cdots=\frac{\pi^2}{6}$

[2 ✎] **Problem 13** (HMMT November 2013) Evaluate

$$\frac{1}{2-\frac{1}{2-\frac{1}{2-\cdots\frac{1}{2-\frac{1}{2}}}}}$$

where the digit 2 appears 2013 times

[2 ✎] **Problem 14** (Mandelbrot) If $\frac{x^2}{y^2}=\frac{8y}{x}=z$, find the sum of all possible z.

[3 ✎] **Problem 15** (MAθ 2018) The solutions to $3\sqrt{2x^2-5x-3}+2x^2-5x=7$ can be written in the form $x=\frac{a\pm\sqrt{b}}{c}$ where a,b,c are positive integers and x is in simplest form. Find $a+b+c$.

[3 ✎] **Problem 16** (BMT 2016) Simplify $\frac{1}{\sqrt[3]{81}+\sqrt[3]{72}+\sqrt[3]{64}}$.

[4 ✎] **Problem 17** (Mandelbrot Nationals 2008) Find the positive real number x for which $5\sqrt{1-x}+5\sqrt{1+x}=7\sqrt{2}$.

[5 ✎] **Problem 18** (2014 November HMMT) Let a,b,c,x be reals with $(a+b)(b+c)(c+a)\neq 0$ that satisfy

$$\frac{a^2}{a+b}=\frac{a^2}{a+c}+20, \frac{b^2}{b+c}=\frac{b^2}{b+a}+14, \text{ and } \frac{c^2}{c+a}=\frac{c^2}{c+b}+x$$

Compute x.

[5 ✎] **Problem 19** (Math Prizes For Girls 2015) Let S be the sum of all distinct real solutions of the equation

$$\sqrt{x+2015}=x^2-2015.$$

Compute $\lfloor 1/S \rfloor$. Recall that if r is a real number, then $\lfloor r \rfloor$ (the floor of r) is the greatest integer that is less than or equal to r

Solution: Let $2015=y$. Then, we have $\sqrt{x+y}=x^2-y \implies x+y=x^4-2x^2y+y^2 \implies y^2+(-2x^2-1)y+x^4-x=0$. Then, $y=\frac{2x^2+1\pm(2x+1)}{2}$. Now, we have $2015=x^2+x+1$ or $2015=x^2-x$. These give $x=\frac{-1\pm\sqrt{8057}}{2}$ and $x=\frac{1\pm\sqrt{8061}}{2}$. Now, note that we have $x+y\geq 0 \implies x\geq -2015$ and $x^2-y\geq 0 \implies |x|\geq\sqrt{2015}$.

We can see that $\frac{-1+\sqrt{8057}}{2}>\sqrt{2015}$ and that $\frac{-1-\sqrt{8057}}{2}<-\sqrt{2015}$. Also, $\frac{1-\sqrt{8061}}{2}>-\sqrt{2015}$ and $\frac{1+\sqrt{8061}}{2}>\sqrt{2015}$. So, we have that our two solutions are $\frac{-1-\sqrt{8057}}{2}$ and $\frac{1+\sqrt{8061}}{2}$.

Then, $\frac{1}{S}=\frac{2}{\sqrt{8061}-\sqrt{8057}}=\frac{\sqrt{8061}+\sqrt{8057}}{2}$. So, $89<\frac{1}{S}<90$ so our answer is $\boxed{89}$

§ 4.4 Telescoping

[3 ✎] **Problem 20** (Purple Comet 2015 HS)

$$\left(1+\frac{1}{1+2^1}\right)\left(1+\frac{1}{1+2^2}\right)\left(1+\frac{1}{1+2^3}\right)\cdots\left(1+\frac{1}{1+2^{10}}\right)=\frac{m}{n},$$

where m and n are relatively prime positive integers. Find $m+n$.

§ 4.5 Trigonometry

[4 ✎] **Problem 21** (MAθ 1992) If A and B are both in $[0, 2\pi)$ and A and B satisfy the equations

$$\sin A + \sin B = \frac{1}{3}$$

$$\cos A + \cos B = \frac{4}{3}$$

find $\cos(A-B)$

[4 ✎] **Problem 22** (TAMU 2019) Simplify $\arctan\frac{1}{1+1+1^2}+\arctan\frac{1}{1+2+2^2}+\arctan\frac{1}{1+3+3^2}\cdots+\arctan\frac{1}{1+n+n^2}$

[6 ✎] **Problem 23** (Purple Comet 2015 HS) Let x be a real number between 0 and $\frac{\pi}{2}$ for which the function $3\sin^2 x + 8\sin x\cos x + 9\cos^2 x$ obtains its maximum value, M. Find the value of $M + 100\cos^2 x$.

§ 4.6 Logarithms

[4 ✎] **Problem 24** (PHS ARML TST 2017) Positive real numbers x, y, and z satisfy the following system of equations:

$$x^{\log(yz)} = 100$$

$$y^{\log(xz)} = 10$$

$$z^{\log(xy)} = 10\sqrt{10}$$

Compute the value of the expression $(\log(xyz))^2$

[4 ✎] **Problem 25** (SLKK AIME 2020) Let x be a real number in the interval $(0, \frac{\pi}{2})$ such that $\log_{\sin^2(x)}\cos(x) + \log_{\cos^2(x)}\sin(x) = \frac{5}{4}$. If $\sin^2(2x)$ can be expressed as $m\sqrt{n}-p$, where m, n, and p are positive integers such that n is not divisible by the square of a prime, find $m+n+p$

§ 4.7 Functions

[3 ✎] **Problem 26** (Mandelbrot Nationals 2009) Let $f(x)$ be a function defined for all positive real numbers satisfying the conditions $f(x) > 0$ for all $x > 0$ and $f(x-y) = \sqrt{f(xy)+1}$ for all $x > y > 0$. Determine $f(2009)$.

§ 4.8 Inequalities

[4 ✎] **Problem 27** (HMMT February 2014) Suppose that x and y are positive real numbers such that $x^2 - xy + 2y^2 = 8$. Find the maximum possible value of $x^2 + xy + 2y^2$.

[4 ✎] **Problem 28** (HMMT November 2013) Find the largest real number λ such that $a^2 + b^2 + c^2 + d^2 \geq ab + \lambda bc + cd$ for all real numbers a, b, c, d.

[5 ✎] **Problem 29** (BMT 2019) Find the number of ordered integer triplets x, y, z with absolute value less than or equal to 100 such that $2x^2 + 3y^2 + 3z^2 + 2xy + 2xz - 4yz < 5$

§ 4.9 Fake Algebra

[3 ✎] **Problem 30** (BMT 2019) Find the maximum value of $\frac{x}{y}$ if x and y are real numbers such that $x^2 + y^2 - 8x - 6y + 20 = 0$.

[6 ✎] **Problem 31** (HMMT February 2014) Given that a, b, and c are complex numbers satisfying

$$a^2 + ab + b^2 = 1 + i$$

$$b^2 + bc + c^2 = 2$$

$$c^2 + ca + a^2 = 1,$$

compute $(ab + bc + ca)^2$

§ 5 Geometry

§ 5.1 Coordinate Geometry

[2 ✎] **Problem 1** (CRMT Individuals 2019) Let S be the set of all distinct points in the coordinate plane that form an acute isosceles triangle with the points $(32, 33)$ and $(63, 63)$. Given that a line L crosses S a finite number of times, find the maximum number of times L can cross S.

Solution: Replace $(32, 33)$ and $(63, 63)$ by A and B. Then, we do casework on $AC = BC$ or $CB = AB$ or $CA = BA$. We get a line and two semicircles. A line can intersect a semicircle two times and a line one time.

[3 ✎] **Problem 2** (HMMT November 2013) Plot points A, B, C at coordinates (0, 0), (0, 1), and (1, 1) in the plane, respectively. Let S denote the union of the two line segments AB and BC. Let X1 be the area swept out when Bobby rotates S counterclockwise 45 degrees about point A. Let X2 be the area swept out when Calvin rotates S clockwise 45 degrees about point A. Find $\frac{X_1+X_2}{2}$

[5 ✎] **Problem 3** (SLKK AIME 2020) Mr. Duck draws points $A = (a, 0), B = (0, b), C = (3, 5)$ and $O = (0, 0)$ such that $a, b > 0$ and $\angle ACB = 45°$. If the maximum possible area of $\triangle AOB$ can be expressed as $m - n\sqrt{p}$ where m, n, and p are positive integers such that p is not divisible by the square of a prime, find $m + n + p$

§ 5.2 3D Geometry

[4 ✎] **Problem 4** (HMMT February 2014) Let C be a circle in the xy plane with radius 1 and center $(0,0,0)$, and let P be a point in space with coordinates $(3,4,8)$. Find the largest possible radius of a sphere that is contained entirely in the slanted cone with base C and vertex P.

§ 5.3 General

[1 ✎] **Problem 5** (MAθ 2018) A parallelograms has diagonals of length 10 and 20. Find the area inclosed by the circle inscribed in the parallelogram.

[1 ✎] **Problem 6** (TAMU 2019) An acute isosceles triangle ABC is inscribed in a circle. Through B and C, tangents to the circle are drawn, meeting at D. If $\angle ABC = 2\angle CDB$, then find the radian measure of $\angle BAC$.

[1 ✎] **Problem 7** (PHS PuMAC TST 2017) In triangle ABC, let D and E be the midpoints of BC and AC. Suppose AD and BE meet at F. If the area of $\triangle DEF$ is 50, then what is the area of $\triangle CDE$?

[2 ✎] **Problem 8** (TAMU 2019) Let AA_1 be an altitude of triangle $\triangle ABC$, and let A_2 be the midpoint of the side BC. Suppose that AA_1 and AA_2 divide angle $\angle BAC$ into three equal angles. Find the product of the angles of $\triangle ABC$ when the angles are expressed in degrees.

Solution: Let $\angle BAA_1 = \angle A_1AA_2 = \angle A_2AC = \alpha$. We have that $\triangle ABA_2$ is an isosceles triangle as $\angle ABA_1 = \angle AA_2A_1 = 90 - \alpha$. Then, as AA_1 is an altitude, $BA_1 = A_1A_2$. Let $BA_1 = A_1A_2 = x$, then $A_2C = BA_2 = 2x$. Consider triangles BAA_1 and $\triangle A_1AC$. We have that $\tan\alpha = \frac{x}{AA_1}$ and $\tan 2\alpha = \frac{3x}{AA_1}$. So, $\frac{\tan 2\alpha}{\tan\alpha} = 3 \implies \tan\alpha = \frac{1}{\sqrt{3}} \implies \alpha = 30$. So, $\angle BAC = 90$, $\angle ABC = 60$, and $\angle ACB = 30$. The product is $\boxed{162000}$.

[2 ✎] **Problem 9** (PHS PuMAC TST 2017) A trapezoid has area 32, and the sum of the lengths of its two bases and altitude is 16. If one of the diagonals is perpendicular to both bases, then what is the length of the other diagonal?

[3 ✎] **Problem 10** (AHSME 1984/28) Triangle ABC has area 10. Points D, E, and F, all distinct from A, B, and C, are on sides AB, BC, and CA, respectively, and AD = 2, DB = 3. Triangle ABE and quadrilateral DBEF have equal areas s. Find s.

[3 ✎] **Problem 11** (HMMT February 2014) In quadrilateral ABCD, $\angle DAC = 98, \angle DBC = 82, \angle BCD = 70$, and $BC = AD$. Find $\angle ACD$.

Solution: Reflect.

[3 ✎] **Problem 12** (HMMT November 2013) Let ABC be an isosceles triangle with $AB = AC$. Let D and E be the midpoints of segments AB and AC, respectively. Suppose that there exists a point F on ray $\overrightarrow{DE}$ outside of ABC such that triangle BFA is similar to triangle ABC. Compute $\frac{AB}{BC}$.

[4 ✎] **Problem 13** (Mandelbrot Nationals 2009) Triangle ABC has sides of length $AB = \sqrt{41}, AC = 5$,and $BC = 8$. Let O be the center of the circumcircle of $\triangle ABC$, and let A' be the point diametrically opposite A, as shown. Determine the area of $\triangle A'BC$.

[4✎] **Problem 14** (HMMT February 2014) Triangle ABC has sides $AB = 14, BC = 13$, and $CA = 15$. It is inscribed in circle, which has center O. Let M be the midpoint of AB, let B' be the point on diametrically opposite B, and let X be the intersection of AO and MB'. Find the length of AX.

Solution: AX is the centroid of ABB'.

[4✎] **Problem 15** (AIME 1989) Triangle ABC has an right angle at B and contains a point P such that $AP = 10$, $BP = 6$, and $\angle APC = \angle CPB = \angle BPA$. Find CP.

Solution: Law of Cosines and Pythagorean Theorem gives $CP = 33$.

[4✎] **Problem 16** (106 Geometry Problems) In triangle ABC, medians BB_1 and CC_1 are perep-ndicular. Given that $AC = 19$ and $AB = 22$, find BC.

Solution: Let $BG = 2x, GB_1 = x$ and $CG = 2y, GC_1 = y$. Set systems of equations and solve.

[4✎] **Problem 17** (PHS ARML TST 2017) An algorithm starts with an equilateral triangle A0B0C0 of side length 1. At step k, points A_k, B_k,and C_k are cosehn on line segments $B_{k-1}C_{k-1}, C_{k-1}A_{k-1}$ and $A_{k-1}B_{k-1}$ respectively, such that

$$B_{k-1}A_k : A_kC_{k-1} = 1 : 1$$
$$C_{k-1}B_k : B_kA_{k-1} = 1 : 2$$
$$A_{k-1}C_k : C_kB_{k-1} = 1 : 3$$

What is the value of the infinite series:

$$\sum_{i=0}^{\infty} \text{Area}[\triangle A_kB_kC_k]$$

[4✎] **Problem 18** (AIME 2005) In quadrilateral $ABCD$, let $BC = 8, CD = 12, AD = 10$ and $\angle A = \angle B = 60°$.

Solution: Extend AD and BC to make an equilateral triangle and then Law of Cosines.

[4✎] **Problem 19** (HMMT November 2013) Let ABC be a triangle and D a point on BC such that $AB = \sqrt{2}, BC = \sqrt{3}, \angle BAD = 30°$, and$\angle CAD = 45°$. Find AD.

[4✎] **Problem 20** (PHS HMMT TST 2020) △ ABC has side lengths $AB = 11, BC = 13, CA = 20$. A circle is drawn with diameter AC. Line AB intersects the circle at $D \neq A$, and line BC intersects the circle at $E \neq B$. Find the length of DE.

[6✎] **Problem 21** (SLKK AIME 2020) Cyclic quadrilateral $AXBY$ is inscribed in circle ω such that AB is a diameter of ω. M is the midpoint of XY and $AM = 13, BM = 5$, and $AB = 16$. If the area of AXBY can be expressed as $m\sqrt{p} + n$, where m, n, and p are positive integers such that m and n are relatively prime and p is not divisible by the square of a prime, find the remainder when $m + n + p$ is divided by 1000.

§ 6 Misc

These are problems that don't really fall into any other category at all.

§ 6.1 Games

[3✎] **Problem 1** (BmMT 2016) Suppose you have a 20×16 bar of chocolate squares. You want to break the bar into smaller chunks, so that after some sequence of breaks, no piece has an area of more than 5. What is the minimum possible number of times that you must break the bar?

§ 6.2 Logic

[5✎] **Problem 2** (Berkeley Math Circle 2013) Ten people sit side by side at a long table, all facing the same direction. Each of them is either a knight (and always tells the truth) or a knave (and always lies). Each of the people announces: "There are more knaves on my left than knights on my right." How many knaves are in the line?

§ 7 Associated Solutions

§ 7.1 Combinatorics

§ 7.2 Number Theory

§ 7.3 Algebra

§ 7.4 Geometry