Collected Problems: Computational

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§ 1 Introduction

I use the following scheme: 1 point is roughly AMC 10 8-14 level. 2 points is roughly AMC 10 # 15-17 level, 3 points are AMC 10 # 18-21 level, 4 points are AMC 10 # 22-23 level and 5 points are # 24-25 level. Furthermore, 6 points are #6-8 AIME level, 7 points are # 9-11 AIME level, 8 points are #12-13 AIME level, 9 points are #14-15 AIME level, and 10 points are a hypothetical #16-18 AIME level. 10 pointers are usually olympiad-style problems that require several major lemmas and have very long solutions.

Most of these problems are from more obscure contests that will serve as good AIME and AMC practice.

§ 2 Combinatorics

§ 2.1 Casework

[1] **Problem 1** (ARML 2014) Let A, B, and C be randomly chosen (not necessarily distinct) integers between 0 and 4 inclusive. Pat and Chris compute the value of $A + B \cdot C$ by two different methods. Pat follows the proper order of operations, computing $A + (B \cdot C)$. Chris ignores order of operations, choosing instead to compute $(A + B) \cdot C$. Compute the probability that Pat and Chris get the same answer.

[2] **Problem 2** (BmMT 2014) Call a positive integer top-heavy if at least half of its digits are in the set $\{7, 8, 9\}$. How many three digit top-heavy numbers exist? (No number can have a leading zero.)

[3] **Problem 3** (PuMAC 2019) Suppose Alan, Michael, Kevin, Igor, and Big Rahul are in a running race. It is given that exactly one pair of people tie (for example, two people both get second place), so that no other pair of people end in the same position. Each competitor has equal skill; this means that each outcome of the race, given that exactly two people tie, is equally likely. The probability that Big Rahul gets first place (either by himself or he ties for first) can be expressed in the form m/n, where m, n are relatively prime, positive integers. Compute $m + n$.

[3] **Problem 4** (2018 Memorial Day Mock AMC 10) In the xy-coordinate plane, three distinct points with integer x- and y- coordinates are randomly chosen within the area bounded by $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. What is the probability that the three points, when connected, form a triangle with an area of 1?

[3] **Problem 5** (PuMAC 2019) Prinstan Trollner and Dukejukem are competing at the game show WASS. Both players spin a wheel which chooses an integer from 1 to 50 uniformly at random, and this number becomes their score. Dukejukem then flips a weighted coin that lands heads with probability 3/5. If he flips heads, he adds 1 to his score. A player wins the game if their score is higher than the other player's score. The probability Dukejukem defeats the Trollner to win WASS equals m/n where m, n are co-prime positive integers. Compute $m + n$.

[3] **Problem 6** (Scrabber94 Mock AMC 10 2020) Robert writes all positive divisors of the number 216 on separate slips of paper, then places the slips into a hat. He randomly selects three slips from the hat, with replacement. What is the probability that the product of the numbers on the three slips Robert selects is a divisor of 216?

[4] **Problem 7** (Scrabber94 Mock AMC 10 2020) How many ways can six people of different heights stand in line such that for all $1 \leq k \leq 6$, the kth tallest person must stand next to either the (k + 1)th or (k - 1)th tallest person (or both)? In particular, the tallest person must stand next to the second tallest person, and the shortest person must stand next to the second shortest person.

[4] **Problem 8** (Purple Comet 2015 HS) Seven people of seven different ages are attending a meeting. The seven people leave the meeting one at a time in random order. Given that the youngest person leaves the meeting sometime before the oldest person leaves the meeting, the probability that the third, fourth, and fifth people to leave the meeting do so in order of their ages (youngest to oldest) is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

[4] **Problem 9** (Payback Mock AMC 10) Let T be the set of positive integer divisors of 27000. Find the number of positive integers that can be expressed as a product of 3 pairwise distinct elements of T.

[4] **Problem 10** (PersonPsychopath's 2nd 2015-2016 Mock AMC 10) A number is called elegant if, for all $0 \leq d \leq 9$, the digit d does not appear more than d times. For example, the number 3 cannot appear more than 3 times in an elegant number (and the digit 0 cannot appear at all). How many positive 4-digit elegant numbers exist? Some numbers to include are $1345, 5999, 6554$ and 7828.

[4] **Problem 11** (AIME I 2020/5) Six cards numbered 1 through 6 are to be lined up in a row. Find the number of arrangements of these six cards where one of the cards can be removed leaving the remaining five cards in either ascending or descending order.

[5] **Problem 12** (AMC 10A 2020) Jason rolls three fair standard six-sided dice. Then he looks at the rolls and chooses a subset of the dice (possibly empty, possibly all three dice) to reroll. After rerolling, he wins if and only if the sum of the numbers face up on the three dice is exactly 7. Jason always plays to optimize his chances of winning. What is the probability that he chooses to reroll exactly two of the dice?

[5] **Problem 13** (AMC 12B 2017/22) Abby, Bernardo, Carl, and Debra play a game in which each of them starts with four coins. The game consists of four rounds. In each round, four balls

are placed in an urn—one green, one red, and two white. The players each draw a ball at random without replacement. Whoever gets the green ball gives one coin to whoever gets the red ball. What is the probability that, at the end of the fourth round, each of the players has four coins?

[5] **Problem 14** (HMMT November 2013) Find the number of positive integer divisors of 12! that leave a remainder of 1 when divided by 3.

[6] **Problem 15** (HMMT November 2014) Consider the set of 5-tuples of positive integers at most 5. We say the tuple $(a_1, a_2, a_3, a_4, a_5)$ is perfect if for any distinct indices i, j, k, the three numbers a_i, a_j, a_k do not form an arithmetic progression (in any order). Find the number of perfect 5-tuples.

§ 2.1.1 PIE

[1] **Problem 16** How many integers from 1 to 100 (inclusive) are multiples of 2 or 3?

[1] **Problem 17** (AMC 10B 2017/13) There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

[8] **Problem 18** (SLKK AIME 2020) Andy the Banana Thief is trying to hide from Sheriff Buffkin in a row of 6 distinct houses labeled 1 through 6. Andy and Sheriff Buffkin each pick a permutation of the 6 houses, chosen uniformly at random. On the n^{th} day, with $1 \leq n \leq 6$, Andy and Sheriff Buffkin visit the n^{th} house in their respective permutations, and Andy is caught by the Sheriff on the first day they visit the same house. For example, if Andy's permutation is $1, 3, 4, 5, 6, 2$ and Sheriff Buffkin's permutation is $3, 4, 1, 5, 6, 2$, Andy is caught on day 4. Given that Sheriff Buffkin catches Andy within 6 days and the expected number of days it takes to catch Andy can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b, find the remainder when $a + b$ is divided by 1000.

§ 2.2 Perspectives

[1] **Problem 19** (AIME I 2002/1) Many states use a sequence of three letters followed by a sequence of three digits as their standard license-plate pattern. Given that each three-letter three-digit arrangement is equally likely, the probability that such a license plate will contain at least one palindrome (a three-letter arrangement or a three-digit arrangement that reads the same left-to-right as it does right-to-left) is $\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

[1] **Problem 20** (PuMAC 2019) How many ways can you arrange 3 Alice's, 1 Bob, 3 Chad's, and 1 David in a line if the Alice's are all indistinguishable, the Chad's are all indistinguishable, and Bob and David want to be adjacent to each other? (In other words, how many ways can you arrange 3 A's, 1 B, 3 C's, and 1 D in a row where the B and D are adjacent?)

[2] **Problem 21** (MAθ 2016) The product of any two of the elements of the set $\{30, 54, N\}$ is divisible by the third. Find the number of possible values of N.

[2✎] **Problem 22** (2017 AMC 10B/17) Call a positive integer *monotonous* if it is a one-digit number or its digits, when read from left to right, form either a strictly increasing or a strictly decreasing sequence. For example, 3, 23578, and 987620 are monotonous, but 88, 7434, and 23557 are not. How many monotonous positive integers are there?

[2✎] **Problem 23** (AIME II 2002/9) Let $\mathcal{S}$ be the set $\{1, 2, 3, \ldots, 10\}$ Let n be the number of sets of two non-empty disjoint subsets of $\mathcal{S}$. (Disjoint sets are defined as sets that have no common elements.) Find the remainder obtained when n is divided by 1000.

[2✎] **Problem 24** (AMC 10B 2018/22) Real numbers x and y are chosen independently and uniformly at random from the interval $[0, 1]$. What is the probability that x, y, and 1 are the side lengths of an obtuse triangle?

[2✎] **Problem 25** (AMC 10B 2020/23) Square $ABCD$ in the coordinate plane has vertices at the points $A(1,1), B(-1,1), C(-1,-1)$, and $D(1,-1)$. Consider the following four transformations:

- L, a rotation of $90°$ counterclockwise around the origin;
- R, a rotation of $90°$ clockwise around the origin;
- H, a reflection across the x-axis; and
- V, a reflection across the y-axis.

Each of these transformations maps the squares onto itself, but the positions of the labeled vertices will change. For example, applying R and then V would send the vertex A at $(1,1)$ to $(-1,-1)$ and would send the vertex B at $(-1,1)$ to itself. How many sequences of 20 transformations chosen from $\{L, R, H, V\}$ will send all of the labeled vertices back to their original positions? (For example, R, R, V, H is one sequence of 4 transformations that will send the vertices back to their original positions.)

[2✎] **Problem 26** (CNCM Online Round 1) Pooki Sooki has 8 hoodies, and he may wear any of them throughout a 7 day week. He changes hishoodie exactly 2 times during the week, and will only do so at one of the 6 midnights. Once he changes out of a hoodie, he never wears it for the rest of the week. The number of ways he can wear his hoodies throughout the week can be expressed as $\frac{8!}{2^k}$. Find k.

[3✎] **Problem 27** (BmMT 2014) If you roll three regular six-sided dice, what is the probability that the three numbers showing will form an arithmetic sequence? (The order of the dice does matter, but we count both (1, 3, 2) and (1, 2, 3) as arithmetic sequences.)

[4✎] **Problem 28** (2018 Memorial Day Mock AMC 10) How many integer values of n between 1 and 100, inclusive, cause the value of the following expression to be an integer?

$$\frac{\sqrt[2]{n^{3n}}}{\sqrt[3]{n^{2n}}}$$

[4✎] **Problem 29** (PuMAC 2019) Keith has 10 coins labeled 1 through 10, where the ith coin has weight 2^i. The coins are all fair, so the probability of flipping heads on any of the coins is $\frac{1}{2}$. After flipping all of the coins, Keith takes all of the coins which land heads and measures their total weight, W. If the probability that $137 \leq W \leq 1061$ is m/n for coprime positive integers m, n, determine $m + n$

[4✎] **Problem 30** (PHS HMMT TST 2016) Compute the number of ordered triples of sets (A_1, A_2, A_3) that satisfy the following:

1. $A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6\}$
2. $A_1 \cap A_2 \cap A_3 = \emptyset$

[4✎] **Problem 31** (Magic Math AMC 10 2020) The county of Tropolis has five towns, with no roads built between any two of them. How many ways are there for the mayor of Tropolis to build five roads between five different pairs of towns such that it is possible to get from any town to any other town using the roads?

[5✎] **Problem 32** (HMMT Feburary 2014) We have a calculator with two buttons that displays an integer x. Pressing the first button replaces x by $\lfloor \frac{x}{2} \rfloor$, and pressing the second button replaces x by $4x + 1$. Initially, the calculator displays 0. How many integers less than or equal to 2014 can be achieved through a sequence of arbitrary button presses? (It is permitted for the number displayed to exceed 2014 during the sequence. Here, byc denotes the greatest integer less than or equal to the real number y.)

[7✎] **Problem 33** (MMATHS 2019) Alice wishes to walk from the point (0, 0) to the point (6, 4) in increments of (1, 0) and (0, 1), and Bob wishes to walk from the point (0, 1) to the point (6, 5) in increments of (1, 0) and (0, 1). How many ways are there for Alice and Bob to get to their destinations if their paths never pass through the same point (even at different times)?

§ 2.2.1 Stars and Bars

[1✎] **Problem 34** (AMC 8 2019/25) Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each person (include Alice) has at least 2 apples?

[1✎] **Problem 35** (AMC 10A 2003/21) Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

[1✎] **Problem 36** (AMC 10A 2018/11) When 7 fair standard 6-sided dice are thrown, the probability that the sum of the numbers on the top faces is 10 can be written as

$$\frac{n}{6^7},$$

where n is a positive integer. What is n?

[4✎] **Problem 37** (AMC 12A 2006/25) How many non- empty subsets S of $\{1, 2, 3, \ldots, 15\}$ have the following two properties?

1. No two consecutive integers belong to S.
2. If S contains k elements, then S contains no number less than k.

[4✐] **Problem 38** (PersonPsychopath's 2nd 2015-2016 Mock AMC 10) Define an increasing sequence $a_1, a_2 \ldots a_k$ of integers to be n-true if it satis
es the following conditions for positive integers n and k:

- The difference between any two consecutive terms is less than n.
- The sequence must start with 0 and end with 10.

How many 5-true sequences exist?

§ 2.2.2 Expected Value

[3✐] **Problem 39** (Math Prizes For Girls 2018) Maryam has a fair tetrahedral die, with the four faces of the die labeled 1 through 4. At each step, she rolls the die and records which number is on the bottom face. She stops when the current number is greater than or equal to the previous number. (In particular, she takes at least two steps.) What is the expected number (average number) of steps that she takes? Express your answer as a fraction in simplest form.

[4✐] **Problem 40** (BMT 2020) Three lights are placed horizontally on a line on the ceiling. All the lights are initially on. Every second, Neil picks one of the three lights uniformly at random to switch: if it is off, he switches it on; if it is on, he switches it off. When a light is switched, any lights directly to the left or right of that light also get turned on (if they were off) or off (if they were on). The expected number of lights that are on after Neil has flipped switches three times can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$

[5✐] **Problem 41** (SLKK AIME 2020) Woulard forms a 8 letter word by picking each letter from the set $\{w, o, u\}$ with equal probability. The score of a word is the nonnegative difference between the number of distinct occurrences of the three-letter word "uwu" and the number of distinct occurrences of the three-letter word "owo". For example, the string "owowouwu" has a score of 2 - 1 = 1. If the expected score of Woulard's string can be expressed as $\frac{a}{b}$ for relatively prime positive integers a and b, find the remainder when $a + b$ is divided by 1000.

[6✐] **Problem 42** (PuMAC 2019) . Marko lives on the origin of the Cartesian plane. Every second, Marko moves 1 unit up with probability 2/9, 1 unit right with probability 2/9, 1 unit up and 1 unit right with probability 4/9, and he doesn't move with probability 1/9. After 2019 seconds, Marko ends up on the point (A, B). What is the expected value of A · B?

[6✐] **Problem 43** (PuMAC 2019) Kelvin and Quinn are collecting trading cards; there are 6 distinct cards that could appear in a pack. Each pack contains exactly one card, and each card is equally likely. Kelvin buys packs until he has at least one copy of every card, then he stops buying packs. If Quinn is missing exactly one card, the probability that Kelvin has at least two copies of the card Quinn is missing is expressible as m/n for coprime positive integers m, n. Determine $m + n$.

[7✐] **Problem 44** (OMO Spring 2019) The sum

$$\sum_{i=0}^{1000} \frac{\binom{1000}{i}}{\binom{2019}{i}}$$

can be expressed in the form $\frac{p}{q}$, where p and q are relatively prime positive integers. Compute $p+q$

[7✎] **Problem 45** (2017 AMC 12B/25) A set of n people participate in an online video basketball tournament. Each person may be a member of any number of 5-player teams, but no two teams may have exactly the same 5 members. The site statistics show a curious fact: The average, over all subsets of size 9 of the set of n participants, of the number of complete teams whose members are among those 9 people is equal to the reciprocal of the average, over all subsets of size 8 of the set of n participants, of the number of complete teams whose members are among those 8 people. How many values n, $9 \leq n \leq 2017$, can be the number of participants?

§ 2.2.3 Recursion

[3✎] **Problem 46** (MMATHS 2019) Noah has an old-style M&M machine. Each time he puts a coin into the machine, he is equally likely to get 1 M&M or 2 M&M's. He continues putting coins into the machine and collecting M&M's until he has at least 6 M&M's. What is the probability that he actually ends up with 7 M&M's?

[3✎] **Problem 47** (OMO Spring 2019) Susan is presented with six boxes $B_1, \cdots, B_6$, each of which is initially empty, and two identical coins of denomination 2^k for each $k = 0, \cdots, 5$. Compute the number of ways for Susan to place the coins in the boxes such that each box Bk contains coins of total value 2^k.

[4✎] **Problem 48** (Math Prizes For Girls 2019) A 1×5 rectangle is split into five unit squares (cells) numbered 1 through 5 from left to right. A frog starts at cell 1. Every second it jumps from its current cell to one of the adjacent cells. The frog makes exactly 14 jumps. How many paths can the frog take to finish at cell 5?

[4✎] **Problem 49** (OMO Spring 2019) When two distinct digits are randomly chosen in $N = 123456789$ and their places are swapped, one gets a new number N_0 (for example, if 2 and 4 are swapped, then $N_0 = 143256789$). The expected value of N_0 is equal to $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute the remainder when $m+n$ is divided by 10^6.

[7✎] **Problem 50** (CNCM Online Round 2) On a chessboard with 6 rows and 9 columns, the Slow Rook is placed in the bottom-left corner and the Blind King is placed on the top-left corner. Then, 8 Sleeping Pawns are placed such that no two Sleeping Pawns are in the same column, no Sleeping Pawn shares a row with the Slow Rook or the Blind King, and no Sleeping Pawn is in the rightmost column. The Slow Rook can move vertically or horizontally 1 tile at a time, the Slow Rook cannot move into any tile containing a Sleeping Pawn, and the Slow Rook takes the shortest path to reach the Blind King. How many ways are there to place the Sleeping Pawns such that the Slow Rook moves exactly 15 tiles to get to the space containing the Blind King?

§ 2.2.4 Burnside's

[2✎] **Problem 51** (AlcumusGuy Mock AMC 10 2016-2017) George wants to construct a bracelet with 2 identical red beads, 2 identical white beads, and 2 identical blue beads, spaced equally around a circle. How many different bracelets can George make if rotations and reflections of a bracelet are not distinct from each other?

§ 2.3 Sequences

[4] **Problem 52** (ARML 2009) For $k \geq 3$, we define an ordered k-tuple of real numbers $(x_1, x_2, ..., x_k)$ to be special if, for every i such that $1 \leq i \leq k$, the product $x_1 \cdot x_2 \cdot \ldots \cdot x_k = x_i^2$. Compute the smallest value of k such that there are at least 2009 distinct special k-tuples.

§ 2.4 Miscellaneous

[1] **Problem 53** (MMATHS 2019) An ant starts at the top vertex of a triangular pyramid (tetrahedron). Each day, the ant randomly chooses an adjacent vertex to move to. What is the probability that it is back at the top vertex after three days?

[1] **Problem 54** (PersonPsychopath's 2nd 2015-2016 Mock AMC 10) Ten different people are attending a party. Three of them have black shirts, four of them have red shirts, two people are wearing blue shirts, and only one person wears a white shirt. Each person then shakes hands with everyone wearing a shirt of different color than himself, and two people with the same shirt color do not shake hands. How many distinct handshakes take place?

[1] **Problem 55** (Mandelbrot Nationals Sample Test) Michael Jordan's probability of hitting any basketball shot is three times greater than mine, which never exceeds a third. To beat him in a game, I need to hit a shot myself and have Jordan miss the same shot. If I pick my shot optimally, what is the maximum probability of winning which I can attain?

[2] **Problem 56** (CNCM Online Round 2) Adi the Baller is shooting hoops, and makes a shot with probability p. He keeps shooting hoops until he misses. The value of p that maximizes the chance that he makes between 35 and 69 (inclusive) buckets can be expressed as $\frac{1}{\sqrt[a]{b}}$ for a prime a and positive integer b. Find $a + b$.

[2] **Problem 57** (PHS ARML TST 2017) Consider a group of eleven high school students. To create a middle school math contest, they must pick a four-person committee to write problems and a four-person committee to proofread. Every student can be on neither committee, one committee, or both committees, except for one student who does not want to be on both. How many combinations of committees are possible?

[2] **Problem 58** (Mandelbrot Regionals 2009) Mr. Strump has formed three person groups in his math class for working on projects. Every student is in exactly two groups, and any two groups have at most one person in common. In fact, if two groups are chosen at random then the probability that they have exactly one person in common is one-third. How many students are there in Mr. Strump's class?

[3] **Problem 59** (ARML 2014) Bobby, Peter, Greg, Cindy, Jan, and Marcia line up for ice cream. In an acceptable lineup, Greg is ahead of Peter, Peter is ahead of Bobby, Marcia is ahead of Jan, and Jan is ahead of Cindy. For example, the lineup with Greg in front, followed by Peter, Marcia, Jan, Cindy, and Bobby, in that order, is an acceptable lineup. Compute the number of acceptable lineups.

[3] **Problem 60** (AMC 12B 2017/17) A coin is biased in such a way that on each toss the probability of heads is $\frac{2}{3}$ and the probability of tails is $\frac{1}{3}$. The outcomes of the tosses are independent. A player has the choice of playing Game A or Game B. In Game A she tosses the coin three times

and wins if all three outcomes are the same. In Game B she tosses the coin four times and wins if both the outcomes of the first and second tosses are the same and the outcomes of the third and fourth tosses are the same. Find the probability of winning game A minus the probability of winning game B.

[3✎] **Problem 61** (ARML 2009) Some students in a gym class are wearing blue jerseys, and the rest are wearing red jerseys. There are exactly 25 ways to pick a team of three players that includes at least one player wearing each color. Compute the number of students in the class.

[3✎] **Problem 62** (MMATHS 2019) Erik wants to divide the integers 1 through 6 into nonempty sets A and B such that no (nonempty) sum of elements in A is a multiple of 7 and no (nonempty) sum of elements in B is a multiple of 7. How many ways can he do this? (Interchanging A and B counts as a different solution.)

[4✎] **Problem 63** (MMATHS 2019) How many ways are there to tile a 4×6 grid with L-shaped triominoes? (A triomino consists of three connected 1×1 squares not all in a line.

[4✎] **Problem 64** (MMATHS 2019) A subset of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ of size 3 is called special if whenever a and b are in the set, the remainder when $a + b$ is divided by 8 is not in the set. (a and b can be the same.) How many special subsets exist?

[5✎] **Problem 65** (BMT 2018) Suppose there are 2017 spies, each with $\frac{1}{2017}$ th of a secret code. They communicate by telephone; when two of them talk, they share all information they know with each other. What is the minimum number of telephone calls that are needed for all 2017 people to know all parts of the code

[6✎] **Problem 66** (CRMT Team 2019) A deck of the first 100 positive integers is randomly shuffled. Find the expected number of draws it takes to get a prime number if there is no replacement.

[6✎] **Problem 67** (CNCM PoTD) Find the remainder when $\sum_{n=0}^{333} \sum_{k=3n}^{999} \binom{k}{3n}$ is divided by 70.

§ 3 Number Theory

§ 3.1 Divisors

[1✎] **Problem 1** (ARML 2013) Compute the smallest positive integer n such that $n^2 + n^0 + n^1 + n^3$ is a multiple of 13.

[1✎] **Problem 2** (LMT Fall 2019) a and b are positive integers and $8^a 9^b$ has 578 factors. Find ab.

[1✎] **Problem 3** (MAθ 2018) How many distinct prime numbers are in the first 50 rows of Pascal's Triangle?

[2✎] **Problem 4** (AHSME 1984) How many triples (a, b, c) of positive integers satisfy the simultaneous equations:

$$ab + bc = 44$$

$$ac + bc = 23$$

[2✎] **Problem 5** (PHS ARML TST 2017) Compute the greatest prime factor of

$$3^8 + 2 \cdot 3^4 \cdot 4^4 + 2^{16}$$

[2✎] **Problem 6** (HMMT November 2014) Compute the greatest common divisor of $4^8 - 1$ and $8^{12} - 1$

[2✎] **Problem 7** (BMT 2018) . Suppose for some positive integers, that $\frac{p+\frac{1}{q}}{q+\frac{1}{p}} = 17$. What is the greatest integer n such that $\frac{p+q}{n}$ is always an integer?

[2✎] **Problem 8** (LMT Spring 2020) Let LMT represent a 3-digit positive integer where L and M are nonzero digits. Suppose that the 2-digit number MT divides LMT. Compute the difference between the maximum and minimum possible values of LMT .

[2✎] **Problem 9** (BMT 2018) How many multiples of 20 are also divisors of 17!?

[2✎] **Problem 10** (Scrabbler94 Mock AMC 10 2020) Let n be the smallest positive integer with the property that $\text{lcm}(n, 2020!) = 2021!$, where $\text{lcm}(a, b)$ denotes the least common multiple of a and b. How many positive factors does n have?

[2✎] **Problem 11** (NanoMath Fall Meet 2020) If $\phi(n)$ is the number of integers $1 \le p \le n$ such that p is relatively prime to n, then find the number of even values n between 1 and 100 inclusive for which $\phi(n) = \phi(\frac{n}{2})$

[2✎] **Problem 12** (LMT Spring 2020) Suppose there are n ordered pairs of positive integers (a_i, b_i) such that $a_i + b_i = 2020$ and a_i, b_i is a multiple of 2020, where $1 \le i \le n$. Compute the sum

$$\sum_{i=1}^{n} a_i + b_i.$$

[3✎] **Problem 13** (HMMT Feburary 2018) Distinct prime numbers p, q, r satisfy the equation

$$2pqr + 50pq = 7pqr + 55pr = 8pqr + 12qr = A$$

for some positive integer A. What is A?

[3✎] **Problem 14** (Math Prizes For Girls 2019) How many positive integers less than 4000 are not divisible by 2, not divisible by 3, not divisible by 5, and not divisible by 7?

[3✎] **Problem 15** (MAθ 2018) The number $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + 100 \cdot 100!$ ends with a string of 9s. How many consecutive 9s are at the end of the number?

[3✎] **Problem 16** (AMC 12B 2017/16) The number $21! = 51,090,942,171,709,440,000$ has over $60,000$ positive integer divisors. One of them is chosen at random. What is the probability that it is odd?

[3✎] **Problem 17** (MMATHS 2019) If n is an integer between 4 and 1000, what is the largest possible power of 2 that $n^4 - 13n^2 + 36$ could be divisible by?

[3✐] **Problem 18** (BMT 2015) There exists a unique pair of positive integers k, n such that k is divisible by 6, and $\sum_{i=1}^{k} i^2 = n^2$. Find (k, n).

[4✐] **Problem 19** (LMT Spring 2020) Let $\phi(k)$ denote the number of positive integers less than or equal to k that are relatively prime to k. For example, $\phi(2) = 1$ and $\phi(10) = 4$. Compute the number of positive integers $n \leq 2020$ such that $\phi(n^2) = 2\phi(n)^2$.

[4✐] **Problem 20** (Magic Math AMC 10 2020) Find the number of ordered pairs of positive integers (a, b) with $a < b < 2017$ such that $10a$ is divisible by b and $10b$ is divisible by a.

[4✐] **Problem 21** (HMMT Feburary 2016) For which integers $n \in \{1, 2, \ldots, 15\}$ is $n^n + 1$ a prime number?

[5✐] **Problem 22** (CNCM Online Round 1) Consider all possible pairs of positive integers (a, b) such that $a \geq b$ and both $\frac{a^2+b}{a-1}$ and $\frac{b^2+a}{b-1}$ are integers. Find the sum of all possible values of the product ab.

[6✐] **Problem 23** (HMMT Feburary 2017) Find all pairs (a, b) of positive integers such that $a^{2017}+b$ is a multiple of ab.

[7✐] **Problem 24** (HMMT Feburary 2017) . Kelvin the Frog was bored in math class one day, so he wrote all ordered triples (a, b, c) of positive integers such that $abc = 2310$ on a sheet of paper. Find the sum of all the integers he wrote down. In other words, compute

$$\sum_{\substack{abc=2310 \\ a,b,c\in\mathbb{N}}} (a + b + c),$$

where $\mathbb{N}$ denotes the set of positive integers.

§ 3.2 Modulo

[1✐] **Problem 25** (MAθ 2018) The number $4^{14} - 1$ is divisible by 29 but $2^{14} - 1$ is not. What is the remainder when $2^{14} - 1$ is divided by 29?

[1✐] **Problem 26** (AMC 12A 2003/18) Let n be a 5-digit number, and let q and r be the quotient and the remainder, respectively, when n is divided by 100. For how many values of n is $q + r$ divisible by 11?

[1✐] **Problem 27** (BMT 2018) Find the minimal N such that any N-element subset of {1, 2, 3, 4, . . . 7} has a subset S such that the sum of elements of S is divisible by 7.

[2✐] **Problem 28** (AMC 10B 2019/14) The base-ten representation for 19! is

$$121, 6T5, 100, 40M, 832, H00,$$

where T, M, and H denote digits that are not given. What is $T + M + H$?

[2✐] **Problem 29** (BMT 2018) What is the remainder when 201820182018 . . . [2018 times] is divided by 15?

[2✐] **Problem 30** (NEMO 2017) Let a, b, c be distinct integers from the set {3, 5, 7, 8}. What is the greatest possible value of the units digit of $a^{(b^c)}$?

[2✐] **Problem 31** (BMT 2020) Compute the remainder when 98! is divided by 101

[2✐] **Problem 32** (CNCM PoTD) Find the number of positive integer x less than 100 such that

$$3^x + 5^x + 7^x + 11^x + 13^x + 17^x + 19^x$$

is prime.

[3✐] **Problem 33** (AMC 10B 2018/16) Let $a_1, a_2, \ldots, a_{2018}$ be a strictly increasing sequence of positive integers such that

$$a_1 + a_2 + \cdots + a_{2018} = 2018^{2018}.$$

What is the remainder when $a_1^3 + a_2^3 + \cdots + a_{2018}^3$ is divided by 6?

[3✐] **Problem 34** (AMC 10A 2020/18) Let (a, b, c, d) be an ordered quadruple of not necessarily distinct integers, each one of them in the set $0, 1, 2, 3$. For how many such quadruples is it true that $a \cdot d - b \cdot c$ is odd? (For example, $(0, 3, 1, 1)$ is one such quadruple, because $0 \cdot 1 - 3 \cdot 1 = -3$ is odd.)

[3✐] **Problem 35** (CNCM Online Round 2) An ordered pair (n, p) is juicy if $n^2 \equiv 1 \pmod{p^2}$ and $n \equiv -1 \pmod{p}$ for positive integer n and odd prime p. How many juicy pairs exist such that $n, p \leq 200$?

[3✐] **Problem 36** (MMATHS 2018) For any prime number p, let S_p be the sum of all the positive divisors of $37^p p^{37}$ (including 1 and $37^p p^3 7$). Find the sum of all primes p such that S_p is divisible by p

[3✐] **Problem 37** (HMMT Feburary 2018) There are two prime numbers p so that $5p$ can be expressed in the form $\lfloor \frac{n^2}{5} \rfloor$ for some positive integer n. What is the sum of these two prime numbers?

[3✐] **Problem 38** (BMT 2019) Compute the remainder when the product of all positive integers less than and relatively prime to 2019 is divided by 2019.

[4✐] **Problem 39** (AMC 12B 2017/21) Last year Isabella took 7 math tests and received 7 different scores, each an integer between 91 and 100, inclusive. After each test she noticed that the average of her test scores was an integer. Her score on the seventh test was 95. What was her score on the sixth test?

[4✐] **Problem 40** (BMT 2018) If r_i are integers such that $0 \leq r_i < 31$ and r_i satisfies the polynomial $x^4 + x^3 + x^2 + x \equiv 30 \pmod{31}$, find

$$\sum_{i=1}^{4} (r_i^2 + 1)^{-1} \pmod{31}.$$

where x^{-1} is the modulo inverse of x, that is, it is the unique integer y such that $0 < y < 31$ and $xy - 1$ is divisible by 31.

[4✐] **Problem 41** (NEMO 2017) Find all positive integers $1 \leq k \leq 289$ such that $k^2 - 32$ is divisible by 289.

[4✐] **Problem 42** (SLLKK AIME 2020) Smush is a huge Kobe Bryant fan. Smush randomly draws n jerseys from his infinite collection of Kobe jerseys, each being either the recent #24 jersey or the throwback #8 jersey with equal probability. Let $p(n)$ be the probability that Smush can divide the n jerseys into two piles such that the sum of all jersey numbers in each pile is the same. If

[5] **Problem 43** (BMT 2018) Ankit wants to create a pseudo-random number generator using modular arithmetic. To do so he starts with a seed x_0 and a function $f(x) = 2x + 25 \pmod{31}$. To compute the kth pseudo random number, he calls $g(k)$ defined as follows:

$$g(k) = \begin{cases} x_0 & \text{if } k = 0 \\ f(g(k-1)) & \text{if } k > 0 \end{cases}$$

If x_0 is 2017, compute $\sum_{j=0}^{2017} g(j) \pmod{31}$.

[5] **Problem 44** (LMT Spring 2020) Compute the maximum integer value of k such that 2^k divides $3^{2n+3} + 40n - 27$ for any positive integer n.

[5] **Problem 45** (PMC Mock AMC 10 2020) For all positive intgers n, let $f(n)$ denote the remainder when $(n+1)^{(n+2)^{\cdots 2n}}$ is divided by n^2. Compute $f(101)$.

[6] **Problem 46** (PHS HMMT TST 2020) Find the largest integer $0 < n < 100$ such that $n^2 + 2n$ divides $4(n-1)! + n + 4$.

[7] **Problem 47** (NEMO 2017) Compute the remainder when $\binom{3^4}{3^2}$ is divided by 3^8.

[7] **Problem 48** (HMMT November 2014) Suppose that m and n are integers with $1 \leq m \leq 49$ and $n \geq 0$ such that m divides $n^{n+1} + 1$. What is the number of possible values of m?

[7] **Problem 49** (BMT 2018) How many $1 < n \leq 2018$ such that the set $\{0, 1, 1+2, \ldots, 1+2+3+\cdots+i, \ldots, 1+2+\cdots+n-1\}$ is a permutation of $\{0, 1, 2, 3, 4, \cdots, n-1\}$ when reduced modulo n?

[8] **Problem 50** (BMT 2018) Determine the number of ordered triples (a, b, c), with $0 \leq a, b, c \leq 10$ for which there exists (x, y) such that $ax^2 + by^2 \equiv c \pmod{11}$

[8] **Problem 51** (BMT 2018) Compute the following:

$$\sum_{i=0}^{99} (x^2 + 1)^{-1} \pmod{199}$$

where x^{-1} is the value $0 \leq y \leq 199$ such that $xy - 1$ is divisible by 199

[10] **Problem 52** (SLKK AIME 2020) : Let p = 991 be a prime. Let S be the set of all lattice points (x, y), with $1 \leq x, y \leq p - 1$. On each point (x, y) in S, Olivia writes the number $x^2 + y^2$. Let $f(x, y)$ denote the product of the numbers written on all points in S that share at least one coordinate with (x, y). Find the remainder when

$$\sum_{i=1}^{p-2} \sum_{j=1}^{p-2} f(i, j)$$

is divided by p.

§ 3.3 Bases

[4✎] **Problem 53** (HMMT November 2014) Mark and William are playing a game with a stored value. On his turn, a player may either multiply the stored value by 2 and add 1 or he may multiply the stored value by 4 and add 3. The first player to make the stored value exceed 2100 wins. The stored value starts at 1 and Mark goes first. Assuming both players play optimally, what is the maximum number of times that William can make a move? (By optimal play, we mean that on any turn the player selects the move which leads to the best possible outcome given that the opponent is also playing optimally. If both moves lead to the same outcome, the player selects one of them arbitrarily.)

[4✎] **Problem 54** (HMMT November 2013) How many of the first 1000 positive integers can be written as the sum of finitely many distinct numbers from the sequence $3^0, 3^1, 3^2 \cdots$?

[6✎] **Problem 55** (HMMT November 2014) For any positive integers a and b, define $a \oplus b$ to be the result when adding a to b in binary (base 2), neglecting any carry-overs. For example, $20 \oplus 14 = 101002 \oplus 11102 = 110102 = 26$. (The operation $\oplus$ is called the exclusive or.) Compute the sum

$$\sum_{k=0}^{2^{2014}-1} (k \oplus \lfloor \frac{k}{2} \rfloor)$$

[3✎] **Problem 56** (CNCM Online Round 1) Akshar is reading a 500 page book, with odd numbered pages on the left, and even numbered pages on the right. Multiple times in the book, the sum of the digits of the two opened pages are 18. Find the sum of the page numbers of the last time this occurs.

[3✎] **Problem 57** (CNCM Online Round 1) Define $S(N)$ to be the sum of the digits of N when it is written in base 10, and take $S^k(N) = S(S(...(N)...))$ with k applications of S. The stability of a number N is defined to be the smallest positive integer K where $S^K(N) = S^{K+1}(N) = S^{K+2}(N) = \ldots$. Let T_3 be the set of all natural numbers with stability 3. Compute the sum of the two least entries of T_3.

[6✎] **Problem 58** (Scrabbler94 Mock AMC 10 2020) How many ordered 11-tuples $(a_0, a_1, a_2, \ldots, a_{10})$ of integers satisfy the equation $a_0 + 2a_1 + 2^2a_2 + \ldots + 2^{10}a_{10} = 2020$ where $0 \le a_i \le 2$ for all $0 \le i \le 10$?

§ 3.4 Diophantine Equations

[4✎] **Problem 59** (2018 Memorial Day Mock AMC 10) q and r are positive integers in the following equation.

$$\frac{q}{r} + \frac{r}{q} = \frac{qr}{144}$$

There is only one possible value of $q+r$. Let $S(n)$ represent the sum of the digits of positive integer n. What is $S(q+r)$?

§ 3.5 Binomial Theorem

[2] **Problem 60** (BMT 2015) Compute the sum of the digits of 1001^{10}.

[3] **Problem 61** (LMT Fall 2019) Determine the remainder when $13^{2020} + 11^{2020}$ is divided by 144.

[3] **Problem 62** (NEMO 2017) Find the last two digits of $17^{(20^{17})}$ in base 9. (Here, the numbers given are in base 10.)

[7] **Problem 63** (AIME I 2020/12) Let n be the least positive integer for which $149^n - 2^n$ is divisible by $3^3 \cdot 5^5 \cdot 7^7$. Find the number of positive integer divisors of n.

§ 3.6 Series and Sequences

[1] **Problem 64** (ARML 2009) Let p be a prime number. If p years ago, the ages of three children formed a geometric sequence with a sum of p and a common ratio of 2, compute the sum of the children's current ages.

[3] **Problem 65** (ARML 2014) For each positive integer k, let S_k denote the infinite arithmetic sequence of integers with first term k and common difference k^2. For example, S_3 is the sequence $3, 12, 21, \ldots$ Compute the sum of all k such that 306 is an element of S_k

[4] **Problem 66** (ARML 2014) The arithmetic sequences $a_1, a_2, a_3, ..., a_{20}$ and $b_1, b_2, b_3, ..., b_{20}$ consist of 40 distinct positive integers, and $a_{20} + b_{14} = 1000$. Compute the least possible value for $b_{20} + a_{14}$.

§ 3.7 Miscellenous

[1] **Problem 67** (LMT Spring 2020) Compute the smallest nonnegative integer that can be written as the sum of 2020 distinct integers.

[2] **Problem 68** (PersonPsychopath's 2nd 2015-2016 Mock AMC 10) A positive integer has the property that the product of its digits is 10!. What is the minimum possible sum of its digits?

[2] **Problem 69** (Reun's Thanksgiving 2017 Mock AMC 10) De
ne the operation $a \circ b$ for positive integers as ab^a For some constant x, the pair (m, n) satis
es $m \circ n = n^{m+2}$ and $m + n = x$. Similarly, the pair (p, q) satis
es $p \circ q = q^{p+2}$ and $p + q = 3x$. If m, n, p and q are distinct, what is the smallest possible value of $mq + np$?

[2] **Problem 70** (OMO Spring 2019) Daniel chooses some distinct subsets of $\{1, ..., 2019\}$ such that any two distinct subsets chosen are disjoint. Compute the maximum possible number of subsets he can choose

[2] **Problem 71** (BMT 2018) How many integers can be expressed in the form: $\pm 1 \pm 2 \pm 3 \cdots \pm 2018$?

[3] **Problem 72** (BMT 2015) Find all integer solutions to

$$x^2 + 2y^2 + 3z^2 = 36$$

$$3x^2 + 2y^2 + z^2 = 84$$
$$xy + xz + yz = -7$$

[3✐] **Problem 73** (Magic Math AMC 10 2020) Let $s(n)$ denote the sum of the digits of a positive integer n. Find the number of three-digit positive integers $k \leq 500$ satisfying $s(k) = s(1000 - k)$.

[3✐] **Problem 74** (OMO Spring 2019) Jay is given 99 stacks of blocks, such that the ith stack has i^2 blocks. Jay must choose a positive integer N such that from each stack, he may take either 0 blocks or exactly N blocks. Compute the value Jay should choose for N in order to maximize the number of blocks he may take from the 99 stacks.

[Reun's Thanksgiving 2017 Mock AMC 10✐] **Problem 75** (Denote by $S(n)$ the sum of the digits of n: Find the sum of all positive integers $n < 100$ such that $S(n^2) = (S(n))^2$)

[4✐] **Problem 76** (BMT 2019) For a positive integer n, define $\phi(n)$ as the number of positive integers less than or equal to n that are relatively prime to n. Find the sum of all positive integers n such that $\phi(n) = 20$

[4✐] **Problem 77** (DMC Mock AMC 10) Joy picks an integer n from the interval $[1, 40]$. She tells Amy the remainder when n is divided by 7 and Sid the number of divisors of n. Amy and Sid both know n is in the interval $[1, 40]$, but they get confused and believe Amy was told the number of divisors and Sid was told the remainder. Amy says, "I know what n is." Sid replies, "If so, then I also know what n is." As it turns out, they thought of the same value but were wrong due to their confusion. If Amy and Sid tell the truth based on their beliefs and can reason perfectly, what is the sum of all possible actual values of n?

[5✐] **Problem 78** (CNCM Online Round 2) Let S be the set of all ordered pairs (x, y) of integer solutions to the equation
$$6x^2 + y^2 + 6x = 3xy + 6y + x^2y.$$
S contains a unique ordered pair (a, b) with a maximal value of b. Compute $a + b$.

[5✐] **Problem 79** (PHS HMMT TST 2020) Find the unique triplet of integers (a, b, c) with $a > b > c$ such that $a + b + c = 65$ and $a^2 + b^2 + c^2 = 3083$.

[5✐] **Problem 80** (BMT 2019) 0. Let S(n) be the sum of the squares of the positive integers less than and coprime to n. For example, $S(5) = 1^2 + 2^2 + 3^2 + 4^2$, but $S(4) = 1^2 + 3^2$. Let $p = 2^7 - 1 = 127$ and $q = 2^5 - 1 = 31$ be primes. The quantity $S(pq)$ can be written in the form
$$\frac{p^2q^2}{6}(a - \frac{b}{c})$$
where $a, b,$ and c are positive integers, with b and c coprime and $b < c$. Find a.

[6✐] **Problem 81** (CNCM PoTD) How many positive integers k are there such that $101 \leq k \leq 10000$ and $\lfloor\sqrt{k - 100}\rfloor$ is a divisor of k?

[6✐] **Problem 82** (Compute the number of ordered triples of positive integers (a, b, c) such that $a + b + c + ab + bc + ac = abc + 1$)

§ 4 Algebra

§ 4.1 Polynomials

Generally uses the following techniques: Vieta's, Binomial Theorem, Multinomial Theorem, Remainder Theorem, Newton's Sums, Reciprocal Roots Trick, Quadratic Formula (including using Determinant), Finite Differences, $x + \frac{k}{x}$ subsitution, Polynomial Interpolation

[1] **Problem 1** (Scrabber94 Mock AMC 10 2020) Let $f(x) = |x + 4|$ and $g(x) = x^2$ for all real numbers x. How many real numbers x satisfy $f(g(x)) = g(f(x))$?

[1] **Problem 2** (CRMT Math Bowl 2019) Find the sum of all real numbers such that

$$\sqrt[4]{16x^4 - 32x^3 + 24x^2 - 8x + 1} = 5$$

[2] **Problem 3** (TAMU 2019) In the expansion of $(1 + ax - x^2)^8$ where a is a positive constant, the coefficient of x^2 is 244. Find the value of a

[2] **Problem 4** (BMT 2020) Let a and b be the roots of the polynomial $x^2 + 2020x + c$. Given that $\frac{a}{b} + \frac{b}{a} = 98$, compute $\sqrt{c}$.

[3] **Problem 5** (HMMT November 2014) Let $f(x) = x^2 + 6x + 7$. Determine the smallest possible value of $f(f(f(f(x))))$ over all real numbers x.

[3] **Problem 6** (HMMT February 2014) Find the sum of all real numbers x such that $5x^4 + 10x^3 + 10x^2 + 5x + 11 = 0$

[3] **Problem 7** (BmMT 2014) Consider the graph of $f(x) = x^3 + x + 2014$. A line intersects this cubic at three points, two of which have x-coordinates 20 and 14. Find the x-coordinate of the third intersection point

[3] **Problem 8** (ARML 2017) Compute the number of ordered pairs of integers (a, b) such that the polynomials $x^2 - ax + 24$ and $x^2 - bx + 36$ have one root in common

[3] **Problem 9** (Reun's Thanksgiving 2017 Mock AMC 10) For integers a and b, the three roots X_1, X_2 and X_3 of the cubic function

$$f(x) = ax^3 - bx^2 + 286x - 120$$

exist such that $X_i = \frac{m+i-1}{m+i-2}$ for some positive integer constant m What is the sum of the digits of b?

[3] **Problem 10** (AMC 10A 2015/23) The zeroes of the function $f(x) = x^2 - ax + 2a$ are integers. What is the sum of the possible values of a?

[4] **Problem 11** (Payback Mock AMC 10) Find the number of pairs of integers (a, c) such that the function $f(x) = x^2 - ax + ac$ has two integer roots(not necessarily distinct) and $1 \le c \le 5$.

[4] **Problem 12** (2018 Memorial Day Mock AMC 10) The quadratic equations $y = ax^2 + 20x + 32$ and $y = x^2 + bx + 16$ share the same vertex in the xy-coordinate plane. If a and b are integers, what is $a + b$?

[4 ✎] **Problem 13** (HMMT Feburary 2017) Let $Q(x) = a_0 + a_1x + \ + a_nx^n$ be a polynomial with integer coefficients, and $0 \leq ai < 3$ for all $0 \leq i \leq n$. Given that $Q(\sqrt{3}) = 20 + 17\sqrt{3}$, compute $Q(2)$

[4 ✎] **Problem 14** (TAMU 2018) Suppose f is a cubic polynomial with roots a, b, c such that

$$a = \frac{1}{3 - bc}$$

$$b = \frac{1}{5 - ac}$$

$$c = \frac{1}{7 - ab}$$

If $f(0) = 1$, find $f(abc + 1)$.

[4 ✎] **Problem 15** (PuMAC 2019) Let $f(x) = x^2 + 4x + 2$. Let r be the difference between the largest and smallest real solutions of the equation $f(f(f(f(x)))) = 0$. Then $r = a^{\frac{p}{q}}$ for some positive integers a, p, q so a is square-free and p, q are relatively prime positive integers. Compute $a + p + q$

[4 ✎] **Problem 16** (HMMT February 2014) Find all real numbers k such that $r^4 + kr^3 + r^2 + 4kr + 16 = 0$ is true for exactly one real number r.

[4 ✎] **Problem 17** (Scrabbler94 Mock AMC 10 2020) Let $f^1(x) = x^2 - 20$ for all real numbers x, and let $f^k(x) = f^1(f^{k-1}(x))$ for all integers $k \geq 2$. Let x_0 and x_1 be the smallest and largest real solutions to the equation $f^{2020}(x) = 0$, respectively. What is the largest integer less than or equal to $x_0^2 + x_1^2$?

[4 ✎] **Problem 18** (PHS HMMT TST 2020) Let a, b, c be the distinct real roots of $x^3 + 2x + 5$. Find $(8 - a^3)(8 - b^3)(8 - c^3)$.

[4 ✎] **Problem 19** (PuMAC 2019) Let Q be a quadratic polynomial. If the sum of the roots of $Q^{100}(x)$ (where $Q^i(x)$ is defined by $Q^1(x) = Q(x), Q^i(x) = Q(Q^{i-1}(x))$ for integers $i \geq 2$) is 8 and the sum of the roots of Q is S, compute $|\log_2(S)|$.

[7 ✎] **Problem 20** (HMMT Feburary 2017) A polynomial P of degree 2015 satisfies the equation $P(n) = \frac{1}{n^2}$ for $n = 1, 2, \ldots, 2016$. Find $\lfloor 2017P(2017) \rfloor$

[7 ✎] **Problem 21** (AIME I 2014/14) Let m be the largest real solution to the equation

$$\frac{3}{x-3} + \frac{5}{x-5} + \frac{17}{x-17} + \frac{19}{x-19} = x^2 - 11x - 4$$

There are positive integers a, b, and c such that $m = a + \sqrt{b + \sqrt{c}}$. Find $a + b + c$.

[8 ✎] **Problem 22** (AoPS Forums) Given x_1, x_2, x_3, x_4 are the roots of $P(x) = 2x^4 - 5x + 1$, find the value of $\sum_{i=1}^{4} \frac{1}{(1-x_i)^3}$.

§ 4.1.1 Newton's Sums

[3 ✎] **Problem 23** (AoPS Forums) Let a_1, a_2, a_3 be the roots to the polynomial $x^3 - 2x^2 - 3x + 16$; find

$$a_1^3(a_2 + a_3) + a_2^3(a_1 + a_3) + a_3^3(a_1 + a_2)$$

[3] **Problem 24** (BMT 2015) Let r, s, and t be the three roots of the equation $8x^3+1001x+2008 = 0$. Find $(r+s)^3 + (s+t)^3 + (t+r)^3$

[3] **Problem 25** (NanoMath Fall Meet 2020) If $x + y = 6$ and $x^3 + y^3 = 108$, find $x^5 + y^5$.

[4] **Problem 26** (BMT 2019) Let r_1, r_2, r_3 be the (possibly complex) roots of the polynomial $x^3 + ax^2 + bx + \frac{4}{3}$. How many pairs of integers a, b exist such that $r_1^3 + r_2^3 + r_3^3 = 0$?

[6] **Problem 27** (SLKK AIME 2020) Let a, b, and c be the three distinct solutions to $x^3 - 4x^2 + 5x + 1 = 0$. Find

$$(a^3 + b^3)(a^3 + c^3)(b^3 + c^3).$$

§ 4.1.2 Roots of Unity

[2] **Problem 28** (AMC 12B 2017/12) What is the sum of the roots of $z^{12} = 64$ that have a positive real part?

[5] **Problem 29** (BMT 2019) Let a_n be the product of the complex roots of $x^{2n} = 1$ that are in the first quadrant of the complex plane. That is, roots of the form $a + bi$ where $a, b > 0$. Let $r = a_1 \cdot a_2 \cdot \ldots \cdot a_{10}$. Find the smallest integer k such that r is a root of $x^k = 1$

[6] **Problem 30** (BMT 2015) Evaluate $\sum_{k=0}^{37} (-1)^k \binom{75}{2k}$.

§ 4.1.3 Polynomial Interpolation

[3] **Problem 31** (BMT 2020) The graph of the degree 2021 polynomial $P(x)$, which has real coefficients and leading coefficient 1, meets the x-axis at the points $(1, 0), (2, 0), (3, 0), \ldots, (2020, 0)$ and nowhere else. The mean of all possible values of $P(2021)$ can be written in the form $\frac{a!}{b}$, where a and $b$are positive integers and a is as small as possible. Compute $a + b$.

[6] **Problem 32** (AMC 12B 2017/23) The graph of $y = f(x)$, where $f(x)$ is a polynomial of degree 3, contains points $A(2, 4)$, $B(3, 9)$, and $C(4, 16)$. Lines AB, AC, and BC intersect the graph again at points D, E, and F, respectively, and the sum of the x-coordinates of D, E, and F is 24. What is $f(0)$?

§ 4.1.4 Generating Functions

[7] **Problem 33** (BMT 2018) Find the value of

$$\frac{1}{\sqrt{2}} + \frac{2^2}{\sqrt{4}} + \frac{3^2}{\sqrt{8}} \cdots$$

[7] **Problem 34** (BMT 2018) Let $F_1 = 0, F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$. Compute

$$\sum_{n=1}^{\infty} \frac{\sum_{i=1}^{n} F_i}{3^n}$$

§ 4.2 Complex Numbers

[4] **Problem 35** (HMMT Feburary 2016) Let z be a complex number such that $|z| = 1$ and $|z - 1.45| = 1.05$. Compute the real part of z.

[4] **Problem 36** (2019 AMC 12B) How many nonzero complex numbers z have the property that $0, z$, and z^3, when represented by points in the complex plane, are the three distinct vertices of an equilateral triangle?

[7] **Problem 37** (hukilau17 FAIME 2020) If x, y are real numbers such that

$$x^3 = 3xy^2 + 18$$

$$y^3 = 3x^2y + 26$$

Find $x^2 + y^2$.

[1] **Problem 38** (2018 AMC 10A/10) Suppose that real number x satisfies

$$\sqrt{49 - x^2} - \sqrt{25 - x^2} = 3.$$

What is the value of $\sqrt{49 - x^2} + \sqrt{25 - x^2}$?

[2] **Problem 39** (2018 BmMT) Let x be a positive real number so that $x - \frac{1}{x} = 1$. Compute $x^8 - \frac{1}{x^8}$.

[2] **Problem 40** (LMT Spring 2020) Let a,b be real numbers satisfying $a^2 + b^2 = 3ab = 75$ and $a > b$. Compute $a^3 - b^3$.

[2] **Problem 41** (HMMT November 2013) Evaluate

$$\cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cdots \cfrac{1}{2 - \frac{1}{2}}}}}$$

where the digit 2 appears 2013 times

[2] **Problem 42** (Mandelbrot) If $\frac{x^2}{y^2} = \frac{8y}{x} = z$, find the sum of all possible z.

[2] **Problem 43** (AMC 10A 2018/14) What is the greatest integer less than or equal to

$$\frac{3^{100} + 2^{100}}{3^{96} + 2^{96}}?$$

[3] **Problem 44** (BMT 2016) Simplify $\frac{1}{\sqrt[3]{81} + \sqrt[3]{72} + \sqrt[3]{64}}$.

[3] **Problem 45** (MAθ 2018) The solutions to $3\sqrt{2x^2 - 5x - 3} + 2x^2 - 5x = 7$ can be written in the form $x = \frac{a \pm \sqrt{b}}{c}$ where a, b, c are positive integers and x is in simplest form. Find $a + b + c$.

[3✎] **Problem 46** (Mandelbrot Nationals 2008) Find the positive real number x for which $5\sqrt{1-x}+5\sqrt{1+x}=7\sqrt{2}$.

[4✎] **Problem 47** (NEMO 2019) Suppose x and y are positive real numbers satisfying

$$\sqrt{xy}=x-y=\frac{1}{x+y}=k$$

Determine k.

[4✎] **Problem 48** (2014 November HMMT) Let a,b,c,x be reals with $(a+b)(b+c)(c+a)\neq 0$ that satisfy

$$\frac{a^2}{a+b}=\frac{a^2}{a+c}+20, \frac{b^2}{b+c}=\frac{b^2}{b+a}+14, \text{ and } \frac{c^2}{c+a}=\frac{c^2}{c+b}+x$$

Compute x.

[5✎] **Problem 49** (Magic Math AMC 10 2020) Determine the remainder when $\sum(-1)^{m+n}mn$ is divided by 1009, where the sum is taken over all pairs of integers (m,n)satisfying $1\leq m<n\leq 2020$

[5✎] **Problem 50** (Math Prizes For Girls 2015) Let S be the sum of all distinct real solutions of the equation

$$\sqrt{x+2015}=x^2-2015.$$

Compute $\lfloor 1/S\rfloor$. Recall that if r is a real number, then $\lfloor r\rfloor$ (the floor of r) is the greatest integer that is less than or equal to r

[7✎] **Problem 51** (BMT 2020) Let a,b and c be real numbers such that $a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ and $abc=5$. The value of

$$(a-\frac{1}{b})^3+(b-\frac{1}{c})^3+(c-\frac{1}{a})^3$$

can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integeres. Compute $m+n$.

§ 4.3 Series and Sequences

[1✎] **Problem 52** (djmathman Mock AMC 2013/9) Let p and q be numbers with $|p|<1$ and $|q|<1$ such that

$$p+pq+pq^2+pq^3+\cdots=2 \quad\text{and}\quad q+qp+qp^2+\cdots=3.$$

What is $100pq$?

[1✎] **Problem 53** (PHS HMMT TST 2020) What is the value of $\frac{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\cdots}{\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\cdots}$? Remember that $\frac{1}{1^2}+\frac{1}{2^2}\cdots=\frac{\pi^2}{6}$

[1✎] **Problem 54** (OMO Spring 2019) Compute

$$\lfloor\sum_{k=2018}^{\infty}\frac{2019!-2018!}{k!}\rfloor.$$

(The notation $\lfloor x\rfloor$ denotes the least integer n such that $n\geq x$.)

[1] **Problem 55** (NEMO 2017) I am thinking of a geometric sequence with 9600 terms, $a_1, a_2, \dots, a_{9600}$. The sum of the terms with indices divisible by three (i.e. $a_3 + a_6 + \cdots + a_{9600}$) is $\frac{1}{56}$ times the sum of the other terms (i.e. $a_1 + a_2 + a_4 + a_5 + \;+ a_{9598} + a_{9599}$). Given that the terms with even indices sum to 10, what is the smallest possible sum of the whole sequence?

[3] **Problem 56** (NanoMath Fall Meet 2020) Let $a_0, a_1, a_2, \dots$ be an arithmetic sequence of positive integers. If $a_0 + a_1 + \cdots + a_{10} = 209$ and $a_{a_0} + a_{a_1} + \cdots + a_{a_{10}} = 671$, then find a_0.

[3] **Problem 57** (BMT 2015) Let $\{a_n\}$ be a sequence of real numbers with $a_1 = -1, a_2 = 2$ and for all $n \geq 3$, $a_{n+1} - a_n - a_{n+2} = 0$. Find $a_1 + a_2 + a_3 + ... + a_{2015}$.

[3] **Problem 58** (MMATHS 2019) Let $F_1 = F_2 = 1$, and let $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$. For each positive integer n, let $g(n)$ be the minimum possible value of $|a_1F_1 + a_2F_2 + \;+ a_nF_n|$, where each a_i is either 1 or -1. Find $g(1) + g(2) + \cdots + g(100)$

[3] **Problem 59** (BMT 2020) Let ϕ be the positive solution to the equation $x^2 = x + 1$. For $n \geq 0$, let an be the unique integer such that $\phi^n - a_n\phi$ is also an integer. Compute

$$\sum_{i=0}^{10} a_n$$

[3] **Problem 60** (Reun's Thanksgiving Mock AMC 10) Let $N = 3 + 66 + 333 + 6666 + 33333 + \dots + \underbrace{666\cdots666}_{2016\text{ 6's}} + \underbrace{333\cdots333}_{2017\text{ 3's}}$. Find the sum of the digits of N

[3] **Problem 61** (2018 AMC 12A/18) Let A be the set of positive integers that have no prime factors other than 2, 3, or 5. The infinite sum

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{12} + \frac{1}{15} + \frac{1}{16} + \frac{1}{18} + \frac{1}{20} + \cdots$$

of the reciprocals of the elements of A can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

[4] **Problem 62** (Math Prizes For Girls 2019) For each integer from 1 through 2019, Tala calculated the product of its digits. Compute the sum of all 2019 of Tala's products.

[4] **Problem 63** (HMMT Feburary 2017) Find the value of

$$\sum_{1 \leq a < b < c} \frac{1}{2^a 3^b 5^c}$$

(i.e the sum of $\frac{1}{2^a 3^b 5^c}$ over all triples of positive integers (a, b, c) satisfying $a < b < c$)

[5] **Problem 64** (MMATHS 2018) Compute

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m\cos^2(n) + n\sin^2(m)}{3^{m+n}(m+n)}$$

[5 ✐] **Problem 65** (HMMT Feburary 2016) Let A denote the set of all integers n such that $1 \leq n \leq 10000$, and moreover the sum of the decimal digits of n is 2. Find the sum of the squares of the elements of A.

[5 ✐] **Problem 66** (HMMT Feburary 2016) Determine the remainder when

$$\sum_{i=0}^{2015} \lfloor \frac{2^i}{25} \rfloor$$

is divided by 100, where $\lfloor x \rfloor$ denotes the largest integer not greater than x.

§ 4.3.1 Telescoping

[3 ✐] **Problem 67** (Purple Comet 2015 HS)

$$\left(1+\frac{1}{1+2^1}\right)\left(1+\frac{1}{1+2^2}\right)\left(1+\frac{1}{1+2^3}\right)\cdots\left(1+\frac{1}{1+2^{10}}\right)=\frac{m}{n},$$

where m and n are relatively prime positive integers. Find $m+n$.

[5 ✐] **Problem 68** (BMT 2020) Given that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, the value of

$$\sum_{n=3}^{10} \frac{\binom{n}{2}}{\binom{n}{3}\binom{n+1}{3}}$$

can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$

[5 ✐] **Problem 69** (MMATHS 2020) Let $p(x)$ be the monic cubic polynomial with roots $\sin^2(1^\circ)$, $\sin^2(3^\circ)$, and $\sin^2(9^\circ)$. Suppose that $p\left(\frac{1}{4}\right) = \frac{\sin(a^\circ)}{n\sin(b^\circ)}$, where $0 < a, b \leq 90$ and a, b, n are positive integers. What is $a+b+n$?

§ 4.4 Trigonometry

[2 ✐] **Problem 70** (ARML Local 2020) A real number x is selected uniformly at random between 0 and π. Compute the probability that $\sin(x)\cos(x) < \frac{1}{4}$

[4 ✐] **Problem 71** (MAθ 1992) If A and B are both in $[0, 2\pi)$ and A and B satisfy the equations

$$\sin A + \sin B = \frac{1}{3}$$

$$\cos A + \cos B = \frac{4}{3}$$

find $\cos(A-B)$

[4 ✐] **Problem 72** (TAMU 2019) Simplify $\arctan\frac{1}{1+1+1^2} + \arctan\frac{1}{1+2+2^2} + \arctan\frac{1}{1+3+3^2}\cdots + \arctan\frac{1}{1+n+n^2}$

[6✐] **Problem 73** (Purple Comet 2015 HS) Let x be a real number between 0 and $\frac{\pi}{2}$ for which the function $3\sin^2 x + 8\sin x\cos x + 9\cos^2 x$ obtains its maximum value, M. Find the value of $M + 100\cos^2 x$.

[6✐] **Problem 74** (ARML 2009) If $6\tan^{-1} x + 4\tan^{-1}(3x) = \pi$, compute x^2

§ 4.5 Logarithms

[2✐] **Problem 75** (AMC 12A 2018/14) The solutions to the equation $\log_{3x} 4 = \log_{2x} 8$, where x is a positive real number other than $\frac{1}{3}$ or $\frac{1}{2}$, can be written as $\frac{p}{q}$ where p and q are relatively prime positive integers. What is $p + q$?

[2✐] **Problem 76** (MAθ 1991) Given that $\log_{10} 2 \approx 0.3010$, how many digits are in 5^{44}?

[3✐] **Problem 77** (ARML 2009) Compute all real values of x such that $\log_2(\log_2 x) = \log_4(\log_4 x)$.

[3✐] **Problem 78** (PuMAC 2019) If x is a real number so $3^x = 27x$, compute $\log_3(\frac{3^{3^x}}{x^{3^3}})$.

[3✐] **Problem 79** (AMC 12B 2017/20) Real numbers x and y are chosen independently and uniformly at random from the interval $(0, 1)$. What is the probability that $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor$?

[4✐] **Problem 80** (MAθ 1991) Suppose a and b are positive numbers for which $\log_{4n} 40\sqrt{3} = \log_{3n} 45$, find n^3

[4✐] **Problem 81** (MAθ 1992) Suppose a and b are positive numbers for which

$$\log_9 a = \log_{15} b = \log_{25}(a + 2b)$$

What is the value of $\frac{b}{a}$?

[4✐] **Problem 82** (PHS ARML TST 2017) Positive real numbers x, y, and z satisfy the following system of equations:

$$x^{\log(yz)} = 100$$
$$y^{\log(xz)} = 10$$
$$z^{\log(xy)} = 10\sqrt{10}$$

Compute the value of the expression $(\log(xyz))^2$

[4✐] **Problem 83** If $60^a = 3$ and $60^b = 5$, then find $12^{\frac{1-a-b}{2-2b}}$.

[4✐] **Problem 84** (NanoMath Fall Meet 2020) Let a be a positive integer. If

$$(\log_{(\log_2(a^{\log_2 a^4}))} a)(\log_2 a^{\log_4 a^{\frac{3}{2}}}) = 8$$

, what is the value of a?

[5✐] **Problem 85** (SLKK AIME 2020) Let x be a real number in the interval $(0, \frac{\pi}{2})$ such that $\log_{\sin^2(x)} \cos(x) + \log_{\cos^2(x)} \sin(x) = \frac{5}{4}$. If $\sin^2(2x)$ can be expressed as $m\sqrt{n} - p$, where m, n, and p are positive integers such that n is not divisible by the square of a prime, find $m + n + p$

§ 4.6 Distance, Rate, Work

[2✐] **Problem 86** (AlcumusGuy Mock AMC 10 2016-2017) Two cars simultaneously leave Lincoln at noon along the same route, with the
first traveling at a constant rate of 50 miles per hour, and the second at a constant rate of 40 miles per hour. A third car leaves Lincoln half an hour later, traveling along the same route at constant speed s miles per hour. It passes the
first car t minutes after noon, which was precisely an hour and a half after passing the second car. What is $s + t$?

§ 4.7 Arbitary Functions

[2✐] **Problem 87** (LMT Spring 2020) A function $f(x)$ is such that for any integer x, $f(x)+xf(2-x) = 6$. Compute $-2019f(2020)$.

[3✐] **Problem 88** (Mandelbrot Nationals 2009) Let $f(x)$ be a function defined for all positive real numbers satisfying the conditions $f(x) > 0$ for all $x > 0$ and $f(x - y) = \sqrt{f(xy) + 1}$ for all $x > y > 0$. Determine $f(2009)$.

[4✐] **Problem 89** (BMT 2020) Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function such that for all $x, y \in \mathbb{R}^+$, $f(x)f(y) = f(xy) + f(\frac{x}{y})$ where $\mathbb{R}^+$ represents the positive real numbers. Given that $f(2) = 3$, compute the last two digits of $f(2^{2^{2020}})$.

[5✐] **Problem 90** (HMMT Feburary 2017) Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(x)f(y) = f(x - y)$. Find all possible values of $f(2017)$.

[7✐] **Problem 91** (MMATHS 2019) The continuous function $f(x)$ satisfies $9f(x + y) = f(x)f(y)$ for all real numbers x and y. If $f(1) = 3$, what is $f(-3)$?

§ 4.8 Inequalities

[1✐] **Problem 92** (PuMAC 2019) Let a, b be positive integers such that $a + b = 10$. Let $\frac{p}{q}$ be the difference between the maximum and minimum possible values of $\frac{1}{a} + \frac{1}{b}$, where p and q are relatively prime positive integers. Compute $p + q$.

[2✐] **Problem 93** (Math Prizes For Girls 2019) The degree measures of the six interior angles of a convex hexagon form an arithmetic sequence (not necessarily in cyclic order). The common difference of this arithmetic sequence can be any real number in the open interval $(-D, D)$. Compute the greatest possible value of D.

[3✐] **Problem 94** (Math Prizes For Girls 2019) Find the least real number K such that for all real numbers x and y, we have $(1 + 20x^2)(1 + 19y^2) \geq Kxy$. Express your answer in simplified radical form.

[3✐] **Problem 95** (PMC Mock AMC 10 2020) An infinite geometric series with initial term a and common ratio b sums to $b - a$. What is the maximum possible value of a?

[4✐] **Problem 96** (HMMT November 2013) Find the largest real number λ such that $a^2 + b^2 + c^2 + d^2 \geq ab + \lambda bc + cd$ for all real numbers a, b, c, d.

[5✎] **Problem 97** (HMMT February 2014) Suppose that x and y are positive real numbers such that $x^2 - xy + 2y^2 = 8$. Find the maximum possible value of $x^2 + xy + 2y^2$.

[5✎] **Problem 98** (BMT 2019) Find the number of ordered integer triplets x, y, z with absolute value less than or equal to 100 such that $2x^2 + 3y^2 + 3z^2 + 2xy + 2xz - 4yz < 5$

§ 4.9 Floors, Ceilings, Fractional Parts

[4✎] **Problem 99** (Czech And Slovak MO) Solve the equation $x \cdot \lfloor x \cdot \lfloor x \cdot \lfloor x \rfloor \rfloor \rfloor = 88$ in the set of real numbers.

§ 4.10 Fake Algebra

[3✎] **Problem 100** (BMT 2019) Find the maximum value of $\frac{x}{y}$ if x and y are real numbers such that $x^2 + y^2 - 8x - 6y + 20 = 0$.

[3✎] **Problem 101** (BMT 2015) Let x and y be real numbers satisfying the equation $x^2 - 4x + y^2 + 3 = 0$. If the maximum and minimum values of $x^2 + y^2$ are M and m respectively, compute the numerical value of $M - m$.

[3✎] **Problem 102** (Magic Math AMC 10 2020) Find the number of ordered pairs of real numbers (x, y) satisfying $x = \frac{2y}{y^2-1}$ and $y = \frac{2x}{x^2-1}$.

[4✎] **Problem 103** (ARML 2014) Compute the area of the region defined by $x^2 + y^2 \leq |x| + |y|$.

[4✎] **Problem 104** (PuMAC 2019) Let x and y be positive real numbers that satisfy $(\log x)^2 + (\log y)^2 = \log x^2 + \log y^2$. Compute the maximum possible value of $(\log xy)^2$.

[5✎] **Problem 105** (PersonPyschopath's 2nd 2015-2016 Mock AMC 10) What is the square root of the value of

$$(\sqrt{85} + \sqrt{205} + \sqrt{218})(-\sqrt{85} + \sqrt{205} + \sqrt{218})(\sqrt{85} - \sqrt{205} + \sqrt{218})(\sqrt{85} + \sqrt{205} - \sqrt{218})$$

[7✎] **Problem 106** (HMMT February 2014) Given that a, b, and c are complex numbers satisfying

$$a^2 + ab + b^2 = 1 + i$$

$$b^2 + bc + c^2 = 2$$

$$c^2 + ca + a^2 = 1,$$

compute $(ab + bc + ca)^2$

§ 4.11 Graphing

[4✎] **Problem 107** (Reun's Thanksgiving 2017 Mock AMC 10) Two functions f and g exist such that

$$f(x) = -\frac{x - 15}{x + 1}$$

and

$$g(x) = -\frac{x+17}{x+1}$$

intersect the circle $x^2 + y^2 = c$ for some constant c at exactly 8 points. What is the least possible integer value of c?

§ 4.12 Miscellaneous

[2✎] **Problem 108** (NEMO 2017) Find all ordered 8-tuples of positive integers (a, b, c, d, e, f, g, h) such that:

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3} + \frac{1}{e^3} + \frac{1}{f^3} + \frac{1}{g^3} + \frac{1}{h^3} = 1$$

[3✎] **Problem 109** (PersonPsychopath's 2nd 2015-2016 Mock AMC 10) A certain post office only uses rectangular prism shaped packages that come in two sizes: small and large. All small packages have a surface area of 200 and a volume of 160. Doubling the height in a small package will result in a large package. If the surface area of each large package is equal to its volume, what is the sum of the length, width, and height of the small package?

[3✎] **Problem 110** (Payback Mock AMC 10) A 20×20 multiplication table has twenty numbers representing each individual row and column, namely $2^0, 2^1, 2^2 \ldots 2^{19}$ A number is formed by multiplying its row- number and column-number. What is the remainder when the sum of all numbers in the multiplication table is divided by 5?

[4✎] **Problem 111** (NEMO 2017) The value of the expression

$$\sqrt{1 + \sqrt{\sqrt[3]{32} - \sqrt[3]{16}}} + \sqrt{1 - \sqrt{\sqrt[3]{32} - \sqrt[3]{16}}}$$

can be written as $\sqrt[m]{n}$, where m and n are positive integers. Compute the smallest possible value of $m + n$.

§ 5 Geometry

§ 5.1 Coordinate Geometry

[1✎] **Problem 1** (BmMT 2014) Find the area of the convex quadrilateral with vertices at the points $(-1, 5), (3, 8), (3, -1)$, and $(-1, -2)$.

[2✎] **Problem 2** (CRMT Individuals 2019) Let S be the set of all distinct points in the coordinate plane that form an acute isosceles triangle with the points $(32, 33)$ and $(63, 63)$. Given that a line L crosses S a finite number of times, find the maximum number of times L can cross S.

[2✎] **Problem 3** (HMMT November 2013) Plot points A, B, C at coordinates (0, 0), (0, 1), and (1, 1) in the plane, respectively. Let S denote the union of the two line segments AB and BC. Let

X1 be the area swept out when Bobby rotates S counterclockwise 45 degrees about point A. Let X2 be the area swept out when Calvin rotates S clockwise 45 degrees about point A. Find $\frac{X_1+X_2}{2}$

[2✎] **Problem 4** (AMC 12B 2017/9) A circle has center $(-10,-4)$ and radius 13. Another circle has center $(3,9)$ and radius $\sqrt{65}$. The line passing through the two points of intersection of the two circles has equation $x+y=c$. What is c?

[3✎] **Problem 5** (Math Prizes For Girls 2019) Two ants sit at the vertex of the parabola $y=x^2$. One starts walking northeast (i.e., upward along the line $y=x$) and the other starts walking northwest (i.e., upward along the line $y=-x$). Each time they reach the parabola again, they swap directions and continue walking. Both ants walk at the same speed. When the ants meet for the eleventh time (including the time at the origin), their paths will enclose 10 squares. What is the total area of these squares?

[4✎] **Problem 6** (AMC 12A 2018/22) The solutions to the equations $z^2=4+4\sqrt{15}i$ and $z^2=2+2\sqrt{3}i$, where $i=\sqrt{-1}$, form the vertices of a parallelogram in the complex plane. The area of this parallelogram can be written in the form $p\sqrt{q}-r\sqrt{s}$, where p, q, r, and s are positive integers and neither q nor s is divisible by the square of any prime number. What is $p+q+r+s$?

[5✎] **Problem 7** (BMT 2019) A regular hexagon has positive integer side length. A laser is emitted from one of the hexagon's corners, and is reflected off the edges of the hexagon until it hits another corner. Let a be the distance that the laser travels. What is the smallest possible value of a^2 such that $a>2019$? You need not simplify/compute exponents.

[5✎] **Problem 8** (SLKK AIME 2020) Mr. Duck draws points $A=(a,0), B=(0,b), C=(3,5)$ and $O=(0,0)$ such that $a,b>0$ and $\angle ACB=45°$. If the maximum possible area of $\triangle AOB$ can be expressed as $m-n\sqrt{p}$ where m,n, and p are positive integers such that p is not divisible by the square of a prime, find $m+n+p$

§ 5.2 3D Geometry

[1✎] **Problem 9** (AMC 12A 2009/5) One dimension of a cube is increased by 1, another is decreased by 1, and the third is left unchanged. The volume of the new rectangular solid is 5 less than that of the cube. What was the volume of the cube?

[1✎] **Problem 10** (AMC 12A 2008/8) What is the volume of a cube whose surface area is twice that of a cube with volume 1?

[1✎] **Problem 11** (Reun's Thanksgiving 2017 Mock AMC 10) A silo is a three-dimensional structure, made up of a cylinder and a cone with equal radii of 3, that stores fodder. The height of the conic part of the silo is 3 units. When the entire silo is half full of fodder, the cylindrical part of the silo is $\frac{2}{3}$ full of fodder. What is the volume of the silo?

[1✎] **Problem 12** (BMT 2018) A cube has side length 5. Let S be its surface area and V its volume. Find $\frac{S^3}{V^2}$.

[2✎] **Problem 13** (AMC 12B 2008/18) On a sphere with a radius of 2 units, the points A and B are 2 units away from each other. Compute the distance from the center of the sphere to the line segment AB.

[2✎] **Problem 14** (AMC 12B 2005/16) Eight spheres of radius 1, one per octant, are each tangent to the coordinate planes. What is the radius of the smallest sphere, centered at the origin, that contains these eight spheres?

[3✎] **Problem 15** (AIME 1984/9) In tetrahedron $ABCD$, edge AB has length 3 cm. The area of face ABC is 15cm^2 and the area of face ABD is 12 cm^2. These two faces meet each other at a $30°$ angle. Find the volume of the tetrahedron in cm^3.

[3✎] **Problem 16** (Scrabber94 Mock AMC 10 2020) A right triangular prism has all edges of length 2. An ant crawls on the exterior of the prism from point A to point B, where B is the midpoint of the edge opposite A as shown. What is the shortest possible distance the ant crawls?

[3✎] **Problem 17** (HMMT February 2014) Let C be a circle in the xy plane with radius 1 and center $(0, 0, 0)$, and let P be a point in space with coordinates $(3, 4, 8)$. Find the largest possible radius of a sphere that is contained entirely in the slanted cone with base C and vertex P.

[3✎] **Problem 18** (AMC 12B 2017/14) An ice-cream novelty item consists of a cup in the shape of a 4-inch-tall frustum of a right circular cone, with a 2-inch-diameter base at the bottom and a 4-inch-diameter base at the top, packed solid with ice cream, together with a solid cone of ice cream of height 4 inches, whose base, at the bottom, is the top base of the frustum. What is the total volume of the ice cream, in cubic inches?

[4✎] **Problem 19** (ARML 2009) A cylinder with radius r and height h has volume 1 and total surface area 12. Compute $\frac{1}{r} + \frac{1}{h}$

[6✎] **Problem 20** (BMT 2018) A rectangular prism has three distinct faces of area 24, 30, and 32. The diagonals of each distinct face of the prism form sides of a triangle. What is the triangle's area?

§ 5.3 Inequalties

[1✎] **Problem 21** (AlcumusGuy Mock AMC 10 2016-2017) Triangle AMC has area 2017 with $\angle M = 90°$. What is the closest integer to the minimum possible length of AC?

[2✎] **Problem 22** (Math Prizes For Girls) A paper equilateral triangle with area 2019 is folded over a line parallel to one of its sides. What is the greatest possible area of the overlap of folded and unfolded parts of the triangle?

[3✎] **Problem 23** (BMT 2018) If A is the area of a triangle with perimeter 1, what is the largest possible value of A^2?

[4✎] **Problem 24** (PersonPsychopath's 2nd 2015-2016 Mock AMC 10) Two unit circles are externally tangent to each other at point P. A pair of perpendicular lines is drawn such that each line is tangent to a different circle. Let the intersection of these lines be I. What is the minimum possible length of PI?

[4✎] **Problem 25** (MMATHS 2018) A, B, C, D all lie on a circle with $AB = BC = CD$. If the distance between any two of these points is a positive integer, what is the smallest possible perimeter of quadrilateral $ABCD$

§ 5.4 Transformations

[3✎] **Problem 26** (HMMT February 2014) In quadrilateral ABCD, $\angle DAC = 98, \angle DBC = 82, \angle BCD = 70$, and $BC = AD$. Find $\angle ACD$.

[6✎] **Problem 27** (LMT Spring 2020) Let ABC be a triangle such that such that $AB = 14, BC = 13$, and $AC = 15$. Let X be a point inside triangle ABC. Compute the minimum possible value of $(\sqrt{2}AX + BX + CX)^2$.

[8✎] **Problem 28** (SLKK AIME 2020) Squares $ABCD$ and $DEFG$ are drawn in the plane with both sets of vertices A, B, C, D and D, E, F, G labeled counterclockwise. Let P be the intersection of lines AE and CG. If $DA = 35, DG = 20$, and $BF = 25\sqrt{2}$, find DP^2.

§ 5.5 Trigonometry

[3✎] **Problem 29** (AMC 12B 2017/15) Let ABC be an equilateral triangle. Extend side $\overline{AB}$ beyond B to a point B' so that $BB' = 3 \cdot AB$. Similarly, extend side $\overline{BC}$ beyond C to a point C' so that $CC' = 3 \cdot BC$, and extend side $\overline{CA}$ beyond A to a point A' so that $AA' = 3 \cdot CA$. What is the ratio of the area of $\triangle A'B'C'$ to the area of $\triangle ABC$?

[3✎] **Problem 30** (ARML 2014) In triangle $ABC, a = 12, b = 17$, and $c = 13$. Compute $bcosC - ccosB$.

[4✎] **Problem 31** (AMC 12A 2018/20) Triangle ABC is an isosceles right triangle with $AB = AC = 3$. Let M be the midpoint of hypotenuse $\overline{BC}$. Points I and E lie on sides $\overline{AC}$ and $\overline{AB}$, respectively, so that $AI > AE$ and $AIME$ is a cyclic quadrilateral. Given that triangle EMI has area 2, the length CI can be written as $\frac{a-\sqrt{b}}{c}$, where a, b, and c are positive integers and b is not divisible by the square of any prime. What is the value of $a + b + c$?

[4✎] **Problem 32** (ARML 2014) In triangle $\triangle ABC$, $BC = 2$. Point D is on AC such that $AD = 1$ and $CD = 2$. If $m\angle BDC = 2m\angle A$, compute $\sin A$.

[5✎] **Problem 33** (AMC 12A 2018/23) In $\triangle PAT$, $\angle P = 36°$, $\angle A = 56°$, and $PA = 10$. Points U and G lie on sides $\overline{TP}$ and $\overline{TA}$, respectively, so that $PU = AG = 1$. Let M and N be the midpoints of segments $\overline{PA}$ and $\overline{UG}$, respectively. What is the degree measure of the acute angle formed by lines MN and PA?

§ 5.6 General

[1✎] **Problem 34** (MAθ 2018) A parallelograms has diagonals of length 10 and 20. Find the area inclosed by the circle inscribed in the parallelogram.

[1✎] **Problem 35** (TAMU 2019) An acute isosceles triangle ABC is inscribed in a circle. Through B and C, tangents to the circle are drawn, meeting at D. If $\angle ABC = 2\angle CDB$, then find the radian measure of $\angle BAC$.

[1✎] **Problem 36** (PHS PuMAC TST 2017) In triangle ABC, let D and E be the midpoints of BC and AC. Suppose AD and BE meet at F. If the area of $\triangle DEF$ is 50, then what is the area of $\triangle CDE$?

[1✎] **Problem 37** (PersonPsychopath's 2nd Mock AMC 10) In convex pentagon $ABCDE, AB = 40, BC = 20, CD = 5$, and $DE = 37$. If $\angle A, \angle B$, and $\angle C$ are all right angles, what is the length of EA?

[1✎] **Problem 38** (AMC 10A 2019/13) Let $\triangle ABC$ be an isosceles triangle with $BC = AC$ and $\angle ACB = 40°$. Construct the circle with diameter $\overline{BC}$, and let D and E be the other intersection points of the circle with the sides $\overline{AC}$ and $\overline{AB}$, respectively. Let F be the intersection of the diagonals of the quadrilateral $BCDE$. What is the degree measure of $\angle BFC$?

[2✎] **Problem 39** (AMC 10B 2011/17) In the given circle, the diameter $\overline{EB}$ is parallel to $\overline{DC}$, and $\overline{AB}$ is parallel to $\overline{ED}$. The angles AEB and ABE are in the ratio $4 : 5$. What is the degree measure of angle BCD?

[2✎] **Problem 40** (Brazil 2007) Let ABC be a triangle with circumcenter O. Let P be the intersection of straight lines BO and AC and ω be the circumcircle of triangle AOP. Suppose that $BO = AP$ and that the measure of the arc OP in ω, that does not contain A, is $40°$. Determine the measure of the angle $\angle OBC$.

[2✎] **Problem 41** (BMT 2018) A 1 by 1 square ABCD is inscribed in the circle m. Circle n has radius 1 and is centered around A. Let S be the set of points inside of m but outside of n. What is the area of S?

[2✎] **Problem 42** (AMC 10B 2011/18) Rectangle $ABCD$ has $AB = 6$ and $BC = 3$. Point M is chosen on side AB so that $\angle AMD = \angle CMD$. What is the degree measure of $\angle AMD$?

[2✎] **Problem 43** (TAMU 2019) Let AA_1 be an altitude of triangle $\triangle ABC$, and let A_2 be the midpoint of the side BC. Suppose that AA_1 and AA_2 divide angle $\angle BAC$ into three equal angles. Find the product of the angles of $\triangle ABC$ when the angles are expressed in degrees.

[2✎] **Problem 44** (PHS PuMAC TST 2017) A trapezoid has area 32, and the sum of the lengths of its two bases and altitude is 16. If one of the diagonals is perpendicular to both bases, then what is the length of the other diagonal?

[2✎] **Problem 45** (Autumn Mock AMC 10) Equilateral triangle ABC has side length 6. Points D, E, F lie within the lines AB, BC and AC such that $BD = 2AD$, $BE = 2CE$, and $AF = 2CF$. Let N be the numerical value of the area of triangle DEF. Find N^2.

[2✎] **Problem 46** (OMO Spring 2019) In triangle ABC, side AB has length 10, and the A- and B-medians have length 9 and 12, respectively. Compute the area of the triangle.

[2✎] **Problem 47** (LMT Spring 2020) Three mutually externally tangent circles are internally tangent to a circle with radius 1. If two of the inner circles have radius $\frac{1}{3}$, the largest possible radius of the third inner circle can be expressed in the form $\frac{a+b\sqrt{c}}{d}$ where c is squarefree and $\gcd(a, b, d) = 1$. Find $a + b + c + d$.

[2✎] **Problem 48** (2018 AMC 12A/17) Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square S so that from the air it looks like the right angle symbol. The rest of the field is planted. The shortest distance from S to the hypotenuse is 2 units. What fraction of the field is planted?

[2] **Problem 49** (2018 AMC 12A/18) Triangle ABC with $AB = 50$ and $AC = 10$ has area 120. Let D be the midpoint of $\overline{AB}$, and let E be the midpoint of $\overline{AC}$. The angle bisector of $\angle BAC$ intersects $\overline{DE}$ and $\overline{BC}$ at F and G, respectively. What is the area of quadrilateral $FDBG$?

[3] **Problem 50** (AHSME 1984/28) Triangle ABC has area 10. Points D, E, and F, all distinct from A, B, and C, are on sides AB, BC, and CA, respectively, and $AD = 2, DB = 3$. Triangle ABE and quadrilateral $DBEF$ have equal areas s. Find s.

[3] **Problem 51** (FARML 2012/6) In triangle ABC, $AB = 7$, $AC = 8$, and $BC = 10$. D is on AC and E is on BC such that $\angle AEC = \angle BED = \angle B + \angle C$. Compute the length AD.

[3] **Problem 52** (HMMT November 2013) Let ABC be an isosceles triangle with $AB = AC$. Let D and E be the midpoints of segments AB and AC, respectively. Suppose that there exists a point F on ray $\overrightarrow{DE}$ outside of ABC such that triangle BFA is similar to triangle ABC. Compute $\frac{AB}{BC}$.

[3] **Problem 53** (Magic Math AMC 10 2020) Two concentric circles have radii $\sqrt{2}$ and 2. Three points A, B, C are chosen on the larger circle such that $\triangle ABC$ is equilateral. Find the area of the region in the smaller circle that is also inside the triangle.

[3] **Problem 54** (CNCM Online Round 2) There is a rectangle $ABCD$ such that $AB = 12$ and $BC = 7$. E and F lie on sides AB and CD respectively such that $\frac{AE}{EB} = 1$ and $\frac{CF}{FD} = \frac{1}{2}$. Call X the intersection of AF and DE. What is the area of pentagon $BCFXE$?

[3] **Problem 55** (LMT Spring 2020) Let $\triangle ABC$ be a triangle such that $AB = 6, BC = 8$, and $AC = 10$. Let M be the midpoint of BC. Circle ω passes through A and is tangent to BC at M. Suppose ω intersects segments AB and AC again at points X and Y, respectively. If the area of $\triangle AXY$ can be expressed as $\frac{p}{q}$ where p, q are relatively prime integers, compute $p + q$.

[3] **Problem 56** (AMC 12B 2017/18) The diameter AB of a circle of radius 2 is extended to a point D outside the circle so that $BD = 3$. Point E is chosen so that $ED = 5$ and line ED is perpendicular to line AD. Segment AE intersects the circle at a point C between A and E. What is the area of $\triangle ABC$?

[3] **Problem 57** (MMATHS 2018) In rectangle $ABCD$, let E lie on CD, and let F be the intersection of AC and BE. If the area of $\triangle ABF$ is 45 and the area of $\triangle CEF$ is 20, find the area of the quadrilateral $ADEF$.

[3] **Problem 58** (PHS ARML TST 2017) An algorithm starts with an equilateral triangle $A_0B_0C_0$ of side length 1. At step k, points A_k, B_k,and C_k are chosen on line segments $B_{k-1}C_{k-1}, C_{k-1}A_{k-1}$ and $A_{k-1}B_{k-1}$ respectively, such that

$$B_{k-1}A_k : A_kC_{k-1} = 1 : 1$$

$$C_{k-1}B_k : B_kA_{k-1} = 1 : 2$$

$$A_{k-1}C_k : C_kB_{k-1} = 1 : 3$$

What is the value of the infinite series:

$$\sum_{i=0}^{\infty} \text{Area}[\triangle A_kB_kC_k]$$

[3✐] **Problem 59** (PersonPsychopath's 2nd 2015-2016 Mock AMC 10) In $\triangle ABC$, $\angle A = 90°$. Points X and Y lie on AB and AC respectively such that $\frac{AX}{AB} = \frac{1}{2}$ and $\frac{AY}{AC} = \frac{1}{3}$. BY and CX intersect at I. What is the ratio of the area of $AXIY$ to $\triangle ABC$?

[4✐] **Problem 60** (Mandelbrot Nationals 2009) Triangle ABC has sides of length $AB = \sqrt{41}, AC =$ 5,and $BC = 8$. Let O be the center of the circumcircle of $\triangle ABC$, and let A' be the point diametrically opposite A, as shown. Determine the area of $\triangle A'BC$.

[4✐] **Problem 61** (DMC Mock AMC 10) In trapezoid $ABCD$ with $\overline{AD}||\overline{BC}$ and side lengths $AD = 18$, $BC = 20$, and $AB = CD = 8$, let X be the intersection of line AB and the bisector of $\angle ADC$, and let Y be the intersection of line CD and the bisector of $\angle DAB$. What is XY ?

[4✐] **Problem 62** (HMMT February 2014) Triangle ABC has sides $AB = 14, BC = 13$, and $CA = 15$. It is inscribed in circle, which has center O. Let M be the midpoint of AB, let B' be the point on diametrically opposite B, and let X be the intersection of AO and MB'. Find the length of AX.

[4✐] **Problem 63** (AIME 1989) Triangle ABC has an right angle at B and contains a point P such that $AP = 10$, $BP = 6$, and $\angle APC = \angle CPB = \angle BPA$. Find CP.

[4✐] **Problem 64** (106 Geometry Problems) In triangle ABC, medians BB_1 and CC_1 are perpendicular. Given that $AC = 19$ and $AB = 22$, find BC.

[4✐] **Problem 65** (AIME 2005) In quadrilateral $ABCD$, let $BC = 8, CD = 12, AD = 10$ and $\angle A = \angle B = 60°$. Given that $AB = p + \sqrt{q}$, where p and q are positive integers, find $p + q$.

[4✐] **Problem 66** (HMMT November 2013) Let ABC be a triangle and D a point on BC such that $AB = \sqrt{2}, BC = \sqrt{3}, \angle BAD = 30°$, and$\angle CAD = 45°$. Find AD.

[4✐] **Problem 67** (CMIMC 2020) Let ABC be a triangle with centroid G and $BC = 3$. If ABC is similar to GAB, compute the area of ABC.

[4✐] **Problem 68** (Magic Math AMC 10 2020) Isosceles triangle $\triangle ABC$ has $AB = AC$ and circumcenter O. The circle with diameter AO intersects segment BC at P and Q, with P closer to B, such that $BP = QC = 4$ and $PQ = 6$. Compute the area of $\triangle A$

[5✐] **Problem 69** (PHS HMMT TST 2020) $\triangle$ ABC has side lengths $AB = 11, BC = 13, CA = 20$. A circle is drawn with diameter AC. Line AB intersects the circle at $D \neq A$, and line BC intersects the circle at $E \neq B$. Find the length of DE.

[5✐] **Problem 70** (NEMO 2017) In convex equilateral hexagon $ABCDEF$, $AC = 13$, $CE = 14$, and $EA = 15$. It is given that the area of $ABCDEF$ is twice the area of triangle ACE. Compute AB.

[5✐] **Problem 71** (CMIMC 2020) Let ABC be a triangle with centroid G and $BC = 3$. If ABC is similar to GAB, compute the area of ABC.

[5✐] **Problem 72** (ARML Local 2020) Triangle ABC has side lengths $AB = 8, BC = 5$, and $AC = 7$. Point P lies inside $\triangle ABC$ so that $\angle APB = \angle BPC = \angle CPA = 120°$. Compute $AP + BP + CP$.

[5✐] **Problem 73** (Scrabber94 Mock AMC 10 2020) Parallelogram $ABCD$ has $AB = CD = 10$, $BC = AD = 6$, and $BD = 8$. Let O_1, O_2, O_3, and O_4 be the circumcenters of $\triangle ABC, \triangle ABD, \triangle ACD$, and $\triangle BCD$, respectively. What is the area of quadrilateral $O_1O_2O_3O_4$?

[5✎] **Problem 74** (Magic Math AMC 10 2020) Six points A, B, C, D, E, F are selected on a circle with center O such that $ABCDEF$ is an equiangular hexagon with $AB = CD = EF < BC = DE = FA$. Diagonals AD, BE, FC are drawn, intersecting at points X, Y, Z. The three circles passing through O and two of X, Y, Z are each internally tangent to O. Find BC/AB.

[6✎] **Problem 75** (AMC 12B 2017/24) Quadrilateral $ABCD$ has right angles at B and C, $\triangle ABC \sim \triangle BCD$, and $AB > BC$. There is a point E in the interior of $ABCD$ such that $\triangle ABC \sim \triangle CEB$ and the area of $\triangle AED$ is 17 times the area of $\triangle CEB$. What is $\frac{AB}{BC}$?

[6✎] **Problem 76** (SLKK AIME 2020) Cyclic quadrilateral $AXBY$ is inscribed in circle ω such that AB is a diameter of ω. M is the midpoint of XY and $AM = 13, BM = 5$, and $AB = 16$. If the area of AXBY can be expressed as $m\sqrt{p} + n$, where m, n, and p are positive integers such that m and n are relatively prime and p is not divisible by the square of a prime, find the remainder when $m + n + p$ is divided by 1000.

§ 6 Misc

These are problems that don't really fall into any other category at all.

§ 6.1 General

[1✎] **Problem 1** (Math Prizes For Girls 2019) Let $a_1, a_2, \ldots, a_{2019}$ be a sequence of real numbers. For every five indices i, j, k, and m from 1 through 2019, at least two of the numbers a_i, a_j, a_k and a_m have the same absolute value. What is the greatest possible number of distinct real numbers in the given sequence?

[1✎] **Problem 2** (AlcumusGuy Mock AMC 10 2016-2017) How many of the following four statements are true?

1. It is possible to find an even number of even integers such that the sum of their reciprocals is 1.
2. It is possible to find an even number of odd integers such that the sum of their reciprocals is 1.
3. It is possible to find an odd number of even integers such that the sum of their reciprocals is 1.
4. It is possible to find an odd number of odd integers such that the sum of their reciprocals is 1.

[2✎] **Problem 3** (ARML 2014) The sequence of words $\{a_n\}$ is defined as follows: $a_1 = X, a_2 = O$, and for $n \geq 3$, an is a_{n-1} followed by the reverse of a_{n-2}. For example, $a_3 = OX, a_4 = OXO, a_5 = OXOXO$, and $a_6 = OXOXOOXO$. Compute the number of palindromes in the first 1000 terms of this sequence.

[3✎] **Problem 4** (MMATHS 2019) S is a set of positive integers with the following properties:

1. There are exactly 3 positive integers missing from S.
2. If a and b are elements of S, then a + b is an element of S. (We allow a and b to be the same.)

How many possibilities are there for the set S?

§ 6.2 Games

[2✎] **Problem 5** (2018 Memorial Day Mock AMC 10) Ashley and Ben are playing a game. At the beginning of the game, $n = 1$. Starting with Ashley, the players then take turns multiplying n by an integer between 2 and 9, inclusive. The first player to get a product of at least 100 wins. If neither player makes any mistakes, how many different integers can Ashley multiply n by on her first turn in order to win?

[2✎] **Problem 6** (BmMT 2016) Suppose you have a 20×16 bar of chocolate squares. You want to break the bar into smaller chunks, so that after some sequence of breaks, no piece has an area of more than 5. What is the minimum possible number of times that you must break the bar?

[3✎] **Problem 7** (BMT 2015) Two players play a game with a pile with N coins is on a table. On a player's turn, if there are n coins, the player can take at most $\frac{n}{2} + 1$ coins, and must take at least one coin. The player who grabs the last coin wins. For how many values of N between 1 and 100 (inclusive) does the first player have a winning strategy?

§ 6.3 Logic

[2✎] **Problem 8** (BmMT 2014) Alice, Bob, Carl, and Dave are either lying or telling the truth. If the four of them make the following statements, who has the coin?

Alice: I have the coin.
Bob: Carl has the coin.
Carl: Exactly one of us is telling the truth.
Dave: The person who has the coin is male.

[2✎] **Problem 9** (Berkeley Math Circle 2013) Ten people sit side by side at a long table, all facing the same direction. Each of them is either a knight (and always tells the truth) or a knave (and always lies). Each of the people announces: "There are more knaves on my left than knights on my right." How many knaves are in the line?

[2✎] **Problem 10** (Scrabbler94 Mock AMC 10 2020) If today is rainy, then Erin wears a raincoat. If today is rainy and Erin is wearing a raincoat, then Erin will not get wet outside. Which of the following statement(s) is logically implied from the above two statements?
I: If today is not rainy, then Erin is not wet outside.
II: If Erin is not wet outside, then today is not rainy.
III: If Erin is wet outside, then today is not rainy.
IV: If Erin is wet outside, then Erin is not wearing a raincoat.